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Asymptotic Stability Analysis of Linear Time-Delay Systems: Delay Dependent Approach

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1. Introduction

The problem of investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability.

During the last three decades, the problem of stability analysis of time delay systems has received considerable attention and many papers dealing with this problem have appeared (Hale & Lunel, 1993). In the literature, various stability analysis techniques have been utilized to derive stability criteria for asymptotic stability of the time delay systems by many researchers (Yan, 2001; Su, 1994; Wu & Muzukami, 1995; Xu, 1994; Oucheriah, 1995; Kim, 2001).

The developed stability criteria are classified often into two categories according to their dependence on the size of the delay: delay-dependent and delay-independent stability criteria (Hale, 1997; Li & de Souza, 1997; Xu et al., 2001). It has been shown that delay-dependent stability conditions that take into account the size of delays, are generally less conservative than delay-independent ones which do not include any information on the size of delays.

Further, the delay-dependent stability conditions can be classified into two classes: frequency-domain (which are suitable for systems with a small number of heterogeneous delays) and time-domain approaches (for systems with a many heterogeneous delays).

In the first approach, we can include the two or several variable polynomials (Kamen 1982; Hertz et al. 1984; Hale et al. 1985) or the small gain theorem based approach (Chen & Latchman 1994).

In the second approach, we have the comparison principle based techniques (Lakshmikantam & Leela 1969) for functional differential equations (Niculescu et al. 1995a; Goubet-Bartholomeus et al. 1997; Richard et al. 1997) and respectively the Lyapunov stability approach with the Krasovskii and Razumikhin based methods (Hale & Lunel 1993; Kolmanovskii & Nosov 1986). The stability problem is thus reduced to one of finding solutions to Lyapunov (Su 1994) or Riccati equations (Niculescu et al., 1994), solving linear matrix inequalities (LMIs) (Boyd et al. 1994; Li & de Souza, 1995; Niculescu et al., 1995b; Gu 1997) or analyzing eigenvalue distribution of appropriate finite-dimensional matrices (Su

1995) or matrix pencils (Chen et al., 1994). For further remarks on the methods see also the guided tours proposed by (Niculescu et al., 1997a; Niculescu et al., 1997b; Kharitonov, 1998; Richard, 1998; Niculescu & Richard, 2002; Richard, 2003).

It is well-known (Kolmanovskii & Richard, 1999) that the choice of an appropriate Lyapunov–Krasovskii functional is crucial for deriving stability conditions. The general form of this functional leads to a complicated system of partial differential equations (Malek-Zavareiand & Jamshidi, 1987). Special forms of Lyapunov–Krasovskii functionals lead to simpler delay-independent (Boyd et al., 1994; Verriest & Niculescu, 1998; Kolmanovskii & Richard, 1999) and (less conservative) delay-dependent conditions (Li & de Souza, 1997; Kolmanovskii et al., 1999; Kolmanovskii & Richard, 1999; Park, 1999; Lien et al., 2000; Niculescu, 2001). Note that the latter simpler conditions are appropriate in the case of unknown delay, either unbounded (delay-independent conditions) or bounded by a known upper bound (delay-dependent conditions).

In the delay-dependent stability case, special attention has been focused on the first delay interval guaranteeing the stability property, under some appropriate assumptions on the system free of delay. Thus, algorithms for computing optimal (or suboptimal) bounds on the delay size are proposed in (Chiasson, 1988; Chen et al., 1994) (frequency-based approach), in (Fu et al., 1997) (integral quadratic constraints interpretations), in (Li & de Souza, 1995; Niculescu et al., 1995b; Su, 1994) (Lyapunov-Razumikhin function approach) or in (Gu, 1997) (discretization schemes for some Lyapunov- Krasovskii functionals). For computing general delay intervals, see, for instance, the frequency based approaches proposed in (Chen, 1995).

In the past few years, there have been various approaches to reduce the conservatism of delay-dependent conditions by using new bounding for cross terms or choosing new Lyapunov–Krasovskii functional and model transformation. The delay-dependent stability criterion of (Park et al., 1998; Park, 1999) is based on a so-called Park's inequality for bounding cross terms. However, major drawback in using the bounding of (Park et al., 1998) and (Park, 1999) is that some matrix variables should be limited to a certain structure to obtain controller synthesis conditions in terms of LMIs. This limitation introduces some conservatism. In (Moon et al., 2001) a new inequality, which is more general than the Park's inequality, was introduced for bounding cross terms and controller synthesis conditions were presented in terms of nonlinear matrix inequalities in order to reduce the conservatism. It has been shown that the bounding technique in (Moon et al., 2001) is less conservative than earlier ones. An iterative algorithm was developed to solve the nonlinear matrix inequalities (Moon et al., 2001).

Further, in order to reduce the conservatism of these stability conditions, various model transformations have been proposed. However, the model transformation may introduce additional dynamics. In (Fridman & Shaked, 2003) the sources for the conservatism of the delay-dependent methods under four model transformations, which transform a system with discrete delays into one with distributed delays are analyzed. It has been demonstrated that descriptor transformation, that has been proposed in (Fridman & Shaked, 2002a), leads to a system which is equivalent to the original one, does not depend on additional assumptions for stability of the transformed system and requires bounding of fewer cross-terms. In order to reduce the conservatism, (Han, 2005a; Han, 2005b) proposed some new methods to avoid using model transformation and bounding technique for cross terms.

In (Fridman & Shaked, 2002b) both the descriptor system approach and the bounding technique using by (Moon et al., 2001) are utilized and the delay-dependent stability results are performed. The derived stability criteria have been demonstrated to be less conservative than existing ones in the literature.
Delay-dependent stability conditions in terms of linear matrix inequalities (LMIs) have been obtained for retarded and neutral type systems. These conditions are based on four main model transformations of the original system and application mentioned inequalities.
The majority of stability conditions in the literature available, of both continual and discrete time delay systems, are sufficient conditions. Only a small number of works provide both necessary and sufficient conditions, (Lee & Diant, 1981; Xu et al., 2001; Boutayeb & Darouach, 2001), which are in their nature mainly dependent of time delay. These conditions do not possess conservatism but often require more complex numerical computations. In our paper we represent some necessary and sufficient stability conditions.
Less attention has been drawn to the corresponding results for discrete-time delay systems (Verriest & Ivanov, 1995; Kapila & Haddad, 1998; Song et al., 1999; Mahmoud, 2000; Lee & Kwon, 2002; Fridman & Shaked, 2005; Gao et al., 2004; Shi et al., 2000). This is mainly due to the fact that such systems can be transformed into augmented high dimensional systems (equivalent systems) without delay (Malek-Zavarei & Jamshidi, 1987; Gorecki et al., 1989). This augmentation of the systems is, however, inappropriate for systems with unknown delays or systems with time varying delays. Moreover, for systems with large known delay amounts, this augmentation leads to large-dimensional systems. Therefore, in these cases the stability analysis of discrete time delay systems can not be to reduce on stability of discrete systems without delay.
In our paper we present delay-dependent stability criteria for particular classes of time delay systems: continuous and discrete time delay systems and continuous and discrete time delay large-scale systems. Thereat, these stability criteria are express in form necessary and sufficient conditions.
The organization of this chapter is as follows. In section 2 we present necessary and sufficient conditions for delay-dependent asymptotic stability of particular class of continuous and discrete time delay systems. Moreover, we show that in the paper of (Lee & Diant, 1981) there are some mistakes in formulation of particular theorems. We correct these errors and extend derived results on discrete time delay systems. Further extensions of these results to the class of continuous and discrete large scale time delay systems are presented in the section 3. All theoretical results are supported by suitable chosen numerical examples. And section 4 discuss and summarizes contributions.

2. Time delay systems

Throughout this chapter we use the following notation. $\mathbb{R}$ and $\mathbb{C}$ denote real (complex) vector space or the set of real (complex) numbers, $\mathbb{T}^+$ denotes the set of all non-negative integers, λ^* means conjugate of $\lambda \in \mathbb{C}$ and F^* conjugate transpose of matrix $F \in \mathbb{C}^{n\times n}$. Re(s) is the real part of $s \in \mathbb{C}$. The superscript T denotes transposition. For real matrix F the notation $F > 0$ means that the matrix F is positive definite. $\lambda_i(F)$ is the eigenvalue of matrix F. Spectrum of matrix F is denoted with $\sigma(F)$ and spectral radius with $\rho(F)$.

2.1 Continuous time delay systems

For the sake of completeness, we present the following result (Lee & Diant, 1981). Considers class of continuous time-delay systems described by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \quad x(t) = \varphi(t), \quad -\tau \le t < 0 \tag{1}$$

Theorem 2.1.1 (Lee & Diant, 1981) Let the system be described by (1). If for any given matrix $Q = Q^* > 0$ there exist matrix $P = P^* > 0$, such that

$$P\left(A_0 + T(0)\right) + \left(A_0 + T(0)\right)^T P = -Q \tag{2}$$

where $T(t)$ is continuous and differentiable matrix function which satisfies

$$\dot{T}(t) = \begin{cases} \left(A_0 + T(0)\right)T(t), & 0 \le t \le \tau, \quad T(\tau) = A_1 \\ 0 \qquad\qquad , & t > \tau \end{cases} \tag{3}$$

then the system (1) is asymptotically stable.

In paper (Lee & Diant, 1981) it is emphasized that the key to the success in the construction of a Lyapunov function corresponding to the system (1) is the existence of at least one solution $T(t)$ of (3) with boundary condition $T(\tau) = A_1$. In other words, it is required that the nonlinear algebraic matrix equation

$$e^{\left(A_0 + T(0)\right)\tau} T(0) = A_1 \tag{4}$$

has at least one solution for $T(0)$. It is asserted, there, that asymptotic stability of the system (*Theorem* 2.1.1) can be determined based on the knowledge of *only one* or *any*, solution of the particular nonlinear matrix equation.

We now demonstrate that *Theorem* 2.1.1 should be improved since it does not take into account all possible solutions for (4). The counterexample, based on our approach and supported by the Lambert function application, is given in (Stojanovic & Debeljkovic, 2006).

Conclusion 2.1.1 (Stojanovic & Debeljkovic, 2006) If we introduce a new matrix,

$$R \triangleq A_1 + T(0) \tag{5}$$

then condition (2) reads

$$PR + R^* P = -Q \tag{6}$$

which presents a well-known Lyapunov's equation for the system without time delay.

This condition will be fulfilled if and only if R is a stable matrix i.e. if

$$\mathrm{Re}\,\lambda_i(R) < 0 \tag{7}$$

holds.

Let Ω_T and Ω_R denote sets of all solutions of eq. (4) per T(0) and (6) per R, respectively.

Conclusion 2.1.2 (Stojanovic & Debeljkovic, 2006) Eq. (4) expressed through matrix R can be written in a different form as follows,

$$R - A_0 - e^{-R\tau}A_1 = 0 \tag{8}$$

and there follows

$$\det\left(R - A_0 - e^{-R\tau}A_1\right) = 0 \tag{9}$$

Substituting a matrix variable R by scalar variable s in (7), the characteristic equation of the system (1) is obtained as

$$f(s) = \det\left(sI - A_0 - e^{-s\tau}A_1\right) = 0 \tag{10}$$

Let us denote

$$\Sigma \triangleq \left\{s \mid f(s) = 0\right\} \tag{11}$$

a set of all characteristic roots of the system (1). The necessity for the correctness of desired results, forced us to propose new formulations of *Theorem* 2.1.1.

Theorem 2.1.2 (Stojanovic & Debeljkovic, 2006) Suppose that there exist(s) the solution(s) $T(0) \in \Omega_T$ of (4). Then, the system (1) is asymptotically stable if and only if any of the two following statements holds:

1. For any matrix $Q = Q^* > 0$ there exists matrix $P_0 = P_0^* > 0$ such that (2) holds for all solutions $T(0) \in \Omega_T$ of (4).
2. The condition (7) holds for all solutions $R = A_1 + T(0) \in \Omega_R$ of (8).

Conclusion 2.1.3 (Stojanovic & Debeljkovic, 2006) Statement *Theorem* 2.1.2 require that condition (2) is fulfilled for all solutions $T(0) \in \Omega_T$ of (4). In other words, it is requested that condition (7) holds for all solution R of (8) (especially for $R = R_{max}$, where the matrix $R_m \in \Omega_R$ is maximal solvent of (8) that contains eigenvalue with a maximal real part $\lambda_m \in \Sigma$: $\mathrm{Re}\lambda_m = \max_{s\in\Sigma} \mathrm{Re}\, s$). Therefore, from (7) follows condition $\mathrm{Re}\lambda_i(R_m) < 0$. These matrix condition is analogous to the following known scalar condition of asymptotic stability: System (1) is asymptotically stable if and only if the condition $\mathrm{Re}\, s < 0$ holds for all solutions s of (10) (especially for $s = \lambda_m$).

On the basis of *Conclusion 2.1.3,* it is possible to reformulate *Theorem 2.1.2* in the following way.

Theorem 2.1.3 (Stojanovic & Debeljkovic, 2006) Suppose that there exists maximal solvent R_m of (8). Then, the system (1) is asymptotically stable if and only if any of the two following equivalent statements holds:

1. For any matrix $Q = Q^* > 0$ there exists matrix $P_0 = P_0^* > 0$ such that (6) holds for the solution $R = R_m$ of (8).
2. $\mathrm{Re}\lambda_i(R_m) < 0$.

2.2 Discrete time delay systems

2.2.1 Introduction

Basic inspiration for our investigation in this section is based on paper (Lee & Diant, 1981), however, the stability of discrete time delay systems is considered herein.

We propose necessary and sufficient conditions for delay dependent stability of discrete linear time delay system, which as distinguished from the criterion based on eigenvalues of the matrix of equivalent system (Gantmacher, 1960), use matrices of considerably lower dimension. The time-dependent criteria are derived by Lyapunov's direct method and are exclusively based on the maximal and dominant solvents of particular matrix polynomial equation. Obtained stability conditions do not possess conservatism but require complex numerical computations. However, if the dominant solvent can be computed by Traub's or Bernoulli's algorithm, it has been demonstrated that smaller number of computations are to be expected compared with a traditional stability procedure based on eigenvalues of matrix A_{eq} of equivalent (augmented) system (see (14)).

2.2.2. Preliminaries

A linear, discrete time-delay system can be represented by the difference equation

$$x(k+1) = A_0 x(k) + A_1 x(k-h) \tag{12}$$

with an associated function of initial state

$$x(\theta) = \psi(\theta), \qquad \theta \in \{-h, -h+1, \dots, 0\} \tag{13}$$

The equation (12) is referred to as homogenous or the unforced state equation.

Vector $x(k) \in \mathbb{R}^n$ is a state vector and $A_0, A_1 \in \mathbb{R}^{n\times n}$ are constant matrices of appropriate dimensions, and pure system time delay is expressed by integers $h \in \mathbb{T}^+$. System (12) can be expressed with the following representation without delay, (Malek-Zavarei & Jamshidi, 1987; Gorecki et al., 1989).

$$\begin{gathered} x_{eq}(k) = \left[x^T(k-h) \quad x^T(k-h+1) \quad \cdots x^T(k) \right] \in \mathbb{R}^N, \qquad N \hat{=} n(h+1) \\ x_{eq}(k+1) = A_{eq} x_{eq}(k), \quad A_{eq} = \begin{bmatrix} 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n \\ A_1 & 0 & \cdots & A_0 \end{bmatrix} \in \mathbb{R}^{N\times N} \end{gathered} \tag{14}$$

The system defined by (14) is called the equivalent (augmented) system, while matrix A_{eq}, the matrix of equivalent (augmented) system. Characteristic polynomial of system (12) is given with:

$$f(\lambda) \hat{=} \det M(\lambda) = \sum_{j=0}^{n(h+1)} a_j \lambda^j, \quad a_j \in \mathbb{R}, \qquad M(\lambda) = I_n \lambda^{h+1} - A_0 \lambda^h - A_1 \tag{15}$$

Denote with

$$\Omega \hat{=} \{ \lambda \mid f(\lambda) = 0 \} = \lambda(A_{eq}) \tag{16}$$

the set of all characteristic roots of system (12). The number of these roots amounts to $n(h+1)$. A root λ_m of Ω with maximal module:

$$\lambda_m \in \Omega : |\lambda_m| = \max\left|\lambda\left(A_{eq}\right)\right| \tag{17}$$

let us call maximal root (eigenvalue). If scalar variable λ in the characteristic polynomial is replaced by matrix $X \in \mathbb{C}^{n\times n}$ the two following monic matrix polynomials are obtained

$$M(X) = X^{h+1} - A_0 X^h - A_1 \tag{18}$$

$$F(X) = X^{h+1} - X^h A_0 - A_1 \tag{19}$$

It is obvious that $F(\lambda) = M(\lambda)$. For matrix polynomial $M(X)$, the matrix of equivalent system A_{eq} represents block companion matrix.

A matrix $S \in \mathbb{C}^{n\times n}$ is a *right solvent* of $M(X)$ if

$$M(S) = 0 \tag{20}$$

If

$$F(R) = 0 \tag{21}$$

then $R \in \mathbb{C}^{n\times n}$ is a *left solvent* of $M(X)$, (Dennis et al., 1976).

We will further use matrix S to denote right solvent and matrix R to denote left solvent of $M(X)$.

In the present paper the majority of presented results start from left solvents of $M(X)$. In contrast, in the existing literature right solvents of $M(X)$ were mainly studied. The mentioned discrepancy can be overcome by the following *Lemma*.

Lemma 2.2.1 (Stojanovic & Debeljkovic, 2008.b). Conjugate transpose value of left solvent of $M(X)$ is also, at the same time, right solvent of the following matrix polynomial

$$\mathcal{M}(X) = X^{h+1} - A_0^T X^h - A_1^T \tag{22}$$

Conclusion 2.2.1 Based on *Lemma* 2.2.1, all characteristics of left solvents of $M(X)$ can be obtained by the analysis of conjugate transpose value of right solvents of $\mathcal{M}(X)$.

The following proposed factorization of the matrix $M(\lambda)$ will help us to better understand the relationship between eigenvalues of left and right solvents and roots of the system.

Lemma 2.2.2 (Stojanovic & Debeljkovic, 2008.b). The matrix $M(\lambda)$ can be factorized in the following way

$$M(\lambda) = \left(\lambda^h I_n + (S - A_0)\sum_{i=1}^{h}\lambda^{h-i}S^{i-1}\right)(\lambda I_n - S) = (\lambda I_n - R)\left(\lambda^h I_n + \sum_{i=1}^{h}\lambda^{h-i}R^{i-1}(R - A_0)\right) \tag{23}$$

Conclusion 2.2.2 From (15) and (23) follows $f(S)=f(R)=0$, e.g. the characteristic polynomial $f(\lambda)$ is *annihilating polynomial* for right and left solvents of $M(X)$. Therefore, $\lambda(S)\subset\Omega$ and $\lambda(R)\subset\Omega$ hold.

Eigenvalues and eigenvectors of the matrix have a crucial influence on the existence, enumeration and characterization of solvents of the matrix equation (20), (Dennis et al., 1976; Pereira, 2003).

Definition 2.2.1 (Dennis et al., 1976; Pereira, 2003). Let $M(\lambda)$ be a matrix polynomial in λ. If $\lambda_i\in\mathbb{C}$ is such that $\det M(\lambda_i)=0$, then we say that λ_i is a *latent root* or an *eigenvalue* of $M(\lambda)$. If a nonzero $v_i\in\mathbb{C}^n$ is such that $M(\lambda_i)v_i=0$ then we say that v_i is a (right) *latent vector* or a (right) *eigenvector* of $M(\lambda)$, corresponding to the eigenvalue λ_i.

Eigenvalues of matrix $M(\lambda)$ correspond to the characteristic roots of the system, i.e. eigenvalues of its block companion matrix A_{eq}, (Dennis et al., 1976). Their number is $n\cdot(h+1)$. Since $F^*(\lambda)=\mathcal{M}(\lambda^*)$ holds, it is not difficult to show that matrices $M(\lambda)$ and $\mathcal{M}(\lambda)$ have the same spectrum.

In papers (Dennis et al., 1976, Dennis et al., 1978; Kim, 2000; Pereira, 2003) some sufficient conditions for the existence, enumeration and characterization of right solvents of $M(X)$ were derived. They show that the number of solvents can be zero, finite or infinite.

For the needs of system stability (12) only the so called maximal solvents are usable, whose spectrums contain maximal eigenvalue λ_m. A special case of maximal solvent is the so called dominant solvent, (Dennis et al., 1976; Kim, 2000), which, unlike maximal solvents, can be computed in a simple way.

Definition 2.2.2 Every solvent S_m of $M(X)$, whose spectrum $\sigma(S_m)$ contains maximal eigenvalue λ_m of Ω is a *maximal solvent*.

Definition 2.2.3 (Dennis et al., 1976; Kim, 2000). Matrix A dominates matrix B if all the eigenvalues of A are greater, in modulus, then those of B. In particular, if the solvent S_1 of $M(X)$ dominates the solvents $S_2,\ldots,S_l$ we say it is a *dominant solvent*.

Conclusion 2.2.3 The number of maximal solvents can be greater than one. Dominant solvent is at the same time maximal solvent, too. The dominant solvent S_1 of $M(X)$, under certain conditions, can be determined by the Traub, (Dennis et al., 1978) and Bernoulli iteration (Dennis et al., 1978; Kim, 2000).

Conclusion 2.2.4 Similar to the definition of right solvents S_m and S_1 of $M(X)$, the definitions of both maximal left solvent, R_m, and dominant left solvent, R_1, of $M(X)$ can be provided. These left solvents of $M(X)$ are used in a number of theorems to follow. Owing to *Lemma* 2.2.1, they can be determined by proper right solvents of $\mathcal{M}(X)$.

2.2.3. Main results

Theorem 2.2.1 (Stojanovic & Debeljkovic, 2008.b). Suppose that there exists at least one left solvent of $M(X)$ and let R_m denote one of them. Then, linear discrete time delay system (12) is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$R_m^* P R_m - P = -Q \tag{24}$$

Proof. S*ufficient condition.* Define the following vector discrete functions

$$x_k = x(k+\theta), \quad \theta \in \{-h, -h+1, \dots, 0\}, \quad z(x_k) = x(k) + \sum_{j=1}^{h} T(j)x(k-j) \tag{25}$$

where, $T(k) \in \mathbb{C}^{n\times n}$ is, in general, some time varying discrete matrix function. The conclusion of the theorem follows immediately by defining Lyapunov functional for the system (12) as

$$V(x_k) = z^*(x_k) P z(x_k), \quad P = P^* > 0 \tag{26}$$

It is obvious that $z(x_k) = 0$ if and only if $x_k = 0$, so it follows that $V(x_k) > 0$ for $\forall x_k \neq 0$. The forward difference of (26), along the solutions of system (12) is

$$\Delta V(x_k) = \Delta z^*(x_k) P z(k) + z^*(x_k) P \Delta z(x_k) + \Delta z^*(x_k) P \Delta z(x_k) \tag{27}$$

A difference of $\Delta z(x_k)$ can be determined in the following manner

$$\begin{aligned} &\Delta z(x_k) = \Delta x(k) + \sum_{j=1}^{h} T(j)\Delta x(k-j), \quad \Delta x(k) = (A_0 - I_n)x(k) + A_1 x(k-h) \\ &\sum_{j=1}^{h} T(j)\Delta x(k-j) = T(1)\left[x(k) - x(k-1)\right] + \cdots + T(h)\left[x(k-h+1) - x(k-h)\right] \\ &\quad = T(1)x(k) - T(h)x(k-h) + \left(T(2) - T(1)\right)x(k-1) + \cdots \\ &\quad + \left(T(h) - T(h-1)\right)x(k-h+1) \end{aligned} \tag{28}$$

Define a new matrix R by

$$R = A_0 + T(1) \tag{29}$$

If

$$\Delta T(h) = A_1 - T(h) \tag{30}$$

then $\Delta z(x_k)$ has a form

$$\Delta z(x_k) = (R - I_n)x(k) + \sum_{j=1}^{h}\left[\Delta T(j)\cdot x(k-j)\right] \tag{31}$$

If one adopts

$$\Delta T(j) = (R - I_n)T(j), \quad j = 1,2,\dots,h \tag{32}$$

then (27) becomes

$$\Delta V(x_k) = z^*(x_k)\left(R^* P R - P\right)z(x_k) \tag{33}$$

It is obvious that if the following equation is satisfied

$$R^* P R - P = -Q, \quad Q = Q^* > 0 \tag{34}$$

then $\Delta V(x_k) < 0,\ x_k \neq 0$.

In the Lyapunov matrix equation (34), of all possible solvents R of $M(X)$, only one of maximal solvents is of importance, for it is the only one that contains maximal eigenvalue $\lambda_m \in \Omega$, which has dominant influence on the stability of the system. So, (24) represent stability sufficient condition for system given by (12).

Matrix $T(1)$ can be determined in the following way. From (32), follows

$$T(h+1) = R^h T(1) \tag{35}$$

and using (29)-(30) one can get (21), and for the sake of brevity, instead of matrix $T(1)$, one introduces simple notation T.

If solvent which is not maximal is integrated into Lyapunov equation, it may happen that there will exist positive definite solution of Lyapunov matrix equation (24), although the system is not stable.

Necessary condition. If the system (12) is asymptotically stable then all roots $\lambda_i \in \Omega$ are located within unit circle. Since $\sigma(R_m) \subset \Omega$, follows $\rho(R_m) < 1$, so the positive definite solution of Lyapunov matrix equation (24) exists.

Corollary 2.2.1 Suppose that there exists at least one maximal left solvent of $M(X)$ and let R_m denote one of them. Then, system (12) is asymptotically stable if and only if $\rho(R_m) < 1$, (Stojanovic & Debeljkovic, 2008.b).

Proof. Follows directly from *Theorem* 2.2.1.

Corollary 2.2.2 (Stojanovic & Debeljkovic, 2008.b) Suppose that there exists dominant left solvent R_1 of $M(X)$. Then, system (12) is asymptotically stable if and only if $\rho(R_1) < 1$.

Proof. Follows directly from *Corollary* 2.2.1, since dominant solution is, at the same time, maximal solvent.

Conclusion 2.2.5 In the case when dominant solvent R_1 may be deduced by Traub's or Bernoulli's algorithm, *Corollary* 2.2.2 represents a quite simple method. If aforementioned algorithms are not convergent but still there exists at least one of maximal solvents R_m, then one should use *Corollary* 2.2.1. The maximal solvents may be found, for example, using the concept of eigenpars, Pereira (2003). If there is no maximal solvent R_m, then proposed necessary and sufficient conditions *can not be used* for system stability investigation.

Conclusion 2.2.6 For some time delay systems it holds

$$\dim\left(\mathrm{R}_1\right)=\dim\left(\mathrm{R}_{\mathrm{m}}\right)=\dim\left(\mathrm{A}_{\mathrm{i}}\right)=\mathrm{n}\ll\dim\left(\mathrm{A}_{\mathrm{eq}}\right)=\mathrm{n}(\mathrm{h}+1)$$

For example, if time delay amounts to $\mathrm{h}=100$, and the row of matrices of the system is $\mathrm{n}=2$, then: $\mathrm{R}_1,\mathrm{R}_{\mathrm{m}}\in\mathbb{C}^{2\times2}$ and $\mathrm{A}_{\mathrm{eq}}\in\mathbb{C}^{202\times202}$.

To check the stability by eigenvalues of matrix A_{eq}, it is necessary to determine 202 eigenvalues, which is not numerically simple. On the other hand, if dominant solvent can be computed by Traub's or Bernoulli's algorithm, *Corollary* 2.2.2 requires a relatively small number of additions, subtractions, multiplications and inversions of the matrix format of only 2×2.

So, in the case of great time delay in the system, by applying *Corollary* 2.2.2, a smaller number of computations are to be expected compared with a traditional procedure of examining the stability by eigenvalues of companion matrix A_{eq}. An accurate number of computations for each of the mentioned method require additional analysis, which is not the subject-matter of our considerations herein.

2.2.4. Numerical examples

Example 2.2.1 (Stojanovic & Debeljkovic, 2008.b). Let us consider linear discrete systems with delayed state (12) with

$$\mathrm{A}_0=\begin{bmatrix}7/10 & -1/2\\ 1/2 & 17/10\end{bmatrix},\quad \mathrm{A}_1=\begin{bmatrix}1/75 & 1/3\\ -1/3 & -49/75\end{bmatrix},$$

A. For $\mathrm{h}=1$ there are two left solvents of matrix polynomial equation (21) ($\mathrm{R}^2-\mathrm{RA}_0-\mathrm{A}_1=0$):

$$\mathrm{R}_1=\begin{bmatrix}19/30 & -1/6\\ 1/6 & 29/30\end{bmatrix},\quad \mathrm{R}_2=\begin{bmatrix}1/15 & -1/3\\ 1/3 & 11/15\end{bmatrix},$$

Since $\lambda(\mathrm{R}_1)=\{4/5,4/5\}$, $\lambda(\mathrm{R}_2)=\{2/5,2/5\}$, dominant solvent is R_1. As we have $\mathrm{V}\left(\mathrm{R}_1,\mathrm{R}_2\right)$ nonsingular, Traub's or Bernoulli's algorithm may be used. Only after $(4+3)$ iterations for Traub's algorithm (Dennis et al., 1978) and 17 iterations for Bernoulli algorithm (Dennis et al., 1978), dominant solvent can be found with accuracy of 10^{-4}. Since $\rho(\mathrm{R}_1)=4/5<1$, based on *Corollary* 2.2.2, it follows that the system under consideration is asymptotically stable.

B. For $\mathrm{h}=20$ applying Bernoulli or Traub's algorithm for computation the dominant solvent R_1 of matrix polynomial equation (21) ($\mathrm{R}^{21}-\mathrm{R}^{20}\mathrm{A}_0-\mathrm{A}_1=0$), we obtain

$$\mathrm{R}_1=\begin{bmatrix}0.6034 & -0.5868\\ 0.5868 & 1.7769\end{bmatrix}$$

Based on *Corollary* 2.2.2, the system is not asymptotically stable because $\rho(R_1) = 1.1902 > 1$.

Finally, let us check stability properties of the system using his maximal eigenvalue:

$$\lambda_{max}\{A_{eq}\} = \lambda_{max}\left[\begin{array}{c|c} 0_{40x2} & I_{40x40} \\ \hline A_0 & 0_{2x2} \ \ldots \ 0_{2x2} \ A_1 \end{array}\right] = 1.1902 > 1$$

Evidently, the same result is obtained as above.

3. Large scale time delay systems

3.1 Continuous large scale time delay systems

3.1.1 Introduction

There exist many real-world systems that can be modeled as large-scale systems: examples are power systems, communication systems, social systems, transportation systems, rolling mill systems, economic systems, biological systems and so on. It is also well known that the control and analysis of large-scale systems can become very complicated owing to the high dimensionality of the system equation, uncertainties, and time-delays. During the last two decades, the stabilization of uncertain large-scale systems becomes a very important problem and has been studied extensively (Siljak, 1978; Mahmoud et al., 1985). Especially, many researchers have considered the problem of stability analysis and control of various large-scale systems with time-delays (Wu, 1999; Park, 2002 and references therein).

Recently, the stabilization problem of large-scale systems with delays has been considered by (Lee & Radovic, 1988; Hu, 1994; Trihn & Aldeen 1995a; Xu, 1995). However, the results in (Lee & Radovic, 1988; Hu, 1994) apply only to a very restrictive class of systems for which the number of inputs and outputs is equal to or greater than the number of states. Also, since the sufficient conditions of (Trinh & Aldeen 1995a; Xu, 1995) are expressed in terms of the matrix norm of the system matrices, usually the matrix norm operation makes the criteria more conservative.

The paper (Xu, 1995) provides a new criterion for delay-independent stability of linear large scale time delay systems by employing an improved Razumikhin-type theorem and M-matrix properties. In (Trinh & Aldeen, 1997), by employing a Razumikhin-type theorem, a robust stability criterion for a class of linear system subject to delayed time-varying nonlinear perturbations is given.

The basic aim of the above mentioned works was to obtain only sufficient conditions for stability of large scale time delay systems. It is notorious that those conditions of stability are more or less conservative.

In contrast, the major results of our investigations are necessary and sufficient conditions of asymptotic stability of continuous large scale time delay autonomous systems. The obtained conditions are expressed by nonlinear system of matrix equations and the Lyapunov matrix equation for an ordinary linear continuous system without delay. Those conditions of stability are delay-dependent and do not possess conservatism. Unfortunately, viewed mathematically, they require somewhat more complex numerical computations.

3.1.2 Main Results

Consider a linear continuous large scale time delay autonomous systems composed of N interconnected subsystems. Each subsystem is described as:

$$\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} A_{ij} x_j\left(t - \tau_{ij}\right),\ 1 \le i \le N \tag{36}$$

with an associated function of initial state $x_i(\theta) = \varphi_i(\theta)$, $\theta \in \left[-\tau_{m_i},\ 0\right]$, $1 \le i \le N$. $x_i(t) \in \mathbb{R}^{n_i}$ is state vector, $A_i \in \mathbb{R}^{n_i \times n_i}$ denote the system matrix, $A_{ij} \in R^{n_i \times n_j}$ represents the interconnection matrix between the i-th and the j-th subsystems, and τ_{ij} is constant delay. For the sake of brevity, we first observe system (36) made up of two subsystems ($N = 2$). For this system, we derive new necessary and sufficient delay-dependent conditions for stability, by Lyapunov's direct method. The derived results are then extended to the linear continuous large scale time delay systems with multiple subsystems.

a) Large scale systems with two subsystems

Theorem 3.1.1. (Stojanovic & Debeljkovic, 2005). Given the following system of matrix equations (SME)

$$\mathcal{R}_1 - A_1 - e^{-\mathcal{R}_1 \tau_{11}} A_{11} - e^{-\mathcal{R}_1 \tau_{21}} S_2 A_{21} = 0 \tag{37}$$

$$\mathcal{R}_1 S_2 - S_2 A_2 - e^{-\mathcal{R}_1 \tau_{12}} A_{12} - e^{-\mathcal{R}_1 \tau_{22}} S_2 A_{22} = 0 \tag{38}$$

where A_1, A_2, A_{12}, A_{21} and A_{22} are matrices of system (36) for $N = 2$, n_i subsystem orders and τ_{ij} pure time delays of the system. If there exists solution of SME (37)-(38) upon unknown matrices $\mathcal{R}_1 \in C^{n_1 \times n_1}$ and $S_2 \in C^{n_1 \times n_2}$, then the eigenvalues of matrix $\mathcal{R}_1$ belong to a set of roots of the characteristic equation of system (36) for $N = 2$.

Proof. By introducing the time delay operator $e^{-\tau s}$, the system (36) can be expressed in the form

$$\dot{x}(t) = \begin{bmatrix} A_1 + A_{11} e^{-\tau_{11} s} & A_{12} e^{-\tau_{12} s} \\ A_{21} e^{-\tau_{21} s} & A_2 + A_{22} e^{-\tau_{22} s} \end{bmatrix} x(t) = A_e(s) x(t),\quad x(t) = \left[x_1^{\,T}(t) \quad x_2^{\,T}(t) \right]^{T} \tag{39}$$

Let us form the following matrix

$$F(s) = \left[F_{ij}(s) \right] = sI_{n_1 + n_2} - A_e(s) = \begin{bmatrix} sI_{n_1} - A_1 - A_{11} e^{-\tau_{11} s} & -A_{12} e^{-\tau_{12} s} \\ -A_{21} e^{-\tau_{21} s} & sI_{n_2} - A_2 - A_{22} s^{-\tau_{22} s} \end{bmatrix} \tag{40}$$

Its determinant is

$$\begin{aligned} \det F(s) &= \det \begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} = \det \begin{bmatrix} F_{11}(s) + S_2 F_{21}(s) & F_{12}(s) + S_2 F_{22}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} \\ &= \det \begin{bmatrix} G_{11}(s, S_2) & G_{12}(s, S_2) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \det G(s, S_2) \end{aligned} \tag{41}$$

$$G_{11}(s,S_2)=sI_{n1}-A_1-A_{11}e^{-\tau_{11}s}-S_2A_{21}e^{-\tau_{21}s} \tag{42}$$

$$G_{12}(s,S_2)=sS_2-S_2A_2-A_{12}e^{-\tau_{12}s}-S_2A_{22}e^{-\tau_{22}s} \tag{43}$$

Transformational matrix S_2 is unknown for the time being, but condition determining this matrix will be derived in a further text.
The characteristic polynomial of system (36) for $N=2$, defined by

$$f(s)\triangleq\det\left(sI_N-A_e(s)\right)=\det G(s,S_2) \tag{44}$$

is independent of the choice of matrix S_2, because the determinant of matrix $G(s,S_2)$ is invariant with respect to elementary row operation of type 3. Let us designate a set of roots of the characteristic equation of system (36) by $\sum\triangleq\left\{s\,|\,f(s)=0\right\}$. Substituting scalar variable s by matrix X in $G(s,S_2)$ we obtain

$$G(X,S_2)=\begin{bmatrix}G_{11}(X,S_2) & G_{12}(X,S_2)\\ G_{21}(X) & G_{22}(X)\end{bmatrix} \tag{45}$$

If there exist transformational matrix S_2 and matrix $\mathcal{R}_1\in C^{n_1\times n_1}$ such that $G_{11}(\mathcal{R}_1,S_2)=0$ and $G_{12}(\mathcal{R}_1,S_2)=0$ is satisfied, i.e. if (37)-(38) hold, then

$$f(\mathcal{R}_1)=\det G_{11}(\mathcal{R}_1,S_2)\cdot\det G_{22}(\mathcal{R}_1)=0 \tag{46}$$

So, the characteristic polynomial (44) of system (36) is annihilating polynomial (Lancaster & Tismenetsky, 1985) for the square matrix $\mathcal{R}_1$, defined by (37)-(38). In other words, $\sigma(\mathcal{R}_1)\subset\sum$.

Theorem 3.1.2 (Stojanovic & Debeljkovic, 2005) Given the following SME

$$\mathcal{R}_2-A_2-e^{-\mathcal{R}_2\tau_{12}}S_1A_{12}-e^{-\mathcal{R}_2\tau_{22}}A_{22}=0 \tag{47}$$

$$\mathcal{R}_2S_1-S_1A_1-e^{-\mathcal{R}_2\tau_{11}}S_1A_{11}-e^{-\mathcal{R}_2\tau_{21}}A_{21}=0 \tag{48}$$

where A_1, A_2, A_{12}, A_{21} and A_{22} are matrices of system (36) for $N=2$, n_i subsystem orders and τ_{ij} time delays of the system. If there exists solution of SME (47)-(48) upon unknown matrices $\mathcal{R}_2\in C^{n_2\times n_2}$ and $S_1\in C^{n_2\times n_1}$, then the eigenvalues of matrix $\mathcal{R}_2$ belong to a set of roots of the characteristic equation of system (36) for $N=2$.
Proof. Proof is similarly with the proof of *Theorem* 3.1.1.
Corollary 3.1.1 If system (36) is asymptotically stable, then matrices $\mathcal{R}_1$ and $\mathcal{R}_2$, defined by SME (37)-(38) and (47)-(48), respectively, are stable ($\mathrm{Re}\,\lambda(\mathcal{R}_i)<0$, $1\le i\le 2$).

Proof. If system (36) is asymptotically stable, then $\forall s \in \Sigma, \mathrm{Re}\, s < 0$. Since $\sigma(\mathcal{R}_i) \subset \Sigma$, $1 \le i \le 2$, it follows that $\forall \lambda \in \sigma(\mathcal{R}_i)$, $\mathrm{Re}\lambda < 0$, i.e. matrices $\mathcal{R}_1$ and $\mathcal{R}_2$ are stable.

Definition 3.1.1 The matrix $\mathcal{R}_1$ ($\mathcal{R}_2$) is referred to as *solvent* of SME (37)-(38) or (47)-(48).

Definition 3.1.2 Each root λ_m of the characteristic equation (44) of the system (36) which satisfies the following condition: $\mathrm{Re}\lambda_m = \max \mathrm{Re}\, s,\ s \in \Sigma$ will be referred to as *maximal root* (*eigenvalue*) of system (36).

Definition 3.1.3 Each solvent $\mathcal{R}_{1m}$ ($\mathcal{R}_{2m}$) of SME (37)-(38) or (47)-(48), whose spectrum contains maximal eigenvalue λ_m of system (36), is referred to as *maximal solvent* of SME (37)-(38) or (47)-(48).

Theorem 3.1.3 (Stojanovic & Debeljkovic, 2005) Suppose that there exists at least one maximal solvent of SME (47)-(48) and let $\mathcal{R}_{1m}$ denote one of them. Then, system (36), for $N = 2$, is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$\mathcal{R}_{1m}^{*} P + P \mathcal{R}_{1m} = -Q \tag{49}$$

Proof. *Sufficient condition.* Similarly (Lee & Diant, 1981), define the following vector continuous functions

$$x_{ti} = x_i(t+\theta), \quad \theta \in \left[-\tau_{m_i},\ 0\right], \quad z(x_{t1}, x_{t2}) = \sum_{i=1}^{2} S_i \left(x_i(t) + \sum_{j=1}^{2} \int_0^{\tau_{ji}} T_{ji}(\eta) x_i(t-\eta) d\eta \right) \tag{50}$$

where $T_{ji}(t) \in C^{n_i \times n_i}$, $j = 1,2$ are some time varying continuous matrix functions and $S_1 = I_{n_1}$, $S_2 \in C^{n_1 \times n_2}$.

The proof of the theorem follows immediately by defining Lyapunov functional for system (36) as

$$V(x_{t1}, x_{t2}) = z^*(x_{t1}, x_{t2}) P z(x_{t1}, x_{t2}), \quad P = P^* > 0 \tag{51}$$

Derivative of (51), along the solutions of system (36) is

$$\dot{V}(x_{t1}, x_{t2}) = \dot{z}^*(x_{t1}, x_{t2}) P z(x_{t1}, x_{t2}) + z^*(x_{t1}, x_{t2}) P \dot{z}(x_{t1}, x_{t2}) \tag{52}$$

$$\dot{z}(x_{t1}, x_{t2}) = \sum_{i=1}^{2} S_i \left(\dot{x}_i(t) + \sum_{j=1}^{2} \frac{d}{dt} \int_o^{\tau_{ji}} T_{ji}(\eta) x_i(t-\eta) d\eta \right) \tag{53}$$

$$\frac{d}{dt} \int_0^{\tau_{ji}} T_{ji}(\eta)\, x_i(t-\eta)\, d\eta = \int_0^{\tau_{ji}} T_{ji}'(\eta)\, x_i(t-\eta)\, d\eta + T_{ji}(0)\, x_i(t) - T_{ji}(\tau_{ji})\, x_i(t-\tau_{ji}) \tag{54}$$

Therefore

$$\dot{z}(x_{t1},x_{t2})=\sum_{i=1}^{2}\left\{S_i\left(A_i+\sum_{j=1}^{2}T_{ji}(0)\right)x_i(t)\right.$$
$$\left.+\sum_{j=1}^{2}\left(S_jA_{ji}-S_iT_{ji}(\tau_{ji})\right)x_i(t-\tau_{ji})+\sum_{j=1}^{2}\int_0^{\tau_{ji}}S_iT_{ji}'(\eta)x_i(t-\eta)d\eta\right\} \tag{55}$$

If we define new matrices

$$\mathcal{R}_i=A_i+\sum_{j=1}^{2}T_{ji}(0),\ i=1,2 \tag{56}$$

and if one adopts

$$S_iT_{ji}(\tau_{ji})=S_jA_{ji}\ ,\ i,j=1,2 \tag{57}$$

$$S_iT_{ji}'(\eta)=\mathcal{R}_1S_iT_{ji}(\eta),\quad S_i\mathcal{R}_i=\mathcal{R}_1S_i,\quad i,j=1,2 \tag{58}$$

then

$$\dot{z}(x_{t1},x_{t2})=\mathcal{R}_1z(x_{t1},x_{t2}),\quad \dot{V}(x_{t1},x_{t2})=z^*(x_{t1},x_{t2})\left(\mathcal{R}_1{}^*P+P\mathcal{R}_1\right)z(x_{t1},x_{t2}) \tag{59}$$

It is obvious that if the following equation is satisfied

$$\mathcal{R}_1{}^*P+P\mathcal{R}_1=-Q<0\,, \tag{60}$$

then $\dot{V}(x_{t1},x_{t2})<0$, $\forall x_{ti}\neq 0$.

In the Lyapunov matrix equation (49), of all possible solvents $\mathcal{R}_1$ only one of maximal solvents $\mathcal{R}_{1m}$ is of importance, because it is containing maximal eigenvalue $\lambda_m\in\Sigma$, which has dominant influence on the stability of the system.

If a solvent, which is not maximal, is integrated into Lyapunov equation (49), it may happen that there will exist positive definite solution of this equation, although the system is not stable.

Necessary condition. Let us assume that system (36) for $N=2$ is asymptotically stable, i.e. $\forall s\in\Sigma$, $\mathrm{Re}\,s<0$ hold. Since $\sigma(\mathcal{R}_{1m})\subset\Sigma$ follows $\mathrm{Re}\,\lambda(\mathcal{R}_{1m})<0$ and the positive definite solution of Lyapunov matrix equation (49) exists.

From (57)-(58) follows

$$S_jA_{ji}=e^{\mathcal{R}_1\tau_{ji}}S_iT_{ji}(0),\ S_1=I_{n_1},\ \ i=1,2,j=1,2 \tag{61}$$

Using (56) and (61), for $i=1$, we obtain (37).

Multiplying (56) (for $i=2$) from the left by matrix S_2 and using (58) and (61) we obtain (38)

Taking a solvent with eigenvalue $\lambda_m \in \Sigma$ (if it exists) as a solution of the system of equations (37)-(38), we arrive at a maximal solvent $\mathcal{R}_{1m}$.

Theorem 3.1.4 (Stojanovic & Debeljkovic 2005) Suppose that there exists at least one maximal solvent of SME (47)-(48) and let $\mathcal{R}_{2m}$ denote one of them. Then, system (36), for $N = 2$, is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$\mathcal{R}_{2m}^* P + P \mathcal{R}_{2m} = -Q \tag{62}$$

Proof. Proof is almost identical to that exposed for *Theorem* 3.1.3.

Conclusion 3.1.1 The proposed criteria of stability are expressed in the form of necessary and sufficient conditions and as such do not possess conservatism unlike the existing sufficient criteria of stability.

Conclusion 3.1.2 To the authors' knowledge, in the literature available, there are no adequate numerical methods for direct computations of maximal solvents $\mathcal{R}_{1m}$ or $\mathcal{R}_{2m}$. Instead, using various initial values for solvents $\mathcal{R}_i$, we determine $\mathcal{R}_{im}$ by applying minimization methods based on nonlinear least squares algorithms (see *Example* 3.1.1).

b) Large scale system with multiple subsystems

Theorem 3.1.5. (Stojanovic & Debeljkovic, 2005) Given the following system of matrix equations

$$\mathcal{R}_k S_i - S_i A_i - \sum_{j=1}^{N} e^{-\mathcal{R}_k \tau_{ji}} S_j A_{ji} = 0, \quad S_i \in C^{n_k \times n_i}, \quad S_k = I_{n_k}, \quad 1 \le i \le N \tag{63}$$

for a given k, $1 \le k \le N$, where A_i and A_{ji}, $1 \le i \le N$, $1 \le j \le N$ are matrices of system (36) and τ_{ji} is time delay in the system. If there is a solvent of (63) upon unknown matrices $\mathcal{R}_k \in C^{n_k \times n_k}$ and S_i, $1 \le i \le N$, $i \ne k$, then the eigenvalues of matrix $\mathcal{R}_k$ belong to a set of roots of the characteristic equation of system (36).

Proof. Proof of this theorem is a generalization of proof of *Theorem* 3.1.1 or *Theorem* 3.1.2.

Theorem 3.1.6 (Stojanovic & Debeljkovic, 2005) Suppose that there exists at least one maximal solvent of (63) *for given* k, $1 \le k \le N$ and let $\mathcal{R}_{km}$ denote one of them. Then, linear discrete large scale time delay system (36) is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$\mathcal{R}_{km}^* P + P \mathcal{R}_{km} = -Q \tag{64}$$

Proof. Proof is based on generalization of proof for *Theorem* 3.1.3 and *Theorem* 3.1.4.
It is sufficient to take arbitrary *N* instead of $N = 2$.

3.1.3 Numerical example

Example 3.1.1 Consider following continuous large scale time delay system with delay interconnections

$$\begin{aligned}\dot{x}_1(t) &= A_1x_1(t) + A_{12}x_2(t-\tau_{12})\\ \dot{x}_2(t) &= A_2x_2(t) + A_{21}x_1(t-\tau_{21}) + A_{23}x_3(t-\tau_{23})\\ \dot{x}_3(t) &= A_3x_3(t) + A_{31}x_1(t-\tau_{31}) + A_{32}x_2(t-\tau_{32})\end{aligned} \tag{65}$$

$$A_1 = \begin{bmatrix} -6 & 2 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -10.9 \end{bmatrix},\ A_{12} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 0 & 3 \\ -2 & 1 & 2 \end{bmatrix},\ A_2 = \begin{bmatrix} -1.87 & 4.91 & 10.30 \\ -2.23 & -16.51 & -24.11 \\ 1.87 & -3.91 & -10.30 \end{bmatrix},\ A_{21} = \begin{bmatrix} -1 & 0 & -2 \\ 3 & 0 & 5 \\ 1 & 0 & 2 \end{bmatrix},$$

$$A_{23} = \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix},\quad A_3 = \begin{bmatrix} -18.5 & -17.5 \\ -13.5 & -18.5 \end{bmatrix},\quad A_{31} = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix},\quad A_{32} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 0 \end{bmatrix},$$

Applying *Theorem* 3.1.5 to a given system, for $k = 1$, the following SME is obtained

$$\begin{aligned}&\mathcal{R}_1 - A_1 - e^{-\mathcal{R}_1\tau_{21}}S_2A_{21} - e^{-\mathcal{R}_1\tau_{31}}S_3A_{31} = 0\\ &\mathcal{R}_1S_2 - S_2A_2 - e^{-\mathcal{R}_1\tau_{12}}A_{12} - e^{-\mathcal{R}_1\tau_{32}}S_3A_{32} = 0\\ &\mathcal{R}_1S_3 - S_3A_3 - e^{-\mathcal{R}_1\tau_{23}}S_2A_{23} = 0\end{aligned} \tag{66}$$

If for pure system time delays we adopt the following values: $\tau_{12} = 5$, $\tau_{21} = 2$, $\tau_{23} = 4$, $\tau_{31} = 5$ and $\tau_{32} = 3$, by applying the nonlinear least squares algorithms, we obtain a great number of solutions upon $\mathcal{R}_1$ which satisfy SME (66):
Among those solutions is a maximal solution:

$$\mathcal{R}_{1m} = \begin{bmatrix} -0.0484 & -0.0996 & 0.0934 \\ 0.2789 & -0.3123 & 0.2104 \\ 1.1798 & -1.1970 & -0.3798 \end{bmatrix}$$

The eigenvalues of matrix $\mathcal{R}_{1m}$ amount to: $\lambda_1 = -0.2517$, $\lambda_{2,3} = -0.2444 \pm j\,0.3726$.
Therefore, for a maximal eigenvalue λ_m one of the values from the set $\{\lambda_2, \lambda_3\}$ can be adopted. Based on *Theorem* 3.1.6, it follows that the large scale time delay system is asymptotically stable.

3.2 Discrete large scale time delay systems

3.2.1 Introduction

Recently, the stability and stabilization problem of large-scale systems with delays has been considered by (Lee & Radovic, 1987, 1988), (Hu, 1994), (Trinh & Aldeen, 1995b), (Xu, 1995), (Huang et al., 1995), (Lee & Hsien 1997), (Wang & Mau 1997) and (Park, 2002).
Most related works treated the stabilization problem in the continuous-time case. Since most modern control systems are controlled by a digital computer, it is natural to deal with the problem in a discrete-time domain.

Based on the Lyapunov stability theorem associated with norm inequality techniques, in (Lee & Hsien, 1997) the stability testing problem for discrete large-scale uncertain systems with time delays in the interconnections is investigated. Three classes of uncertainties are treated: nonlinear, linear unstructured and linear highly structured uncertainties. A criterion to guarantee the robust stabilization and the state estimation for perturbed discrete time-delay large-scale systems is proposed in (Wang & Mau, 1997). This criterion is independent of time delay and does not need the solution of a Lyapunov equation or Riccati equation.
In paper (Park, 2002) the synthesis of robust decentralized controllers for uncertain large-scale discrete-time systems with time delays in the subsystem interconnections is considered. Based on the Lyapunov method, a sufficient condition for robust stability is derived in terms of a linear matrix inequality. Further, (Park et al., 2004) was discussed how to solve dynamic output feedback controller design problem for decentralized guaranteed cost stabilization of large-scale discrete-delay system by convex optimization. The problems of robust non-fragile control for uncertain discrete-delay large-scale systems under state feedback gain variations are investigated in (Park, 2004).
In this section the necessary and sufficient conditions for the asymptotic stability of a particular class of large-scale linear discrete time-delay systems are considered. The obtained conditions of stability are derived by Lyapunov's direct method and expressed by system of matrix polynomial equations. The conditions are not conservative against the majority of results reported in the literature available. In the case of great time delays in the system and a great number of subsystems, by applying the derived results it has been demonstrated that a smaller number of computations are to be expected compared with a classical stability criteria based on eigenvalues of matrix of equivalent system.

3.2.2. Preliminaries

Consider a large-scale linear discrete time-delay systems composed of N interconnected S_i. Each subsystem S_i, $1 \le i \le N$ is described as

$$S_i: \quad x_i(k+1) = A_i x_i(k) + \sum_{j=1}^{N} A_{ij} x_j\left(k - h_{ij}\right) \tag{67}$$

with an associated function of initial state

$$x_i(\theta) = \psi_i(\theta), \quad \theta \in \left\{-h_{m_i}, -h_{m_i}+1, \ldots, 0\right\} \tag{68}$$

where $x_i(k) \in \mathbb{R}^{n_i}$ is state vector, $A_i \in \mathbb{R}^{n_i \times n_i}$ denotes the system matrix, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ represents the interconnection matrix between the i-th and the j-th subsystems and the constant delay $h_{ij} \in \mathbb{T}^+$.

In the following lemma necessary and sufficient condition for asymptotic stability of system (67) has been given, expressed via eigenvalues the so called *equivalent matrix* $\mathcal{A}$. This condition is based upon the fact that the observed system is finite-dimensional. The order of this system is very high and time delay dependent.
Lemma 3.2.1 System (67) will be asymptotically stable if and only if

$$\rho(\mathcal{A}) < 1 \tag{69}$$

holds, where matrix

$$\mathcal{A}=\left[\mathcal{A}_{ij}\right]\in\mathbb{R}^{N_e\times N_e},\ N_e=\sum_{i=1}^{N}N_i,\quad N_i=n_i\left(h_{m_i}+1\right),\quad h_{m_i}=\max_j h_{ji} \tag{70}$$

is defined in the following way

$$\mathcal{A}_{ii}=\left[\begin{array}{cccccc|c} \underset{\downarrow}{1} & & \cdots & \underset{\downarrow}{h_{ii}+1} & & & \\ A_i & 0 & \cdots & A_{ii} & \cdots & 0 & 0 \\ \hline I_{n_i} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \vdots & 0 & \vdots & I_{n_i} & 0 \end{array}\right]\in\mathbb{R}^{N_i\times N_i},\quad \mathcal{A}_{ij}=\left[\begin{array}{ccccc} \underset{\downarrow}{1} & \cdots & \underset{\downarrow}{h_{ij}+1} & & \\ 0 & \cdots & A_{ij} & \cdots & 0 \\ \hline 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{array}\right]\in\mathbb{R}^{N_i\times N_i} \tag{71}$$

where A_i and A_{ij}, $1\le i\le N$, $1\le j\le N$, are matrices of system (67).

Proof. It is not difficult to demonstrate that system (67) can be given in the following equivalent form

$$\begin{gathered}\hat{x}(k+1)=\mathcal{A}\hat{x}(k),\quad \hat{x}(k)=\left[\begin{array}{cccc}\hat{x}_1^{\,T}(k) & \hat{x}_2^{\,T}(k) & \cdots & \hat{x}_N^{\,T}(k)\end{array}\right]^T \qquad 1\le i\le N \\ \hat{x}_i(k)=\left[\begin{array}{ccccc}x_i^{\,T}(k) & x_i^{\,T}(k-1) & \cdots & x_i^{\,T}\left(k-h_{m_i}\right)\end{array}\right]^T\end{gathered} \tag{72}$$

wherefrom a given condition for asymptotic stability follows directly.

3.2.3. Main results

Using Lyapunov's direct method, necessary and sufficient conditions for delay-dependent stability for system (67), are derived.

Prior to it, we demonstrate that the spectrum of matrix, which is integrated into Lyapunov equation, is a subset of spectrum of matrix $\mathcal{A}$, i.e. a set of characteristic roots of system (67).

Theorem 3.2.1. (Stojanovic & Debeljkovic, 2008.a) Given the following system of monic matrix polynomial equations (SMPE)

$$\mathcal{R}_\ell^{\,h_{m_i}+1}S_i-\mathcal{R}_\ell^{\,h_{m_i}}S_iA_i-\sum_{j=1}^{N}\mathcal{R}_\ell^{\,h_{m_i}-h_{ji}}S_jA_{ji}=0,\quad S_i\in\mathbb{C}^{n_\ell\times n_i},\quad S_\ell=I_{n_\ell} \tag{73}$$

for a given ℓ, $1\le\ell\le N$, , where A_i and A_{ji}, $1\le i\le N$, $1\le j\le N$ are matrices of system (67) and h_{ji} is time delay in the system, $h_{m_i}=\max_j h_{ji}$, $1\le i\le N$.

If there is a solution of SMPE (73) upon unknown matrices $\mathcal{R}_\ell\in\mathbb{C}^{n_\ell\times n_\ell}$ and S_i, $1\le i\le N$, $i\ne\ell$, then $\lambda(\mathcal{R}_\ell)\subset\lambda(\mathcal{A})$ holds, where matrix $\mathcal{A}$ is defined by (70)-(71).

Proof. By introducing time-delay operator z^{-h}, system (67) can be expressed in the following form

$$x(k+1)=A_e(z)x(k),\quad x(k)=\left[x_1^{\,T}(k)\quad x_2^{\,T}(k)\quad\cdots\quad x_N^{\,T}(k)\right]^T$$
$$A_e(z)=\begin{bmatrix}A_1+A_{11}z^{-h_{11}} & \cdots & A_{1N}z^{-h_{1N}}\\ \vdots & \ddots & \vdots\\ A_{N1}z^{-h_{N1}} & \cdots & A_N+A_{NN}z^{-h_{NN}}\end{bmatrix} \tag{74}$$

Let us form the following matrix.

$$F(z)=zI_{N_e}-A_e(z)=\left[F_{ij}(z)\right]=\begin{bmatrix}zI_{n_1}-A_1-A_{11}z^{-h_{11}} & \cdots & -A_{1N}z^{-h_{1N}}\\ \vdots & \ddots & \vdots\\ -A_{N1}z^{-h_{N1}} & \cdots & zI_{n_N}-A_N-A_{NN}z^{-h_{NN}}\end{bmatrix} \tag{75}$$

If we add to the arbitrarily chosen ℓ - th block row of this matrix the rest of its block rows previously multiplied from the left by the matrices $S_j\neq 0$, $1\le j\le N$, $j\neq\ell$ respectively, we obtain

$$\det F(z)=\det\begin{bmatrix}F_{11}(z) & \cdots & F_{1N}(z)\\ \vdots & \cdots & \vdots\\ F_1(z)+\sum\limits_{\substack{j=1\\ j\neq\ell}}^{N}S_jF_{j1}(z) & \cdots & F_{\ell N}(z)+\sum\limits_{\substack{j=1\\ j\neq\ell}}^{N}S_jF_{jN}(z)\\ \vdots & \ddots & \vdots\\ F_{N1}(z) & \cdots & F_{NN}(z)\end{bmatrix} \tag{76}$$

After multiplying i -th of the block column, $1\le i\le N$, of the preceding matrix by $z^{h_{m_i}}$ and after integrating the matrix $S_\ell=I_{n_\ell}$, the determinant of matrix $F(z)$ equals

$$z^{\sum_{i=1}^{N}n_ih_{m_i}}\det F(z)=$$
$$=\det\begin{bmatrix}z^{h_{m_1}}F_{11}(z) & \cdots & z^{h_{m_N}}F_{1N}(z)\\ \vdots & \vdots & \vdots\\ z^{h_{m_1}}\sum\limits_{j=1}^{N}S_jF_{j1}(z) & \cdots & z^{h_{m_N}}\sum\limits_{j=1}^{N}S_jF_{jN}(z)\\ \vdots & \ddots & \vdots\\ z^{h_{m_1}}F_{N1}(z) & \cdots & z^{h_{m_N}}F_{NN}(z)\end{bmatrix}=\det\begin{bmatrix}G_{11}(z) & \cdots & G_{1N}(z)\\ \vdots & \vdots & \vdots\\ G_{\ell 1}(z,S) & \cdots & G_{\ell N}(z,S)\\ \vdots & \vdots & \vdots\\ G_{N1}(z) & \cdots & G_{NN}(z)\end{bmatrix} \tag{77}$$
$$=\det G(z,S),\ S=\{S_1,\cdots,S_N\}$$

The ℓ -th block row of the $N\times N$ block matrix $G(z,S)$ is defined by

$$G_{\ell i}(z,S)=z^{h_{m_i}+1}S_i-z^{h_{m_i}}S_iA_i-\sum_{j=1}^{N}z^{h_{m_i}-h_{ji}}S_jA_{ji},\quad 1\le i\le N,\quad S_\ell=I_{n_\ell} \tag{78}$$

The relation (76) was obtained by applying a finite sequence of elementary row operations of type 3 over matrix $F(z)$, (Lancaster & Tismenetsky, 1985). Transformation matrices $S_1,\cdots,S_N$, with the exception of matrix $S_\ell=I_{n_\ell}$, are unknown for the time being, but in a further text a condition will be derived that the unknown matrices are determined upon. The characteristic polynomial of system (67), (Gorecki et al., 1989)

$$g(z)\triangleq \det G(z,S)=\sum_{j=0}^{N_e}a_jz^j,\quad N_e=\sum_{i=1}^{N}n_i\left(h_{m_i}+1\right),\ a_j\in\mathbb{R},\ 0\le j\le N_e \tag{79}$$

does not depend on the choice of transformation matrices $S_1,\cdots,S_N$), (Lancaster & Tismenetsky, 1985).
Let us denote

$$\Sigma\triangleq\left\{z\,|\,g(z)=0\right\} \tag{80}$$

a set of all characteristic roots of system (67). This set of roots equals the set $\lambda(\mathcal{A})$. Substituting a scalar variable z by matrix $X\in\mathbb{C}^{n_\ell\times n_\ell}$ in $G(z,S)$, a new block matrix is obtained $G(X,S)$. If there exist the transformation matrices S_i, $1\le i\le N$, $i\ne\ell$ and solvent $\mathcal{R}_\ell\in\mathbb{C}^{n_\ell\times n_\ell}$ such that for the ℓ-th block row of $G(X,S)$ holds $G_{\ell i}(\mathcal{R}_\ell,S)=0,\ 1\le i\le N$ i.e. holds (73), then

$$g(\mathcal{R}_\ell)=0 \tag{81}$$

Therefore, the characteristic polynomial of system (67) is annihilating polynomial for the square matrix $\mathcal{R}_\ell$ and $\lambda(\mathcal{R}_\ell)\subset\Sigma$ holds. The mentioned assertion holds $\forall\ell,\ 1\le\ell\le N$.

Definition 3.2.1 The matrix $\mathcal{R}_\ell$ is referred to as *solvent* of equations (73) for the given ℓ, $1\le\ell\le N$.

From (73) for the given ℓ, $1\le\ell\le N$, transformation matrices S_j $1\le j\le N$ and solvent $\mathcal{R}_\ell$ are computed, the latter being used further for examining the stability of system (67).

Definition 3.2.2 The characteristic root λ_m of system (67) with maximal module:

$$\lambda_m\in\Sigma:\ \ |\lambda_m|=\max|\Sigma|=\max_i|\lambda_i(\mathcal{A})| \tag{82}$$

will be referred to as *maximal root* (*eigenvalue*) of system (67).

Definition 3.2.3 Each solvent $\mathcal{R}_{\ell m}$ of SMPE (73), for the given ℓ, $1\le\ell\le N$, whose spectrum contains maximal eigenvalue λ_m of system (67), is referred to as *maximal solvent* of (73).

Theorem 3.2.2 (Stojanovic & Debeljkovic, 2008.a) Suppose that there exist at least one ℓ, $1 \le \ell \le N$, that there exists *at least one* maximal solvent of SMPE (73) and let $\mathcal{R}_{\ell m}$ denote one of them. Then, linear discrete large-scale time-delay system (67) is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$\mathcal{R}_{\ell m}^{*} P \mathcal{R}_{\ell m} - P = -Q . \tag{83}$$

Proof. S*ufficient condition.* Define the following vector discrete functions

$$v(x_{k1}, \cdots, x_{kN}) = \sum_{i=1}^{N} S_i \left[x_i(k) + \sum_{j=1}^{N} \sum_{l=1}^{h_{ji}} T_{ji}(l) x_i(k-l) \right], \; x_{ki} = x_i(k+\theta), \; \theta \in \left\{ -h_{m_i}, \ldots, 0 \right\} \tag{84}$$

where $T_{ji}(k) \in \mathbb{C}^{n_i \times n_i}$, $1 \le j \le N$, $1 \le i \le N$ are, in general, some time-varying discrete matrix functions and $S_\ell = I_{n_\ell}$, $S_i \in \mathbb{C}^{n_\ell \times n_i}$, $1 \le i \le N$, $i \ne \ell$. The conclusion of the theorem follows immediately by defining Lyapunov functional for system (67) as

$$V(x_{k1}, \cdots, x_{kN}) = v^*(\cdot, \cdots, \cdot) P v(\cdot, \cdots, \cdot), \quad P = P^* > 0 \tag{85}$$

It is obvious that $V(\cdot, \cdots, \cdot) > 0$ for $\forall x_{ki} \ne 0$, $1 \le i \le N$. The forward difference of (85), along the solutions of system (67) is

$$\Delta V(\cdot, \cdots, \cdot) = \Delta v^*(\cdot, \cdots, \cdot) P v(\cdot, \cdots, \cdot) + v^*(\cdot, \cdots, \cdot) P \Delta v(\cdot, \cdots, \cdot) + \Delta v^*(\cdot, \cdots, \cdot) P \Delta v(\cdot, \cdots, \cdot) \tag{86}$$

A difference of $v(\cdot, \cdots, \cdot)$ can be determined in the following manner

$$\Delta v(\cdot, \cdots, \cdot) = \sum_{i=1}^{N} S_i \left[\Delta x_i(k) + \sum_{j=1}^{N} \sum_{l=1}^{h_{ji}} T_{ji}(l) \, \Delta x_i(k-l) \right] \tag{87}$$

$$\Delta x_i(k) = \left(A_i - I_{n_i} \right) x_i(k) + \sum_{j=1}^{N} A_{ij} x_j\left(k - h_{ij}\right) \tag{88}$$

Then

$$\begin{aligned} \Delta v(\cdot, \cdots, \cdot) = \sum_{i=1}^{N} S_i \Bigg[& \left(A_i - I_{n_i} + \sum_{j=1}^{N} T_{ji}(1) \right) x_i(k) + \sum_{j=1}^{N} T_{ji}\left(h_{ji}\right) x_i\left(k - h_{ji}\right) \\ & + \sum_{j=1}^{N} \sum_{l=1}^{h_{ji}-1} \Delta T_{ji}(l) x_i(k-l) + \sum_{j=1}^{N} A_{ij} x_j\left(k - h_{ij}\right) \Bigg] \end{aligned} \tag{89}$$

If we define new matrices

$$\mathcal{R}_i = A_i + \sum_{j=1}^{N} T_{ji}(1),\ 1 \le i \le N \tag{90}$$

then $\Delta v(\cdot,\cdots,\cdot)$ has a form

$$\Delta v(\cdot,\cdots,\cdot) = \sum_{i=1}^{N}\Bigg[S_i\left(\mathcal{R}_i - I_{n_i}\right)x_i(k) + \sum_{j=1}^{N}\left(S_j A_{ji} - S_i T_{ji}\left(h_{ji}\right)\right)x_i\left(k-h_{ji}\right) + \sum_{j=1}^{N}\sum_{l=1}^{h_{ji}-1} S_i \Delta T_{ji}(l)\, x_i(k-l)\Bigg] \tag{91}$$

If

$$S_j A_{ji} - S_i T_{ji}\left(h_{ji}\right) = S_i \Delta T_{ji}\left(h_{ji}\right),\quad 1 \le i \le N,\quad 1 \le j \le N \tag{92}$$

$$S_i\left(\mathcal{R}_i - I_{n_i}\right) = \left(\mathcal{R}_\ell - I_{n_\ell}\right)S_i,\quad 1 \le i \le N \tag{93}$$

$$S_i \Delta T_{ji}(l) = \left(\mathcal{R}_\ell - I_{n_\ell}\right)S_i\, T_{ji}(l),\quad 1 \le i \le N,\quad 1 \le j \le N \tag{94}$$

then

$$\Delta v(\cdot,\cdots,\cdot) = \left(\mathcal{R}_\ell - I_{n_\ell}\right)v(\cdot,\cdots,\cdot),\quad \Delta V(\cdot,\cdots,\cdot) = v^*(\cdot,\cdots,\cdot)\left(\mathcal{R}_\ell{}^* P\,\mathcal{R}_\ell - P\right)v(\cdot,\cdots,\cdot) \tag{95}$$

It is obvious that if the following equation is satisfied

$$\mathcal{R}_\ell{}^* P\,\mathcal{R}_\ell - P = -Q,\qquad Q = Q^* > 0 \tag{96}$$

then $\Delta V(\cdot,\cdots,\cdot) < 0$, $\forall x_{ki} \ne 0$, $1 \le i \le N$.

In the Lyapunov matrix equation (83), of all possible solvents $\mathcal{R}_\ell$ of (73), only one of maximal solvents $\mathcal{R}_{\ell m}$ is of importance, for it is the only one that contains maximal eigenvalue $\lambda_m \in \Sigma$ (*Definition* 3.2.3), which has dominant influence on the stability of the system. If a solvent which is not maximal is integrated into Lyapunov equation (83), it may happen that there will exist a positive definite solution of this equation, although the system is not stable. Accordingly, condition (83) represents sufficient condition of the stability of system (67).

If it exists, maximal solvent $\mathcal{R}_{\ell m}$ can be determined in the following way. From (92) and (94) we obtain

$$S_j A_{ji} = \mathcal{R}_\ell{}^{h_{ji}} S_i T_{ji}(1),\quad S_\ell = I_{n_\ell},\quad 1 \le i \le N,\ 1 \le j \le N \tag{97}$$

Multiplying i-th equation of the system of matrix equations (90) from the left by matrix $\mathcal{R}_\ell{}^{h_{mi}} S_i$ and using (93) and (97), we obtain equation (73). Taking solvent with eigenvalue

$\lambda_m \in \Sigma$ (if it exists) as a solution of the system of equations (73), we arrive at maximal solvent $\mathcal{R}_{\ell m}$.

Necessary condition. If system (67) is asymptotically stable, then $\forall \lambda_i \in \Sigma$, $|\lambda_i| < 1$. Since $\lambda(\mathcal{R}_{\ell m}) \subset \Sigma$,it follows that $\rho(\mathcal{R}_{\ell m}) < 1$, therefore the positive definite solution of Lyapunov matrix equation (67) exists.

Corollary 3.2.1 Suppose that for the given ℓ, $1 \le \ell \le N$, there exists matrix $\mathcal{R}_\ell$ being solution of SMPE (73). If system (67) is asymptotically stable, then matrix $\mathcal{R}_\ell$ is discrete stable ($\rho(\mathcal{R}_\ell) < 1$).

Proof. If system (67) is asymptotically stable, then $\forall z \in \Sigma \ |z| < 1$. Since $\lambda(\mathcal{R}_\ell) \subset \Sigma$, it follows that $\forall \lambda \in \lambda(\mathcal{R}_\ell)$, $|\lambda| < 1$, i.e. matrix $\mathcal{R}_\ell$ is discrete stable.

Conclusion 3.2.1 It follows from the aforementioned, that it makes no difference which of the matrices $\mathcal{R}_{\ell m}$, $1 \le \ell \le N$ we are using for examining the asymptotic stability of system (67). The only condition is that there exists at least one matrix for at least one ℓ. Otherwise, it is impossible to apply *Theorem* 3.2.2.

Conclusion 3.2.2 The dimension of system (67) amounts to $N_e = \sum_{j=1}^{N} n_j \left(h_{m_j} + 1\right)$. Conversely, if there exists a maximal solvent, the dimension of $\mathcal{R}_{\ell m}$ is multiple times smaller and amounts to n_ℓ. That is why our method is superior over a traditional procedure of examining the stability by eigenvalues of matrix $\mathcal{A}$.

The disadvantage of this method reflects in the probability that the obtained solution need not be a maximal solvent and it can not be known ahead if maximal solvent exists at all.

Hence the proposed methods are at present of greater theoretical than of practical significance.

3.2.4 Numerical example

Example 3.2.1 Consider a large-scale linear discrete time-delay systems, consisting of three subsystems described by Lee, Radovic (1987)

$$S_1:\ x_1(k+1) = A_1 x_1(k) + B_1 u_1(k) + A_{12} x_2(k - h_{12}),$$

$$S_2:\ x_2(k+1) = A_2 x_2(k) + B_2 u_2(k) + A_{21} x_1(k - h_{21}) + A_{23} x_3(k - h_{23}),$$

$$S_3:\ x_3(k+1) = A_3 x_3(k) + B_3 u_3(k) + A_{31} x_1(k - h_{31}),$$

$$A_1 = \begin{bmatrix} 0.8 & 0.6 \\ 0.4 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0.7 & 0 & -0.5 \\ -0.1 & 6 & -0.1 \\ -0.6 & 1 & 0.8 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & -0.1 \\ 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -0.1 & -0.2 \\ 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, A_{23} = \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.2 \\ 0.1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 0.8 \end{bmatrix}, B_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, A_{31} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{bmatrix},$$

The overall system is stabilized by employing a local memory-less state feedback control for each subsystem

$$u_i(k) = K_i x_i(k),\ K_1 = [-6 \quad -7], K_2 = \begin{bmatrix} -7 & -45 & 10 \\ 4 & -4 & -4 \end{bmatrix}, K_3 = \begin{bmatrix} -5 & -1 \\ 1 & -4 \end{bmatrix}$$

Substituting the inputs into this system, we obtain the equivalent closed loop system representations

$$S_i:\ x_i(k+1) = \hat{A}_i\, x_i(k) + \sum_{j=1}^{3} A_{ij} x_j\left(k - h_{ij}\right),\ \ 1 \le i \le 3,\ \ \hat{A}_i = A_i + B_i K_i$$

For time delay in the system, let us adopt: $h_{12} = 5$, $h_{21} = 2$, $h_{23} = 4$ and $h_{31} = 5$. Applying *Theorem* 3.2.1 to a given closed loop system, we obtain the following SMPE for $\ell = 1$

$$\mathcal{R}_1^{\,6} - \mathcal{R}_1^{\,5}\hat{A}_1 - \mathcal{R}_1^{\,3} S_2 A_{21} - S_3 A_{31} = 0,$$

$$\mathcal{R}_1^{\,6} S_2 - \mathcal{R}_1^{\,5} S_2 \hat{A}_2 - A_{12} = 0,$$

$$\mathcal{R}_1^{\,5} S_3 - \mathcal{R}_1^{\,4} S_3 \hat{A}_3 - S_2 A_{23} = 0.$$

Solving this SMPE by minimization methods, we obtain

$$\mathcal{R}_1 = \begin{bmatrix} 0.6001 & 0.3381 \\ 0.6106 & 0.3276 \end{bmatrix},\ \ S_2 = \begin{bmatrix} 0.0922 & 1.3475 & 0.5264 \\ 0.0032 & 1.3475 & 0.4374 \end{bmatrix},\ \ S_3 = \begin{bmatrix} 0.6722 & -0.3969 \\ 1.3716 & -1.0963 \end{bmatrix}.$$

Eigenvalue with maximal module of matrix $\mathcal{R}_1$ equals 0.9382. Since eigenvalue λ_m of $\mathcal{A} \in \mathbb{R}^{40\times 40}$ also has the same value, we conclude that solvent $\mathcal{R}_1$ is maximal solvent ($\mathcal{R}_{1m} = \mathcal{R}_1$). Applying *Theorem* 3.2.2, we arrive at condition $\rho(\mathcal{R}_{1m}) = 0.9382 < 1$ wherefrom we conclude that the observed closed loop large-scale time-delay system is asymptotically stable.

The difference in dimensions of matrices $\mathcal{R}_1 \in \mathbb{R}^{2\times 2}$ and $\mathcal{A} \in \mathbb{R}^{40\times 40}$ is rather high, even with relatively small time delays (the greatest time delay in our example is 5). So, in the case of great time delays in the system and a great number of subsystems N, by applying the derived results, a smaller number of computations are to be expected compared with a traditional procedure of examining the stability by eigenvalues of matrix $\mathcal{A}$.

An accurate number of computations for each of the mentioned method require additional analysis, which is not the subject-matter of our considerations herein.

4. Conclusion

In this chapter, we have presented new, necessary and sufficient, conditions for the asymptotic stability of a particular class of linear continuous and discrete time delay systems. Moreover, these results have been extended to the large scale systems covering the cases of two and multiple existing subsystems.

The time-dependent criteria were derived by Lyapunov's direct method and are exclusively based on the maximal and dominant solvents of particular matrix polynomial equation. It can be shown that these solvents exist only under some conditions, which, in a sense, limits the applicability of the method proposed. The solvents can be calculated using generalized Traub's or Bernoulli's algorithms. Both of them possess significantly smaller number of computation than the standard algorithm.

Improving the converging properties of used algorithms for these purposes, may be a particular research topic in the future.

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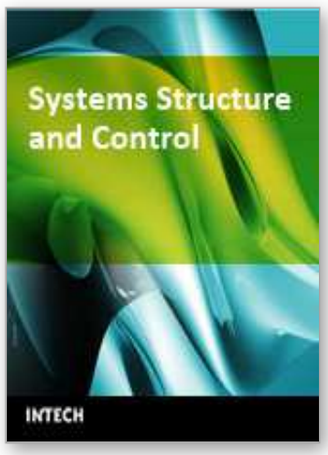

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