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On Nonoscillatory Solutions of Two-Dimensional Nonlinear Dynamical Systems

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Abstract

During the past years, there has been an increasing interest in studying oscillation and nonoscillation criteria for dynamical systems on time scales that harmonize the oscillation and nonoscillation theory for the continuous and discrete cases in order to combine them in one comprehensive theory and eliminate obscurity from both. We not only classify nonoscillatory solutions of two-dimensional systems of first-order dynamic equations on time scales but also guarantee the existence of such solutions using the Knaster, Schauder-Tychonoff and Schauder's fixed point theorems. The approach is based on the sign of components of nonoscillatory solutions. A short introduction to the time scale calculus is given as well. Examples are significant in order to see if nonoscillatory solutions exist or not. Therefore, we give several examples in order to highlight our main results for the set of real numbers $\mathbb{R}$, the set of integers $\mathbb{Z}$ and $q^{\mathbb{N}_0} = \{1, q, q^2, q^3, \ldots\}$, $q > 1$, which are the most well-known time scales.

Keywords: dynamical systems, dynamic equations, differential equations, difference equations, time scales, oscillation

1. Introduction

In this chapter, we investigate the existence and classification of nonoscillatory solutions of two-dimensional (2D) nonlinear time-scale systems of first-order dynamic equations. The method we follow is based on the sign of components of nonoscillatory solutions and the most well-known fixed point theorems. The motivation of studying dynamic equations on time scales is to unify continuous and discrete analysis and harmonize them in one comprehensive theory and eliminate obscurity from both. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The most well-known examples for time scales are $\mathbb{R}$ (which leads to

differential equations, see [1]), $\mathbb{Z}$ (which leads to difference equations, see Refs. [2, 3]) and $q^{\mathbb{N}_0} := \{1, q, q^2, \cdots\}$, $q > 1$ (which leads to q-difference equations, see Ref. [4]). In 1988, the theory of time scales was initiated by Stefan Hilger in his Ph.D. thesis [5]. We assume that most readers are not familiar with the calculus of time scales and therefore we give a brief introduction to time scales calculus in Section 2. In fact, we refer readers books [6, 7] by Bohner and Peterson for more details.

The study of 2D dynamic systems in nature and society has been motivated by their applications. Especially, a system of delay dynamic equations, considered in Section 4, take a lot of attention in all areas such as population dynamics, predator-prey epidemics, genomic and neuron dynamics and epidemiology in biological sciences, see [8, 9]. For instance, when the birth rate of preys is affected by the previous values rather than current values, a system of delay dynamic equations is utilized, because the rate of change at any time depends on solutions at prior times. Another novel application of delay dynamical systems is time delays that often arise in feedback loops involving actuators. A major issue faced in engineering is an unavoidable time delay between measurement and the signal received by the controller. In fact, the delay should be taken into consideration at the design stage to avoid the risk of instability, see Refs. [10, 11].

Another special case of 2D systems of dynamic equations is the Emden-Fowler type, which is covered in Section 5 of this chapter. The equation has several interesting applications, such as in astrophysics, gas dynamics and fluid mechanics, relativistic mechanics, nuclear physics and chemically reacting systems, see Refs. [12–15]. For example, the fundamental problem in studying the stellar structure for gaseous dynamics in astrophysics was to look into the equilibrium formation of the mass of spherical clouds of gas for the continuous case, proposed by Kelvin and Lane, see Refs. [16, 17]. Such an equation is called Lane-Emden equation in literature. Much information about the solutions of Lane-Emden equation was provided by Ritter, see Ref. [18], in a series of 18 papers, published during 1878–1889. The mathematical foundation for the study of such an equation was made by Fowler in a series of four papers during 1914–1931, see Refs. [19–22].

2. Preliminaries

The set of real numbers $\mathbb{R}$, the set of integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the nonnegative integers $\mathbb{N}_0$ and the Cantor set, $q^{\mathbb{N}_0}$, $q > 1$ and $[0, 1] \cup [2, 3]$ are some examples of time scales. However, the set of rational numbers $\mathbb{Q}$, the set of irrational numbers $\mathbb{R} \backslash \mathbb{Q}$, the complex numbers $\mathbb{C}$, and the open interval $(0, 1)$ are not considered as time scales.

Definition 2.1. [6, Definition 1.1] Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is given by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \quad s > t\} \quad \text{for all} \quad t \in \mathbb{T}$$

whereas the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : \quad s < t\} \quad \text{for all} \quad t \in \mathbb{T}.$$

Finally, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is given by $\mu(t) := \sigma(t) - t$ for all $t \in \mathbb{T}$.

We define $\inf \varnothing = \sup \mathbb{T}$. If $\sigma(t) > t$, then t is called *right-scattered*, whereas if $\rho(t) < t$, t is called *left-scattered*. If t is right- and left-scattered at the same time, then we say that t is *isolated*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, we say that t is *left-dense*. Also, if t is right- and left-dense at the same time, then we say that t is *dense*.

Table 1 shows some examples of the forward and backward jump operators and the graininess function for most known time scales.

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$\mathbb{Z}$	$t+1$	$t-1$	1
$q^{\mathbb{N}_0}$	tq	$\dfrac{t}{q}$	$(q-1)t$

Table 1. Examples of most known time scales.

If $\sup \mathbb{T} < \infty$, then $\mathbb{T}^\kappa = \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$ and $\mathbb{T}^\kappa = \mathbb{T}$ if $\sup \mathbb{T} = \infty$. Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a function. Then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 2.2. [6, Definition 1.10] For any ε, if there exists a $\delta > 0$ such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in (t - \delta, t + \delta) \cap \mathbb{T},$$

then f is called *delta (or Hilger) differentiable* on $\mathbb{T}^\kappa$ and f^Δ is called *delta derivative* of f.

Theorem 2.3 [6, Theorem 1.16] *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a function with* $t \in \mathbb{T}^\kappa$. *Then*

a. *If f is differentiable at t, f is continuous at t.*

b. *If f is continuous at t and t is right-scattered, then f is differentiable at t and*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

c. *If t is right dense, then f is differentiable at t if and only if*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number.

d. *If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$.*

If $\mathbb{T} = \mathbb{R}$, then f^Δ turns out to be the usual derivative f' while f^Δ is reduced to forward difference operator Δf if $\mathbb{T} = \mathbb{Z}$. Finally, if $\mathbb{T} = q^{\mathbb{N}_0}$, then the delta derivative turns out to be

q-difference operator Δ_q. The following theorem presents the sum, product and quotient rules on time scales.

Theorem 2.4 [6, Theorem 1.20] *Let $f, g : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^\kappa$. Then*

a. *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

b. *If $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t, then*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

c. *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

The following concepts must be introduced in order to define delta-integrable functions.

Definition 2.5. [6, Definition 1.58] $f : \mathbb{T} \to \mathbb{R}$ is called *right dense continuous* (rd-continuous), denoted by $C_{rd}, C_{rd}(\mathbb{T})$, or $C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at right dense points in $\mathbb{T}$ and its left-sided limits exist as a finite number at left dense points in $\mathbb{T}$. We denote continuous functions by C throughout this chapter.

Theorem 2.6 [6, Theorem 1.60] *Let $f : \mathbb{T} \to \mathbb{R}$.*

a. *If f is continuous, then f is rd-continuous.*

b. *The jump operator σ is rd-continuous.*

Also, the Cauchy integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all} \quad a, b \in \mathbb{T}.$$

The following theorem presents the existence of antiderivatives.

Theorem 2.7 [6, Theorem 1.74] *Every rd-continuous function has an antiderivative. Moreover, F given by*

$$F(t) = \int_{t_0}^t f(s)\Delta s \quad for \quad t \in \mathbb{T}$$

is an antiderivative of f.

Theorem 2.8 [6, Theorems 1.76–1.77] *Let $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f, g \in C_{rd}$. Then we have:*

1. *If $f^\Delta \geq 0$, then f is nondecreasing.*

2. *If $f(t) \geq 0$ for all $a \leq t \leq b$, then $\int_a^b f(t)\Delta t \geq 0$.*

3. $\displaystyle\int_a^b [(\alpha f(t)) + (\alpha g(t))]\Delta t = \alpha \int_a^b f(t)\Delta t + \alpha \int_a^b g(t)\Delta t.$

4. $\displaystyle\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t.$

5. $\displaystyle\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$

6. $\displaystyle\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$

7. $\displaystyle\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t$

8. $\displaystyle\int_a^a f(t)\Delta t = 0.$

Table 2 shows the derivative and integral definitions for the most known time scales for $a, b \in \mathbb{T}$.

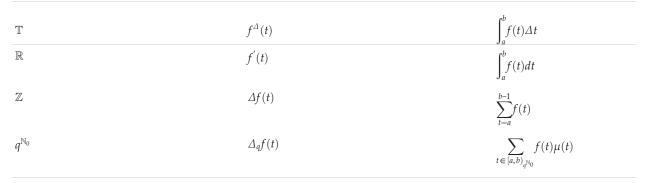| $\mathbb{T}$	$f^\Delta(t)$	$\displaystyle\int_a^b f(t)\Delta t$
$\mathbb{R}$	$f'(t)$	$\displaystyle\int_a^b f(t)dt$
$\mathbb{Z}$	$\Delta f(t)$	$\displaystyle\sum_{t=a}^{b-1} f(t)$
$q^{\mathbb{N}_0}$	$\Delta_q f(t)$	$\displaystyle\sum_{t \in [a,b)_{q^{\mathbb{N}_0}}} f(t)\mu(t)$

Table 2. Derivatives and integrals for most common time scales.

Finally, we finish the section by the following fixed point theorems.

Theorem 2.9 (Schauder's Fixed Point Theorem) [23, Theorem 2.A] *Let S be a nonempty, closed, bounded, convex subset of a Banach space X and suppose that $T : S \to S$ is a compact operator. Then, T has a fixed point.*

The Schauder fixed point theorem was proved by Juliusz Schauder in 1930. In 1934, Tychonoff proved the same theorem for the case when S is a compact convex subset of a locally convex space X. In the literature, this version is known as the Schauder-Tychonoff fixed point theorem, see Ref. [24].

Theorem 2.10 (Schauder-Tychonoff Fixed Point Theorem). *Let S be a compact convex subset of a locally convex (linear topological) space X and T a continuous map of S into itself. Then, T has a fixed point.*

Finally, we provide the Knaster fixed point theorem, see Ref. [25].

Theorem 2.11 (Knaster Fixed Point Theorem) *If $(S, \leq)$ is a complete lattice and $T : S \to S$ is order-preserving (also called monotone or isotone), then T has a fixed point. In fact, the set of fixed points of T is a complete lattice.*

3. Dynamical Systems on Time Scales

In this section, we consider the following system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(t)), \end{cases} \tag{1}$$

where $f, g \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing such that $uf(u) > 0$, $ug(u) > 0$ for $u \neq 0$ and $a, b \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$. The main results in this section come from Ref. [26]. If $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, Eq. (1) turns out to be a system of first-order differential equations and difference equations, see Refs. [27] and [28], respectively. Recent advances in oscillation and nonoscillation criteria for two-dimensional time scale systems have been studied in Refs. [29–31].

Throughout this chapter, we assume that $\mathbb{T}$ is unbounded above. Whenever we write $t \geq t_1$, we mean $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$. We call (x, y) a *proper solution* if it is defined on $[t_0, \infty)_{\mathbb{T}}$ and $\sup\{|x(s)|, |y(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$ for $t \geq t_0$. A solution (x, y) of Eq. (1) is said to be *nonoscillatory* if the component functions x and y are both nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise, it is said to be *oscillatory*. The definitions above are also valid for systems considered in the next sections.

Assume that (x, y) is a nonoscillatory solution of system (1) such that x oscillates but y is eventually positive. Then the first equation of system (1) yields $x^{\Delta}(t) = a(t)f(y(t)) > 0$ eventually one sign for all large $t \geq t_0$, a contradiction. The case where y is eventually negative is similar. Therefore, we have that the component functions x and y are themselves nonoscillatory. In other words, any nonoscillatory solution (x, y) of system (1) belongs to one of the following classes:

$$M^+ := \{(x, y) \in M : \ xy > 0 \quad \text{eventually }\}$$

$$M^- := \{(x, y) \in M : \ xy < 0 \quad \text{eventually }\},$$

where M is the set of all nonoscillatory solutions of system (1).

In this section, we only focus on the existence of nonoscillatory solutions of system (1) in M^-, whereas M^+ is considered together with delay system (12) in the following section.

For convenience, let us set

$$Y(t) = \int_t^{\infty} a(s)\Delta s \quad \text{and} \quad Z(t) = \int_t^{\infty} b(s)\Delta s. \tag{2}$$

We begin with the following results playing an important role in this chapter.

Lemma 3.1 *Let (x, y) be a nonoscillatory solution of system (1) and $t_0 \in \mathbb{T}$. Then we have the followings:*

a. *[29, Lemma 2.3] If $Y(t_0) < \infty$ and $Z(t_0) < \infty$, then system (1) is nonoscillatory.*

b. [29, Lemma 2.2] *If $Y(t_0) = \infty$ and $Z(t_0) = \infty$, then system (1) is oscillatory.*

c. *If $Y(t_0) < \infty$ and $Z(t_0) = \infty$, then $M^+ = \emptyset$.*

d. *If $Y(t_0) = \infty$ and $Z(t_0) < \infty$, then $M^- = \emptyset$.*

e. *Let $Y(t_0) < \infty$. Then x has a finite limit.*

f. *If $Y(t_0) = \infty$ or $Z(t_0) < \infty$, then y has a finite limit.*

Proof. Here, we only prove (a), (c) and (e) and the reader is asked to finish the proof in Exercise 3.2. To prove (a), choose $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\int_{t_1}^{\infty} a(t) f \left(1 + g(2) \int_t^{\infty} b(s) \Delta s \right) \Delta t < 1.$$

Let X be the space of all continuous functions on $\mathbb{T}$ with the norm $\|x\| = \sup\limits_{t \in [t_1, \infty)_{\mathbb{T}}} |x(t)|$ and with the usual point-wise ordering $\leq$. Define a subset Ω of X as

$$\Omega := \{x \in X : \quad 1 \leq x(t) \leq 2, \quad t \geq t_1\}.$$

For any subset S of Ω, we have $\inf S \in \Omega$ and $\sup S \in \Omega$. Define an operator $F : \Omega \to X$ such that

$$(Fx)(t) = 1 + \int_{t_1}^{t} a(s) f \left(1 + \int_s^{\infty} b(u) g(x(u)) \Delta u \right) \Delta s, \quad t \geq t_1.$$

By using the monotonicity and the fact that $x \in \Omega$, we have

$$1 \leq (Fx)(t) \leq 1 + \int_{t_1}^{t} a(s) f \left(1 + g(2) \int_s^{\infty} b(u) \Delta u \right) \Delta s \leq 2, \quad t \geq t_1.$$

It is also easy to show that F is an increasing mapping. So by Theorem 2.11, there exists $\overline{x} \in \Omega$ such that $F\overline{x} = \overline{x}$. Then we have

$$\overline{x}^{\Delta}(t) = a(t) f \left(1 + \int_t^{\infty} b(u) g(\overline{x}(u)) \Delta u \right).$$

Setting

$$\overline{y}(t) = 1 + \int_t^{\infty} b(u) g(\overline{x}(u)) \Delta u > 0, \quad t \geq t_1$$

gives us

$$\overline{y}^{\Delta}(t) = -b(t) g(\overline{x}(t)) \quad \text{and} \quad \overline{x}^{\Delta}(t) = a(t) f(\overline{y}(t)),$$

that is, $(\overline{x}, \overline{y})$ is a nonoscillatory solution of Eq. (1). In order to prove part (c), assume that there exists a nonoscillatory solution (x, y) of system (1) in M^+ such that $x(t) > 0$ for $t \geq t_1$. Then by

monotonicity of x and g, there exists a number $k > 0$ such that $g(x(t)) \geq k$ for $t \geq t_1$. Integrating the second equation of system from t_1 to t gives us

$$y(t) \leq y(t_1) - k \int_{t_1}^{t} b(s) \Delta s.$$

As $t \to \infty$, it follows $y(t) \to -\infty$. But this contradicts that y is eventually positive. Finally for part (e), without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. If $(x, y) \in M^-$, then by the first equation of system (1), $x^\Delta(t) < 0$ for $t \geq t_1$. Hence, the limit of x exists. So let us show that the assertion follows if $(x, y) \in M^+$. Suppose $(x, y) \in M^+$. Then from the first equation of system (1), we have $x^\Delta(t) > 0$ for $t \geq t_1$. Now let us show that $\lim_{t \to \infty} x(t) = \infty$ cannot happen. Integrating the first equation of system (1) from t_1 to t and using the monotonicity of y and f yield

$$x(t) \leq x(t_1) + f(y(t_1)) \int_{t_1}^{t} a(s) \Delta s.$$

Taking the limit as $t \to \infty$, it follows that x has a finite limit. This completes the proof.

Exercise 3.2. Prove the remainder of Lemma 3.1.

Throughout this section, we assume $Y(t_0) < \infty$ and $Z(t_0) = \infty$. Note that Lemma 3.1 (c) indicates $M^+ = \emptyset$. Therefore, every nonoscillatory solution of system (1) belongs to M^-. Let (x, y) be a nonoscillatory solution of system (1) such that the component function x of solution (x, y) is eventually positive. Then, the second equation of system (1) yields $y < 0$ and eventually decreasing. Then for $k < 0$, we have that y approaches k or $-\infty$. In view of Lemma 3.1 (e), x has a finite limit. So in light of this information, any nonoscillatory solution of system (1) in M^- belongs to one of the following subclasses for $0 < c < \infty$ and $0 < d < \infty$:

$$M_{0,B}^- = \{(x, y) \in M^- : \lim_{t \to \infty} |x(t)| = 0, \quad \lim_{t \to \infty} |y(t)| = d\},$$

$$M_{B,B}^- = \{(x, y) \in M^- : \lim_{t \to \infty} |x(t)| = c, \quad \lim_{t \to \infty} |y(t)| = d\},$$

$$M_{0,\infty}^- = \{(x, y) \in M^- : \lim_{t \to \infty} |x(t)| = 0, \quad \lim_{t \to \infty} |y(t)| = \infty\},$$

$$M_{B,\infty}^- = \{(x, y) \in M^- : \lim_{t \to \infty} |x(t)| = c, \quad \lim_{t \to \infty} |y(t)| = \infty\}.$$

Nonoscillatory solutions in $M_{0,\infty}^-$ is called *slowly decaying solutions* in literature, see [32]. The following theorems show the existence of nonoscillatory solutions in subclasses of M^- given above. Our approach for the next two theorems is based on the Schauder fixed point theorem, see Theorem 2.9.

Theorem 3.3 $M_{0,B}^- \neq \emptyset$ *if and only if*

$$\int_{t_0}^{\infty} b(t) g\left(c_1 \int_{t}^{\infty} a(s) \Delta s\right) \Delta t < \infty, \quad c_1 \neq 0. \tag{3}$$

Proof. Suppose that there exists a solution $(x, y) \in M_{0,B}^-$ such that $x(t) > 0$ for $t \geq t_0$, $x(t) \to 0$ and

$y(t) \to -d$ as $t \to \infty$, where $d > 0$. Integrating the first equation of system (1) from t to ∞ and the monotonicity of f yield that there exists $c > 0$ such that

$$x(t) \geq c \int_t^\infty a(s)\Delta s, \quad t \geq t_0. \tag{4}$$

By integrating the second equation from t_0 to t, using inequality (4) with $c = c_1$ and the monotonicity of g, we have

$$y(t) = y(t_0) - \int_{t_0}^t b(s)g(x(s))\Delta s \leq -\int_{t_0}^t b(s)g\left(c_1 \int_s^\infty a(u)\Delta u\right)\Delta s.$$

So as $t \to \infty$, the assertion follows since y has a finite limit. (For the case $x < 0$ eventually, the proof can be shown similarly with $c_1 < 0$.)

Conversely, suppose that Eq. (3) holds for some $c_1 > 0$. (For the case $c_1 < 0$ can be shown similarly.) Then there exist $t_1 \geq t_0$ and $d > 0$ such that

$$\int_{t_1}^\infty b(t)g\left(c_1 \int_t^\infty a(s)\Delta s\right)\Delta t < d, \quad t \geq t_1, \tag{5}$$

where $c_1 = -f(-3d)$. Let X be the space of all continuous and bounded functions on $[t_1, \infty)_{\mathbb{T}}$ with the norm $\|y\| = \sup\limits_{t \in [t_1, \infty)_{\mathbb{T}}} |y(t)|$. Then X is a Banach space, see Ref. [33]. Let Ω be the subset of X such that

$$\Omega := \{y \in X : \quad -3d \leq y(t) \leq -2d, \quad t \geq t_1\}$$

and define an operator $T : \Omega \to X$ such that

$$(Ty)(t) = -3d + \int_t^\infty b(s)g\left(-\int_s^\infty a(u)f(y(u))\Delta u\right)\Delta s.$$

It is easy to see that T maps into itself. Indeed, we have

$$-3d \leq (Ty)(t) \leq -3d + \int_t^\infty b(s)g\left(-\int_s^\infty a(u)f(-3d)\Delta u\right)\Delta s \leq -2d$$

by Eq. (5). Let us show that T is continuous on Ω. To accomplish this, let y_n be a sequence in Ω such that $y_n \to y \in \Omega = \overline{\Omega}$. Then

$$|(Ty_n)(t) - (Ty)(t)|$$
$$\leq \int_{t_1}^\infty b(s)|[g\left(-\int_s^\infty a(u)f(y_n(u))\Delta u\right) - g\left(-\int_s^\infty a(u)f(y(u))\Delta u\right)]|\Delta s.$$

Then the Lebesque dominated convergence theorem and the continuity of g give $\|(Ty_n) - (Ty)\| \to 0$ as $n \to \infty$, i.e., T is continuous. Also, since

$$0 < -(Ty)^{\Delta}(t) = b(t)g\left(-\int_t^{\infty} a(u)f(y(u))\Delta u\right) < \infty,$$

it follows that $T(\Omega)$ is relatively compact. Then by Theorem 2.9, we have that there exists $\overline{y} \in \Omega$ such that $\overline{y} = T\overline{y}$. So as $t \to \infty$, we have $\overline{y}(t) \to -3d < 0$. Setting

$$\overline{x}(t) = -\int_t^{\infty} a(u)f(\overline{y}(u))\Delta u > 0, \quad t \geq t_1$$

gives that $\overline{x}(t) \to 0$ as $t \to \infty$ and implies $\overline{x}^{\Delta} = af(\overline{y})$, i.e., $(\overline{x}, \overline{y})$ is a nonoscillatory solution in $M_{0,B}^-$.

In the following example, we apply Theorem 3.3 to show the nonemptiness of $M_{0,B}^-$.

Example 3.4 *Let* $\mathbb{T} = q^{\mathbb{N}_0}, q > 1$ *and consider the system*

$$\begin{cases} \Delta_q x(t) = \dfrac{t^{\frac{1}{3}}}{(t+1)(tq+1)(2t-1)^{\frac{1}{3}}} y^{\frac{1}{3}}(t) \\ \Delta_q y(t) = -\dfrac{(t+1)^{\frac{5}{3}}}{qt^2} x^{\frac{5}{3}}(t). \end{cases} \tag{6}$$

Since

$$\int_1^T a(s)\Delta s = (q-1) \sum_{s \in [1,T)_{q^{\mathbb{N}_0}}} \frac{s^{\frac{4}{3}}}{(s+1)(sq+1)(2s-1)^{\frac{1}{3}}} \leq (q-1) \sum_{s \in [1,T)_{q^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}},$$

where $t = q^n$ *and* $s = tq^m, n, m \in \mathbb{N}_0$, *we obtain*

$$Y(1) \leq (q-1) \sum_{n=0}^{\infty} \left(\frac{1}{q^{\frac{2}{3}}}\right)^n < \infty.$$

Also,

$$\int_1^T b(s)\Delta s = \sum_{s \in [1,T)_{q^{\mathbb{N}_0}}} \frac{(s+1)^{\frac{5}{3}}}{qs^2}(q-1)s \geq \frac{q-1}{q} \sum_{s \in [1,T)_{q^{\mathbb{N}_0}}} s^{\frac{2}{3}} \quad implies \quad Z(1) \geq \frac{q-1}{q} \sum_{m=0}^{\infty} (q^{\frac{2}{3}})^m = \infty. \quad Now \quad let \quad us$$

show that Eq. (3) holds. First,

$$\int_t^T a(s)\Delta s \leq (q-1) \sum_{s \in [t,T)_{q^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}} \quad implies \quad \int_t^{\infty} a(s)\Delta s \leq (q-1) \sum_{s \in [t,\infty)_{q^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}} = \frac{q^{\frac{2}{3}}(q-1)}{(q^{\frac{2}{3}}-1)t^{\frac{2}{3}}}.$$

Therefore,

$$\int_1^T b(t)g\left(c_1 \int_t^{\infty} a(s)\Delta s\right)\Delta t \leq \alpha \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{(t+1)^{\frac{5}{3}}}{t^{\frac{19}{10}}},$$

where $\alpha = \dfrac{(q-1)^2 q^{\frac{1}{9}}}{(q^{\frac{2}{3}}-1)^{\frac{5}{3}}}$. *So as* $T \to \infty$, *we have that Eq. (3) holds by the Ratio test. One can also show that* $\left(\frac{1}{t+1}, -2 + \frac{1}{t}\right)$ *of system (6) such that* $x(t) \to 0$ *and* $y(t) \to -2$ *as* $t \to \infty$, *i.e.,* $M_{0,B}^- \neq \emptyset$.

The proof of the following theorem is similar to the proof of Theorem 3.3.

Theorem 3.5 $M_{B,B}^- \neq \emptyset$ if and only if

$$\int_{t_0}^{\infty} b(t)g\left(d_1 - c_1 \int_t^{\infty} a(s)\Delta s\right)\Delta t < \infty$$

for some $c_1 < 0$ and $d_1 > 0$. (Or $c_1 > 0$ and $d_1 < 0$.)

Exercise 3.6. Prove Theorem 3.5 by means of Theorem 2.9.

The following theorem follows from the Knaster fixed point theorem, see Theorem 2.11.

Theorem 3.7 $M_{B,\infty}^- \neq \emptyset$ if and only if

$$\int_{t_0}^{\infty} a(s)f\left(c_1 \int_{t_0}^s b(u)\Delta u\right)\Delta s < \infty \tag{7}$$

for some $c_1 \neq 0$, where f is an odd function.

Proof. Suppose that there exists a nonoscillatory solution $(x,y) \in M_{B,\infty}^-$ such that $x > 0$ eventually, $x(t) \to c_2$ and $y(t) \to -\infty$ as $t \to \infty$, where $0 < c_2 < \infty$. Because of the monotonicity of x and the fact that x has a finite limit, there exist $t_1 \geq t_0$ and $c_3 > 0$ such that

$$c_2 \leq x(t) \leq c_3 \quad \text{for} \quad t \geq t_1. \tag{8}$$

Integrating the first equation from t_1 to t gives us

$$c_2 \leq x(t) = x(t_1) + \int_{t_1}^t a(s)f(y(s))\Delta s \leq c_3, \quad t \geq t_1.$$

So by taking the limit as $t \to \infty$, we have

$$\int_{t_1}^{\infty} a(s)|f(y(s))|\Delta s < \infty. \tag{9}$$

The monotonicity of g, Eq. (8) and integrating the second equation from t_1 to t yield

$$y(t) \leq y(t_1) - g(c_2)\int_{t_1}^t b(s)\Delta s \leq -g(c_2)\int_{t_1}^t b(s)\Delta s.$$

Since $f(-u) = -f(u)$ for $u \neq 0$ and by the monotonicity of f, we have

$$|f(y(t))| \geq f\left(g(c_2)\int_{t_1}^t b(s)\Delta s\right), \quad t \geq t_1. \tag{10}$$

By Eqs. (9) and (10), we have

$$\int_{t_1}^t a(s)|f(y(s))|\Delta s \geq \int_{t_1}^t a(s)f\left(g(c_2)\int_{t_1}^s b(u)\Delta u\right)\Delta s, \quad \text{where} \quad g(c_2) = c_1.$$

As $t \to \infty$, the proof is finished. (The case $x < 0$ eventually can be proved similarly with $c_1 < 0$.)

Conversely, suppose $\int_{t_0}^{\infty} a(s)f\left(c_1\int_{t_0}^{s} b(u)\Delta u\right)\Delta s < \infty$ for some $c_1 \neq 0$. Without loss of generality, assume $c_1 > 0$. (The case $c_1 < 0$ can be done similarly.) Then, we can choose $t_1 \geq t_0$ and $d > 0$ such that

$$\int_{t_1}^{\infty} a(s)f\left(c_1\int_{t_1}^{s} b(u)\Delta u\right)\Delta s < d, \quad t \geq t_1,$$

where $c_1 = g(2d) > 0$. Let X be the partially ordered Banach space of all real-valued continuous functions endowed with supremum norm $\|x\| = \sup\limits_{t \in [t_1, \infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega =: \{x \in X : \quad d \leq x(t) \leq 2d, \quad t \geq t_1\}.$$

For any subset B of Ω, $\inf B \in \Omega$ and $\sup B \in \Omega$, i.e., $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \rightarrow X$ as

$$(Fx)(t) = d + \int_{t}^{\infty} a(s)f\left(\int_{t_1}^{s} b(u)g(x(u))\Delta u\right)\Delta s, \quad t \geq t_1.$$

The rest of the proof can be completed similar to the proof of Lemma 3.1(a). So, it is omitted.

Exercise 3.8 Let $\mathbb{T} = \mathbb{Z}$. Use Theorem 3.7 to justify that $(x_n, y_n) = (1 + 2^{-n}, -2^n)$ is a nonoscillatory solution in $M_{B,\infty}^-$ of

$$\begin{cases} \Delta x_n = 2^{\frac{-6n}{5}-1}(y_n)^{\frac{1}{5}} \\ \Delta y_n = -\dfrac{4^n}{1+2^n}(x_n). \end{cases}$$

For convenience, set

$$I = \int_{t_0}^{\infty} a(t)f\left(k\int_{t}^{\infty} b(s)\Delta s\right)\Delta t, \quad k \neq 0. \tag{11}$$

In order to obtain the nonemptiness of $M_{0,\infty}^-$, we apply Theorem 2.11 and use the similar discussion as in Lemma 3.1(a).

Theorem 3.9 $M_{0,\infty}^- \neq \varnothing$ *if for some* $k > 0$ *and any* $d_1 > 0$ *($k < 0$ and $d_1 < 0$)*

$$I < \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(t)g\left(d_1\int_{t}^{\infty} a(s)\Delta s\right)\Delta t = \infty,$$

where I is defined as in Eq. (11) and f is an odd function.

Exercise 3.10. Prove Theorem 3.9.

We reconsider system (1) in the next section to emphasize the existence of nonoscillatory solutions in M^+.

4. Delay Dynamical Systems on Time Scales

This section is concerned with the delay system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(\tau(t))) \end{cases} \tag{12}$$

with $a, b \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [t_0, \infty)_{\mathbb{T}})$, $\tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$, $f, g \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing functions such that $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$. Motivated by Ref. [34] in which $\tau(t) = t - \eta$, $\eta > 0$, our purpose in this section is to obtain the criteria for the existence of nonoscillatory solutions of Eq. (12) based on $Y(t_0)$ and $Z(t_0)$. However, note that the results in Ref. [34] do not hold for any time scale, e.g., $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, because $t - \eta$ is not necessarily in $\mathbb{T}$. In fact, theoretical claims in this section follow from Ref. [35].

Since system (12) is oscillatory for the case $Y(t_0) = \infty$ and $Z(t_0) = \infty$, the existence results on any time scale are obtained in the next subsections based on the other three cases of $Y(t_0)$ and $Z(t_0)$. Let (x, y) be a nonoscillatory solution of system (12) in M^+ such that the component function x is eventually positive. Then by the second equation of system (12), y is eventually decreasing. In addition, using the first equation of system (12), we have that $x(t) \to c$ or ∞ and $y(t) \to d$ or 0 as $t \to \infty$ for $0 < c < \infty$ and $0 < d < \infty$. Therefore, we have the following subclasses of M^+:

$$M_{B,B}^+ = \{(x, y) \in M^+ : \lim_{t \to \infty}|x(t)| = c, \quad \lim_{t \to \infty}|y(t)| = d\},$$

$$M_{B,0}^+ = \{(x, y) \in M^+ : \lim_{t \to \infty}|x(t)| = c, \quad \lim_{t \to \infty}|y(t)| = 0\},$$

$$M_{\infty,B}^+ = \{(x, y) \in M^+ : \lim_{t \to \infty}|x(t)| = \infty, \quad \lim_{t \to \infty}|y(t)| = d\},$$

$$M_{\infty,0}^+ = \{(x, y) \in M^+ : \lim_{t \to \infty}|x(t)| = \infty, \quad \lim_{t \to \infty}|y(t)| = 0\}.$$

In the literature, solutions in $M_{B,0}^+$, $M_{\infty,B}^+$ and $M_{\infty,0}^+$ are called *subdominant*, *dominant* and *intermediate solutions*, respectively, see Ref. [36]. Any nonoscillatory solution of system (12) belongs to M^+ or M^- given in Section 3. Also, it is important to emphasize that Lemma 3.1 holds for system (12) as well.

4.1. The case $Y(t_0) = \infty$ and $Z(t_0) < \infty$

We restrict our attention to M^+ in this subsection because $M^- = \varnothing$ when $Y(t_0) = \infty$ and $Z(t_0) < \infty$. The following lemma specifies the limit behavior of the component functions of nonoscillatory solutions (x, y) under the case $Y(t_0) = \infty$ and $Z(t_0) < \infty$.

Lemma 4.1 *If $|x(t)| \to c$, then $y(t) \to 0$ as $t \to \infty$ for $0 < c < \infty$.*

Proof. Assume to the contrary. So $y(t) \to d$ for $0 < d < \infty$ as $t \to \infty$. Then since $y(t) > 0$ and decreasing eventually, there exists $t_1 \geq t_0$ such that $f(y(\tau(t))) \geq f(d) = k$ for $t \geq t_1$. By the same discussion as in the proof of Theorem 3.3, we obtain

$$x(t) \geq k \int_{t_1}^{t} a(s)\Delta s, \quad t \geq t_1.$$

However, this gives us a contradiction to the fact that $x(t) \to c$ as $t \to \infty$. So the assertion follows.

Remark 4.2. The discussion above and Lemma 4.1 yield us $M_{B,B}^{+} = \emptyset$.

Theorem 4.3. $M_{B,0}^{+} \neq \emptyset$ if and only if $I < \infty$.

Proof. Suppose that there exists a solution $(x,y) \in M_{B,0}^{+}$ such that $x(t) > 0$, $x(\tau(t)) > 0$ for $t \geq t_0$, $x(t) \to c_1$ and $y(t) \to 0$ as $t \to \infty$. Because x is eventually increasing, there exist $t_1 \geq t_0$ and $c_2 > 0$ such that $c_2 \leq g(x(\tau(t)))$ for $t \geq t_1$. Integrating the second equation from t to ∞ gives

$$y(t) = \int_{t}^{\infty} b(s)g(x(\tau(s)))\Delta s, \quad t \geq t_1. \tag{13}$$

Also, integrating the first equation from t_1 to t, Eq. (13) and the monotonicity of g result in

$$x(t) \geq \int_{t_1}^{t} a(s)f\left(\int_{s}^{\infty} b(u)g(x(\tau(u)))\Delta u\right)\Delta s \geq \int_{t_1}^{t} a(s)f\left(c_2\int_{s}^{\infty} b(u)\Delta u\right)\Delta s.$$

Setting $c_2 = k$ and taking the limit as $t \to \infty$ prove the assertion. (For the case $x < 0$ eventually, the proof can be shown similarly with $k < 0$.)

Conversely, suppose $I < \infty$ for some $k > 0$. (For the case $k < 0$ can be shown similarly.) Then, choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^{\infty} a(t)f\left(k\int_{t}^{\infty} b(s)\Delta s\right)\Delta t < \frac{c_1}{2}, \quad t \geq t_1,$$

where $k = g(c_1)$. Let X be the space of all continuous and bounded functions on $[t_1,\infty)_{\mathbb{T}}$ with the norm $\|y\| = \sup\limits_{t \in [t_1,\infty)_{\mathbb{T}}} |y(t)|$. Then, X is a Banach space. Let Ω be the subset of X such that

$$\Omega := \{x \in X : \quad \frac{c_1}{2} \leq x(\tau(t)) \leq c_1, \quad \tau(t) \geq t_1\},$$

and define an operator $F : \Omega \to X$ such that

$$(Fx)(t) = c_1 - \int_{t}^{\infty} a(s)f\left(\int_{s}^{\infty} b(u)g(x(\tau(u)))\Delta u\right)\Delta s, \quad \tau(t) \geq t_1.$$

It is easy to see that Ω is bounded, convex and a closed subset of X. It can also be shown that F maps into itself, relatively compact and continuous on Ω by the Lebesques dominated convergence theorem. Then, Theorem 2.9 gives that there exists $\overline{x} \in \Omega$ such that $\overline{x} = F\overline{x}$. As $t \to \infty$, we get $\overline{x}(t) \to c_1 > 0$. Setting

$$\overline{y}(t) = \int_t^\infty b(u)g(\overline{x}(\tau(u)))\Delta u > 0, \quad \tau(t) \geq t_1$$

shows $\overline{y}(t) \to 0$ as $t \to \infty$. Taking the derivatives of $\overline{x}$ and $\overline{y}$ yield that $(\overline{x}, \overline{y})$ is a solution of system (12). Hence, $M_{B,0}^+ \neq \emptyset$.

We demonstrate the following example to highlight Theorem 4.3.

Example 4.4 *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$ *and consider the system*

$$\begin{cases} \Delta_2 x(t) = \dfrac{1}{2t^{\frac{4}{5}}}\left(y(t)\right)^{\frac{3}{5}} \\ \Delta_2 y(t) = -\dfrac{3}{4t^2(8t-4)}x\left(\dfrac{t}{4}\right). \end{cases} \tag{14}$$

First, it must be shown $Y(t_0) = \infty$ *and* $Z(t_0) < \infty$. *Indeed,*

$$\int_{t_0}^t a(s)\Delta s = \frac{1}{2} \sum_{s \in [4,t)_{2^{\mathbb{N}_0}}} s^{\frac{1}{5}} \quad implies \quad Y(t_0) = \frac{1}{2}\lim_{n\to\infty}\sum_{m=2}^{n-1}(2^m)^{\frac{1}{5}} = \infty$$

and

$$\int_{t_0}^t b(s)\Delta s \leq \frac{3}{16}\sum_{s \in [4,t)_{2^{\mathbb{N}_0}}}\frac{1}{s} \quad implies \quad Z(t_0) \leq \frac{3}{16}\lim_{n\to\infty}\sum_{m=2}^{n-1}\frac{1}{2^m} < \infty$$

by the geometric series, where $t = 2^n$, $s = 2^m$, $m, n \geq 2$. *Note that*

$$\int_t^T b(s)\Delta s \leq \frac{3}{16}\sum_{s \in [t,T)_{2^{\mathbb{N}_0}}}\frac{1}{s} \quad implies \quad Z(t) \leq \frac{3}{16}\lim_{n\to\infty}\sum_{m=2}^{n-1}\frac{1}{2^m} = \frac{3}{8}\lim_{n\to\infty}\left(\frac{1}{t}-\frac{1}{t2^n}\right) = \frac{3}{8t}.$$

Letting $k = 1$ *and using the last inequality gives*

$$\int_{t_0}^T a(t)f\left(k\int_t^\infty b(s)\Delta s\right)\Delta t \leq \int_{t_0}^T \frac{1}{2t^{\frac{4}{5}}}\left(\frac{3}{8t}\right)^{\frac{3}{5}}\Delta t = \left(\frac{3}{8}\right)^{\frac{3}{5}}\frac{1}{2}\sum_{t \in [1,T)_{2^{\mathbb{N}_0}}}\frac{1}{t^{\frac{2}{5}}}.$$

Therefore, we have

$$\int_{t_0}^\infty a(t)f\left(k\int_t^\infty b(s)\Delta s\right)\Delta t \leq \left(\frac{3}{8}\right)^{\frac{3}{5}}\frac{1}{2}\sum_{n=0}^\infty \frac{1}{2^{\frac{2n}{5}}} < \infty$$

by the geometric series. It can be seen that $(x,y) = \left(8-\dfrac{1}{t}, \dfrac{1}{t^2}\right)$ *is a nonoscillatory solution of Eq.* (14) *such that* $x(t) \to 8$ *and* $y(t) \to 0$ *as* $t \to \infty$, *i.e.,* $M_{B,0}^+ \neq \emptyset$.

The existence in subclasses $M^+_{\infty,B}$ and $M^+_{\infty,0}$ is not obtained on general time scales. The main reason is that setting an operator including a delay function gives a struggle when the fixed points theorems are applied. In fact, when we restrict the delay function to $\tau(t) = t - \eta$ for $\eta \geq 0$, it was shown $M^+_{\infty,B} \neq \varnothing$, see Ref. [34]. Nevertheless, the existence in $M^+_{\infty,B}$ and $M^+_{\infty,0}$ for system (1) is shown in Subsection 4.4.

4.2. The case $Y(t_0) < \infty$ and $Z(t_0) < \infty$

Because the component functions x and y have finite limits by Lemma 3.1(e) and (f), the subclasses $M^+_{\infty,B}$ and $M^+_{\infty,0}$ are empty. Since the existence of nonoscillatory solutions in $M^+_{B,0}$ is shown in Theorem 4.3, we only focus on $M^+_{B,B}$ in this subsection.

The Knaster fixed point theorem is utilized in order to prove the following theorem.

Theorem 4.5 $M^+_{B,B} \neq \varnothing$ if and only if

$$\int_{t_0}^{\infty} a(s) f\left(d_1 + k \int_s^{\infty} b(u) \Delta u\right) \Delta s < \infty, \quad k, d_1 \neq 0. \tag{15}$$

Proof. The proof of the necessity part is very similar to those of previous theorems. So for sufficiency, suppose Eq. (15) holds. Choose $t_1 \geq t_0$, $k > 0$ and $d_1 > 0$ such that

$$\int_{t_1}^{\infty} a(s) f\left(d_1 + k \int_s^{\infty} b(u) \Delta u\right) \Delta s < d_1,$$

where $k = g(2d_1)$. (The case $k, d_1 < 0$ can be done similarly.) Let X be the Banach space of all continuous real-valued functions endowed with the norm $\|x\| = \sup\limits_{t \in [t_1, \infty)_{\mathbb{T}}} |x(t)|$ and with usual point-wise ordering $\leq$. Define a subset Ω of X as

$$\Omega := \{x \in X : \quad d_1 \leq x(\tau(t)) \leq 2d_1, \quad \tau(t) \geq t_1\}.$$

For any subset B of Ω, it is clear that $\inf B \in \Omega$ and $\sup B \in \Omega$. An operator $F : \Omega \to X$ is defined as

$$(Fx)(t) = d_1 + \int_{t_1}^{t} a(s) f\left(d_1 + \int_s^{\infty} b(u) g(x(\tau(u))) \Delta u\right) \Delta s, \quad \tau(t) \geq t_1.$$

It is obvious that F is an increasing mapping into itself. Therefore,

$$d_1 \leq (Fx)(t) \leq d_1 + \int_{t_1}^{t} a(s) f\left(d_1 + g(2d_1) \int_s^{\infty} b(u) \Delta u\right) \Delta s \leq 2d_1, \quad \tau(t) \geq t_1.$$

Then, by Theorem 2.11, there exists $\overline{x} \in \Omega$ such that $\overline{x} = F\overline{x}$. By setting

$$\overline{y}(t) = d_1 + \int_t^{\infty} b(u) g(\overline{x}(\tau(u))), \quad \tau(t) \geq t_1,$$

we get that

$$\overline{y}^{\Delta}(t) = -b(t)g(\overline{x}(\tau(t))). \tag{16}$$

Also taking the derivative of $\overline{x}$ and Eq. (16) give that $(\overline{x}, \overline{y})$ is a solution of system (12). Hence, we conclude that $\overline{x}(t) \to \alpha$ and $\overline{y}(t) \to d_1$ as $t \to \infty$, where $0 < \alpha < \infty$, i.e., $M^+_{B,B} \neq \emptyset$. Note that a similar proof can be done for the case $k < 0$ and $d_1 < 0$ with $x < 0$.

Example 4.6 *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$ *and consider the system*

$$\begin{cases} \Delta_2 x(t) = \dfrac{1}{2t^{\frac{5}{3}}(3t+1)^{\frac{1}{3}}} y^{\frac{1}{3}}(t) \\ \Delta_2 y(t) = -\dfrac{1}{2t(6t-4)} x\left(\dfrac{t}{4}\right). \end{cases} \tag{17}$$

We first demonstrate $Y(t_0) < \infty$ *and* $Z(t_0) < \infty$.

$$\int_{t_0}^t a(s)\Delta s = \frac{1}{2} \sum_{s \in [4,t)_{2^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}(3s+1)^{\frac{1}{3}}} \quad implies \quad Y(t_0) = \frac{1}{2} \lim_{n \to \infty} \sum_{m=2}^{n-1} \frac{1}{(2^m)^{\frac{2}{3}}(3 \cdot 2^m + 1)^{\frac{1}{3}}} < \infty$$

by the Ratio test for $t = 2^n$, $s = 2^m$, $n \geq 2$. *Similarly,*

$$\int_{t_0}^t b(s)\Delta s = \frac{1}{2} \sum_{s \in [4,t)_{2^{\mathbb{N}_0}}} \frac{1}{6s-4} \quad implies \quad Z(t_0) = \frac{1}{2} \lim_{n \to \infty} \sum_{m=2}^{n-1} \frac{1}{6.2^m - 4} < \infty.$$

Because $Y(t_0) < \infty$ *and* $Z(t_0) < \infty$, *it is easy to show that Eq. (15) holds. One can also verify that* $\left(6 - \frac{1}{t}, 3 + \frac{1}{t}\right)$ *is a nonoscillatory solution of system (17) such that* $x(t) \to 6$ *and* $y(t) \to 3$ *as* $t \to \infty$, *i.e.,* $M^+_{B,B} \neq \emptyset$ *by Theorem 4.5.*

4.3. The case $Y(t_0) < \infty$ and $Z(t_0) = \infty$

Lemma 3.1(c) yields $M^+ = \emptyset$ for the case $Y(t_0) < \infty$ and $Z(t_0) = \infty$. Thus, we pay our attention to M^- in this subsection. The proof of the following remark is similar to that of Theorem 3.7.

Remark 4.7 $M^-_{B,\infty} \neq \emptyset$ *if and only if integral condition (7) holds.*

Exercise 4.8 Prove Remark 4.7 and also show that $\left(3 + \frac{1}{t}, -t - \frac{1}{t}\right)$ is a nonoscillatory solution of

$$\begin{cases} \Delta_2 x(t) = \dfrac{1}{2t^{\frac{7}{5}}(t^2+1)^{\frac{3}{5}}} (y(t))^{\frac{3}{5}} \\ \Delta_2 y(t) = -\dfrac{2t^2-1}{2t^{\frac{9}{5}}(3t+4)^{\frac{1}{5}}} \left(x\left(\dfrac{t}{4}\right)\right)^{\frac{1}{5}} \end{cases}$$

in $M^-_{B,\infty} \neq \emptyset$ when $\mathbb{T} = 2^{\mathbb{N}_0}$.

4.4. Dominant and intermediate solutions of Eq. (1)

Note that the existence of nonoscillatory solutions of system (1) in $M^-_{0,\infty}, M^-_{B,B}$ and $M^-_{0,B}$ is not shown on a general time scale. In fact, the existence in these subclasses is obtained for system

(1) in Section 3. Since system (12) is reduced to system (1) when $\tau(t) = t$, notice that the results obtained for system (12) in Section 4 also hold for system (1). Therefore, we only need to show the existence of nonoscillatory solutions for Eq. (1) in $M_{\infty,B}^{+}$ and $M_{\infty,0}^{+}$, which are not acquired for Eq. (12) on a general time scale. To achieve the goal, we assume $Y(t_0) = \infty$ and $Z(t_0) < \infty$.

Theorem 4.9 $M_{\infty,B}^{+} \neq \varnothing$ *if and only if*

$$\int_{t_0}^{\infty} b(s)g\left(c_1 \int_{t_0}^{s} a(u)\Delta u\right)\Delta s < \infty, \quad c_1 \neq 0. \tag{18}$$

Proof. The necessity part is left to readers as an exercise. Therefore, for sufficiency, suppose that Eq. (18) holds. Choose $t_1 \geq t_0$, $c_1 > 0$ and $d_1 > 0$ such that

$$\int_{t_1}^{\infty} b(s)g\left(c_1 \int_{t_1}^{s} b(u)\Delta u\right)\Delta s < d_1, \quad t \geq t_1, \tag{19}$$

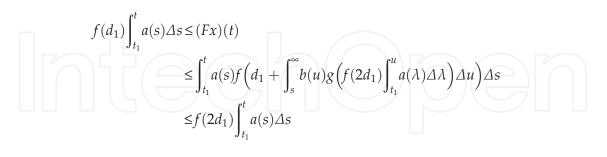where $c_1 = f(2d_1) > 0$. (The case $c_1 < 0$ can be done similarly.) Let X be the partially ordered Banach space of all real-valued continuous functions endowed with supremum norm $\|x\| = \sup\limits_{t \in [t_1,\infty)_{\mathbb{T}}} \frac{|x(t)|}{\int_{t_1}^{t} a(s)\Delta s}$ and with the usual point-wise ordering $\leq$. Define a subset Ω of X such that

$$\Omega =: \{x \in X : \quad f(d_1)\int_{t_1}^{t} a(s)\Delta s \leq x(t) \leq f(2d_1)\int_{t_1}^{t} a(s)\Delta s, \quad t \geq t_1\}.$$

For any subset B of Ω, $\inf B \in \Omega$ and $\sup B \in \Omega$, i.e., $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = \int_{t_1}^{t} a(s)f\left(d_1 + \int_{t}^{\infty} b(u)g(x(u))\Delta u\right)\Delta s, \quad t \geq t_1.$$

It is obvious that it is an increasing mapping, so let us show $F := \Omega \to \Omega$.

$$f(d_1)\int_{t_1}^{t} a(s)\Delta s \leq (Fx)(t)$$

$$\leq \int_{t_1}^{t} a(s)f\left(d_1 + \int_{s}^{\infty} b(u)g\left(f(2d_1)\int_{t_1}^{u} a(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s$$

$$\leq f(2d_1)\int_{t_1}^{t} a(s)\Delta s$$

by Eq. (19). Then, by Theorem 2.11, there exists $\overline{x} \in \Omega$ such that $\overline{x} = F\overline{x}$ and so

$$\overline{x}^{\Delta}(t) = a(t)f\left(d_1 + \int_{t}^{\infty} b(u)g(\overline{x}(u))\Delta u\right), \quad t \geq t_1.$$

Setting $\overline{y}(t) = d_1 + \int_{t}^{\infty} b(u)g(\overline{x}(u))\Delta u$ leads us $\overline{y}^{\Delta} = -bg(\overline{x})$ and so, $(\overline{x}, \overline{y})$ is a solution of system (1) such that $\overline{x}(t) > 0$ and $\overline{y}(t) > 0$ for $t \geq t_1$ and $\overline{x}(t) \to \infty$ and $\overline{y}(t) \to d_1 > 0$ as $t \to \infty$, i.e., $M_{\infty,B}^{+} \neq \varnothing$.

Theorem 4.10 $M_{\infty,0}^+ \neq \emptyset$ *if*

$$I = \infty \quad and \quad \int_{t_0}^{\infty} b(t)g\left(l\int_{t_0}^{\infty} a(s)\Delta s\right)\Delta t < \infty,$$

where I is defined as in Eq. (11), for any $k > 0$ and some $l > 0$ ($k < 0$ and $l < 0$).

Exercise 4.11 Prove Theorem 4.10 using Theorem 2.11.

5. Emden-Fowler Dynamical Systems on Time Scales

Motivated by the papers [28, 36, 37], we deal with the classification and existence of nonoscillatory solutions of the Emden-Fowler dynamical system
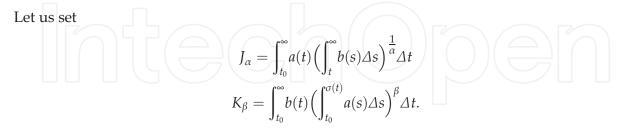
$$\begin{cases} x^{\Delta}(t) = a(t)|y(t)|^{\frac{1}{\alpha}}\text{sgn } y(t) \\ y^{\Delta}(t) = -b(t)|x^{\sigma}(t)|^{\beta}\text{sgn } x^{\sigma}(t), \end{cases} \tag{20}$$

where $\alpha, \beta > 0$ $a, b \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $x^{\sigma}(t) = x(\sigma(t))$. The main results of this section follow from Ref. [38]. If $\mathbb{T} = \mathbb{Z}$, system (20) is reduced to a Emden-Fowler system of difference equations while it is reduced to a Emden-Fowler system of differential equations when $\mathbb{T} = \mathbb{R}$, see Refs. [32, 39, 40], respectively. We also refer readers to Refs. [41–46] for quasilinear and Emden-Fowler dynamic equations on time scales.

Note that any nonoscillatory solution of system (20) belongs to M^+ or M^- given in Section 3. Also, it could be shown that Lemma 3.1 holds for system (20) as well.

5.1. The case $Y(t_0) = \infty$ and $Z(t_0) < \infty$

In this case, we have $M^- = \emptyset$, see Lemma 3.1(d). By a similar discussion as in Subsection 4.1, solutions in M^+ belongs to one of the subclasses $M_{B,0}^+$, $M_{\infty,B}^+$ and $M_{\infty,0}^+$.

Let us set

$$J_{\alpha} = \int_{t_0}^{\infty} a(t)\left(\int_{t}^{\infty} b(s)\Delta s\right)^{\frac{1}{\alpha}}\Delta t$$

$$K_{\beta} = \int_{t_0}^{\infty} b(t)\left(\int_{t_0}^{\sigma(t)} a(s)\Delta s\right)^{\beta}\Delta t.$$

Note that integral I, defined as in Eq. (11), is reduced to J_{α} by replacing $f(z) = z^{\frac{1}{\alpha}}$ and $g(z) = z^{\beta}$. The following theorem can be proven similar to Theorem 4.3.

Theorem 5.1 $M_{B,0}^+ \neq \emptyset$ *if and only if* $J_{\alpha} < \infty$.

Exercise 5.2 Prove Theorem 5.1.

Next, we provide the existence of dominant and intermediate solutions of system (20) along with examples.

Theorem 5.3 $M^+_{\infty,B} \neq \emptyset$ *if and only if* $K_\beta < \infty$.

Proof. Suppose that there exists $(x,y) \in M^+$ such that $x > 0$ eventually, $x(t) \to \infty$ and $y(t) \to d$ as $t \to \infty$ for $0 < d < \infty$. Integrating the first equation from t_1 to $\sigma(t)$, using the monotonicity of y and integrating the second equation from t_1 to t of system (20) give us

$$x^\sigma(t) = x^\sigma(t_1) + \int_{t_1}^{\sigma(t)} a(s)y^{\frac{1}{\alpha}}(s)\Delta s > d^{\frac{1}{\alpha}}\int_{t_1}^{\sigma(t)} a(s)\Delta s. \tag{21}$$

and

$$y(t_1) - y(t) = \int_{t_1}^{t} b(s)\left(x^\sigma(s)\right)^\beta \Delta s, \tag{22}$$

respectively. Then, by Eqs. (21) and (22), we have

$$\int_{t_1}^{t} b(s)\left(\int_{t_1}^{\sigma(s)} a(u)\Delta u\right)^\beta \Delta s < d^{\frac{-\beta}{\alpha}}\int_{t_1}^{t} b(s)\left(x^\sigma(s)\right)^\beta \Delta s = d^{\frac{-\beta}{\alpha}}\left(y(t_1) - y(t)\right)$$

So as $t \to \infty$, it follows $K_\beta < \infty$.

Conversely, suppose $K_\beta < \infty$. Choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^{\infty} b(s)\left(\int_{t_1}^{\sigma(s)} a(u)\Delta u\right)^\beta \Delta s < \frac{d^{1-\beta}}{2^\beta}$$

for arbitrarily given $d > 0$. Let X be the partially ordered Banach Space of all real-valued continuous functions with the norm $\|x\| = \sup\limits_{t > t_1} \dfrac{|x(t)|}{\int_{t_1}^{t} a(s)\Delta s}$ and the usual point-wise ordering $\leq$.

Define a subset Ω of X as follows:

$$\Omega : \{x \in X : \quad d^{\frac{1}{\alpha}}\int_{t_1}^{t} a(s)\Delta s \leq x(t) \leq (2d)^{\frac{1}{\alpha}}\int_{t_1}^{t} a(s)\Delta s \quad \text{for} \quad t > t_1\}.$$

First, since every subset of Ω has a supremum and infimum in Ω, $(\Omega, \leq)$ is a complete lattice. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = \int_{t_1}^{t} a(s)\left(d + \int_{s}^{\infty} b(u)\left(x^\sigma(u)\right)^\beta \Delta\tau\right)^{\frac{1}{\alpha}}\Delta s.$$

The rest of the proof can be finished via the Knaster fixed point theorem, see Theorem 4.9 and thus is left to readers.

Example 5.4 *Let* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ *and consider the system*

$$\begin{cases} x^\Delta = \dfrac{t}{1 + 2t} |y| \operatorname{sgn} y \\ y^\Delta = -\dfrac{1}{q^{1+\beta} t^{\beta+2}} |x^\sigma|^\beta \operatorname{sgn} x. \end{cases} \tag{23}$$

It is left to readers to show $Y(t_0) = \infty$ *and* $Z(t_0) < \infty$. *In order to show* $K_\beta < \infty$, *we first calculate*

$$\int_{t_0}^T b(t) \left(\int_{t_0}^{\sigma(t)} a(s) \Delta s \right)^\beta \Delta t = \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{1}{q^{1+\beta} t^{\beta+2}} \left(\sum_{s \in [1,\sigma(t))_{q^{\mathbb{N}_0}}} \frac{s^2(q-1)}{1+2s} \right)^\beta (q-1)t$$

$$< \frac{(q-1)^{\beta+1}}{q^{1+\beta}} \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{1}{t^{1+\beta}} \left(\sum_{s \in [1,\sigma(t))_{q^{\mathbb{N}_0}}} s \right)^\beta < \frac{q-1}{q} \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{1}{t},$$

where $s = q^m$ *and* $t = q^n$ *for* $m, n \in \mathbb{N}_0$. *Since*

$$\lim_{T \to \infty} \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{1}{t} = \sum_{n=0}^\infty \frac{1}{q^n} < \infty$$

by the geometric series, we have $K_\beta < \infty$. *It can be verified that* $(t, \frac{1}{t} + 2)$ *is a nonoscillatory solution of system* (23) *in* $M_{\infty,B}^+$.

Theorem 5.5 $M_{\infty,0}^+ \neq \emptyset$ *if* $J_\alpha = \infty$ *and* $K_\beta < \infty$.

Proof. Suppose that $J_\alpha = \infty$ and $K_\beta < \infty$ hold. Since $Y(t_0) = \infty$, we can choose t_1 and t_2 so large that

$$\int_{t_2}^\infty b(t) \left(\int_{t_0}^{\sigma(t)} a(s) \Delta s \right)^\beta \Delta t \leq 1 \quad \text{and} \quad \int_{t_1}^{t_2} a(s) \Delta s \geq 1, \quad t \geq t_2 \geq t_1.$$

Let X be the Fréchet Space of all continuous functions on $[t_1, \infty)_\mathbb{T}$ endowed with the topology of uniform convergence on compact subintervals of $[t_1, \infty)_\mathbb{T}$. Set

$$\Omega := \left\{ x \in X : \quad 1 \leq x(t) \leq \int_{t_1}^t a(s) \Delta s \quad \text{for} \quad t \geq t_1 \right\}$$

and define an operator $T : \Omega \to X$ by

$$(Tx)(t) = 1 + \int_{t_2}^t a(s) \left(\int_s^\infty b(u) \left(x^\sigma(u) \right)^\beta \Delta u \right)^{\frac{1}{\alpha}}. \tag{24}$$

We can show that $T : \Omega \to \Omega$ is continuous on $\Omega \subset X$ by the Lebesque dominated convergence theorem. Since

$$0 \le [(Tx)(t)]^\Delta = a(t)\left(\int_t^\infty b(u)\left(x^\sigma(u)\right)^\beta \Delta u\right)^{\frac{1}{\alpha}}$$

$$\le a(t)\left(\int_t^\infty b(u)\left(\int_{t_1}^{\sigma(u)} a(\lambda)\Delta\lambda\right)^\beta \Delta u\right)^{\frac{1}{\alpha}} < \infty,$$

it follows that T is equibounded and equicontinuous. Then by Theorem 2.10, there exists $\overline{x} \in \Omega$ such that $\overline{x} = T\overline{x}$. Thus, it follows that $\overline{x}$ is eventually positive, i.e nonoscillatory. Then differentiating $\overline{x}$ and the first equation of system (20) give us

$$\overline{y}(t) = \left(\frac{1}{a(t)}\right)^\alpha \left(\overline{x}^\Delta(t)\right)^\alpha = \int_t^\infty b(u)\left(\overline{x}^\sigma(u)\right)^\beta \Delta u > 0, \quad t \ge t_1. \tag{25}$$

This results in that $\overline{y}$ is eventually positive and hence $(\overline{x}, \overline{y})$ is a nonoscillatory solution of system (20) in M^+. Also by monotonicity of $\overline{x}$, we have

$$\overline{x}(t) = 1 + \int_{t_2}^t a(s)\left(\int_s^\infty b(u)\left(\overline{x}^\sigma(u)\right)^\beta \Delta u\right)^{\frac{1}{\alpha}} \ge \left(\overline{x}(t_2)\right)^\beta \int_{t_2}^t a(s)\left(\int_s^\infty b(u)\Delta u\right)^{\frac{1}{\alpha}}.$$

Hence as $t \to \infty$, it follows $\overline{x}(t) \to \infty$. And by Eq. (25), we have $\overline{y}(t) \to 0$ as $t \to \infty$. Therefore $M_{\infty,0}^+ \ne \varnothing$.

Example 5.6 *Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ and $\beta < 1$. Consider the system*

$$\begin{cases} x^\Delta = (1+t)|y|^{\frac{1}{\alpha}}\mathrm{sgn}\, y \\ y^\Delta = -\dfrac{1}{(1+t)(1+tq)^{\beta+1}}|x^\sigma|^\beta \mathrm{sgn}\, x. \end{cases} \tag{26}$$

It is easy to verify $Y(t_0) = \infty$ and $Z(t_0) < \infty$. Letting $s = q^m$ and $t = q^n$, where $m, n \in \mathbb{N}_0$ gives

$$\int_{t_0}^T a(t)\left(\int_t^T b(s)\Delta s\right)^{\frac{1}{\alpha}}\Delta t = \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} (1+t)\left(\sum_{s \in [t,T)_{q^{\mathbb{N}_0}}} \frac{(q-1)s}{(1+s)(1+sq)^{\beta+1}}\right)(q-1)t$$

$$\ge (q-1)^2 \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} (1+t)\left(\frac{t}{(1+t)(1+tq)^{\beta+1}}\right)t = (q-1)^2 \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{t^2}{(1+tq)^{\beta+1}}.$$

So we have

$$\lim_{T \to \infty} \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{t^2}{(1+tq)^{\beta+1}} = \sum_{n=0}^\infty \frac{q^{2n}}{(1+q^{n+1})^{\beta+1}} = \infty$$

by the Test for Divergence and $\beta < 1$. Now let us show that $K_\beta < \infty$. Since

$$\int_{t_0}^{\sigma(t)} a(s)\Delta s = \sum_{s \in [1,t)_{q^{\mathbb{N}_0}}} (1+s)(q-1)s \le tq(1+tq),$$

we have

$$\int_{t_0}^{T} b(t) \left(\int_{t_0}^{\sigma(t)} a(s) \Delta s \right)^{\beta} \Delta t \leq \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{1}{(1+t)(1+tq)^{\beta+1}} \left(tq(1+tq) \right)^{\beta} t(q-1)$$

$$\leq q^{\beta}(q-1) \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{t^{\beta}}{1+t}.$$

Therefore by the Ratio test,

$$\lim_{T \to \infty} q^{\beta}(q-1) \sum_{t \in [1,T)_{q^{\mathbb{N}_0}}} \frac{t^{\beta}}{1+t} = q^{\beta}(q-1) \sum_{n=0}^{\infty} \frac{(q^n)^{\beta}}{(1+q^n)} < \infty$$

gives $K_{\beta} < \infty$. It can also be verified that $\left(1+t, \frac{1}{t+1} \right)$ is a nonoscillatory solution of Eq. (26) in $M_{\infty,0}^{+}$.

Exercise 5.7 Show that the following system

$$\begin{cases} x' = e^{2t} |y|^{\frac{1}{\alpha}} \mathrm{sgn}\, y \\ y' = -\alpha e^{-t(\alpha+\beta)} |x|^{\beta} \mathrm{sgn}\, x \end{cases}$$

has a nonoscillatory solution $(e^t, e^{-\alpha t})$ in $M_{\infty,0}^{+}$.

Next, we intend to derive a conclusion for the existence of nonoscillatory solutions of system (20) based on α and β. The proof of the following lemma is similar to the proofs of Lemmas 1.1, 3.2, 3.3, 3.6 and 3.7 in [47].

Lemma 5.8

a. *If $J_{\alpha} < \infty$, or $K_{\beta} < \infty$ then $Z_b < \infty$.*

b. *If $K_{\beta} = \infty$, then $Y(t_0) = \infty$ or $Z(t_0) = \infty$.*

c. *If $J_{\alpha} = \infty$, then $Y(t_0) = \infty$ or $Z(t_0) = \infty$.*

d. *Let $\alpha \geq 1$. If $J_{\alpha} < \infty$, then $K_{\alpha} < \infty$.*

e. *Let $\beta \leq 1$. If $K_{\beta} < \infty$, then $J_{\beta} < \infty$.*

f. *Let $\alpha < \beta$. If $K_{\beta} < \infty$, then $J_{\alpha} < \infty$ and $K_{\alpha} < \infty$.*

g. *Let $\alpha > \beta$. If $J_{\alpha} < \infty$, then $K_{\beta} < \infty$ and $J_{\beta} < \infty$.*

Exercise 5.9 Prove Lemma 5.8.

The following corollary summarizes the existence of subdominant and dominant solutions of system (20) in this subsection by means of Lemma 5.8.

Corollary 5.10 *Suppose that $Y(t_0) = \infty$ and $Z(t_0) < \infty$. Then*

a. *$M_{B,0}^{+} \neq \emptyset$ if any of the followings hold:*

   (i) $J_{\alpha} < \infty$, (ii) $\alpha < \beta$, $\beta \geq 1$ and $J_{\beta} < \infty$,

(iii) $\alpha < \beta$ and $K_\beta < \infty$, **(iv)** $\alpha \leq 1$ and $K_\alpha < \infty$.

b. $M^+_{\infty, B} \neq \varnothing$ if any of the followings hold:

(i) $K_\beta < \infty$, **(ii)** $\alpha \geq 1$ and $J_\beta < \infty$,

(iii) $\alpha > \beta$ and $J_\alpha < \infty$.

5.2. The Case $Y(t_0) < \infty$ and $Z(t_0) < \infty$

With the similar discussion as in Subsection 4.2, we concentrate on $M^+_{B,B}$ and $M^+_{B,0}$. Actually, the existence in $M^+_{B,0}$ is shown in Subsection 5.1. Also, we use the same argument of the proof of Lemma 3.1(a) so that the criteria for the existence of nonoscillatory solutions of system (20) in $M^+_{B,B}$ is $Y(t_0) < \infty$ and $Z(t_0) < \infty$.

The most important question that arose in this section is about the existence of nonoscillatory solutions of the Emden-Fowler system in M^-. The existence of such solutions in $M^-_{B,\infty}$, $M^-_{0,\infty}$ can similarly be shown as in Theorems 3.7 and 3.9. When concerns about and $M^-_{0,B}$ come to our attention, we need to assume that σ must be differentiable, which is not necessarily true on arbitrary time scales, see Example 1.56 in [6]. The following exercise is a great observation about the discussion mentioned above.

Exercise 5.11 Consider the system

$$\begin{cases} x^\Delta(t) = \dfrac{t^{\frac{1}{2}}}{2(t+1)(t+2)(3t-1)^{\frac{1}{2}}} |y(t)|^{\frac{1}{2}} \mathrm{sgn}\, y(t) \\ y^\Delta(t) = -\dfrac{(t+1)^{\frac{1}{3}}}{2^{\frac{2}{3}}t^2(4t+5)^{\frac{1}{3}}} |x^\sigma(t)|^{\frac{1}{3}} \mathrm{sgn}\, x^\sigma(t) \end{cases} \tag{27}$$

in $\mathbb{T} = 2^{\mathbb{N}_0}$ and show that $(2 + \frac{1}{t+2}, -3 + \frac{1}{t})$ is a nonoscillatory solution of system (27) in $M^-_{B,B}$. Note that $\sigma(t) = 2t$ is differentiable on $\mathbb{T} = 2^{\mathbb{N}_0}$.

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