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A Sampled-data Regulator using Sliding Modes and Exponential Holder for Linear Systems

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Abstract

In a general command tracking and disturbance rejection problem, it is known that a sampled-data controller using zero-order hold may only guarantee asymptotic tracking at the sampling instances, but in general cannot guarantee the absence of ripples between the sampling instants. In this paper, a discrete robust regulator and a sampled-data robust regulator using slide modes techniques and exponential holder are presented. In particular, it is shown that the controller proposed for the sampled-data system ensures asymptotic tracking when applied to the continuous-time system.

1. Introduction

The extensive use of digital computers has introduced a great flexibility on the implementation of control laws but has also, in some cases, given rise to some problems related to the dynamic behavior to the coupling of continuous-time systems with digital devices via A/D and D/A converters. In fact, when a control law is implemented via digital devices, two ways are possible. The first is to design a continuous control law and use sufficiently small sampling periods with respect to the plant dynamics, to approximate by a discrete system the original continuous controller. The second approach consists in discretizing the plant dynamics and to design a digital control law on the basis of the sampled measurements. The output of the digital controller is then converted to continuous signal generally using zero orders holders. This second solution is in general more adequate since some of the structural properties may be ensured, even if only at the sampling instants, since in the intersampling time the system is in open-loop. In particular, for nonconstant reference signals, a digital control law applied via zero order holders to a continuous time system may cause the presence of ripple in the output tracking error signal. This means that the asymptotic output tracking is guaranteed only at the sampling instants, where the steady-state output error is zero. This can be explained by the fact that a necessary and sufficient condition for guaranteeing a ripple-free tracking is that an internal model of the reference and/or disturbance is present in the controller structure ([2], [3], [5], [11]). Clearly, when using zero-order holders, it is not possible to reconstruct the internal model, except for the constant signals.

For sampled-data linear systems, in [5] among others, a hybrid controller was presented; pointing out that a continuous internal model is necessary and sufficient to provide ripple-free response. Along the same lines, in [4], a hybrid robust controller consisting of a discrete-time linear controller and an analog linear immersion which guarantees a ripple free behavior was presented. In [6] a more general setting using a so-called exponential holder for nonlinear systems was presented.
Based on these ideas, in this work we present a ripple-free sampled-data robust regulator with sliding modes control scheme for linear systems. We formulate the design of a robust controller on the basis of sampling a continuous-time linear systems and then introducing the sliding mode approach, which permits to guarantee the stabilization property relaxing the requirements of the existence of a linear stabilizing control law and using the exponential holder to guarantee the existence of the internal model inside the controller structure... The paper is organized as follows: in Section 2 we give some preliminaries on the robust regulator by sliding modes techniques, while in Section 3 we introduce the main result of the paper. Section 4 is devoted to an illustrative example and finally, some conclusions are drawn.

2. Basic results on Robust Regulation

A central problem in control theory is that of manipulating the inputs of a system in such a way that the outputs track, at least asymptotically, a defined reference signals, preserving at the same time some desired stability property of the close-loop system. In [14], a discontinuous regulator using a sliding modes control technique is proposed, where the underlying idea is to design a sliding surface on which the dynamics of the system are constrained to evolve by means of a discontinuous control law, instead of designing a continuous stabilizing feedback, as in the case of the classical regulator problem. The sliding surface is constructed with the steady-state surface, and the state of the system is forced to reach the sliding surface in finite time with a sliding control.
To precise the ideas, let us consider a continuous-time linear system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + Pw(t) \tag{1}$$

$$\dot{w}(t) = Sw(t) \tag{2}$$

$$e(t) = Cx(t) - Rw(t) \tag{3}$$

where $u \in \Re^m$ is the input signal, $x \in \Re^n$ is the state of the system, $w \in \Re^p$ represents the state of an external signal generator, described by (2), which provides the reference and/or perturbation signals. Equation (3) describes the output tracking error $e \in \Re^q$ defined as the difference between the system output and the reference signal.
For this system, the mentioned problem has been treated under different approaches, among which is the regulator theory by sliding modes techniques. In general terms, this problem consists in finding a submanifold (the steady state submanifold) on which the output tracking error is zeroed, as well as an input signal (the steady state input) which makes this submanifold invariant and attractive. The sliding regulator problem approach has been studied in the linear case ([Louk:99],[Louk:99b]).

Since we are concerned with a discrete controller, the discretization of the continuous system (1)-(3) can be described by

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d u_k + P_d w_k \\ w_{k+1} &= S_d w_k \\ e_k &= C x_k - R w_k \end{aligned}$$

where

$$A_d = e^{\delta A} = \sum_{i=0}^{\infty} \frac{\delta^i}{i!} A^i;$$

$$B_d = \int_0^{\delta} e^{sA} B ds = \sum_{i=0}^{\infty} \frac{\delta^i}{i!} A^{i-1} B;$$

$$S_d = e^{\delta S} = \sum_{i=0}^{\infty} \frac{\delta^i}{i!} S^i;$$

$$C_d = C; R_d = R; P_d = \int_0^{\delta} e^{sA} P ds = \sum_{i=0}^{\infty} \frac{\delta^{i+1}}{(i+1)!} P_i;$$

where P_i can be computed iteratively from

$$P_0 = P; P_i = AP_{i-1} + PS^i; i = 1, 2, \ldots.$$

The classical **Robust Regulator Problem with Measurement of the Output** for system (1)-(3) consists in finding a dynamic controller

$$\begin{aligned} \dot{\xi}(t) &= F\xi(t) + Ge(t) \\ u &= H_e \xi(t) \end{aligned}$$

such that the following requirements hold:

S) The equilibrium point $(x, \xi) = (0, 0)$ of the closed loop system without disturbances

$$\dot{x}(t) = Ax(t) + BH_e \xi(t)$$

$$\dot{\xi}(t) = F\xi(t) + GCx(t)$$

is exponentially stable.

R) For each initial condition (x_0, w_0, ξ_0) , the dynamics of the system

$$\dot{x}(t) = Ax(t) + BH_e\xi(t) + Pw(t)$$
$$\dot{\xi}(t) = F\xi(t) + G(Cx(t) - Rw(t))$$
$$\dot{w}(t) = Sw(t)$$

satisfy that

$$\lim_{t\to\infty} e(t) = 0.$$

A solution to this problem can be found in [1]. This solution is stated in terms of the existence of mappings $x_{ss} = \Pi w; \xi_{ss} = \Sigma w$ satisfying the Francis equations

$$\begin{aligned} \Pi S &= A\Pi + BH_e\Sigma + P \\ \Sigma S &= F\Sigma \\ 0 &= C\Pi - R \end{aligned} \tag{4}$$

for all admissible values of the systems parameters. More precisely, the solution can be stated in terms of the existence of mappings $x_{ss} = \Pi w, u_{ss} = \Gamma w$ solving the equations

$$\Pi S = A\Pi + B\Gamma + P \tag{5}$$

$$0 = C\Pi - R \tag{6}$$

from which we reckon

$$\Sigma = \begin{pmatrix} \Gamma \\ \Gamma S \\ \vdots \\ \Gamma S^{q-1} \\ -a_0\Gamma - a_1\Gamma S - .... - a_{q-1}\Gamma S^{q-1} \end{pmatrix}$$

where the polynomial

$$s^q + a_{q-1}s^{q-1} + ... + a_1 s + a_0 = 0$$

is the characteristic polynomial of S. The mapping $x_{ss} = \Pi w$ represents the steady state zero output subspace and $u_{ss} = \Gamma w$ is the steady-state input which make invariant that subspace. This steady-state input can be generated, independently of the values of the parameters of the system and thanks to the Cayley-Hamilton Theorem, by the linear dynamical system

$$\dot{\eta} = \Phi\eta \tag{7a}$$

$$u_{ss} = H\eta \tag{7b}$$

where $\Phi = diag\{\Phi_1,...\Phi_m\}; H = diag\{H_1,..., H_m\}$ and

$$\Phi_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{q-1} \end{pmatrix};$$

$$H_i = (1 \quad 0 \quad \cdots \quad 0)_{1\times q}.$$

Defining the transformation $z_1 = x - \Pi w; z_2 = \eta,$ the system can be rewritten as

$$\dot{z}_1 = Az_1 - BHz_2 + Bu \tag{8}$$

$$\dot{z}_2 = \Phi z_2 \tag{9}$$

$$e(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{10}$$

Finally, a controller which solves the problem can be constructed as an observer for system (8)-(9), namely

$$\begin{aligned} \dot{\xi}_1 &= (A_0 - G_1C_0)\xi_1 - B_0H\xi_2 + B_0u + G_1e \\ \dot{\xi}_2 &= -G_2C_0\xi_1 + \Phi\xi_2 + G_2e \\ u &= K\xi_1 + H\xi_2 \end{aligned} \tag{11}$$

where A_0, B_0, C_0 are the nominal values of the matrices of the system (1)-(3) and K and G_1, G_2 make stable the matrices $(A_0 + B_0K)$ and

$$\begin{pmatrix} A_0 & -B_0H \\ 0 & \Phi \end{pmatrix} - \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \begin{pmatrix} C_0 & 0 \end{pmatrix}. \tag{12}$$

When dealing with controllers implemented via digital devices and zero order holders, the sampled data version of the controller could render unstable the closed-loop system. In this work we will take the approach of designing a hybrid controller consisting in two parts: a discrete sliding mode controller ensuring the stabilization of the closed-loop system, and a continuous part containing the internal model dynamics (internal model) obtained from the continuous model.

3. The Continuous Sliding Robust Regulator

Analogously to the case of the Robust Regulator Problem, we formulate the ***Sliding Mode Robust Regulator Problem*** ([13], [14], [15]) as the problem of finding a sliding surface

$$\sigma = \sigma(\xi) = 0,\ \sigma = col(\sigma_1(\xi),...,\sigma_m(\xi)) \tag{13}$$

and a dynamic compensator

$$\dot{\xi} = g(\xi, e) \tag{14}$$

with the control action defined as

$$u_i = \begin{cases} u_i^+(\xi) & \sigma_i(\xi) > 0 \\ u_i^-(\xi) & \sigma_i(\xi) < 0 \end{cases}, \quad i = 1,...,m \tag{15}$$

where the mappings $u_i^+(\xi)$, $u_i^-(\xi)$ and $\sigma_i(\xi)$ are calculated in order to induce an asymptotic convergence to the sliding surface $\sigma_i(\xi) = 0$ and such that, for all admissible parameter values in a suitable neighborhood $\mathcal{P}$ of the nominal parameter vector, the following conditions hold:

(**SS**$_c$) The equilibrium point $(x, \xi) = (0, 0)$ of the closed-loop system is asymptotically stable.

(**SM**$_c$) The sliding surface is attractive, namely the state of the closed loop system converges to the manifold $\sigma(\xi) = 0$.

(**SR**$_c$) The output tracking error tends asymptotically to zero, namely

$$\lim_{t \to \infty} e(t) = 0$$

Now, to introduce the sliding mode approach into the regulator problem, we will chose the control input $u(t)$ as

$$u(t) = u_{slid} + u_{eq}$$

instead of $u(t) = K\xi_1 + H\xi_2$ as taken in the controller (11), where we impose that u_{eq} must be equal to $H\xi_2$ when $\sigma(\xi) = 0$. Note that the stabilizing part $K\xi_1$ will now be substituted by the term u_{slid} which will be calculated to make attractive the sliding surface.

To be more precise, let us consider the switching surface

$$\sigma = \begin{bmatrix} \Sigma & 0 \end{bmatrix} \xi = \Sigma \xi_1, \tag{16}$$

where $\sigma \in \Re^m$, $\Sigma \in \Re^{mxn}$ with $rank \quad \Sigma B_0 = m$.
Differentiating this function, and from the first equation of (11) we reckon

$$\begin{aligned} \dot{\sigma} &= \Sigma \dot{\xi}_1 = \Sigma[(A_0 - G_1 C_0)\xi_1 - B_0 H \xi_2 + B_0 u + G_1 e] \\ &= \Sigma(A_0 - G_1 C_0)\xi_1 - \Sigma B_0 H \xi_2 + \Sigma B_0 u + \Sigma G_1 e \end{aligned}$$

from which the equivalent control u_{eq} is obtained from the condition $\dot{\sigma} = 0$ as

$$u_{eq} = -(\Sigma B_0)^{-1} \Sigma \left[(A_0 - G_1 C_0)\xi_1 - B_0 H \xi_2 + G_1 e \right]$$

Defining the estimation errors as $\varepsilon_1 = z_1 - \xi_1$ and $\varepsilon_2 = z_2 - \xi_2$, we may substitute u_{eq} into equation (8) at the nominal values of the parameters to get the sliding motion dynamics

$$\dot{z}_1 = [I_n - B_0(\Sigma B_0)^{-1}\Sigma]A_0 z_1 + B_0(\Sigma B_0)^{-1}\Sigma(A_0 - G_1 C_0)\varepsilon_1 - B_0 H \varepsilon_2$$

where the estimation errors satisfy the dynamics

$$\begin{aligned} \dot{\varepsilon}_1 &= (A_0 - G_1 C_0)\varepsilon_1 - B_0 H \varepsilon_2 \\ \dot{\varepsilon}_2 &= -G_2 C_0 \varepsilon_1 + \Phi \varepsilon_2. \end{aligned}$$

Note that these dynamics are asymptotically stable thanks to the observability assumption of matrix (12).

Lemma 1. [14] *Define the operator* D *as* $D = (I_n - B(\Sigma B)^{-1}\Sigma)$ *. Then the relation*

$$D(A\Pi - \Pi S + P) = 0 \tag{17}$$

is true if and only if there exist matrices Π *and* Γ *such that*

$$A\Pi - \Pi S + P = B\Gamma. \tag{18}$$

Proof. The operator D is a projection operator along the rank of B over the null space of Σ [16], namely

$$\begin{aligned} DB &= (I_n - B(\Sigma B)^{-1}\Sigma)B = 0 \\ Dz_1 &= z_1 \quad \forall z_1 \in \aleph, \aleph = \{z_1 \in \Re^n \mid \Sigma z_1 = 0\} \end{aligned}$$

Thus, if condition (18) holds, then it follows that $D(A\Pi - \Pi S + P) = DB\Sigma = 0.$ Conversely, if condition (17) holds, then $(A\Pi - \Pi S + D)$ must be in the image of B this is, $(A\Pi - \Pi S + D) = B\Gamma$ for some matrix Γ. ■

A condition for the solution of the Sliding Mode Regulator Problem can be given in the following result.

Proposition 2. *Assume the following assumptions:*

H1) *The matrix* S *has all its eigenvalues on the imaginary axis*

H2) *The pair* (A_0, B_0) *is stabilizable*

H3) *The pair* $\begin{bmatrix} C_0 & 0 \end{bmatrix}$, $\begin{bmatrix} A_0 & -B_0 H \\ 0 & \Phi \end{bmatrix}$ *is observable.*

Then the Sliding Mode Regulator Problem is solvable if there exists a matrix Π *solving the equations*

$$A\Pi - \Pi S + P = -B\Gamma \tag{19}$$

$$C\Pi - R = 0 \tag{20}$$

for some matriz Γ, *and or all admissible values of the system parameters.*

Proof . Let us choose the control as

$$u = -Msign(\sigma) + u_{eq},$$

with $M = diag(m_i); m_i > 0,$ and $sign(\sigma) = [sign(\sigma_1), ..., sign(\sigma_m)]^T$. This control action guarantees a sliding mode motion on the surface $\sigma = 0$. Then, assuming that the observer estimation error decays rapidly by appropriate choice of the gains G_1, G_2 we have that

$$\dot{z}_1 = DA_0 z_1 |_{\Sigma z_1 = 0}$$

Since the matrix Σ by assumption H2 can be chosen such that ΣB is invertible, and the $(n-m)$ eigenvalues of DA_0 can be arbitrarily placed in C^-, then $z_1(t) \rightarrow 0$ as $t \rightarrow \infty$ satisfying condition (**SS** $_c$). Now, since the tracking error equation is given by $e(t) = C_0 z_1(t)$, then it follows that $e(t)$ goes to zero asymptotically, satisfying condition (**SR** $_c$). ■

Note that when the state of the system is on the sliding surface, the control signal is exactly u_{eq} which in turn comes to be $u_{eq} = H\xi_2 = u_{ss}$, namely, the steady-state input. This steady -state input guarantees that the output tracking error stays at zero. This property will be used later.

4. A Sliding Robust Regulator for Discrete Systems

For the discrete case, the problem can be formulated in a similar way to the continuous case. To this end, let us consider the discretization of system (8)-(10), this is

$$\begin{bmatrix} z_{1,k+1} \\ z_{2,k+1} \end{bmatrix} = \begin{bmatrix} A_d & -\Lambda \\ 0 & \Phi_d \end{bmatrix} \begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} + \begin{bmatrix} B_d \\ 0 \end{bmatrix} u_k \tag{21}$$

$$e_k = \begin{bmatrix} C_d & 0 \end{bmatrix} \begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} \tag{22}$$

where

$$A_{d0} = e^{A_0 T}, \Lambda = \int_0^T e^{A\theta} B_0 H d\theta,\ C_{d0} = C_0$$
$$\Phi_d = e^{\Phi T}, u(kT+\theta) = u(kT);$$
$$B_{d0} = \int_0^T e^{A_0\theta} B_0 d\theta,\ 0 \le \theta \le T.$$

For this system, the Sliding Regulator Problem can be set as the problem of finding a sliding surface σ_k and a dynamic controller

$$\xi_{k+1} = F_d \xi_k + G_d e_k \tag{23}$$

$$u_k = \alpha_d(\xi_k, e_k) \tag{24}$$

such that, for all admissible parameter values in a suitable neighborhood $\mathcal{P}$ of the nominal parameter vector, the following conditions hold:

(**SS** $_d$) The equilibrium point $(x,\xi) = (0,0)$ of the closed-loop system is asymptotically stable.

(**SM** $_d$) The sliding surface is attractive, namely the state of the closed loop system converges to the manifold $\sigma_k(\xi_k) = 0$.

(**SR** $_d$) For each initial condition (x_0, w_0, ξ_0), the dynamics of the closed-loop system

$$x_{k+1} = A_d x_k + B_d \alpha_d(\xi_k, e_k) + P w_k$$
$$\xi_{k+1} = F\xi_k + G(C_d x_k - R_d w_k)$$
$$w_{k+1} = S_d w_k$$

where $S_d = e^{ST}$ guarantees that $\lim_{k\to\infty} e_k = 0.$
Assume the following conditions hold:
(**H1** $_d$) All the eigenvalues of matrix S_d lie on the unitary circle.

($\mathbf{H2}_d$) The pair $\{A_{d0}, B_{d0}\}$ is stabilizable,

($\mathbf{H3}_d$) There exists a solution Π_d, Γ_d to the regulator equations

$$\Pi_d S_d = A_d \Pi_d + B_d \Gamma_d + P_d \tag{25}$$

$$0 = C_d \Pi_d - R_d \tag{26}$$

($\mathbf{H4}_d$) The pair $\begin{bmatrix} C_{d0} & 0 \end{bmatrix}, \begin{bmatrix} A_{d0} & -\Lambda \\ 0 & \Phi_d \end{bmatrix}$ is observable.

Then, a classic robust regulator can be constructed as

$$\begin{aligned} \xi_{1,k+1} &= (A_{d0} - G_{d1} C_{d0}) \xi_{1,k} - \Lambda \xi_{2,k} + B_{d0} u_k + G_{d1} e_k \\ \xi_{2,k+1} &= -G_{d2} C_{d0} \xi_{1,k} + \Phi_d \xi_{2,k} + G_{d2} e_k \\ u_k &= K_d \xi_{1,k} + H \xi_{2,k} \end{aligned} \tag{27}$$

where K_d and G_{d1}, G_{d2} make stable the matrices $(A_{d0} + B_{d0} K_d)$ and

$$\begin{pmatrix} A_{d0} & -\Lambda \\ 0 & \Phi_d \end{pmatrix} - \begin{pmatrix} G_{d1} \\ G_{d2} \end{pmatrix} \begin{pmatrix} C_{d0} & 0 \end{pmatrix}. \tag{28}$$

respectively.
For the Discrete Sliding Regulator Problem, we can chose a sliding surface

$$\sigma_k = \begin{bmatrix} \Sigma_d & 0 \end{bmatrix} \xi_k = \Sigma_d \xi_{1,k}, \tag{29}$$

and calculate the equivalent control. The following result, which can be proved similarly to the continuous case, gives a solution to the Discrete Sliding Regulator Problem:

Proposition 3. *Assume that assumptions* $H1_d$ *through* $H4_d$ *hold. Then the Discrete Sliding Regulator Problem is solvable. Moreover, the controller solving the problem can be chosen as*

$$u_k = u_{eq,k} = -(\Sigma_d B_{d0})^{-1} \Sigma_d \left[(A_{d0} - G_{d1} C_{d0}) \xi_{1,k} - \Lambda \xi_{2,k} + G_{d1} e_k \right].$$

Proof. Calculating

$$\begin{aligned} \sigma_{k+1} &= \Sigma_d \xi_{1,k+1} \\ &= \Sigma_d [(A_{d0} - G_{d1} C_{d0}) \xi_{1,k} - \Lambda \xi_{2,k} + B_{d0} u_k + G_{d1} e_k] \end{aligned}$$

we can calculate the equivalent control from the condition $\sigma_{k+1} = 0,$ namely:

$$u_{eq,k} = -(\Sigma_d B_{d0})^{-1} \Sigma_d \left[(A_{d0} - G_{d1} C_{d0}) \xi_{1,k} - \Lambda \xi_{2,k} + G_{d1} e_k \right].$$

Note that this control makes also the sliding surface attractive, since the same control guarantees that $\sigma_{k+j}=0$ for $j \geqslant 1$. Now, substituting u_{eq} in the first equation of (21) we obtain

$$z_{1,k+1}=[I_n - B_{d0}(\Sigma_d B_{d0})^{-1}\Sigma_d]A_{d0}z_{1,k} + B_{d0}(\Sigma_d B_{d0})^{-1}\Sigma_d(A_{d0}-G_{d1}C_{d0})\varepsilon_{1,k} - \Lambda\varepsilon_{2,k}$$

where $\epsilon_{1,k}=z_{1,k}-\xi_{1,k}$; $\epsilon_{2,k}=z_{2,k}-\xi_{2,k}$. As in the continuos case, if the gains G_{d1}, G_{d2} are appropriately chosen, the estimation errors $\epsilon_{1,k}$ and $\epsilon_{2,k}$ will converge to zero and then

$$z_{1,k+1}=DA_{d0}z_{1,k}$$

where $D=[I_n - B_{d0}(\Sigma_d B_{d0})^{-1}\Sigma_d]$. Since the matrix Σ_d by assumption H2 $_d$ can be chosen such that $\Sigma_d B_{d0}$ is invertible, and the *(n-m)* eigenvalues of DA_{d0} can be arbitrarily placed inside the unitary circle, then $z_{1,k}\to 0$ as $k\to\infty$ satisfying condition (**SS** $_d$). Now, since the tracking error equation is given by $e_k = C_{d0}z_{1,k}$, then it follows that e_k goes to zero asymptotically, satisfying condition (**SR** $_d$). ■

Note that when the state of the system is on the sliding surface, the control signal is exactly u_{eq} which in turn comes to be $u_{eq}=H\xi_2=u_{ss}$, namely, the steady-state input.

Again note that when the solution of the system is on the sliding surface, the control signal is exactly u_{eq} which in turn, since $\Lambda = B_{d0}H$, comes to be

$$u_{eq}=(\Sigma_d B_{d0})^{-1}\Sigma_d\Lambda\xi_{2,k}=H\xi_{2,k},$$

namely, the steady-state input.

Clearly, this controller guarantees zero output tracking error only at the sampling instants, but not at the intersampling. To force the output tracking error to converge to zero also in the intersampling time, in the following section we will formulate the a ripple-free sliding regulator problem.

5. A Ripple-Free Sliding Robust Regulator for Sampled Data Linear Systems

From the previous discussion it is clear that implementing a Sliding Mode Robust Regulator for the discretization of the continuous linear system, this will guarantee only that the output tracking error will be zeroed only at the sampling instant. In order to eliminate the possible ripple, it is necessary to reproduce the internal model (7) from its discrete time realization. To do this, we note that the solution of (7) can be written as $\xi(t)=e^{\Phi t}\xi(0)$, and setting $t=kT+\theta$ with $\theta\in[0,T)$ we have

$$\xi(k\delta+\theta)=e^{\Phi(k\delta+\theta)}\xi(0)=e^{\Phi\theta}e^{\Phi kT}\xi(0)$$
$$=e^{\Phi\theta}\xi(kT)$$
$$u_{ss}(kT+\theta)=H\xi(kT+\theta)=He^{\Phi\theta}\xi(kT)$$

which describe *exactly* the behavior also in the intersampling. The term $e^{\Phi\theta}$ is known as the ***exponential holder***.

We can now formulate the **Ripple-Free Sliding Robust Regulator Problem** as the problem of finding a sliding surface

$$\sigma_k=\Sigma\xi_k \tag{30}$$

and a dynamic controller

$$\xi_{k+1}=F\xi_k+Ge_k \tag{31}$$

$$u(kT+\theta)=\alpha\left(\xi_k,\theta,e_k\right); \tag{32}$$
$$0\le\theta\le T.$$

such that, for all admissible parameter values in a suitable neighborhood $\mathcal{P}$ of the nominal parameter vector, the following conditions hold:

(**SS** r) The equilibrium point $\left(x_k,\xi_k\right)=\left(0,0\right)$ of the system in closed-loop is asymptotically stable.

(**SM** r) The sliding surface is attractive, namely the state of the closed loop system converges to the manifold $\sigma_k(\xi_k)=0.$

(**SR** r) For each initial condition $\left(x_0,w_0,\xi_0\right)$, the dynamics of the closed-loop system

$$\dot{x}(t)=Ax(t)+B\alpha\left(\xi_k,\theta,e_k\right)+Pw(t)$$
$$\xi_{k+1}=F\xi_k+G(C_dx_k-R_dw_k)$$
$$\dot{w}(t)=Sw(t)$$

guarantees that

$$\lim_{t\to\infty}e(t)=0.$$

In order to solve the Ripple-Free Sliding Robust Regulator Problem, the following assumptions will be considered:

H1) The matrix S has all its eigenvalues on imaginary axis

H2) The pair $\left(A_0,B_0\right)$ is stabilizable

H3) The equations (5), (6) have solution Π,Γ for all admissible values of the system parameters.

H4) The pair $\begin{bmatrix} C_d & 0 \end{bmatrix}$, $\begin{bmatrix} A_d & -M_d \\ 0 & \Phi_d \end{bmatrix}$ is detectable, where $M_d = \int_0^T e^{A_0(T-\theta)} B_0 H e^{\Phi\theta} d\theta.$

For this case, and taking the previous results, we now state the following result.

Theorem 4. *Let us assume assumptions H1) to H4) hold. Then the* ***RFSRRP*** *is solvable. Moreover, the controller which solves the problem is given by*

$$\begin{aligned} \xi_{1,k+1} &= (A_{d0} - G_{d1}C_{d0})\xi_{1,k} - M_d\xi_{2,k} + B_{d0}u_k + G_{d1}e_k \\ \xi_{2,k+1} &= -G_{d2}C_{d0}\xi_{1,k} + \Phi_d\xi_{2,k} + G_{d2}e_k \\ u_k &= -(\Sigma_d B_{d0})^{-1}\Sigma_d\left[(A_{d0} - G_{d1}C_{d0})\xi_{1,k} - B_{d0}He^{\Phi\theta}\xi_{2,k} + G_{d1}e_k\right] \end{aligned} \tag{33}$$

Proof. In order to implement the discretized controller, we consider again the transformed continuous system

$$\dot{z}_1 = Az_1 - BHz_2 + Bu \tag{34}$$

$$\dot{z}_2 = \Phi z_2 \tag{35}$$

$$e(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{36}$$

$$\dot{w} = Sw. \tag{37}$$

Substituting u_k in the equation (34) gives:

$$\dot{z}_1 = Az_1 - BHz_2 - B(\Sigma_d B_{d0})^{-1}\Sigma_d \times \left[(A_{d0} - G_{d1}C_{d0})\xi_{1,k} - B_{d0}He^{\Phi\theta}\xi_{2,k} + G_{d1}e_k\right]$$

whose discretization, together with that of (35) is be given by

$$\begin{aligned} z_{1,k+1} &= [I_n - B_{d0}(\Sigma_d B_{d0})^{-1}\Sigma_d]A_{d0}z_{1,k} + \\ &+ B_{d0}(\Sigma_d B_{d0})^{-1}\Sigma_d(A_{d0} - G_{d1}C_{d0})\varepsilon_{1,k} - \Lambda\varepsilon_{2,k} \\ z_{2,k+1} &= \Phi_d z_{2.k}. \end{aligned}$$

As in the case of discrete sliding regulator, an observer may be constructed as

$$\begin{aligned} \xi_{1,k+1} &= (A_{d0} - G_{d1}C_{d0})\xi_{1,k} - M_d\xi_{2,k} + B_{d0}u_k + G_{d1}e_k \\ \xi_{2,k+1} &= -G_{d2}C_{d0}\xi_{1,k} + \Phi_d\xi_{2,k} + G_{d2}e_k. \end{aligned}$$

Defining a switching function as

$$\sigma_k = \begin{bmatrix} \Sigma_d & 0 \end{bmatrix} \xi_k = \Sigma_d \xi_{1,k}$$

and proceeding as in the discrete case, we may show that by a proper choice of the gains $G_{d1,} G_{d2}$, the estimation errors converge to zero and the matrix $DA_d z_k$ where $D = [I_n - B_{d0}(\Sigma_d B_{d0})^{-1}\Sigma_d]$ has all the eigenvalues inside the unitary circle. Thus $e_k \to 0$ when $k \to \infty$. To see that the error is eliminated also during the interval $kT < \theta \le (k+1)T$, $k = 0,1,2,\ldots$, we observe that when $e_k = 0,$ the control law u_k is

$$\begin{aligned} u(kT+\theta) &= He^{\Phi\theta}\xi_{2,k} \\ &= H\xi_2 \end{aligned}$$

which is exactly the continuous steady-state input needing to zeroing the continuous output tracking error, so requirement SR $_r$ is also fulfilled. ■

6. An illustrative example

Consider the model of a DC motor given by:

$$\begin{aligned} \frac{dw_m}{dt} &= \frac{k_t}{J} i_a - \frac{\tau_1}{J} \\ \frac{di_a}{dt} &= -\frac{\lambda_0}{L} w_m - \frac{R}{L} i_a + \frac{1}{L} u \end{aligned}$$

where i_a is the armature current, w_m is the shaft speed, R is armature resistance, λ_o is the back-EMF constant, τ_1 is the load torque, u is the terminal voltage, J is the inertia of the motor, rotor and load, L is the armature inductance and k_t is the torque constant. Defining $x_1 = w_m$ and $x_2 = i_a$, and assuming that τ_1 is a known constant we have:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_t}{J} \\ -\frac{\lambda_0}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u + \begin{bmatrix} -\frac{\tau}{J} \\ 0 \end{bmatrix}$$

$$\dot{w} = Sw$$

where $w = (w_1, w_2, w_3)^T$, $S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & -\alpha & 0 \end{bmatrix}$, $y = x_{1,}$ $y_{ref} = w_2$, $w_1 = \tau_1$ and

$L = 1mH$, $R = 0.5\Omega$, $J = 0.001Kgm^2$, $\lambda_o = 0.001V \times s \times rad^{-1}$, $\beta = 0.01Nm \times s \times rad^{-1}$, $k_t = 0.008NmA^{-1}$.

From this, we can calculate

$$\Phi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, a_0 = 0, a_1 = \alpha^2, a_2 = 0.$$

Discretizing the system with a sampling of $T=0.3\ s$ and choosing a reference $y_{ref} = 0.1\sin(5t)$, the discrete robust controller with no exponential holder is constructed with:

$$F_d = \begin{bmatrix} -0.5399 & 1.0201 & 0 & 0 & 0 \\ 0.0645 & -0.1218 & 0 & 0 & 0 \\ -0.8072 & 0 & 0 & 1 & 0 \\ 1.3671 & 0 & 0 & 0 & 1 \\ 1.3775 & 0 & 1 & -1.1414 & 1.1414 \end{bmatrix}$$

$$G_d = \begin{bmatrix} 1.802 & 0.1481 & 0.8072 & -1.3671 & -1.3775 \end{bmatrix}^T$$

where

$$\Sigma_d = \begin{bmatrix} 0.1194 & 1 \end{bmatrix}, A_d = \begin{bmatrix} 0.9996 & 2.2284 \\ -0.00027 & 0.8603 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0.343 & 0.279 \end{bmatrix}^T, C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

As is shown in Figure 1, as expected for the Discrete Sliding Regulator, the output tracking error is zero at the sampling instant, but different from zero in the intersampling times. Constructing now the controller (33) with an exponential holder we obtain

$$G_d = \begin{bmatrix} 1.802 & 0.156 & 1.199 & -0.992 & -35.054 \end{bmatrix}^T$$

$$F_d = \begin{bmatrix} -0.531 & 1.0201 & 0 & 0 & 0 \\ 0.0634 & -0.1218 & 0 & 0 & 0 \\ -1.1993 & 0 & 1 & 0.1994 & 0.0371 \\ 0.9922 & 0 & 0 & 0.0707 & 0.1994 \\ 35.0536 & 0 & 0 & -4.9874 & 0.0707 \end{bmatrix}$$

where

$$e^{\Phi\theta} = \begin{bmatrix} 1 & 0.2\sin(5\theta) & -0.04\cos(5\theta)+0.04 \\ 0 & \cos(5\theta) & 0.2\sin(5\theta) \\ 0 & -5\sin(5\theta) & \cos(5\theta) \end{bmatrix}$$

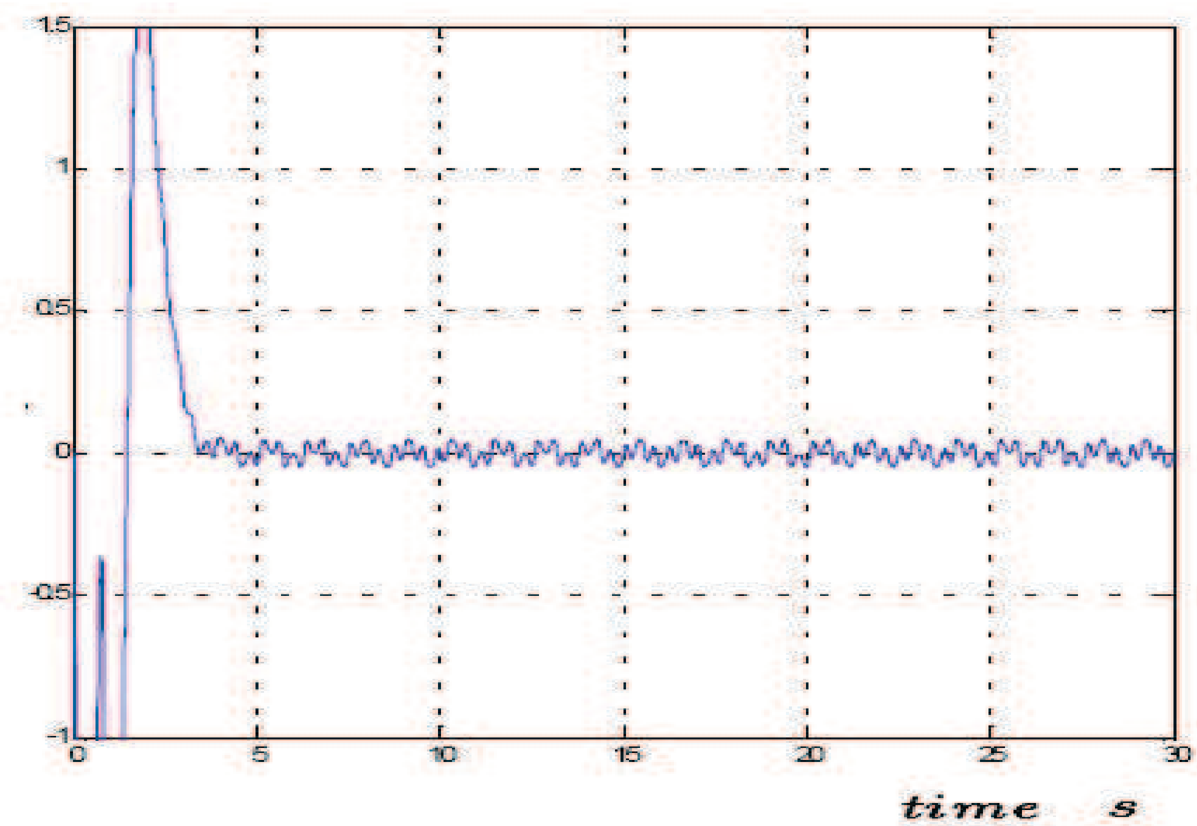

Figure 1. Output tracking error for the Discrete Sliding Robust Regulator

As shown in Figure 2, the sliding discretized controller with exponential holder present a remarkable performance guaranteeing zero output trackin error also in the intersampling.

Finally, variations on the values of the parameters ranging up to $\pm 25\%$ for R and $\pm 12\%$ for L were introduced. As may be observed in Figure 3, the controller is able to cope with these variations, maintaining the asymptotic tracking property as well.

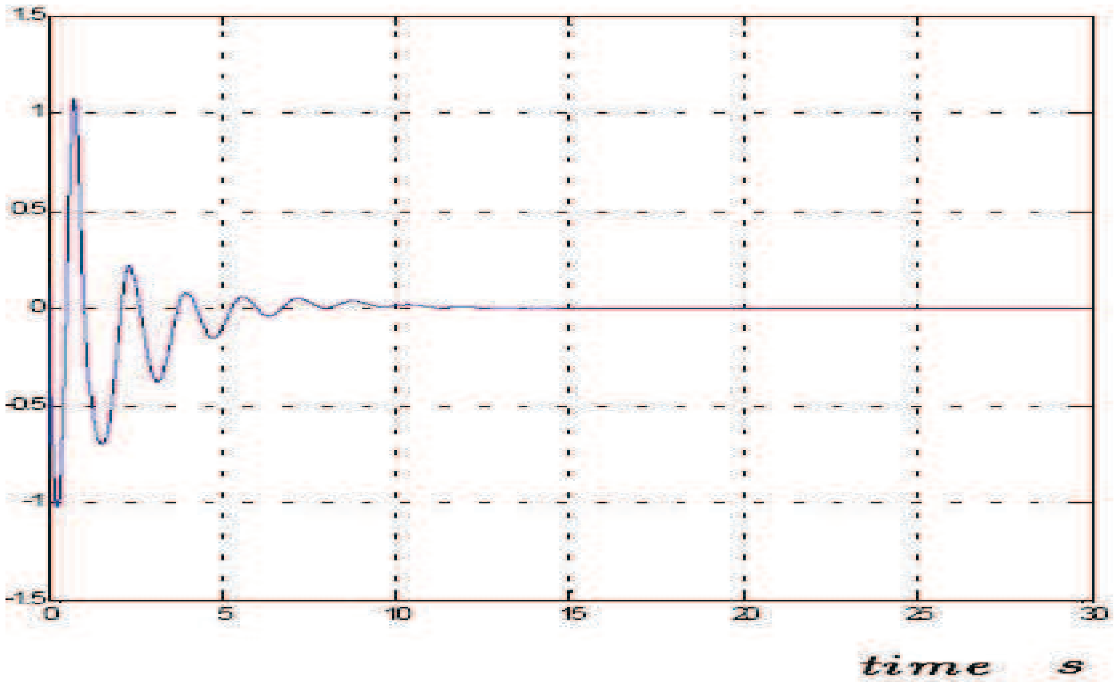

Figure 2. Output tracking error for the Ripple-Free Slidng Robust Regulator

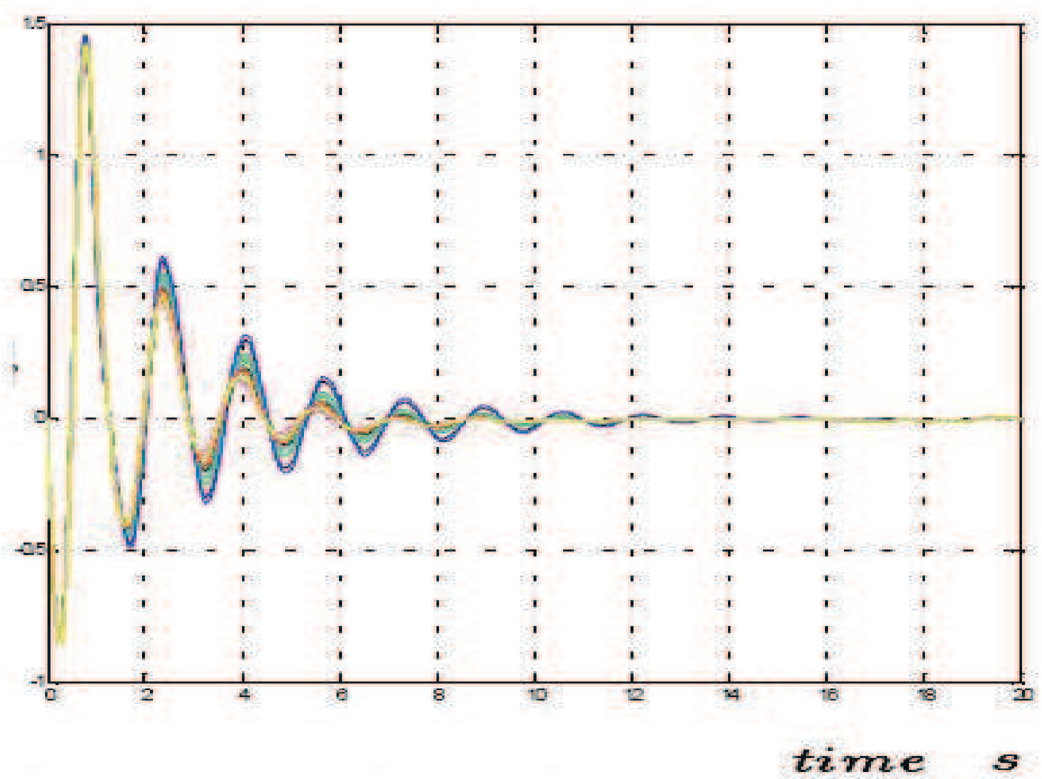

Figure 3. Output tracking error for the Ripple-Free Slidng Robust Regulator with parametric variations

7. Conclusions

In this paper, we presented an extensión to the Continuous Sliding Robust Regulator to the Dicrete case. A Ripple-Free Sliding Robust Regulator which guarantees that the output tracking error is zeroed not only at the sampling instants, but also in the intersampling behavior was alsoformulated and a solution was obtained. The controller has two components: one of them depending of the discrete dynamics of the system, and the other containing the internal model of the reference and/or perturbations generator. This feature allows the implementation of the controller on a digital device. An illustrative example shows the performance of the presented scheme.

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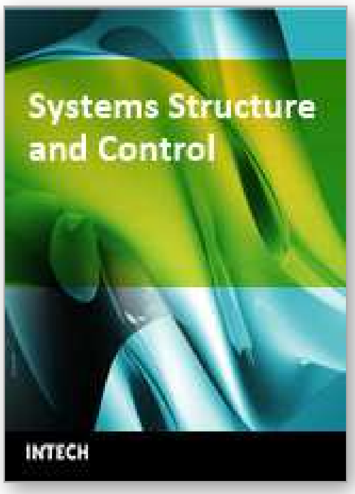

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