Chapter 7

Recent Fixed Point Techniques in Fractional Set-Valued Dynamical Systems

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Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/67069

Abstract

In this chapter, we present a recollection of fixed point theorems and their applications in fractional set-valued dynamical systems. In particular, the fractional systems are used in describing many natural phenomena and also vastly used in engineering. We consider mainly two conditions in approaching the problem. The first condition is about the cyclicity of the involved operator and this one takes place in ordinary metric spaces. In the latter case, we develop a new fundamental theorem in modular metric spaces and apply to show solvability of fractional set-valued dynamical systems.

Keywords: fractional set-valued dynamical system, fixed point theory, contraction, modular metric space

1. Introduction

Dynamical system is a wide area that deals with a system that changes over time. The two main characteristics of the time domain here are identified with the discrete and continuous manners. In discrete time domain, major considerations turn to the difference equations and generating functions. While in the latter one, which we shall be considering mainly for this chapter, the system is usually represented by differential equations. It might be more influential to talk about the inclusion problems if a set-valued system is to be analyzed.

The very first and fundamental dynamical system is known nowadays under the term Cauchy problem. It is represented with the following C^1 initial-valued problem:

$$\begin{cases} u'(t) = f(t, u(t)), \\ u(0) = u_0 \end{cases}$$

In this case, we assume that $f : [0, T] \times R \to R$ is continuous and $u \in C^1([0, T])$. From simple calculus, we may see that this system is equivalent to the following integral equation:

$$u(t) = u_0 + \int_{[0,t]} f(s, u(s))ds \tag{1}$$

This is where Banach got the idea to solve the problem. He proposed his famous fixed point theorem known today as the contraction principle in 1922 [1], mainly to solve this Cauchy problem effectively. Recall that the contraction principle states that if X is a complete metric space and $T : X \to X$ is Lipschitz continuous with constant $0 < L < 1$, then T has a unique fixed point.

Let us consider a map $\Lambda : C^1([0, T]) \to C^1([0, T])$ given by

$$\Lambda(u)(t) := u_0 + \int_{[0,t]} f(s, u(s))ds, \quad \forall u \in C^1([0, T]),\ \forall t \in [0, T]$$

One can notice that $u \in C^1([0, T])$ solves Eq. (1) if and only if it is a fixed point of Λ. With this approach, by considering $C^1([0, T])$ with the supremum norm $\|\cdot\|_\infty$, we end up with the local solvability of the Cauchy problem. To obtain the global solution, we have to apply some techniques to extend the boundary of the local solution.

It is not very obvious that renorming by the L-weighted norm $\|f\|_L := \sup_{t \in [0,T]} e^{-Lt} f(t)$, with $L > 0$, will resolve such difficulty. We shall give the short solvability result of the Cauchy problem with the contraction principle here, to illustrate the concept of how we apply fixed point theorem to continuous dynamical systems. Under the assumption that f must be Lipschitz in the second variable with constant $L > 0$, we have for any $x, y \in C^1([0, T])$ the following:

$$\begin{aligned} e^{-Lt}|\Lambda(x)(t) - \Lambda(y)(t)| &= e^{-Lt}\left|\int_{[0,t]} f(s, x(s)) - f(s, y(s))ds\right| \\ &\le e^{-Lt}\int_{[0,t]} |f(s, x(s)) - f(s, y(s))|ds \\ &\le e^{-Lt}\int_{[0,t]} Le^{Ls}e^{-Ls}|x(s) - y(s)|ds \\ &\le e^{-Lt}\|x-y\|_L \int_{[0,t]} Le^{Ls}e^{-Ls}ds \\ &\le e^{-Lt}(e^{Lt}-1)\|x-y\|_L \\ &\le (1-e^{-LT})\|x-y\|_L. \end{aligned}$$

Taking supremum over $t \in [0, T]$ yields the result and the solvability thus follows.

http://dx.doi.org/10.5772/67069

This is the alternative technique to guarantee the solvability of the Cauchy problem, without obtaining the local solution first. It is important to remark that there are many mathematicians that can later adapt different technique and different direction to obtain the solvability of various classes of dynamical systems, under one unifying fact—by applying fixed point theorems.

It is natural to raise the situation of set-valued integral, which proved itself for its importance in practical applications especially in engineering. In 1965, Aumann [2] introduced the concept of definite set-valued integral on real line and Euclidean spaces. Suppose that Ψ is an interval $[0, T]$, where $T > 0$. Let $F : \Psi \to 2^{\mathbb{R}}$ be a set-valued operator. A selection of F is the function $f : \Psi \to \mathbb{R} \cup \{\pm\infty\}$ such that $f(t) \in F(t)$ a.e. $t \in \Psi$. We write $\mathscr{F}$ to denote the set containing all integrable selections of F. According to Aumann [2], the set-valued integral is determined by the operator J in the following:

$$\mathrm{J}_{\Psi} F(t)dt := \left\{ \int_{\Psi} f(t)dt \,;\, f \in \mathscr{F} \right\}$$

that is, the set of the integrals of integrable selections of F.

On the other hand, in elementary calculus, one deals with derivatives and integrals, including the higher-integer-order iterations. Here, in fractional integral, one looks at a broader concept where the real-order iteration is taken into account. There are many approaches to study this kind of extensions. In our context, we shall use the classical notion introduced by Riemann and Liouville, the latter of which is the first one to point out the possibility of fractional calculus in 1832. Given a function $f \in L^1(\Psi, \mu)$, the fractional integral of order $\alpha > 0$ is given by

$$\mathrm{I}^{\alpha}_{\Psi} f(t)dt := \frac{1}{\Gamma(\alpha)} \int_{\Psi} (t-\tau)^{\alpha-1} f(\tau)d\tau$$

Naturally, we may further consider the following fractional integral:

$$\mathrm{J}^{\alpha}_{\Psi} F(t)dt := \left\{ \mathrm{I}^{\alpha}_{\Psi} f(t)dt \,;\, f \in \mathscr{F} \right\}$$

These two concepts have brought up the studies of new systems, the set-valued dynamical systems and the fractional dynamical systems. Even the combination of the two, the fractional set-valued dynamical systems, is an emerging area in research. We shall be particular with this latter class of systems and give some brief investigations over the problem.

The very concept of set-valued fractional integral operator was first proposed by El-Sayed and Ibrahim [3–5] and this has opened a new universe of investigation to fractional operator equations. It has been reflected that such theory can better describe nonlinear phenomena, compared to the classical theory of differential and integral equations. The extensive use of this theory lays naturally in automatic control theory, network theory and dynamical systems (see, e.g. [6–10]).

The central system that we are going to investigate in this chapter is the following delayed system:

$$u(t)-\sum_{i=1}^{n}\beta_i(t)u(t-\tau_i)\in \mathrm{I}^{\alpha}F(t,u(t))\ ;\quad \alpha\in(0,1],\ t\in J:=[0,T],\ T>0 \tag{2}$$

where $\tau_i\in[0,t]$ for all $i\in\{1,2,\cdots,n\}$, $F:J\times\mathbb{R}\to CB(\mathbb{R})$, $\mathrm{I}^{\alpha}F(t,u(t))$ is the definite integral of order α given by

$$\mathrm{I}^{\alpha}F(t,u(t)):=\left\{\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau\ ;\ f\in S_F(u)\right\}$$

and

$$S_F(u):=\{f\in L^1(J,\mathbb{R})\ ;\ f(t)\in F(t,u(t))\ \text{ a.e. }\ t\in J\}$$

denotes the set of selections of F and $\beta_i:J\to\mathbb{R}$ is continuous for each $i\in\{1,2,\cdots,n\}$. Also, set $B:=\max_{1\leq i\leq n}\sup_{t\in J}\beta_i(t)$.

In this chapter, we shall bring up some recent results in fixed point theory in several approaches and then show how these theorems apply to different classes of dynamical systems. Going precise, in Section 2, we investigate the system (2) in standard metric spaces through a newly developed fixed point theorem. The mentioned fixed point theorem deals with an operator that satisfied the so-called implicit contractivity condition only on a portion of a space, where such partial partition is obtained from the cyclicity behavior that we imposed. We also note the relation between this cyclicity behavior and the one that arises from the partial ordering relation approach. The solvability of the dynamical system (2) in this section is naturally obtained via the cyclicity and implicit contractivity assumptions. For further readings related to this topic, consult [11–17]. In Section 3, we consider a newly emerged approach of studying fixed point theory, i.e., fixed point theory in modular metric spaces. This theory has only been introduced to researchers only a few years ago and has been investigated reasonably in such a short duration. We bring up one of the fundamental fixed point theorem in this modular metric spaces, give appropriate examples and then apply it to guarantee the solvability of, again, the system (2). Even the studies of modular metric spaces are relatively limited at the time, we suggest that further readings from Refs. [18–20] should give some ideas about the theory itself and also how to develop further dynamical systems in this framework.

2. Cyclic operators in metric spaces

In this section, we consider a very general class of operators that satisfy the implicit contractivity condition. Moreover, we also assume the operator to be cyclic over its domain. This cyclicity weakens the contractivity only to a portion of the space. This is a more general case than the contractivity on comparable pairs, as we show later in this chapter. This also allows the coexistence result that is better than the exact solution and the sub-/super-solution.

Note that results in this section are based on our paper [21]. Recall the following notion of cyclic operators.

Definition 2.1. Let X be a nonempty set and $A_1, A_2, \cdots, A_p$ be nonempty subsets of X. An operator $F : \cup_{k=1}^{p} A_k \to 2^{\cup_{k=1}^{p} A_k}$ is called a phset-valued cyclic operator over $\cup_{k=1}^{p} A_k$ if $F(A_i) \subseteq A_{i+1}$ for all $i \in \{1, 2, \cdots, p-1\}$ and $F(A_p) \subseteq A_1$.

There is a special property about the location of fixed point of this operator, as illustrated in the following.

Proposition 2.2. *Let* $_X$ *be a nonempty set and* $A_1, A_2, \cdots, A_p$ *be nonempty subsets of X. If F is a set-valued cyclic operator over* $\cup_{k=1}^{p} A_k$*, then we have the inclusion* $\mathrm{Fix}(F) \subseteq \cap_{k=1}^{p} A_k$*, where* $\mathrm{Fix}(F)$ *denotes the fixed point set of F.*

Proof. If either $Fix(F) = ptyset$ or $\cap_{k=1}^{p} A_k = ptyset$, the conclusion is clear. Thus, let $z \in \cup_{k=1}^{p} A_k$ be a fixed point of F. Then, $z \in A_q$ for some $q \in \{1, 2, \cdots, p\}$ and $z \in Fz \subseteq A_{q+1}$. Consequently, we also have $z \in Fz \subseteq A_{q+2}$. It is easy to see that $z \in A_{q+n}$ for all $n \in \mathbb{N}$. Therefore, it is enough to conclude that $z \in \cap_{k=1}^{p} A_k$.

The following classes of functions are necessary to our further contents.

Definition 2.3. Let Φ be the class of functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(Φ1) φ is right continuous.

(Φ2) $\varphi(0) = 0$.

(Φ3) $\varphi(t) < t$ for all $t > 0$.

Definition 2.4. Let Ψ be the class of functions $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(Ψ1) ψ is continuous.

(Ψ2) ψ is nondecreasing in the first variable and is nonincreasing in the remaining variables.

(Ψ3) There exists a function $\varphi \in \Phi$ such that, for all $u, v \geq 0$, either $\psi(u, v, u, v, 0, u + v) \leq 0$ or $\psi(u, v, 0, 0, u, v) \leq 0$ implies that $u \leq \varphi(v)$.

(Ψ4) $\psi(u, 0, u, 0, 0, u), \psi(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

Remark 2.5. If $\varphi \in \Phi$, then $\varphi^n(t) \to 0$.

Example 2.6 ([22]). The following functions are contained in the class Ψ:

a. $\psi_1(t_1, t_2, \cdots, t_6) := t_1 - \alpha \max\{t_2, t_3, t_4\} - (1-\alpha)[at_5 + bt_6]$, where $\alpha \in [0, 1)$ and $a, b \in \left[0, \frac{1}{2}\right)$.

b. $\psi_2(t_1, t_2, \cdots, t_6) := t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, \frac{1}{2}[t_5 + t_6]\right\}\right)$, where $\varphi \in \Phi$.

c. $\psi_3(t_1, t_2, \cdots, t_6) := t_1^2 - t_1(\alpha t_2 + \beta t_3 \gamma t_4) - \delta t_5 t_6$, where $\alpha > 0$ and $\beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma < 1$ and $\alpha + \delta < 1$.

2.1. Fixed point theorem for cyclic operators

Now, we give the main fixed point theorem for cyclic implicit contractive operators.

THEOREM 2.7. *Let (X, d) be a complete metric space and let $A_1, A_2, \ldots, A_p$ be nonempty closed subsets of X. Suppose that F is a proximal set-valued cyclic operator over $\cup_{k=1}^{p} A_k$ in which there exists some $\psi \in \Psi$ satisfying*

$$\psi(H(Fx, Fy), d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)) \leq 0$$

whenever either $(x, y) \in A_i \times A_{i+1}$ or $(x, y) \in A_{i+1} \times A_i$ holds for some $i \in \{1, 2, \cdots, p\}$. Then, we have the following:

(I) *F has at least one fixed point;*

(II) *F has no fixed point outside $\cap_{k=1}^{p} A_k$.*

PROOF. For (I), let x_0 be chosen arbitrarily from some A_j. Choose any $x_1 \in Fx_0$. Then, we define implicitly a sequence (x_n) by choosing $x_{n+1} \in Fx_n$ satisfying

$$d(x_n, x_{n+1}) = d(x_n, Fx_n).$$

Note that this definition is valid since F is a proximal operator. Also note that by this definition, we may derive that

$$d(x_n, x_{n+1}) \leq H(Fx_{n-1}, Fx_n) \tag{3}$$

Now, since $(x_{n+1}, x_n) \in A_{j+n+1} \times A_{j+n}$, we have

$$\begin{aligned} 0 &\geq \psi\begin{pmatrix} H(Fx_{n+1}, Fx_n), d(x_{n+1}, x_n), d(x_{n+1}, Fx_{n+1}), \\ d(x_n, Fx_n), d(x_{n+1}, Fx_n), d(x_n, Fx_{n+1}) \end{pmatrix} \\ &\geq \psi\begin{pmatrix} H(Fx_n, Fx_{n+1}), d(x_n, x_{n+1}), H(Fx_n, Fx_{n+1}), \\ d(x_n, x_{n+1}), 0, d(x_n, x_{n+1}) + H(Fx_n, Fx_{n+1}) \end{pmatrix} \end{aligned}$$

Suppose that $\varphi \in \Phi$ is chosen according to $(\Psi 3)$. Thus, we have

$$H(Fx_n, Fx_{n+1}) \leq \varphi(d(x_n, x_{n+1}))$$

At this point, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, otherwise a fixed point is already obtained. Together with Eq. (3), we may deduce that

$$d(x_n, x_{n+1}) \leq H(Fx_{n-1}, Fx_n) \leq \varphi(d(x_{n-1}, x_n)) \leq \cdots \leq \varphi^{n-1}(d(x_0, x_1))$$

Therefore, we have immediately that $d(x_n, x_{n+1}) \to 0$.

Next, we show that (x_n) is Cauchy. Suppose to the contrary. So, we may find $\varepsilon_0 > 0$ and two strictly increasing sequences of integers (m_k) and (n_k) in which

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon_0$$

We can assume, without loss of generality, that $n_k > m_k > k$ and n_k is minimal in the sense that $d(x_{m_k}, x_r) < \varepsilon_0$ for all $m_k \leq r < n_k$.

Consequently, $d(x_{m_k}, x_{n_k-1}) < \varepsilon_0$. Moreover, we may obtain that $\varepsilon_0 \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) < \varepsilon_0 + d(x_{n_k-1}, x_{n_k})$. Letting $k \to \infty$, we have $d(x_{m_k}, x_{n_k}) \to \varepsilon_0$.

On the other hand, for each $k \in \mathbb{N}$, we may find $j_k \in \{1, 2, \cdots, p\}$ in which $n_k - m_k + j_k \equiv 1 (\mathrm{mod} p)$. For k sufficiently large, we may see that $m_k - j_k > 0$. Observe that

$$\begin{aligned} |d(x_{m_k-j_k}, x_{n_k}) - d(x_{n_k}, x_{m_k})| &\leq d(x_{m_k-j_k}, x_{m_k}) \\ &\leq \sum_{l=0}^{j_k-1} d(x_{m_k-j_k+l}, x_{m_k-j_k+l+1}) \\ &\leq \sum_{l=0}^{p-1} d(x_{m_k-j_k+l}, x_{m_k-j_k+l+1}) \end{aligned}$$

Letting $k \to \infty$, we have $d(x_{m_k-j_k}, x_{n_k}) \to \varepsilon_0$. Also consider that

$$|d(x_{n_k}, x_{m_k-j_k}) - d(x_{m_k-j_k}, x_{n_k+1})| \leq d(x_{n_k}, x_{n_k+1}).$$

As $k \to \infty$, we have $d(x_{m_k-j_k}, x_{n_k+1}) \to \varepsilon_0$. Similarly, we have

$$|d(x_{m_k-j_k}, x_{n_k}) - d(x_{n_k}, x_{m_k-j_k+1})| \leq d(x_{m_k-j_k}, x_{m_k-j_k+1}).$$

So, we get $d(x_{n_k}, x_{m_k-j_k+1}) \to \varepsilon_0$ as $k \to \infty$. Also observe that

$$|d(x_{n_k}, x_{n_k+1}) - d(x_{n_k+1}, x_{m_k-j_k+1})| \leq d(x_{n_k}, x_{m_k-j_k+1}).$$

Again, letting $k \to \infty$, we obtain that $d(x_{n_k+1}, x_{m_k-j_k+1}) \to \varepsilon_0$. Finally, by the fact that $(x_{m_k-j_k}, x_{n_k}) \in A_i \times A_{i+1}$ for some $i \in \{1, 2, \cdots, p\}$ and Eq. (3), we may obtain that

$$\begin{aligned} 0 &\geq \psi\begin{pmatrix} H(Fx_{m_k-j_k}, Fx_{n_k}), d(x_{m_k-j_k}, x_{n_k}), d(x_{m_k-j_k}, Fx_{m_k-j_k}), \\ d(x_{n_k}, Fx_{n_k}), d(x_{m_k-j_k}, Fx_{n_k}), d(x_{n_k}, Fx_{m_k-j_k}) \end{pmatrix} \\ &\geq \psi\begin{pmatrix} d(x_{m_k-j_k+1}, x_{n_k+1}), d(x_{m_k-j_k}, x_{n_k}), d(x_{m_k-j_k}, x_{m_k-j_k+1}), d(x_{n_k}, x_{n_k+1}), \\ d(x_{m_k-j_k}, x_{n_k+1}), d(x_{n_k}, x_{m_k-j_k}) + d(x_{m_k-j_k}, Fx_{m_k-j_k}) \end{pmatrix} \\ &= \psi\begin{pmatrix} d(x_{m_k-j_k+1}, x_{n_k+1}), d(x_{m_k-j_k}, x_{n_k}), d(x_{m_k-j_k}, x_{m_k-j_k+1}), d(x_{n_k}, x_{n_k+1}), \\ d(x_{m_k-j_k}, x_{n_k+1}), d(x_{n_k}, x_{m_k-j_k}) + d(x_{m_k-j_k}, x_{m_k-j_k+1}) \end{pmatrix} \end{aligned}$$

By the condition $(\Psi 4)$ and letting $k \to \infty$, we may deduce that

$$0 \geq \psi(\varepsilon_0, \varepsilon_0, 0, 0, \varepsilon_0, \varepsilon_0) > 0$$

which is absurd. Hence, the sequence (x_n) is Cauchy. Since $\cup_{k=1}^{p} A_k$ is closed, it is complete and therefore (x_n) converges to some unique point $x_* \in \cup_{k=1}^{p} A_k$.

Next, we shall prove that x_* is, in fact, a fixed point of F. Let us assume now that $d(x_*, Fx_*) > 0$. Note that for any $n \in \mathbb{N}$, $(x_*, x_n) \in A_i \times A_{i+1}$ for some $i \in \{1, 2, \cdots, p\}$. So, it is followed that

$$\begin{aligned} 0 &\geq \psi(H(Fx_*, Fx_n), d(x_*, x_n), d(x_*, Fx_*), d(x_n, Fx_n), d(x_*, Fx_n), d(x_n, Fx_*)) \\ &\geq \psi\begin{pmatrix} d(x_{n+1}, Fx_*), d(x_*, x_n), d(x_*, Fx_*), d(x_n, x_{n+1}), \\ d(x_*, x_n) + d(x_n, Fx_n), d(x_n, Fx_*) \end{pmatrix} \\ &= \psi\begin{pmatrix} d(x_{n+1}, Fx_*), d(x_*, x_n), d(x_*, Fx_*), d(x_n, x_{n+1}), \\ d(x_*, x_n) + d(x_n, x_{n+1}), d(x_n, Fx_*) \end{pmatrix} \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain that

$$0 \geq \psi(d(x_*, Fx_*), 0, d(x_*, Fx_*), 0, 0, d(x_*, Fx_*)) > 0$$

which is absurd. Therefore, $d(x_*, Fx_*) = 0$. Since Fx_* is closed, we conclude that $x_* \in Fx_*$.

To obtain (II), apply Proposition 2.2.

2.2. Ordered spaces as corollaries

Let X be a nonempty set, recall that the binary relation $\overset{\text{Ê}}{\text{a}}$ is said to be a ph(partial) ordering on X if it is reflexive, antisymmetric and transitive. By an phordered set, we shall mean the pair $(X, \sqsubseteq)$ where X is nonempty and $\sqsubseteq$ is an ordering on X. A ph(partially) ordered metric space is the triple $(X, \sqsubseteq, d)$, where $(X, \sqsubseteq)$ is an ordered set and (X, d) is a metric space.

In this part, we show that contractivity on comparable pairs is particularly a cyclic operator over a single set. The following general assumption on the ordered structure is central in the few forthcoming theorems.

Definition 2.8. Let $(X, \sqsubseteq, d)$ is said to satisfies the phcondition (Θ) if every convergent sequence (x_n) in X and every point $z_0 \in X$ such that $z_0 \sqsubseteq x_n$ for all $n \in \mathbb{N}$, there holds the property $z_0 \sqsubseteq x_*$, where $x_* \in X$ is the limit of (x_n).

Theorem 2.9. *Let $(X, \sqsubseteq, d)$ be a complete ordered metric space satisfying the condition (Θ) and let $F : X \to CB(X)$ be a nondecreasing proximal operator in the sense that if $x, y \in X$ satisfies $x \sqsubseteq y$, then $u \sqsubseteq v$ for all $u \in Fx$ and $v \in Fy$. Suppose that there exists $\psi \in \Psi$ such that*

$$\psi(H(Fx, Fy), d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)) \leq 0 \tag{4}$$

for all $x, y \in X$ in which we can find some $z \in X$ satisfying both $z \sqsubseteq x$ and $z \sqsubseteq y$. If there exists $x_0 \in X$ such that $x_0 \sqsubseteq w$ for all $w \in Fx_0$, then F has at least one fixed point.

Proof. By the existence of such a point x_0, we shall now construct a set

$$C(x_0) := \{z \in X\ ;\ x_0 \sqsubseteq z\}$$

Taking any sequence (x_n) in $C(x_0)$. By the condition (Θ) with $z_0 := x_0$, we may see that if (x_n) converges, its limit is also included in $C(x_0)$. Hence, $C(x_0)$ is closed and therefore it is complete.

On the other hand, we define an operator $G : C(x_0) \to CB(X)$ by

$$G := F|_{C(x_0)}.$$

For any $z \in C(x_0)$, observe that $x_0 \sqsubseteq w$ for all $w \in Gz$. Thus, $G(C(x_0)) \subseteq C(x_0)$ so that G is cyclic over $C(x_0)$. Moreover, for any $x, y \in C(x_0)$, we have by definition that $x_0 \sqsubseteq x$ and $x_0 \sqsubseteq y$, so that the inequality (4) holds whenever $(x, y) \in C(x_0) \times C(x_0)$. Therefore, we can now apply Theorem 2.7 to obtain that G has at least one fixed point. Passing this property to F, we have now proved the theorem.

Corollary 2.10. *Let $(X, \sqsubseteq, d)$ be a complete ordered metric space and let $F : X \to CB(X)$ be a nondecreasing proximal operator in the sense that if $x, y \in X$ satisfies $x \sqsubseteq y$, then $u \sqsubseteq v$ for all $u \in Fx$ and $v \in Fy$. Suppose that there exists $\psi \in \Psi$ such that*

$$\psi(H(Fx, Fy), d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)) \leq 0$$

whenever $x, y \in X$ satisfy $x \sqsubseteq y$. Also assume that if the sequence (x_n) in X is nondecreasing and converges to $x_* \in X$, then $x_n \sqsubseteq x_*$ for all $n \in \mathbb{N}$. If there exists $x_0 \in X$ such that $x_0 \sqsubseteq w$ for all $w \in Fx_0$, then F has at least one fixed point.

Proof. Note that if $x, y \in X$ are comparable, then, according to Theorem 2.9, we may choose $z := x \in X$ so that $z \sqsubseteq x$ and $z \sqsubseteq y$.

On the other hand, let (y_n) be a sequence in X which is both nondecreasing and convergent to $y_* \in X$. According to the condition (Θ), set $z_0 := y_1$. We may see easily that, in this case, X satisfies the condition (Θ). We next apply Theorem 2.9 to finish the proof.

2.3. An example

We now give a validating example for our fixed point theorem to help the understanding of the content.

Example 2.11. Consider the Euclidean space E^2 with its standard metric d. For each $t \in \mathbb{R}$, we define

$$\ell_0 := [0, \tfrac{1}{2}] \times \{0\}, \quad \ell_1 := [0, \tfrac{1}{2}] \times \{\tfrac{1}{\sqrt{2}}\}, \quad \text{and} \quad \ell_2 := [0, \tfrac{1}{2}] \times \{-\tfrac{1}{\sqrt{2}}\}.$$

Suppose that A_1 and A_2 are two closed sets defined by

$$A_1 := \ell_0 \cup \ell_1 \quad \text{and} \quad A_2 := \ell_0 \cup \ell_2.$$

Let $F : A_1 \cup A_2 \to 2^{A_1 \cup A_2}$ be an operator defined by

$$Fx := \begin{cases} \{x\}, & \text{if} x \in \ell_0, \\ P_{\ell_1}^{-1}(x) \cap A_2, & \text{if} x \in \ell_1, \\ P_{\ell_2}^{-1}(x) \cap A_1, & \text{if} x \in \ell_2. \end{cases} \tag{5}$$

Note that the notation P as is appeared in Eq. (5) is the metric projection onto the corresponding sets ℓ_1 and ℓ_2, respectively. The cyclicity of F is apparent.

Claim. The operator F satisfies the inequality in Theorem 2.7 with ψ defined as in (c) of Example 2.6 when $\alpha = \frac{9}{20}$, $\beta = \gamma = \frac{1}{4}$ and $\delta = \frac{1}{2}$.

The case $x, y \in \ell_0$ is trivial and so we omit it. For the case $x \in \ell_0$ as $y \in \ell_1$ and $x \in \ell_1$ as $y \in \ell_2$, we consider the following calculation.

From **Table 1**(A), we have

$$\begin{aligned}
&[H(Fx, Fy)]^2 \\
&= (x_1 - y_1)^2 + \tfrac{1}{2} \\
&\leq \left(\tfrac{9}{20} + \tfrac{1}{4\sqrt{2}} + \tfrac{1}{2}\right)\left((x_1 - y_1)^2 + \tfrac{1}{2}\right) \\
&\leq \tfrac{9}{20}\left((x_1 - y_1)^2 + \tfrac{1}{2}\right) + \tfrac{1}{4\sqrt{2}}\sqrt{(x_1 - y_1)^2 + \tfrac{1}{2}} + \tfrac{1}{2}\left((x_1 - y_1)^2 + \tfrac{1}{2}\right) \\
&= \sqrt{(x_1 - y_1)^2 + \tfrac{1}{2}}\left(\tfrac{9}{20}\sqrt{(x_1 - y_1)^2 + \tfrac{1}{2}} + \tfrac{1}{4\sqrt{2}}\right) + \tfrac{1}{2}\left((x_1 - y_1)^2 + \tfrac{1}{2}\right) \\
&= H(Fx, Fy)[\alpha d(x, y) + \beta d(x, Fx) + \gamma d(y, Fy)] + \delta d(x, Fy) d(y, Fx)
\end{aligned}$$

for all $x \in \ell_0$ and $y \in \ell_1$. We can similarly obtain from **Table 1**(B) the following:

(A) $x \in \ell_0$ as $y \in \ell_1$	
$H(Fx, Fy)$	$\sqrt{(x_1 - y_1)^2 + 1/2}$
$d(x, y)$	$\sqrt{(x_1 - y_1)^2 + 1/2}$
$d(x, Tx)$	0
$d(y, Ty)$	$1/\sqrt{2}$
$d(x, Ty)$	$\sqrt{(x_1 - y_1)^2 + 1/2}$
$d(y, Tx)$	$\sqrt{(x_1 - y_1)^2 + 1/2}$
(B) $x \in \ell_1$ as $y \in \ell_2$	
$H(Fx, Fy)$	$\sqrt{(x_1 - y_1)^2 + 1/2}$
$d(x, y)$	$\sqrt{(x_1 - y_1)^2 + 2}$
$d(x, Tx)$	1
$d(y, Ty)$	1
$d(x, Ty)$	$\lvert x_1 - y_1 \rvert$
$d(y, Tx)$	$\lvert x_1 - y_1 \rvert$

Table 1. Distances.

$$
\begin{aligned}
&[H(Fx,Fy)]^2 \\
&= (x_1-y_1)^2 + \tfrac{1}{2} \\
&\leq \left(\tfrac{9}{20}\sqrt{\tfrac{5}{2}} + \sqrt{2}\right)\left((x_1-y_1)^2 + \tfrac{1}{2}\right) \\
&\leq \left(\tfrac{9}{20}\sqrt{\tfrac{5}{2}}\right)\left((x_1-y_1)^2 + \tfrac{1}{2}\right) + \sqrt{2}\cdot\sqrt{(x_1-y_1)^2 + \tfrac{1}{2}} \\
&\leq \tfrac{9}{20}\sqrt{\tfrac{5}{2}\left((x_1-y_1)^2 + \tfrac{1}{2}\right)^2} + \sqrt{2}\cdot\sqrt{(x_1-y_1)^2 + \tfrac{1}{2}} \\
&= \tfrac{9}{20}\sqrt{\left((x_1-y_1)^2 + \tfrac{1}{2}\right)^2 + \tfrac{3}{2}\left((x_1-y_1)^2 + \tfrac{1}{2}\right)^2} + \sqrt{2}\cdot\sqrt{(x_1-y_1)^2 + \tfrac{1}{2}} \\
&\leq \tfrac{9}{20}\sqrt{\left((x_1-y_1)^2 + \tfrac{1}{2}\right)^2 + \tfrac{3}{2}\left((x_1-y_1)^2 + \tfrac{1}{2}\right)} + \sqrt{2}\cdot\sqrt{(x_1-y_1)^2 + \tfrac{1}{2}} \\
&= \tfrac{9}{20}\sqrt{\left((x_1-y_1)^2 + \tfrac{1}{2}\right)\left((x_1-y_1)^2 + \tfrac{1}{2} + \tfrac{3}{2}\right)} + \sqrt{2}\cdot\sqrt{(x_1-y_1)^2 + \tfrac{1}{2}} \\
&= \tfrac{9}{20}\sqrt{\left((x_1-y_1)^2 + \tfrac{1}{2}\right)\left((x_1-y_1)^2 + 2\right)} + \sqrt{2}\cdot\sqrt{(x_1-y_1)^2 + \tfrac{1}{2}} \\
&= \sqrt{(x_1-y_1)^2 + \tfrac{1}{2}}\left(\tfrac{9}{20}\sqrt{(x_1-y_1)^2 + 2} + \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right) \\
&= H(Fx,Fy)[\alpha d(x,y) + \beta d(x,Fx) + \gamma d(y,Fy)] \\
&\leq H(Fx,Fy)[\alpha d(x,y) + \beta d(x,Fx) + \gamma d(y,Fy)] + \delta d(x,Fy)d(y,Fx)
\end{aligned}
$$

for all $x \in \ell_1$ and $y \in \ell_2$. Therefore, we have now proved our claim.

Observe now that $Fix(F) = \ell_0 = A_1 \cap A_2$, coincide with the Theorem 2.7.

2.4. Fractional set-valued dynamical systems

For convenience, we shall always consider the nonempty closed and bounded subspace

$$\Omega \subset C(J,\mathbb{R}) := \{u : J \to \mathbb{R}\,;\ \text{uiscontinuous}\},$$

endowed with the supremum norm $\|\cdot\|$ given by

$$\|u\| := \sup_{t\in J}|u(t)|.$$

The solutions for the problem (2) are assumed to be in Ω under this circumstance. Moreover, we shall need some more notions in order to obtain the existence of solutions for the problem (2).

Definition 2.12. Let (X, d) be a metric space and let J be an interval of $\mathbb{R}$. An operator $F : J \to 2^X$ is said to be measurable if for each $x \in X$ and $t \in J$, the mapping $x \mapsto d(x, F(t))$ is measurable.

Next, we shall define the set-valued operator $\Lambda : \Omega \to 2^\Omega$ given by

$$(\Lambda u)(t) := \left\{ w \in \Omega \; ; \; w(t) = \sum_{i=1}^{n} \beta_i(t) u(t-\tau_i) + \mathbb{U}^\alpha f(t, u(t)), \; f \in S_F(u) \right\}, \tag{6}$$

where $\mathbb{U}$ is the ordinary single-valued fractional integral.

We shall next illustrate that the operator Λ possesses closed values.

LEMMA 2.13. *Suppose that the operator Λ is given as in* (2.4), *then Λu is closed for all $u \in \Omega$.*

PROOF. Let $u \in \Omega$ and let (u_k) be a sequence in Λu which converges to some $u_* \in \Omega$. We shall prove the statement by showing that limits of convergent sequence in Λu are in Λu. Then, there exists a sequence (f_k) in $S_F(u)$ in which

$$u_k(t) = \sum_{i=1}^{n} \beta_i(t) u(t-\tau_i) + \mathbb{U}^\alpha f_k(t, u(t)).$$

Also note that this sequence (f_k) converges to some $f_* \in L^1(J, \mathbb{R})$. Since $F(t, u(t))$ is closed, $f_* \in S_F(u)$. Actually, we have

$$u_*(t) = \sum_{i=1}^{n} \beta_i(t) u(t-\tau_i) + \mathbb{U}^\alpha f_*(t, u(t)) \in \Lambda u.$$

This completes the proof.

Now, we give the solvability of the system (2).

THEOREM 2.14. *According to Eq.* (2), *assume that there exist non-empty closed subsets $\Pi_1, \Pi_2, \cdots, \Pi_p$ in Ω such that $\cup_{k=1}^{p} \Pi_k = \Omega$ and F has the following properties:*

1. *$t \mapsto F(t, u(t))$ is measurable for each $u \in \Omega$;*
2. *there exists a function $\xi : \mathbb{R}_+^5 \to \mathbb{R}_+$ such that*

   $H(F(t, u(t)), F(t, v(t))) \le \xi(\|u-v\|, d(u, \Lambda u), d(v, \Lambda v), d(u, \Lambda v), d(v, \Lambda u))$ whenever either $(u, v) \in \Pi_i \times \Pi_{i+1}$ or $(u, v) \in \Pi_{i+1} \times \Pi_i$ holds for some $i \in \{1, 2, \cdots, p\}$;
3. *Λ is proximal and cyclic over $\cup_{k=1}^{p} \Pi_k = \Omega$.*

If the function $\psi : \mathbb{R}_+^6 \to \mathbb{R}_+$ given by

$$\psi(t_1, t_2, \cdots, t_6) := t_1 - nBt_2 - \frac{T^\alpha}{\Gamma(\alpha+1)} \xi(t_2, t_3, t_4, t_5, t_6)$$

is in the class Ψ, then the problem (1.2) *has at least one solution.*

PROOF. Let $(u, v) \in \Pi_i \times \Pi_{i+1}$ for some $i \in \{1, 2, \cdots, p\}$. By 2, we may choose some $f_1(t, u(t)) \in F(t, u(t))$ and $f_2(t, v(t)) \in F(t, v(t))$ in which

$$|f_1(t,u(t)) - f_2(t,v(t))| \le \xi(\|u-v\|, d(u,\Lambda u), d(v,\Lambda v), d(u,\Lambda v), d(v,\Lambda u))$$

Consider the two functions

$$w_1(t) = \sum_{i=1}^{n} \beta_i(t) u(t-\tau_i) + \mathbb{U}^{\alpha} f_1(t,u(t)) \in \Lambda u$$

and

$$w_2(t) = \sum_{i=1}^{n} \beta_i(t) v(t-\tau_i) + \mathbb{U}^{\alpha} f_2(t,v(t)) \in \Lambda v.$$

Next, observe that

$$\begin{aligned}
&|w_1(t) - w_2(t)| \\
&\quad \le \sum_{i=1}^{n} \beta_i(t) |u(t-\tau_i) - v(t-\tau_i)| + |\mathbb{U}^{\alpha} f_1(t,u(t)) - \mathbb{U}^{\alpha} f_2(t,v(t))| \\
&\quad \le \sum_{i=1}^{n} \beta_i(t) |u(t-\tau_i) - v(t-\tau_i)| + \mathbb{U}^{\alpha} |f_1(t,u(t)) - f_2(t,v(t))| \\
&\quad \le nB\|u-v\| + \frac{T^{\alpha}}{\Gamma(\alpha+1)} |f_1(t,u(t)) - f_2(t,v(t))| \\
&\quad \le nB\|u-v\| + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \xi(\|u-v\|, d(u,\Lambda u), d(v,\Lambda v), d(u,\Lambda v), d(v,\Lambda u))
\end{aligned}$$

It follows that

$$H(\Lambda u, \Lambda v) \le nB\|u-v\| + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \xi(\|u-v\|, d(u,\Lambda u), d(v,\Lambda v), d(u,\Lambda v), d(v,\Lambda u)).$$

Consequently, we have for each $(u,v) \in \Pi_i \times \Pi_{i+1}$, $i \in \{1, 2, \cdots, p\}$, that

$$\psi(H(\Lambda u, \Lambda v), \|u-v\|, d(u,\Lambda u), d(v,\Lambda v), d(u,\Lambda v), d(v,\Lambda u)) \le 0.$$

We may deduce similarly that the above inequality holds also in the case $(u,v) \in \Pi_{i+1} \times \Pi_i$. Apply Theorem 2.7 to obtain the desired result.

We next consider the existence of solutions to Eq. (2) in the case when an ordering $\sqsubseteq$ is defined on Ω in such a way that for $u, v \in \Omega$,

$$u \sqsubseteq v \Leftrightarrow u(t) \le v(t) \quad \text{a.e.} \quad t \in J$$

It is easy to see that if (u_n) is a nondecreasing sequence in Ω which converges to some $u_* \in \Omega$, then $u_n \sqsubseteq u_*$ for all $n \in \mathbb{N}$. In the further step, we shall need in the initial state that a weak solution to Eq. (2) exists.

Definition 2.15. Suppose that $(\Omega, \sqsubseteq)$ is a partially ordered set. A phweak solution for the problem (2) (w.r.t. $\sqsubseteq$) is a function $u \in \Omega$ such that $u \sqsubseteq v$ for all $v \in \Lambda u$.

Corollary 2.16. *According to Eq.* (2)*, assume that there is an ordering* $\sqsubseteq$ *defined on* Ω*. Suppose also that we have the following properties:*

1. $t \mapsto F(t, u(t))$ *is measurable for each* $u \in \Omega$*;*
2. *there exists a function* $\xi : \mathbb{R}_+^5 \to \mathbb{R}_+$ *such that*

   $H(F(t, u(t)), F(t, v(t))) \leq \xi(\|u - v\|, d(u, \Lambda u), d(v, \Lambda v), d(u, \Lambda v), d(v, \Lambda u))$ *whenever* $u, v \in \Omega$ *are comparable;*
3. Λ *is proximal and nondecreasing;*
4. *a weak solution* $u_0 \in \Omega$ *to the problem* (2) *exists.*

If the function $\psi : \mathbb{R}_+^6 \to \mathbb{R}_+$ *given by*

$$\psi(t_1, t_2, \cdots, t_6) := t_1 - nBt_2 - \frac{T^\alpha}{\Gamma(\alpha + 1)} \xi(t_2, t_3, t_4, t_5, t_6)$$

is in the class Ψ*, then the problem* (2) *has at least one solution.*

Proof. As in the proof of the previous theorem, we may similarly derive that

$$\psi(H(\Lambda u, \Lambda v), \|u - v\|, d(u, \Lambda u), d(v, \Lambda v), d(u, \Lambda v), d(v, \Lambda u)) \leq 0$$

whenever $u, v \in \Omega$ are comparable. Therefore, we may apply Corollary 2.10 to obtain the desired result.

3. Fractional set-valued systems in modular metric spaces

In this section, we shall consider on fixed point inclusions that are studied within a modular metric spaces. With certain conditions, we can extend Nadler's theorem to the context of modular metric spaces successfully. A modular metric space is a relatively new concept. It generalizes and unifies both modular and metric spaces. It is therefore not necessarily equipped with a linear structure.

Before we go further, let us first give basic definitions and related properties of a modular metric space.

Definition 3.1. ([23]). Let X be a nonempty set. A function $w : (0, \infty) \times X \times X \to [0, +\infty]$ is said to be a phmetric modular on X if the following conditions are satisfied for any $s, t > 0$ and $x, y, z \in X$:

1. $x = y$ if and only if $w_t(x, y) = 0$ for all $t > 0$.
2. $w_t(x, y) = w_t(y, x)$.
3. $w_{s+t}(x, y) \leq w_s(x, z) + w_t(z, y)$.

Here, we use $w_t(\cdot,\cdot) := w(t, \cdot, \cdot)$. In this case, we say that (X, w) is a phmodular metric space. Notice that the value of a metric modular can be infinite.

Since we are focusing on the generalized metric space approach, we shall not be discussing about modular space theory here. Suppose that (X, d) is a metric space, then $w_t(\cdot,\cdot) := d(\cdot,\cdot)$ is a metric modular on X.

Now, we turn to basic definitions we need in this particular space. We start by giving the topology of the space.

Let (X, w) be a modular metric space. By defining an open ball with $B_w(x;r)$:=$\{z\in X;\ \sup_{t>0} w_t(x,z) < r\}$, we can define a Hausdorff topology on X having the collection of all such open balls as a base. The convergence in this topology can therefore be written by:

$$(x_n) \to \bar{x} \Leftrightarrow \sup_{t>0} w_t(x_n, \bar{x}) \to 0,$$

where $(x_n) \subset X$ and $\bar{x} \in X$. With this characterization, we now have a good hint to define the Cauchy sequence. A sequence $(x_n) \subset X$ is said to be phCauchy if for any given $\varepsilon > 0$, there exists $n_* \in \mathbb{N}$ such that

$$\sup_{t>0} w_t(x_m, x_n) < \varepsilon$$

whenever $m, n > n_*$. Naturally, X is said to be phcomplete if Cauchy sequences in X converges.

We next give another route of investigation of fixed point inclusion in modular metric spaces. This time, we shall apply more on analytical assumptions. Briefly said, we shall use the contractivity assumptions.

Before we could stomp into the main exploration, we need the following knowledge of metric modular of sets.

We write $C(X)$ to denote the set of all nonempty closed subsets of X. For any subset $A \subset X_w$ and point $x \in X$, we denote $w_t(x, A) := \inf_{y\in A} w_t(x, y)$.

Given two subsets $A, B \in C(X)$, define $w_t(A, B) := \sup_{x\in A} w_t(x, B)$. Most importantly, the Hausdorff-Pompieu metric modular $W_t(A, B) := \max\{w_t(A, B), w_t(B, A)\}$.

LEMMA 3.2. *Let (X, w) be a modular metric space, $A \in C(X)$ and $x \in X$. Then,*

$$w_t(x, A) = 0 \ \text{for all} \ t > 0 \ \Leftrightarrow \ x \in A.$$

DEFINITION 3.3. Given a modular metric space (X, w) and an arbitrary point $x \in X$. A subset $Y \subset X$ is said to be phreachable from x if

$$\inf_{y\in Y} \sup_{t>0} w_t(x, y) = \sup_{t>0} w_t(x, Y) < \infty.$$

This lemma gives a simple criterion of when the reachability holds.

LEMMA 3.4. *Let* (X, w) *be a modular metric space with* w *being l.s.c.,* $Y \subset X$ *a nonempty compact subset. For a point* $x \in X$*, if either* $\inf_{y \in Y} \sup_{t>0} w_t(x, y) < \infty$ *or* $\sup_{t>0} w_t(x, Y) < \infty$*, then* Y *is reachable from* x.

The following lemma is essential in showing the solvability of fixed point inclusion for contractivity condition.

LEMMA 3.5. *Suppose that* $Y, Z \in C(X)$ *are nonempty and* $z \in Z$*. If* Y *is reachable from* z*, then for each* $\varepsilon > 0$*, there exists a point* $y_\varepsilon \in Y$ *such that* $\sup_{t>0} w_t(z, y_\varepsilon) \leq \sup_{t>0} W_t(X, Y) + \varepsilon$.

3.1. Fixed point inclusion in modular metric spaces

Now, we state the notion of the contraction and the Kannan's contraction. Make note that these two concepts are not generalizations of one another.

DEFINITION 3.6. Let (X, w) be a modular metric space. A set-valued operator $F : X \rightrightarrows X$ is said to be a phcontraction if there exists a constant $k \in [0, 1)$ such that

$$W_t(Fx, Fy) \leq k w_t(x, y), \tag{7}$$

for all $t > 0$ and $x, y \in X$.

If k is restricted in $[0, \frac{1}{2})$ and Eq. (7) is replaced with the following inequality:

$$W_t(F(x), F(y)) \leq k[w_t(x, F(x)) + w_t(y, F(y))].$$

Then, we call F a phKannan's contraction

Now, we present the main existence theorems.

THEOREM 3.7. *Let* (X, w) *be a complete modular metric space with* w *being l.s.c. and* F *a contraction on* X *having compact values with contraction constant* k*. Suppose that there exists a pair of points* $x_0 \in X$ *and* $x_1 \in F(x_0)$ *with the following properties:*

(A) the set $\{x_0, x_1\}$ *is bounded,*

(B) $F(x_1)$ *is reachable from* x_1.

Then, F *has at least one fixed point.*

PROOF. Since $F(x_1)$ is reachable from x_1, by using Lemma 3.5, we may choose $x_2 \in F(x_1)$ such that

$$\sup_{t>0} w_t(x_1, x_2) \leq \sup_{t>0} w_t(F(x_0), F(x_1)) + k.$$

From the above evidence and the hypothesis that $\{x_0, x_1\}$ is bounded, it comes to the following inequalities:

$$\begin{aligned}
\sup_{w>0} w_t(x_2, F(x_2)) &\leq \sup_{t>0} w_t(F(x_1), F(x_2)) \\
&\leq k \sup_{t>0} w_t(x_1, x_2) \\
&\leq k[\sup_{t>0} W_t(F(x_0), F(x_1)) + k] \\
&\leq k^2 \sup_{t>0} w_t(x_0, x_1) + k^2 \\
&< \infty.
\end{aligned}$$

By the assumptions, we apply Lemma 3.4 to guarantee that $F(x_2)$ is actually reachable from x_2.

Inductively, by this procedure, we define a sequence (x_n) in X, with the supplement from Lemma 3.5, satisfying the following properties for all $n \in \mathrm{N}$:

$$\begin{cases}
x_n \in F(x_{n-1}), \\
\sup_{t>0} w_t(x_n, x_{n+1}) \leq \sup_{t>0} W_t(F(x_{n-1}), F(x_n)) + k^n, \\
\mathrm{F(x_n)} \text{ is reachable from } x_n.
\end{cases}$$

Hence, by the contractivity of F, we have

$$\begin{aligned}
\sup_{t>0} w_t(x_n, x_{n+1}) &\leq \sup_{t>0} W_t(F(x_{n-1}), F(x_n)) + k^n \\
&\leq k \sup_{t>0} w_t(x_{n-1}, x_n) + k^n \\
&\leq k[k \sup_{t>0} w_t(x_{n-2}, x_{n-1}) + k^{n-1}] + k^n \\
&\leq k^2 \sup_{t>0} w_t(x_{n-2}, x_{n-1}) + 2k^n.
\end{aligned}$$

Thus, by induction, we have

$$\sup_{t>0} w_t(x_n, x_{n+1}) \leq k^n \sup_{t>0} w_t(x_0, x_1) + nk^n.$$

Moreover, it follows that

$$\sup_{t>0} \sum_{n \in \mathbb{N}} w_t(x_n, x_{n+1}) \leq \sup_{t>0} w_t(x_0, x_1) \sum_{n \in \mathbb{N}} k^n + \sum_{n \in \mathbb{N}} nk^n < \infty.$$

Without loss of generality, suppose $m, n \in \mathbb{N}$ and $m > n$. Observe that

$$\begin{aligned}
\sup_{t>0} w_t(x_n, x_m) &\leq \sup_{t>0} [w_{\frac{t}{m-n}}(x_n, x_{n+1}) + \ldots + w_{\frac{t}{m-n}}(x_{m-1}, x_m)] \\
&\leq \sup_{t>0} w_t(x_n, x_{n+1}) + \ldots + \sup_{t>0} w_t(x_{m-1}, x_m) \\
&\leq \sum_{n=n_*}^{\infty} \sup_{t>0} w_t(x_n, x_{n+1}) \\
&< \varepsilon,
\end{aligned}$$

for all $m > n \geq n_*$ for some $n_* \in \mathbb{N}$. Hence, (x_n) is a Cauchy sequence so that the completeness of X_w implies that (x_n) converges to some point $x \in X_w$. Consequently, we may conclude from the contractivity of F that the sequence $(F(x_n))$ converges to $F(x)$. Since $x_n \in F(x_{n-1})$, we have for any $t > 0$,

$$0 \leq w_t(x, F(x)) \leq w_{\frac{t}{2}}(x, x_n) + W_{\frac{t}{2}}(F(x_{n-1}), F(x)),$$

which implies that $w_t(x, F(x)) = 0$ for all $t > 0$. Since $F(x)$ is closed, it then follows from Lemma 3.2 that $x \in F(x)$.

Example 3.8. Suppose that $X = [0, 1]$ and $w : (0, +\infty) \times X \times X \to [0, +\infty]$ is defined by

$$w_t(x, y) = \frac{1}{(1+t)}|x-y|.$$

Clearly, w is an l.s.c. metric modular on X. Notice that any two-point subset is bounded. Now, we define a set-valued operator $F : X \rightrightarrows X$ by

$$F(x) := \left[\frac{x+1}{2}, 1\right]$$

for every $x \in X$.

Observe that F has compact values on X. Note that for each $t > 0$ and $x, y \in X$, we have

$$W_t(Fx, Fy) = \frac{1}{2(1+t)}|x-y| \leq \frac{1}{2} w_t(x, y).$$

Therefore, F is a contraction with contraction constant $k = \frac{1}{2}$. Moreover, it is easy to see that the conditions (A) and (B) hold. Finally, we have that 1 is a fixed point of F (and it is unique).

Next, we shall show that the fixed point in the above theorem needs not be unique, as we shall see in the following example:

Example 3.9. Suppose that X is defined as in the previous example. Consider the operator $G : X \rightrightarrows X$ given by

$$G(x) := \left[0, \frac{x+1}{2}\right],$$

for each $x \in X$.

Note that this operator G is also a contraction with constant $k = \frac{1}{2}$ and takes compact values on X. Also, the conditions (A) and (B) hold. However, every point in X is a fixed point of G. This shows the nonuniqueness of fixed points for a set-valued contraction.

Theorem 3.10. *Replacing F in Theorem 3.7 with a Kannan's contraction yields the same existence result.*

Proof. Since $F(x_1)$ is reachable from x_1, by using Lemma 3.5, we may choose $x_2 \in F(x_1)$ such that

$$\sup_{t>0} w_t(x_1, x_2) \leq \sup_{t>0} W_t(F(x_0), F(x_1)) + k.$$

Now, observe that

$$\begin{aligned}
&\sup_{t>0} w_t(x_2, F(x_2)) \\
&\leq \sup_{t>0} W_t(F(x_1), F(x_2)) \\
&\leq k \sup_{t>0} w_t(x_1, F(x_1)) + k \sup_{t>0} w_t(x_2, F(x_2)) \\
&\leq k \sup_{t>0} W_t(F(x_0), F(x_1)) + k \sup_{t>0} w_t(x_2, F(x_2)) \\
&\leq k \sup_{t>0} w_t(x_0, F(x_0)) + k \sup_{t>0} w_t(x_1, F(x_1)) + k \sup_{t>0} w_t(x_2, F(x_2)) \\
&\leq k \sup_{t>0} w_t(x_0, x_1) + k \sup_{t>0} w_t(x_1, F(x_1)) + k \sup_{t>0} w_t(x_2, F(x_2)).
\end{aligned}$$

Writing $\xi := \frac{k}{1-k} < 1$, we obtain, from the boundedness of $\{x_0, x_1\}$ and the reachability of $F(x_1)$ from x_1, that

$$\sup_{t>0} w_t(x_2, F(x_2)) \leq \xi \sup_{t>0} w_t(x_0, x_1) + \xi \sup_{t>0} w_t(x_1, F(x_1)) < \infty.$$

Thus, from the assumptions and Lemma 3.5, we may see that $F(x_2)$ is reachable from x_2.

Inductively, we can construct a sequence (x_n) in X with exactly the same properties appearing in the proof of Theorem 3.7.

Now, consider further that

$$\begin{aligned}
&\sup_{t>0} w_t(x_n, x_{n+1}) \\
&\leq \sup_{t>0} W_t(F(x_{n-1}), F(x_n)) + k^n \\
&\leq k \sup_{t>0} w_t(x_{n-1}, F(x_{n-1})) + k \sup_{t>0} w_t(x_n, F(x_n)) + k^n \\
&\leq k \sup_{t>0} w_t(x_{n-1}, F(x_{n-1})) + k \sup_{t>0} w_t(x_n, x_{n+1}) + k^n.
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
\sup_{t>0} w_t(x_n, x_{n+1}) &\leq \xi \sup_{t>0} w_t(x_{n-1}, x_n) + \frac{k^n}{1-k} \\
&\leq \xi^2 \sup_{t>0} w_t(x_{n-2}, x_{n-1}) + \frac{k^n}{(1-k)^2} + \frac{k^n}{(1-k)} \\
&\leq \xi^2 \sup_{t>0} w_t(x_{n-2}, x_{n-1}) + 2 \cdot \frac{k^n}{(1-k)^2} \\
&\quad \vdots \\
&\leq \xi^n \sup_{t>0} w_t(x_0, x_1) + n\xi^n.
\end{aligned}$$

As in the proof of Theorem 3.7, the sequence (x_n) converges to some $x \in X$. Observe now that

$$\begin{aligned}
&\sup_{t>0} w_t(x, F(x)) \\
&\quad = \sup_{t>0} w_t(\{x\}, F(x)) \\
&\quad \leq \sup_{t>0} w_t(\{x\}, F(x_n)) + \sup_{t>0} w_t(F(x_n), F(x)) \\
&\quad = \sup_{t>0} w_t(x, F(x_n)) + \sup_{t>0} w_t(F(x_n), F(x)) \\
&\quad \leq \sup_{t>0} w_t(x, x_{n+1}) + \sup_{t>0} W_t(F(x_n), F(x)) \\
&\quad \leq \sup_{t>0} w_t(x, x_{n+1}) + k \sup_{t>0} w_t(x_n, F(x_n)) + k \sup_{t>0} w_t(x, F(x)) \\
&\quad = (1+k)\sup_{t>0} w_t(x, x_{n+1}) + k \sup_{t>0} w_t(x, F(x)).
\end{aligned}$$

Thus, we have

$$\sup_{t>0} w_t(x, F(x)) \leq \frac{1+k}{1-k} \sup_{t>0} w_t(x, x_{n+1}).$$

Letting $n \to \infty$ to conclude the theorem.

3.2. Fractional integral inclusion

In this particular subsection, we shall use notations a bit differently than those of earlier sections. This is due to conventional uses of variables and functions that is common to integral and differential equations.

Suppose that Ψ is the interval mentioned in the previous section. Let us assume throughout the section that the real line $\mathbb{R}$ is equipped with the metric modular

$$\omega_\lambda^{\mathbb{R}}(x, y) := \frac{1}{1+\lambda}|x-y|,$$

for $\lambda > 0$ and $x, y \in \mathbb{R}$. Thus, for the space $C(\Psi)$ of all continuous (in $\omega^{\mathbb{R}}$-topology) real-valued functions on Ψ, we shall use the metric modular

$$\omega_\lambda^{C(\Psi)}(\varphi, \psi) := \sup_{t \in \Psi} \omega_\lambda^{\mathbb{R}}(\varphi(t), \psi(t)),$$

for $\lambda > 0$ and $\varphi, \psi \in C(\Psi)$. Note that both $\omega^{\mathbb{R}}$ and $\omega^{C(\Psi)}$ satisfy the Fatou's property. Also note that the set $\mathbb{R}$ is second countable, i.e., it has a countable base, w.r.t. $\omega^{\mathbb{R}}$-topology. Moreover, it is clear that the set $\{\varphi, \psi\}$ is bounded w.r.t. $\omega^{C(\Psi)}$, for any $\varphi, \psi \in C(\Psi)$. Suppose that $F : \Psi \times \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued operator with nonempty compact values and $u \in C(\Psi)$. We shall use the following notation to explain the collection of integrable selections:

$$S_F(u) := \left\{ f \in L^1(\Psi, \mu) \,;\, f(t) \in F(t, u(t)) \text{a.e.} t \in \Psi \right\}.$$

It is clear that $S_F(u)$ is closed. Next, for each $i \in \{0, 1, \cdots, N\}$, $N \in \mathrm{N}$, assume that $\beta_i : \Psi \to R$ is continuous and $\tau_i : \Psi \to \mathbb{R}_+$ is a function with $\tau_i(t) \leq t$. We write $B := \max_{0 \leq i \leq N} \sup_{t \in \Psi} \beta_i(t)$. The main aim of this section is to consider the fractional integral inclusion:

$$u(t)-\sum_{i=0}^{N}\beta_i(t)u(t-\tau_i(t)) \in \mathrm{J}_{\Psi}^{\alpha}F(t,u(t))dt, \quad \alpha \in (0,1]. \tag{FII}$$

In the above inclusion, the summation here is interpreted to be the delay term.

We shall define a set-valued operator $\Lambda : C(\Psi) \to 2^{C(\Psi)}$ by

$$\Lambda(u) := \left\{ w \in C(\Psi)\ ;\ w(t) = \sum_{i=0}^{N}\beta_i(t)u(t-\tau_i(t)) + \mathrm{I}_{\Psi}^{\alpha}f(t,u(t))dt, \quad f \in S_F(u) \right\}.$$

Note here that for any $\varphi \in C(\Psi)$, we have $\Lambda(\varphi)$ is reachable from φ w.r.t. $\omega^{C(\Psi)}$. To restrict the operator Λ with some nice property, we assume that $S_F(u)$ is nonempty.

LEMMA 3.11. *The operator Λ given above is compact valued if $S_F(u)$ is nonempty.*

PROOF. For the proof, we shall show the compactness by its sequential characterization. Suppose that $u \in C(\Psi)$ and (w_n) is an arbitrary sequence in $\Lambda(u)$. By definition, there corresponds a convergent sequence (f_n) in $S_F(u) \subset F(\cdot, u(\cdot))$ satisfying

$$w_n(t) = \sum_{i=0}^{N}\beta_i(t)u(t-\tau_i(t)) + \mathrm{I}_{\Psi}^{\alpha}f_n(t,u(t))dt.$$

The conclusion is then followed.

Now, we shall state now the solvability result for the problem (FII). It is clear that $u \in C(\Psi)$ solves Eq. (FII) if and only if u is a fixed point of Λ.

THEOREM 3.12. *Suppose that F defined above is compact-valued and $S_F(u)$ is nonempty. Assume further that*

(F1) for any given $u, v \in C(\Psi)$ and a selection $f \in S_F(u)$ of F, there corresponds a function $f' \in S_F(v)$ such that

$$\begin{cases} \omega_{\Lambda}^{\mathbb{R}}(f(t,u(t)), f'(t,v(t))) = \omega_{\Lambda}^{\mathbb{R}}(f_1(t,u(t)), F(t,v(t))), \\ \omega_{\Lambda}^{\mathbb{R}}(f(t,u(t)), f'(t,v(t))) \leq L\omega_{\Lambda}^{C(\Psi)}(u,v), \end{cases}$$

for all $t \in \Psi$;

(F2) $\frac{(N+1)B\Gamma(\alpha)+LT^{\alpha}}{\Gamma(\alpha)} < 1.$

Then, Λ has a fixed point.

PROOF. For each $u, v \in C(\Psi)$, we may choose, from the assumption, functions f_1, f_2 such that

$$\begin{cases} f_1 \in S_F(u), \\ f_2 \in S_F(v), \\ \omega_{\Lambda}^{\mathbb{R}}(f_1(t,u(t)), f_2(t,v(t))) = \omega_{\Lambda}^{\mathbb{R}}(f_1(t,u(t)), F(t,v(t))), \\ \omega_{\Lambda}^{\mathbb{R}}(f_1(t,u(t)), f_2(t,v(t))) \leq L\omega_{\Lambda}^{C(\Psi)}(u,v), \end{cases}$$

for each $t \in \Psi$. Consider the two functions $w_1 \in \Lambda(u)$ and $w_2 \in \Lambda(v)$, respectively as follows:

$$\begin{cases} w_1(t) := \sum_{i=0}^{N} \beta_i(t) u(t - \tau_i(t)) + \mathrm{I}_{\Psi}^{\alpha} f_1(t, u(t)) dt, \\ w_2(t) := \sum_{i=0}^{N} \beta_i(t) v(t - \tau_i(t)) + \mathrm{I}_{\Psi}^{\alpha} f_2(t, v(t)) dt. \end{cases}$$

Now, consider the following computation:

$$\begin{aligned} & \omega_{\lambda}^{\mathbb{R}}(w_1(t), w_2(t)) \\ \leq & \sum_{i=0}^{N} \beta_i(t) \omega_{\lambda}^{\mathbb{R}}(u(t - \tau_i(t)), v(t - \tau_i(t)) \\ + & \omega_{\lambda}^{C(\Psi)}(\mathrm{I}_{\Psi}^{\alpha} f_1(t, u(t)) dt, \mathrm{I}_{\Psi}^{\alpha} f_2(t, u(t)) dt) \\ \leq & (N+1) B \omega_{\lambda}^{C(\Psi)}(u, v) + \mathrm{I}_{\Psi}^{\alpha} \omega_{\lambda}^{\mathbb{R}}(f_1(t, u(t)), f_2(t, v(t))) \\ \leq & (N+1) B \omega_{\lambda}^{C(\Psi)}(u, v) + \frac{L T^{\alpha}}{\Gamma(\alpha)} \omega_{\lambda}^{C(\Psi)}(u, v) \\ = & \left[\frac{(N+1) B \Gamma(\alpha) + L T^{\alpha}}{\Gamma(\alpha)} \right] \omega_{\lambda}^{C(\Psi)}(u, v). \end{aligned}$$

It follows that

$$\Omega_{\lambda}^{C(\Psi)}(\Lambda(u), \Lambda(v)) \leq \left[\frac{(N+1) B \Gamma(\alpha) + L T^{\alpha}}{\Gamma(\alpha)} \right] \omega_{\lambda}^{C(\Psi)}(u, v).$$

The proof ends here by applying Theorem 3.7.

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