Chapter 3

Sub-Manifolds of a Riemannian Manifold

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Additional information is available at the end of the chapter

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Abstract

In this chapter, we introduce the theory of sub-manifolds of a Riemannian manifold. The fundamental notations are given. The theory of sub-manifolds of an almost Riemannian product manifold is one of the most interesting topics in differential geometry. According to the behaviour of the tangent bundle of a sub-manifold, with respect to the action of almost Riemannian product structure of the ambient manifolds, we have three typical classes of sub-manifolds such as invariant sub-manifolds, anti-invariant sub-manifolds and semi-invariant sub-manifolds. In addition, slant, semi-slant and pseudo-slant sub-manifolds are introduced by many geometers.

Keywords: Riemannian product manifold, Riemannian product structure, integral manifold, a distribution on a manifold, real product space forms, a slant distribution

1. Introduction

Let $i : M \to \tilde{M}$ be an immersion of an n-dimensional manifold M into an m-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Denote by $g = i^*\tilde{g}$ the induced Riemannian metric on M. Thus, i become an isometric immersion and M is also a Riemannian manifold with the Riemannian metric $g(X, Y) = \tilde{g}(X, Y)$ for any vector fields X, Y in M. The Riemannian metric g on M is called the induced metric on M. In local components, $g_{ij} = g_{AB}B_j^B B_i^A$ with $g = g_{ji}dx^j dx^j$ and $\tilde{g} = g_{BA}dU^B dU^A$.

If a vector field ξ_p of $\tilde{M}$ at a point $p \in M$ satisfies

$$\tilde{g}(X_p, \xi_p) = 0 \tag{1}$$

for any vector X_p of M at p, then ξ_p is called a normal vector of M in $\tilde{M}$ at p. A unit normal vector field of M in $\tilde{M}$ is called a normal section on M [3].

By $T^{\perp}M$, we denote the vector bundle of all normal vectors of M in $\tilde{M}$. Then, the tangent bundle of $\tilde{M}$ is the direct sum of the tangent bundle TM of M and the normal bundle $T^{\perp}M$ of M in $\tilde{M}$, i.e.,

$$T\tilde{M} = TM \oplus T^{\perp}M. \tag{2}$$

We note that if the sub-manifold M is of codimension one in $\tilde{M}$ and they are both orientiable, we can always choose a normal section ξ on M, i.e.,

$$g(X, \xi) = 0,\ g(\xi, \xi) = 1, \tag{3}$$

where X is any arbitrary vector field on M.

By $\tilde{\nabla}$, denote the Riemannian connection on $\tilde{M}$ and we put

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

for any vector fields X, Y tangent to M, where $\nabla_X Y$ and $h(X, Y)$ are tangential and the normal components of $\tilde{\nabla}_X Y$, respectively. Formula (4) is called the Gauss formula for the sub-manifold M of a Riemannian manifold $(\tilde{M}, \tilde{g})$.

Proposition 1.1. ∇ is the Riemannian connection of the induced metric $g = i^{*}\tilde{g}$ on M and $h(X, Y)$ is a normal vector field over M, which is symmetric and bilinear in X and Y.

Proof: Let α and β be differentiable functions on M. Then, we have

$$\begin{aligned}\tilde{\nabla}_{\alpha X}(\beta Y) &= \alpha\{X(\beta)Y + \beta\tilde{\nabla}_X Y\}\\ &= \alpha\{X(\beta)Y + \beta\nabla_X Y + \beta h(X, Y)\}\\ \nabla_{\alpha X}\beta Y + h(\alpha X, \beta Y) &= \alpha\beta\nabla_X Y + \alpha X(\beta)Y + \alpha\beta h(X, Y)\end{aligned} \tag{5}$$

This implies that

$$\nabla_{\alpha X}(\beta Y) = \alpha X(\beta)Y + \alpha\beta\nabla_X Y \tag{6}$$

and

$$h(\alpha X, \beta Y) = \alpha\beta h(X, Y). \tag{7}$$

Eq. (6) shows that ∇ defines an affine connection on M and Eq. (4) shows that h is bilinear in X and Y since additivity is trivial [1].

Since the Riemannian connection $\tilde{\nabla}$ has no torsion, we have

$$0 = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \nabla_X Y + h(X, Y) - \nabla_X Y - h(Y, X) - [X, Y]. \tag{8}$$

By comparing the tangential and normal parts of the last equality, we obtain

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{9}$$

and

$$h(X, Y) = h(Y, X). \tag{10}$$

These equations show that ∇ has no torsion and h is a symmetric bilinear map. Since the metric $\tilde{g}$ is parallel, we can easily see that

$$\begin{aligned}(\nabla_X g)(Y, Z) &= (\tilde{\nabla}_X \tilde{g})(Y, Z) \\ &= \tilde{g}(\tilde{\nabla}_X Y, Z) + \tilde{g}(Y, \tilde{\nabla}_X Z) \\ &= \tilde{g}\Big(\nabla_X Y + h(X, Y), Z\Big) + \tilde{g}(Y, \nabla_X Z + h(X, Z)) \\ &= \tilde{g}(\nabla_X Y, Z) + \tilde{g}(Y, \nabla_X Z) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\end{aligned} \tag{11}$$

for any vector fields X, Y, Z tangent to M, that is, ∇ is also the Riemannian connection of the induced metric g on M.

We recall h the second fundamental form of the sub-manifold M (or immersion i), which is defined by

$$h : \Gamma(TM) \times \Gamma(TM) \to \Gamma(T^{\perp}M). \tag{12}$$

If $h = 0$ identically, then sub-manifold M is said to be totally geodesic, where $\Gamma(T^{\perp}M)$ is the set of the differentiable vector fields on normal bundle of M.

Totally geodesic sub-manifolds are simplest sub-manifolds.

Definition 1.1. Let M be an n-dimensional sub-manifold of an m-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. By h, we denote the second fundamental form of M in $\tilde{M}$.

$H = \frac{1}{n}\mathrm{trace}(h)$ is called the mean curvature vector of M in $\tilde{M}$. If $H = 0$, the sub-manifold is called minimal.

On the other hand, M is called pseudo-umbilical if there exists a function λ on M, such that

$$\tilde{g}\Big(h(X, Y), H\Big) = \lambda g(X, Y) \tag{13}$$

for any vector fields X, Y on M and M is called totally umbilical sub-manifold if

$$h(X, Y) = g(X, Y)H. \tag{14}$$

It is clear that every minimal sub-manifold is pseudo-umbilical with $\lambda = 0$. On the other hand, by a direct calculation, we can find $\lambda = \tilde{g}(H, H)$ for a pseudo-umbilical sub-manifold. So, every totally umbilical sub-manifold is a pseudo-umbilical and a totally umbilical sub-manifold is totally geodesic if and only if it is minimal [2].

Now, let M be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$ and V be a normal vector field on M, X be a vector field on M. Then, we decompose

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \tag{15}$$

where $A_V X$ and $\nabla_X^{\perp} V$ denote the tangential and the normal components of $\nabla_X^{\perp} V$, respectively. We can easily see that $A_V X$ and $\nabla_X^{\perp} V$ are both differentiable vector fields on M and normal bundle of M, respectively. Moreover, Eq. (15) is also called Weingarten formula.

Proposition 1.2. Let M be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. Then

(a) $A_V X$ is bilinear in vector fields V and X. Hence, $A_V X$ at point $p \in M$ depends only on vector fields V_p and X_p.

(b) For any normal vector field V on M, we have

$$g(A_V X, Y) = g\Big(h(X, Y), V\Big). \tag{16}$$

Proof: Let α and β be any two functions on M. Then, we have

$$\begin{aligned} \tilde{\nabla}_{\alpha X}(\beta V) &= \alpha \tilde{\nabla}_X(\beta V) \\ &= \alpha\{X(\beta)V + \beta \tilde{\nabla}_X V\} \\ -A_{\beta V}\alpha X + \nabla_{\alpha X}^{\perp}\beta V &= \alpha X(\beta)V - \alpha\beta A_V X + \alpha\beta \nabla_X^{\perp} V. \end{aligned} \tag{17}$$

This implies that

$$A_{\beta V}\alpha X = \alpha\beta A_V X \tag{18}$$

and

$$\nabla_{\alpha X}^{\perp}\beta V = \alpha X(\beta)V + \alpha\beta \nabla_X^{\perp} V. \tag{19}$$

Thus, $A_V X$ is bilinear in V and X. Additivity is trivial. On the other hand, since $g$is a Riemannian metric,

$$X\tilde{g}(Y, V) = 0, \tag{20}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Eq. (12) implies that

$$\tilde{g}(\tilde{\nabla}_X Y, V) + \tilde{g}(Y, \tilde{\nabla}_X V) = 0. \tag{21}$$

By means of Eqs. (4) and (15), we obtain

$$\tilde{g}\Big(h(X,Y),V\Big) - g(A_V X,Y) = 0. \tag{22}$$

The proof is completed [3].

Let M be a sub-manifold of a Riemannian manifold $(\tilde{M},\tilde{g})$, and h and A_V denote the second fundamental form and shape operator of M, respectively.

The covariant derivative of h and A_V is, respectively, defined by

$$(\tilde{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) \tag{23}$$

and

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{\nabla_X^{\perp} V} Y - A_V \nabla_X Y \tag{24}$$

for any vector fields X,Y tangent to M and any vector field V normal to M. If $\nabla_X h = 0$ for all X, then the second fundamental form of M is said to be parallel, which is equivalent to $\nabla_X A = 0$. By direct calculations, we get the relation

$$g\Big((\nabla_X h)(Y,Z),V\Big) = g\Big((\nabla_X A)_V Y,Z\Big). \tag{25}$$

Example 1.1. We consider the isometric immersion

$$\phi : \mathbb{R}^2 \to \mathbb{R}^4, \tag{26}$$

$$\phi(x_1,x_2) = (x_1,\sqrt{x_1^2-1},x_2,\sqrt{x_2^2-1}) \tag{27}$$

we note that $M = \phi(\mathbb{R}^2) \subset \mathbb{R}^4$ is a two-dimensional sub-manifold of $\mathbb{R}^4$ and the tangent bundle is spanned by the vectors

$TM = S_p\Big\{e_1 = \Big(\sqrt{x_1^2-1},x_1,0,0\Big),e_2 = \Big(0,0,\sqrt{x_2^2-1},x_2\Big)\Big\}$ and the normal vector fields

$$T^{\perp}M = sp\Big\{w_1 = \Big(-x_1,\sqrt{x_1^2-1},0,0),w_2 = (0,0,-x_1,\sqrt{x_2^2-1}\Big)\Big\}. \tag{28}$$

By $\tilde{\nabla}$, we denote the Levi-Civita connection of $\mathbb{R}^4$, the coefficients of connection, are given by

$$\tilde{\nabla}_{e_1} e_1 = \frac{2x_1\sqrt{x_1^2-1}}{2x_1^2-1} e_1 - \frac{1}{2x_1^2-1} w_1, \tag{29}$$

$$\tilde{\nabla}_{e_2} e_2 = \frac{2x_2\sqrt{x_2^2-1}}{2x_2^2-1} e_2 - \frac{1}{2x_2^2-1} w_2 \tag{30}$$

and

$$\nabla_{e_2} e_1 = 0. \tag{31}$$

Thus, we have $h(e_1, e_1) = -\frac{1}{2x_1^2-1} w_1$, $h(e_2, e_2) = -\frac{1}{2x_2^2-1} w_2$ and $h(e_2, e_1) = 0$. The mean curvature vector of $M = \phi(\mathbb{R}^2)$ is given by

$$H = -\frac{1}{2}(w_1 + w_2). \tag{32}$$

Furthermore, by using Eq. (16), we obtain

$$\begin{aligned} g(A_{w_1} e_1, e_1) &= g\Big(h(e_1, e_1), w_1\Big) = -\frac{1}{2x_1^2-1}(x_1^2 + x_1^2-1) = -1, \\ g(A_{w_1} e_2, e_2) &= g\Big(h(e_2, e_2), w_1\Big) = -\frac{1}{2x_2^2-1} g(w_1, w_2) = 0, \\ g(A_{w_1} e_1, e_2) &= 0, \end{aligned} \tag{33}$$

and

$$\begin{aligned} &g(A_{w_2} e_1, e_1) = g\Big(h(e_1, e_1), w_2\Big) = 0, \\ &g(A_{w_2} e_1, e_2) = 0, g(A_{w_2} e_2, e_2) = 1. \end{aligned} \tag{34}$$

Thus, we have

$$A_{w_1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A_{w_2} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{35}$$

Now, let M be a sub-manifold of a Riemannian manifold $(\tilde{M}, g)$, $\tilde{R}$ and R be the Riemannian curvature tensors of $\tilde{M}$ and M, respectively. From then the Gauss and Weingarten formulas, we have

$$\begin{aligned} \tilde{R}(X, Y)Z &= \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z \\ &= \tilde{\nabla}_X \Big(\nabla_Y Z + h(Y, Z)\Big) - \tilde{\nabla}_Y \Big(\nabla_X Z + h(X, Z)\Big) - \nabla_{[X,Y]} Z - h([X, Y], Z) \\ &= \tilde{\nabla}_X \nabla_Y Z + \tilde{\nabla}_X h(Y, Z) - \tilde{\nabla}_Y \nabla_X Z - \tilde{\nabla}_Y h(X, Z) - \nabla_{[X,Y]} Z - h(\nabla_X Y, Z) + h(\nabla_Y X, Z) \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + h(X, \nabla_Y Z) - h(\nabla_X Z, Y) + \nabla^{\perp}_X h(Y, Z) \\ &\quad - A_{h(Y,Z)} X - \nabla^{\perp}_Y h(X, Z) + A_{h(X,Z)} Y - \nabla_{[X,Y]} Z - h(\nabla_X Y, Z) + h(\nabla_Y X, Z) \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) \\ &\quad - h(Y, \nabla_X Z) - \nabla^{\perp}_Y h(X, Z) + h(\nabla_Y X, Z) + h(\nabla_Y Z, X) \\ &\quad + A_{h(X,Z)} Y - A_{h(Y,Z)} X \\ &= R(X, Y)Z + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + A_{h(X,Z)} Y - A_{h(Y,Z)} X \end{aligned} \tag{36}$$

from which

$$\tilde{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{37}$$

for any vector fields X, Y and Z tangent to M. For any vector field W tangent to M, Eq. (37) gives the Gauss equation

$$g\left(\tilde{R}(X,Y)Z, W\right) = g\left(R(X,Y)Z, W\right) + g\left(h(Y,W), h(X,Z)\right) - g\left(h(Y,Z), h(X,W)\right). \tag{38}$$

On the other hand, the normal component of Eq. (37) is called equation of Codazzi, which is given by

$$\left(\tilde{R}(X,Y)Z\right)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z). \tag{39}$$

If the Codazzi equation vanishes identically, then sub-manifold M is said to be curvature-invariant sub-manifold [4].

In particular, if $\tilde{M}$ is of constant curvature, $\tilde{R}(X,Y)Z$ is tangent to M, that is, sub-manifold is curvature-invariant. Whereas, in Kenmotsu space forms, and Sasakian space forms, this not true.

Next, we will define the curvature tensor $R^{\perp}$ of the normal bundle of the sub-manifold M by

$$R^{\perp}(X,Y)V = \nabla^{\perp}_X \nabla^{\perp}_Y V - \nabla^{\perp}_Y \nabla^{\perp}_X V - \nabla^{\perp}_{[X,Y]} V \tag{40}$$

for any vector fields X, Y tangent to sub-manifold M, and any vector field V normal to M. From the Gauss and Weingarten formulas, we have

$$\begin{aligned}
\tilde{R}(X,Y)V &= \tilde{\nabla}_X \tilde{\nabla}_Y V - \tilde{\nabla}_Y \tilde{\nabla}_X V - \tilde{\nabla}_{[X,Y]} V \\
&= \tilde{\nabla}_X(-A_V Y + \nabla^{\perp}_Y V) - \tilde{\nabla}_Y(-A_V X + \nabla^{\perp}_X V) + A_V[X,Y] - \nabla^{\perp}_{[X,Y]} V \\
&= -\tilde{\nabla}_X A_V Y + \tilde{\nabla}_Y A_V X + \tilde{\nabla}_X \nabla^{\perp}_Y V - \tilde{\nabla}_Y \nabla^{\perp}_X V + A_V[X,Y] - \nabla^{\perp}_{[X,Y]} V \\
&= -\nabla_X A_V Y - h(X, A_V Y) + \nabla_Y A_V X + h(Y, A_V X) \\
&\quad + \nabla^{\perp}_X \nabla^{\perp}_Y V - \nabla^{\perp}_Y \nabla^{\perp}_X V - A_{\nabla^{\perp}_Y V} X + A_{\nabla^{\perp}_X V} Y + A_V[X,Y] - \nabla^{\perp}_{[X,Y]} V \\
&= \nabla^{\perp}_X \nabla^{\perp}_Y V - \nabla^{\perp}_Y \nabla^{\perp}_X V - \nabla^{\perp}_{[X,Y]} V - A_{\nabla^{\perp}_Y V} X + A_{\nabla^{\perp}_X V} Y + A_V[X,Y] \\
&\qquad - \nabla_X A_V Y + \nabla_Y A_V X - h(X, A_V Y) + h(Y, A_V X) \\
&= R^{\perp}(X,Y)V + h(A_V X, Y) - h(X, A_V Y) - (\nabla_X A)_V Y + (\nabla_Y A)_V X.
\end{aligned} \tag{41}$$

For any normal vector U to M, we obtain

$$\begin{aligned} g\left(\tilde{R}\,(X,Y)V,U\right) &= g\left(R^{\perp}(X,Y)V,U\right) + g\left(h(A_V X,Y),U\right) - g\left(h(X,A_V Y),U\right) \\ &= g\left(R^{\perp}(X,Y)V,U\right) + g(A_U Y,A_V X) - g(A_V Y,A_U X) \\ &= g\left(R^{\perp}(X,Y)V,U\right) + g(A_V A_U Y,X) - g(A_U A_V Y,X) \end{aligned} \tag{42}$$

Since $[A_U, A_V] = A_U A_V - A_V A_U$, Eq. (42) implies

$$g\left(\tilde{R}(X,Y)V,U\right) = g\left(R^{\perp}(X,Y)V,U\right) + g([A_U,A_V]Y,X). \tag{43}$$

Eq. (43) is also called the Ricci equation.

If $R^{\perp} = 0$, then the normal connection of M is said to be flat [2].

When $\left(\tilde{R}(X,Y)V\right)^{\perp} = 0$, the normal connection of the sub-manifold M is flat if and only if the second fundamental form M is commutative, i.e. $[A_U, A_V] = 0$ for all U, V. If the ambient space $\tilde{M}$ is real space form, then $\left(\tilde{R}(X,Y)V\right)^{\perp} = 0$ and hence the normal connection of M is flat if and only if the second fundamental form is commutative. If $\tilde{R}(X,Y)Z$ tangent to M, then equation of codazzi Eq. (37) reduces to

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z) \tag{44}$$

which is equivalent to

$$(\nabla_X A)_V Y = (\nabla_Y A)_V X. \tag{45}$$

On the other hand, if the ambient space $\tilde{M}$ is a space of constant curvature c, then we have

$$\tilde{R}(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\} \tag{46}$$

for any vector fields X, Y and Z on $\tilde{M}$.

Since $\tilde{R}(X,Y)Z$ is tangent to M, the equation of Gauss and the equation of Ricci reduce to

$$\begin{aligned} g\left(R(X,Y)Z,W\right) &= c\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &\quad + g\left(h(Y,Z),h(X,W)\right) - g\left(h(Y,W),h(X,Z)\right) \end{aligned} \tag{47}$$

and

$$g\left(R^{\perp}(X,Y)V,U\right) = g([A_U,A_V]X,Y), \tag{48}$$

respectively.

Proposition 1.3. A totally umbilical sub-manifold M in a real space form $\tilde{M}$ of constant curvature c is also of constant curvature.

Proof: Since M is a totally umbilical sub-manifold of $\tilde{M}$ of constant curvature c, by using Eqs. (14) and (46), we have

$$\begin{aligned} g\Big(R(X,Y)Z,W\Big) &= c\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\ &\quad +g(H,H)\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\ &= \{c+g(H,H)\}\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}. \end{aligned} \tag{49}$$

This shows that the sub-manifold M is of constant curvature $c + \|H^2\|$ for $n > 2$. If $n = 2$, $\|H\| =$ constant follows from the equation of Codazzi [3].

This proves the proposition.

On the other hand, for any orthonormal basis $\{e_a\}$ of normal space, we have

$$\begin{aligned} g(Y,Z)g(X,W)-g(X,Z)g(Y,W) &= \sum_a \Big[g\Big(h(Y,Z),e_a\Big)g\Big(h(X,W),e_a\Big) \\ &\quad -g\Big(h(X,Z),e_a\Big)g\Big(h(Y,W),e_a\Big)\Big] \\ &= \sum_a g(A_{e_a}Y,Z)g(A_{e_a}X,W)-g(A_{e_a}X,Z)g(A_{e_a}Y,W) \end{aligned} \tag{50}$$

Thus, Eq. (45) can be rewritten as

$$\begin{aligned} g\Big(R(X,Y)Z,W\Big) &= c\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\ &\quad +\sum_a [g(A_{e_a}Y,Z)g(A_{e_a}X,W)-g(A_{e_a}X,Z)g(A_{e_a}Y,W)] \end{aligned} \tag{51}$$

By using A_{e_a}, we can construct a similar equation to Eq. (47) for Eq. (23).

Now, let S- be the Ricci tensor of M. Then, Eq. (47) gives us

$$S(X,Y) = c\{ng(X,Y)-g(e_i,X)g(e_i,Y)\} \tag{52}$$

$$+\sum_{e_a}[g(A_{e_a}e_i,e_i)g(A_{e_a}X,Y)-g(A_{e_a}X,e_i)g(A_{e_a}e_i,Y)]$$

$$= c(n-1)g(X,Y) + \sum_{e_a}[Tr(A_{e_a})g(A_{e_a}X,Y)-g(A_{e_a}X,A_{e_a}Y)], \tag{53}$$

where $\{e_1, e_2, \ldots, e_n\}$ are orthonormal basis of M.

Therefore, the scalar curvature r of sub-manifold M is given by

$$r = cn(n-1)\sum_{e_a} Tr^2(A_{e_a}) - \sum_{e_a} Tr(A_{e_a})^2 \tag{54}$$

$\sum_{e_a} Tr(A_{e_a})^2$ is the square of the length of the second fundamental form of M, which is denoted by $|A_{e_a}|^2$. Thus, we also have

$$\| h^2 \| = \sum_{i,j=1}^{n} g\Big(h(e_i, e_j), h(e_i, e_j)\Big) = \| A^2 \|. \tag{55}$$

2. Distribution on a manifold

An m-dimensional distribution on a manifold $\tilde{M}$ is a mapping $\mathcal{D}$ defined on $\tilde{M}$, which assignes to each point p of $\tilde{M}$ an m-dimensional linear subspace $\mathcal{D}_p$ of $T_{\tilde{M}}(p)$. A vector field X on $\tilde{M}$ belongs to $\mathcal{D}$ if we have $X_p \in \mathcal{D}_p$ for each $p \in \tilde{M}$. When this happens, we write $X \in \Gamma(\mathcal{D})$. The distribution $\mathcal{D}$ is said to be differentiable if for any $p \in \tilde{M}$, there exist m-differentiable linearly independent vector fields $X_j \in \Gamma(\mathcal{D})$ in a neighbordhood of p.

The distribution $\mathcal{D}$ is said to be involutive if for all vector fields $X, Y \in \Gamma(\mathcal{D})$ we have $[X, Y] \in \Gamma(\mathcal{D})$. A sub-manifold M of $\tilde{M}$ is said to be an integral manifold of $\mathcal{D}$ if for every point $p \in M$, $\mathcal{D}_p$ coincides with the tangent space to M at p. If there exists no integral manifold of $\mathcal{D}$ which contains M, then M is called a maximal integral manifold or a leaf of $\mathcal{D}$. The distribution $\mathcal{D}$ is said to be integrable if for every $p \in \tilde{M}$, there exists an integral manifold of $\mathcal{D}$ containing p [2].

Let $\tilde{\nabla}$ and distribution be a linear connection on $\tilde{M}$, respectively. The distribution $\mathcal{D}$ is said to be parallel with respect to $\tilde{M}$, if we have

$$\tilde{\nabla}_X Y \in \Gamma(\mathcal{D}) \text{ for all } X \in \Gamma(T\tilde{M}) \text{ and } Y \in \Gamma(\mathcal{D}) \tag{56}$$

Now, let $(\tilde{M}, \tilde{g})$ be Riemannian manifold and $\mathcal{D}$ be a distribution on $\tilde{M}$. We suppose $\tilde{M}$ is endowed with two complementary distribution $\mathcal{D}$ and $\mathcal{D}^\perp$, i.e., we have $T\tilde{M} = \mathcal{D} \oplus \mathcal{D}^\perp$. Denoted by P and Q the projections of $T\tilde{M}$ to $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively.

Theorem 2.1. All the linear connections with respect to which both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel, are given by

$$\nabla_X Y = P\nabla'_X PY + Q\nabla'_X QY + PS(X, PY) + QS(X, QY) \tag{57}$$

for any $X, Y \in \Gamma(T\tilde{M})$, where ∇' and S are, respectively, an arbitrary linear connection and arbitrary tensor field of type $(1, 2)$ on $\tilde{M}$.

Proof: Suppose ∇' is an arbitrary linear connection on $\tilde{M}$. Then, any linear connection ∇ on $\tilde{M}$ is given by

$$\nabla_X Y = \nabla'_X Y + S(X, Y) \tag{58}$$

for any $X, Y \in \Gamma(T\tilde{M})$. We can put

$$X = PX + QX \tag{59}$$

for any $X \in \Gamma(T\tilde{M})$. Then, we have

$$\begin{aligned}\nabla_X Y &= \nabla_X(PY + QY) = \nabla_X PY + \nabla_X QY = \nabla'_X PY + S(X, PY) \\ &+ \nabla'_X QY + S(X, QY) = \mathrm{P}\nabla'_X PY + Q\nabla'_X PY + PS(X, PY) + QS(X, PY) \\ &+ P\nabla'_X QY + Q\nabla'_X QY + PS(X, QY) + QS(X, QY)\end{aligned} \tag{60}$$

for any $X, Y \in \Gamma(T\tilde{M})$.

The distributions $\mathcal{D}$ and $\mathcal{D}$ are both parallel with respect to ∇ if and only if we have

$$\phi(\nabla_X PY) = 0 \mathrm{and} \mathrm{P}(\nabla_X QY) = 0. \tag{61}$$

From Eqs. (58) and (61), it follows that $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are parallel with respect to ∇ if and only if

$$\mathrm{Q}\nabla'_X PY + QS(X, PY) = 0 \text{ and } \mathrm{P}\nabla'_X QY + PS(X, QY) = 0. \tag{62}$$

Thus, Eqs. (58) and (62) give us Eq. (57).

Next, by means of the projections P and Q, we define a tensor field F of type $(1, 1)$ on $\tilde{M}$ by

$$FX = PX - QX \tag{63}$$

for any $X \in \Gamma(T\tilde{M})$. By a direct calculation, it follows that $F^2 = I$. Thus, we say that F defines an almost product structure on $\tilde{M}$. The covariant derivative of F is defined by

$$(\nabla_X F)Y = \nabla_X FY - \mathrm{F}\nabla_X Y \tag{64}$$

for all $X, Y \in \Gamma(T\tilde{M})$. We say that the almost product structure F is parallel with respect to the connection ∇, if we have $\nabla_X F = 0$. In this case, F is called the Riemannian product structure [2].

Theorem 2.2. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold and $\mathcal{D}$, $\mathcal{D}^{\perp}$ be orthogonal distributions on $\tilde{M}$ such that $T\tilde{M} = \mathcal{D} \oplus \mathcal{D}^{\perp}$. Both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are parallel with respect to ∇ if and only if F is a Riemannian product structure.

Proof: For any $X, Y \in \Gamma(T\tilde{M})$, we can write

$$\tilde{\nabla}_Y PX = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{QY} PX \tag{65}$$

and

$$\tilde{\nabla}_Y X = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{PY} QX + \tilde{\nabla}_{QY} PX + \tilde{\nabla}_{QY} QX, \tag{66}$$

from which

$$g(\tilde{\nabla}_{QY} PX, QZ) = QYg(PX, QZ) - g(\nabla_{QY} QZ, PX) = 0 - g(\tilde{\nabla}_{QY} QZ, PX) = 0, \tag{67}$$

that is, $\nabla_{QY} PX \in \Gamma(\mathcal{D})$ and so $P\tilde{\nabla}_{QY} PX = \tilde{\nabla}_{QY} PX$,

$$\mathrm{Q}\tilde{\nabla}_{QY} PX = 0. \tag{68}$$

In the same way, we obtain

$$g(\tilde{\nabla}_{PY} QX, PZ) = PYg(QX, PZ) - g(QX, \tilde{\nabla}_{PY} PZ) = 0, \tag{69}$$

which implies that

$$\mathrm{P}\tilde{\nabla}_{PY} QX = 0 \text{ and } \mathrm{Q}\tilde{\nabla}_{PY} QX = \tilde{\nabla}_{PY} QX. \tag{70}$$

From Eqs. (66), (68) and (70), it follows that

$$P\tilde{\nabla}_Y X = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{QY} PX. \tag{71}$$

By using Eqs. (64) and (71), we obtain

$$(\tilde{\nabla}_Y P)X = \tilde{\nabla}_Y PX - P\tilde{\nabla}_Y X = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{QY} PX - \tilde{\nabla}_{PY} PX - \tilde{\nabla}_{QY} PX = 0. \tag{72}$$

In the same way, we can find $\tilde{\nabla} Q = 0$. Thus, we obtain

$$\tilde{\nabla} F = \tilde{\nabla}(P - Q) = 0. \tag{73}$$

This proves our assertion [2].

Theorem 2.3. Both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are parallel with respect to Levi-Civita connection ∇ if and only if they are integrable and their leaves are totally geodesic in $\tilde{M}$.

Proof: Let us assume both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are parallel. Since ∇ is a torsion free linear connection, we have

$$[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(\mathcal{D}), \text{for any } X, Y \in \Gamma(\mathcal{D}) \tag{74}$$

and

$$[U, V] = \nabla_U V - \nabla_V U \in \Gamma(\mathcal{D}^{\perp}), \text{for any } U, V \in \Gamma(\mathcal{D}^{\perp}) \tag{75}$$

Thus, $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are integrable distributions. Now, let M be a leaf of $\mathcal{D}$ and denote by h the second fundamental form of the immersion of M in $\tilde{M}$. Then by the Gauss formula, we have

$$\nabla_X Y = \nabla'_X Y + h(X, Y) \tag{76}$$

for any $X, Y \in \Gamma(\mathcal{D})$, where ∇' denote the Levi-Civita connection on M. Since $\mathcal{D}$ is parallel from Eq. (76) we conclude $h = 0$, that is, M is totally in $\tilde{M}$. In the same way, it follows that each leaf of $\mathcal{D}^\perp$ is totally geodesic in $\tilde{M}$.

Conversely, suppose $\mathcal{D}$ and $\mathcal{D}^\perp$ be integrable and their leaves are totally geodesic in $\tilde{M}$. Then by using Eq. (4), we have

$$\nabla_X Y \in \Gamma(\mathcal{D}) \text{ for any } X, Y \in \Gamma(\mathcal{D}) \tag{77}$$

and

$$\nabla_U V \in \Gamma(\mathcal{D}^\perp) \text{ for any } U, V \in \Gamma(\mathcal{D}^\perp). \tag{78}$$

Since g is a Riemannian metric tensor, we obtain

$$g(\nabla_U Y, V) = -g(Y, \nabla_U V) = 0 \tag{79}$$

and

$$g(\nabla_X V, Y) = -g(V, \nabla_X Y) = 0 \tag{80}$$

for any $X, Y \in \Gamma(\mathcal{D})$ and $U, V \in \Gamma(\mathcal{D}^\perp)$. Thus, both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel on $\tilde{M}$.

3. Locally decomposable Riemannian manifolds

Let $(\tilde{M}, \tilde{g})$ be n−dimensional Riemannian manifold and F be a tensor $(1,1)$−type on $\tilde{M}$ such that $F^2 = I$, $F \neq \mp I$.

If the Riemannian metric tensor $\tilde{g}$ satisfying

$$\tilde{g}(X, Y) = \tilde{g}(FX, FY) \tag{81}$$

for any $X, Y \in \Gamma(T\tilde{M})$ then $\tilde{M}$ is called almost Riemannian product manifold and F is said to be almost Riemannian product structure. If F is parallel, that is, $(\tilde{\nabla}_X F)Y = 0$, then $\tilde{M}$ is said to be locally decomposable Riemannian manifold.

Now, let $\tilde{M}$ be an almost Riemannian product manifold. We put

$$P = \frac{1}{2}(I + F), \; Q = \frac{1}{2}(I - F). \tag{82}$$

Then, we have

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 \quad \text{and} \quad F = P - Q. \tag{83}$$

Thus, P and Q define two complementary distributions P and Q globally. Since $F^2 = I$, we easily see that the eigenvalues of F are 1 and −1. An eigenvector corresponding to the eigenvalue 1 is in P and an eigenvector corresponding to −1 is in Q. If F has eigenvalue 1 of multiplicity P and eigenvalue −1 of multiplicity q, then the dimension of P is p and that of Q is q. Conversely, if there exist in $\tilde{M}$ two globally complementary distributions P and Q of dimension p and q, respectively. Then, we can define an almost Riemannian product structure F on $\tilde{M}$ by $\tilde{M}$ by $F = P - Q$ [7].

Let $(\tilde{M}, \tilde{g}, F)$ be a locally decomposable Riemannian manifold and we denote the integral manifolds of the distributions P and Q by M^p and M^q, respectively. Then we can write $\tilde{M} = M^p X M^q$, $(p, q > 2)$. Also, we denote the components of the Riemannian curvature R of $\tilde{M}$ by R_{dcba}, $1 \leq a, b, c, d \leq n = p + q$.

Now, we suppose that the two components are both of constant curvature λ and μ. Then, we have

$$R_{dcba} = \lambda\{g_{da}g_{cb} - g_{ca}g_{db}\} \tag{84}$$

and

$$R_{zyxw} = \mu\{g_{zw}g_{yx} - g_{yw}g_{zx}\}. \tag{85}$$

Then, the above equations may also be written in the form

$$\begin{aligned} R_{kjih} = {} & \frac{1}{4}(\lambda + \mu)\{(g_{kh}g_{ji} - g_{jh}g_{ki}) + (F_{kh}F_{ji} - F_{jh}F_{ki})\} \\ & + \frac{1}{4}(\lambda - \mu)\{(F_{kh}g_{ji} - F_{jh}g_{ki}) + (g_{kh}F_{ji} - g_{jh}F_{ki})\}. \end{aligned} \tag{86}$$

Conversely, suppose that the curvature tensor of a locally decomposable Riemannian manifold has the form

$$\begin{aligned} R_{kjih} = {} & a\{(g_{kh}g_{ji} - g_{jh}g_{ki}) + (F_{kh}F_{ji} - F_{jh}F_{ki})\} \\ & + b\{(F_{kh}g_{ji} - F_{jh}g_{ki}) + (g_{kh}F_{ji} - g_{jh}F_{ki})\}. \end{aligned} \tag{87}$$

Then, we have

$$R_{cdba} = 2(a + b)\{g_{da}g_{cb} - g_{ca}g_{db}\} \tag{88}$$

and

$$R_{zyxw} = 2(a - b)\{g_{zw}g_{yx} - g_{yw}g_{zx}\}. \tag{89}$$

Let $\tilde{M}$ be an $m-$dimensional almost Riemannian product manifold with the Riemannian structure $(F, \tilde{g})$ and M be an $n-$dimensional sub-manifold of $\tilde{M}$. For any vector field X tangent to M, we put

$$FX = fX + wX, \tag{90}$$

where fX and wX denote the tangential and normal components of FX, with respect to M, respectively. In the same way, for $V \in \Gamma(T^{\perp}M)$, we also put

$$FV = BV + CV, \tag{91}$$

where BV and CV denote the tangential and normal components of FV, respectively.

Then, we have

$$f^2 + Bw = I, Cw + wf = 0 \tag{92}$$

and

$$fB + BC = 0, wB + C^2 = I. \tag{93}$$

On the other hand, we can easily see that

$$g(X, fY) = g(fX, Y) \tag{94}$$

and

$$g(X, Y) = g(fX, fY) + g(wX, wY) \tag{95}$$

for any $X, Y \in \Gamma(TM)$ [6].

If $wX = 0$ for all $X \in \Gamma(TM)$, then M is said to be invariant sub-manifold in $\tilde{M}$, i.e., $F(T_M(p)) \subset T_M(p)$ for each $p \in M$. In this case, $f^2 = I$ and $g(fX, fY) = g(X, Y)$. Thus, (f, g) defines an almost product Riemannian on M.

Conversely, (f, g) is an almost product Riemannian structure on M, the $w = 0$ and hence M is an invariant sub-manifold in $\tilde{M}$.

Consequently, we can give the following theorem [7].

Theorem 3.1. Let M be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(F, \tilde{g})$. The induced structure (f, g) on M is an almost Riemannian product structure if and only if M is an invariant sub-manifold of $\tilde{M}$.

Definition 3.1. Let M be a sub-manifold of an almost Riemannian product $\tilde{M}$ with almost product Riemannian structure $(F, \tilde{g})$. For each non-zero vector $X_p \in T_M(p)$ at $p \in M$, we denote the slant angle between FX_p and $T_M(p)$ by $\theta(p)$. Then M said to be slant sub-manifold if the angle $\theta(p)$ is constant, i.e., it is independent of the choice of $p \in M$ and $X_p \in T_M(p)$ [5].

Thus, invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A proper slant immersion is neither invariant nor anti-invariant.

Theorem 3.2. Let M be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost product Riemannian structure $(F, \tilde{g})$. M is a slant sub-manifold if and only if there exists a constant $\lambda \in (0, 1)$, such tha

$$f^2 = \lambda I. \tag{96}$$

Furthermore, if the slant angle is θ, then it satisfies $\lambda = \cos^2\theta$ [9].

Definition 3.2. Let M be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(F, \tilde{g})$. M is said to be semi-slant sub-manifold if there exist distributions $\mathcal{D}^\theta$ and $\mathcal{D}^T$ on M such that

(i) TM has the orthogonal direct decomposition $TM = \mathcal{D} \oplus \mathcal{D}^T$.

(ii) The distribution $\mathcal{D}^\theta$ is a slant distribution with slant angle θ.

(iii) The distribution $\mathcal{D}^T$ is an invariant distribution, .*e*., $F(\mathcal{D}_{\mathrm{T}}) \subseteq \mathcal{D}^T$.

In a semi-slant sub-manifold, if $\theta = \frac{\pi}{2}$, then semi-slant sub-manifold is called semi-invariant sub-manifold [8].

Example 3.1. Now, let us consider an immersed sub-manifold M in $\mathbb{R}^7$ given by the equations

$$x_1^2 + x_2^2 = x_5^2 + x_6^2, x_3 + x_4 = 0. \tag{97}$$

By direct calculations, it is easy to check that the tangent bundle of M is spanned by the vectors

$$\begin{gathered} z_1 = \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} + \cos\beta \frac{\partial}{\partial x_5} + \sin\beta \frac{\partial}{\partial x_6} \\ z_2 = -u\sin\theta \frac{\partial}{\partial x_1} + u\cos\theta \frac{\partial}{\partial x_2}, z_3 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \\ z_4 = -u\sin\beta \frac{\partial}{\partial x_5} + u\cos\beta \frac{\partial}{\partial x_6}, z_5 = \frac{\partial}{\partial x_7}, \end{gathered} \tag{98}$$

where θ, β and u denote arbitrary parameters.

For the coordinate system of $\mathbb{R}^7 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) | x_i \in \mathbb{R}, 1 \le i \le 7\}$, we define the almost product Riemannian structure F as follows:

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, F\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_j}, 1 \le i \le 3 \text{ and } 4 \le j \le 7. \tag{99}$$

Since Fz_1 and Fz_3 are orthogonal to M and Fz_2,Fz_4,Fz_5 are tangent to M, we can choose a $\mathcal{D} = S_p\{z_2, z_4, z_5\}$ and $\mathcal{D}^\perp = S_p\{z_1, z_3\}$. Thus, M is a 5−dimensional semi-invariant sub-manifold of $\mathbb{R}^7$ with usual almost Riemannian product structure $(F, <\,,\,>)$.

Example 3.2. Let M be sub-manifold of $\mathbb{R}^8$ by given

$$(u + v, u - v, u\cos\alpha, u\sin\alpha, u + v, u - v, u\cos\beta, u\sin\beta) \tag{100}$$

where u, v and β are the arbitrary parameters. By direct calculations, we can easily see that the tangent bundle of M is spanned by

$$\begin{gathered} e_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos\alpha\frac{\partial}{\partial x_3} + \sin\alpha\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6} + \cos\beta\frac{\partial}{\partial x_7} + \sin\beta\frac{\partial}{\partial x_8} \\ e_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}, e_3 = -u\sin\frac{\partial}{\partial x_3} + u\cos\alpha\frac{\partial}{\partial x_4}, \\ e_4 = -u\sin\beta\frac{\partial}{\partial x_7} + u\cos\beta\frac{\partial}{\partial x_8}. \end{gathered} \tag{101}$$

For the almost Riemannian product structure F of $\mathbb{R}^8 = \mathbb{R}^4 \mathrm{x} \mathbb{R}^4$, $F(TM)$ is spanned by vectors

$$\begin{gathered} Fe_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos\alpha\frac{\partial}{\partial x_3} + \sin\alpha\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} - \cos\beta\frac{\partial}{\partial x_7} - \sin\beta\frac{\partial}{\partial x_8}, \\ Fe_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6},\ Fe_3 = e_3 \text{ and } Fe_4 = -e_4. \end{gathered} \tag{102}$$

Since Fe_1 and Fe_2 are orthogonal to M and Fe_3 and Fe_4 are tangent to M, we can choose $\mathcal{D}^T = Sp\{e_3, e_4\}$ and $\mathcal{D}^\perp = Sp\{e_1, e_2\}$. Thus, M is a four-dimensional semi-invariant sub-manifold of $\mathbb{R}^8 = \mathbb{R}^4 x \mathbb{R}^4$ with usual Riemannian product structure F.

Definition 3.3. Let M be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(F, \tilde{g})$. M is said to be pseudo-slant sub-manifold if there exist distributions $\mathcal{D}_\theta$ and $\mathcal{D}_\perp$ on M such that

i. The tangent bundle $TM = \mathcal{D}_\theta \oplus \mathcal{D}^\perp$.

ii. The distribution $\mathcal{D}_\theta$ is a slant distribution with slant angle θ.

iii. The distribution $\mathcal{D}^\perp$ is an anti-invariant distribution, i.e., $F(\mathcal{D}^\perp) \subseteq T^\perp M$.

As a special case, if $\theta = 0$ and $\theta = \frac{\pi}{2}$, then pseudo-slant sub-manifold becomes semi-invariant and anti-invariant sub-manifolds, respectively.

Example 3.3. Let M be a sub-manifold of $\mathbb{R}^6$ by the given equation

$$(\sqrt{3}u, v, v\sin\theta, v\cos\theta, s\cos t, -s\cos t) \tag{103}$$

where u, v, s and t arbitrary parameters and θ is a constant.

We can check that the tangent bundle of M is spanned by the tangent vectors

$$\begin{gathered} e_1 = \sqrt{3}\frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial y_1} + \sin\theta\frac{\partial}{\partial x_2} + \cos\theta\frac{\partial}{\partial y_2}, \\ e_3 = \cos t\frac{\partial}{\partial x_3} - \cos t\frac{\partial}{\partial y_3}, e_4 = -s\sin t\frac{\partial}{\partial x_3} + s\sin t\frac{\partial}{\partial y_3}. \end{gathered} \tag{104}$$

For the almost product Riemannian structure F of $\mathbb{R}^6$ whose coordinate systems $(x_1, y_1, x_2, y_2, x_3, y_3)$ choosing

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, 1 \le i \le 3,$$
$$F\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, 1 \le j \le 3, \tag{105}$$

Then, we have

$$Fe_1 = \sqrt{3}\frac{\partial}{\partial y_1}, Fe_2 = -\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial y_2} - \cos\theta\frac{\partial}{\partial x_2}$$
$$Fe_3 = \cos t\frac{\partial}{\partial y_3} + \cos t\frac{\partial}{\partial x_3}, Fe_4 = -s\sin t\frac{\partial}{\partial y_3} - s\sin t\frac{\partial}{\partial x_3}. \tag{106}$$

Thus, $\mathcal{D}_\theta = S_p\{e_1, e_2\}$ is a slant distribution with slant angle $\alpha = \frac{\pi}{4}$. Since Fe_3 and Fe_4 are orthogonal to M, $\mathcal{D}^\perp = S_p\{e_3, e_4\}$ is an anti-invariant distribution, that is, M is a 4-dimensional proper pseudo-slant sub-manifold of $\mathbb{R}^6$ with its almost Riemannian product structure $(F, <\,,\,>)$.

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