Chapter 8

Role of Vortex Dynamics in Relativistic Fluids

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Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/65372

Abstract

Relativistic versions of Helmholtz's three theorems on vorticity flux conservation are derived by invoking an alternative approach based on Greenberg's theory of spacelike congruence generated by vortex lines and (1 + 1) + 2 decomposition of the gradient of fluid's 4-velocity. It is shown that the meridional circulation causes vorticity winding due to the breakdown of gravitational isorotation via differential rotation. A new streamline invariant describing energy conservation associated with vorticity flux is shown to exist.

Keywords: vorticity flux, vorticity winding, fluid helicity, vortex motion

1. Introduction

Vortices are found in classical, relativistic and quantum fluids [1–7]. It is well known from classical theory of vortices that the dynamics of vortices is described by Kelvin-Helmholtz's flux conservation theorem. There are three theorems of Helmholtz on vortex tubes in the classical theory of fluids [2]. The product of the magnitude of fluid's vorticity vector and of vortex tube's cross-sectional area is called the strength of a vortex tube. Helmholtz's first theorem states that the strength of vortex tube remains constant along the vortex tube. This result is purely kinematical in nature in the sense that its derivation does not require Euler's equations which govern the motion of a fluid. The second theorem tells us that the vortex lines are material lines in the case of perfect fluid flows. The third theorem says that the vortex tube's strength multiplied by the fluid's chemical potential remains constant along the flow lines (or streamlines) for perfect fluid flows. Since a two-dimensional surface across which fluid's vorticity flux passes is usually bounded by a closed circuit, Kelvin's circulation theorem is intimately connected with that of Helmholtz's flux conservation theorem for perfect fluid flows. The circulation of a fluid motion around any closed curve on a vertex tube wall is equal to the flux of vorticity along a vertex tube. Kelvin's circulation theorem asserts that the circulation of a perfect fluid motion around a closed material curve is conserved in time as the fluid evolves. This assertion in turn implies that the vortex tube is a material tube and moves with the fluid (i.e., vortex lines are material lines) [1, 2].

Since relativistic fluid dynamics differs from that of Newtonian fluid dynamics in many ways, Greenberg [8] developed a theory of a family of spacelike curves in order to study the kinematical behavior of vortex lines (i.e., spacelike curves) appearing in an isentropic perfect fluid motion. Greenberg [8] formulated a relativistic analogue of Helmboltz's first theorem on the basis of Ehlers' divergence identity of fluid's vorticity vector [9] in the Newtonian limit (i.e., ignoring the fluid's acceleration term). The relativistic divergence identity of fluid's vorticity vector involves fluid's acceleration vector which embodies the curvature of space-time, and thereby, it differs from classical divergence identity of vorticity vector. This is the reason that the exact relativistic version of Helmholtz's first theorem is not obtainable kinematically. Greenberg [8] further extended his analysis to obtain a relativistic version of Helmholtz's third theorem using Euler's equations of motion for an isentropic perfect fluid. In this derivation, the concept of vortex lines to be material lines is automatically implemented because of his transport law (i.e., it is presently known as Greenberg's transport law).

The necessary and sufficient conditions for the existence of conservation of flux associated with a vector field have been given firstly by Bekenstein and Oron [10] and then secondly by Carter [11]. Of two conditions, the first condition is satisfied if a vorticity 2-form is expressed as a curl of momentum covector associated with a fluid and the second condition when imposed on vorticity 2-form gives Euler's equations that govern fluid's motion. Vorticity 2-form satisfying these two conditions led to the formulation of a relativistic version of Kelvin's circulation conservation theorem and Helmholtz's flux conservation theorem [10, 11].

There is a link between Greenberg's transport law [8] and Carter's formulation [11] of vorticity flux conservation. Carter's vorticity flux conservation is based on vorticity 2-form of rank 2 (i.e., simple vorticity bivector field). The matrix of vorticity 2-form admits eigen vectors with zero eigen values. Such eigen vectors are referred to as flux vectors. The flux vectors span two-dimensional tangent subspaces at each point of a four-dimensional space-time. These tangent subspaces mesh together to form a family of timelike 2-surfaces because simple vorticity bivector field satisfies Frobenius condition [12] of 2-surface forming. Such timelike 2-surfaces are called flux surfaces. It is to be noted that Greenberg's transport law is an alternative version of Frobenius condition [12] due to which congruences of fluid flow lines and vortex lines form a family of timelike two-dimensional surfaces in the case of an isentropic perfect fluid flows. Furthermore, vortex lines are material lines. Flux surfaces are also material surfaces.

An intimate connection between Kelvin's circulation theorem and the fluid helicity (i.e., helicity of vorticity vector field) conservation was long ago demonstrated by Moffat [13]. The geometric structure of streamlines and vortex lines (forming a vortex tube) inherent in the helicity conservation was discovered by expressing the conservation of the sum of writhe and twist of vortex tubes undergoing continuous deformation in a fluid flow [14, 15]. The fluid helicity is conserved when vortex lines are frozen lines (i.e., material lines) in a fluid flow [14]. Such topological consequences have been extensively investigated by several authors [16, 17] in classical fluid dynamics.

Although relativistic version of fluid helicity conservation has been formulated by Carter [11] and Bekenstein [18], yet its topological consequences remained unknown. In a relativistic frame work, frozen-in property of spacelike vortex lines is describable by Greenberg's transport law [8], which in turn leads to the fact that an isolated vortex is a timelike two-dimensional connected manifold called vortex world sheet. The deformation of timelike fluid flow lines and spacelike vortex lines caused by gravitation forbids the application of Biot-Savart-like law and Frenet-like transport frame along a spacelike vortex line. Furthermore, the kinematic deformation of a vortex tube is not independent of the kinematic deformation of fluid flow lines forming a stream tube. The deformation of a vortex tube is determined in terms of the expansion, shear and acceleration associated with fluid flow lines and the magnetic part of Weyl curvature tensor representing the free gravitational field [19]. In particular, the variation in cross-sectional area of a vortex tube along the tube is controlled by free gravitational field even if the acceleration vector of the fluid motion and the shear are ignored [19].

An inherent connection between helicity conservation and streamline invariant has already been pointed out by Bekenstein [18] in order to extract relevant information of astrophysical significance from nonlinear equations of relativistic fluid (or magnetofluid) dynamics. But such relationship between fluid helicity conservation and inherently connected streamline invariant is still to be investigated. This is the idea which motivates to work further in order to understand the role of vorticity in the rotational evolution of relativistic fluid, since most, if not all, compact stars composed of fluids rotate. It may be conceived that the gravitational effect on vorticity can be significant in the understanding of the internal structures of compact stars. It is known that differential rotation of fluid elements of which stars are composed arises from the gravitational collapse of massive stellar cores [20, 21]. Differential rotation produces vorticity which twists streamlines in fluid's motion. It is expected that the winding up of vortex lines caused by differential rotation will change the angular velocity profile in a similar way as has been found in magnetized stars due to the presence of magnetic fields [22].

In the present work, we confine our attention to the derivation of streamline invariants which adhere to the fluid helicity conservation and a formulation of vorticity winding due to meridional circulation via differential rotation along poloidal components of vorticity. Furthermore, we derive new conservation law for vorticity flux.

The present paper is organized as follows. In Section 2, relativistic versions of Helmholtz's three theorems on vorticity flux conservation are derived using an alternative approach based on Greenberg's theory of spacelike congruence generated by vortex lines and (1 + 1) + 2 decomposition of gradient of fluid's 4-velocity. Section 3 is devoted to the derivation of streamline invariants and the formulation of vorticity winding. In Section 4, we obtain new conservation law for the vorticity flux.

Convention: Space-time metric is of signature +2. Semi-colon (;) and comma (,) are, respectively, used to denote covariant derivative and partial derivative. Speed of light c is assumed to be unity.

2. Helmholtz's theorems

In this section, we obtain an alternative derivation of exact relativistic version of Helmholtz's theorems. We demonstrate that a relativistic version of Helmholtz's first theorem describes vorticity flux conservation inside a vortex tube when the variation of flux is measured by a comoving observer with the fluid along a vortex tube. Such conservation of flux is independent of proper time. We will also show that a relativistic version of Helmholtz's third theorem describes vorticity flux conservation in a streamline tube whose cross-sectional area lies in a spacelike two-dimensional subspace orthogonal to both stream and vortex lines. Such flux conservation is in proper time as the perfect fluid evolves and vortex lines are material lines. In order to prove these two results, we begin with Euler's equation that governs the motion of a perfect fluid. Euler's equation is as follows [23]:

$$(\rho + p)u'_a = -p_{,a} - p_{,b}u^b u_a \tag{2.1}$$

where $u'_a = u_{a;b} u^b$ denotes the 4-acceleration vector field of the fluid. u_a is the 4-velocity of the fluid and normalized according to. $u^a u_a = -1$. The proper mass density and pressure are, respectively, represented by ρ and p. It is assumed that the fluid is composed of baryons whose proper number density n is conserved

$$\left(nu^a\right)_{;a} = 0. \tag{2.2}$$

The first law of thermodynamics is [23]:

$$d\left(\frac{\rho}{n}\right) = Tds - pd\left(\frac{1}{n}\right), \tag{2.3}$$

where T and s denote, respectively, the local temperature and the entropy per baryon. The relativistic enthalpy per baryon called chemical potential is expressed as:

$$\mu = \frac{\rho + p}{n} \tag{2.4}$$

If we assume that the entropy per baryon s is constant, then it follows from Eqs. (2.3) and (2.4) that

$$dp = nd\mu \tag{2.5}$$

Substituting Eqs. (2.4) and (2.5) in Eq. (2.1), we can reduce Euler's equation in the following form:

$$\mu u'_a + \mu_{,a} + \mu_{,b} u^b u_a = 0. \tag{2.6}$$

It is known from relativistic version of Kelvin's circulation theorem for a perfect fluid that the vorticity flux equals the closed contour line integral $\oint \mu u_a \, dx^a$. This integral is equivalent to a surface integral $\oint W_{ab} \, dS^{ab}$ where W_{ab} is the fluid particle vorticity 2-form [24] and dS^{ab} is an element of area on a two-dimensional surface bounded by the contour C. The conservation of vorticity flux means that the value of this integral is the same for all times when each point in the contour C is dragged along the fluid flow (mathematically, it is Lie transported along u^a) [11, 18]. The conditions required for flux conservation are as follows: (i) W_{ab} satisfies the closure property (i.e., $W_{[ab;c]} = 0$, where the square bracket indicates skew-symmetrization), and (ii) the matrix of W_{ab} admits eigen vectors with zero eigen values [11]. The closure property of W_{ab} is automatically satisfied in the case of a perfect fluid if W_{ab} is expressed as a curl of particle energy-momentum convector μu_a [24]. Its explicit expression is given by

$$W_{ab} = \left(\mu u_b\right)_{;a} - \left(\mu u_a\right)_{;b}. \tag{2.7}$$

Contraction of Eq. (2.7) with u^a yields Euler's equation in the form given as below:

$$u^a W_{ab} = 0. \tag{2.8}$$

Dualizing Eq. (2.7) and contracting the resulting equation with u_b, we get

$$\overset{*}{W}{}^{ab} u_b = -V^a, \tag{2.9}$$

where an overhead star (*) is used for Hodge dualization. $V^a = \mu\omega^a$ and $\omega^a = \eta^{abcd} u_{b;c} u_d$ represents the kinematic vorticity vector of the fluid. It is to be noted that the factor $\frac{1}{2}$ which appears in the usual definition of vorticity vector [9] is ignored only for calculational purpose, and due to this omission, there is no loss of generality. η^{abcd} denotes the Levi-Civita skew-symmetric tensor.

Inverting Eq. (2.9), we get

$$W^{ab} = -\eta^{abcd} V_c u_d \tag{2.10}$$

which satisfies

$$W_{ab} V^b = 0. \tag{2.11}$$

It is evident from Eqs. (2.8) and (2.11) that u^a and V^a are eigen vectors of vorticity 2-form W_{ab} with zero eigen value. Inverting Eq. (2.10), we get

$$^*W^{ab} = V^a u^b - V^b u^a, \tag{2.12}$$

Since $V^a = \mu\omega^a$ is an eigen vector of W_{ab} with zero eigen value, it represents a flux vector of W_{ab} [11]. Hereafter, V^a will be referred to as a "vorticity flux" vector.

Because W_{ab} is a curl, we have

$$^*W^{ab}_{;b} = 0. \tag{2.13}$$

Substituting Eq. (2.12) in Eq. (2.13) and contracting the resulting equation with u^a and m^a, respectively, we get

$$V_{,a} m^a + V\left(m^a_{;a} - u'_a m^a\right) = 0 \tag{2.14}$$

and

$$V_{,a} u^a + V\left(u^a_{;a} - u_{a,b} m^a m^b\right) = 0, \tag{2.15}$$

where m^a is unit spacelike vector field along vortex lines.

A spacelike congruence of vortex lines is generated by unit spacelike vector field m^a with properties $m^a m_a = 1, m_{a;b} m^a = 0$, and $m_a u^a = 0$. In the derivation of Eqs. (2.14) and (2.15), we have used $V^a = V m^a$, where V denotes the magnitude of the vorticity flux vector V^a and is equal to $\mu\omega$.

Since the vorticity flux vector V^a is orthogonal to the 4-velocity of the fluid at all points along a vortex line, a comoving observer with the fluid to observe deformation of a vortex tube erects a screen lying in a spacelike two-dimensional tangent subspace orthogonal to both stream and vortex lines generated by u^a and m^a, respectively. Greenberg's expansion parameter $\hat{\theta}$ of a vortex tube is expressed as:

$$\hat{\theta} = u'_a m^a \tag{2.16}$$

and

$$\hat{\theta} = \frac{1}{\hat{A}} \frac{d\hat{A}}{d\sigma}, \tag{2.17}$$

where $p^a_b = \delta^a_b + u^a u_b - m^a m_b$ is the projection tensor and $\hat{A}$ is the cross-sectional area of a vortex tube that lies in a spacelike two-dimensional tangent subspace orthogonal to both u^a and m^a. The arc-length is denoted by σ and $\frac{d}{d\sigma}$ is the directional derivative along a vortex line. Substituting Eqs. (2.16) and (2.17) in Eq. (2.14), we get

$$\frac{d}{d\sigma}\left(\mu\omega\hat{A}\right) = 0, \tag{2.18}$$

which is an exact relativistic version of Helmholtz's first theorem. It is evident from Eq. (2.18) that the variation in vorticity flux of a vortex tube is measured along the tube and the flux remains constant inside a tube. The proper time τ plays no role in such variation of flux. Since an observer employed to measure flux is comoving along vortex line, a vortex tube is a material tube. In relativistic framework, a vortex tube to be a material tube is automatically associated with Helmholtz's first theorem because its derivation originates from Euler's equation. An exact relativistic analogue of Helmholtz's first theorem is not obtainable kinematically.

In order to obtain Helmholtz's third theorem, we use (1 + 1) + 2 decomposition of the gradient of 4-velocity u_a of the fluid. In this formulation [25], the expansion parameter $\tilde{\theta}$ of a stream tube is expressed as:

$$\tilde{\theta} = u_{a;b} m^a m^b \tag{2.19}$$

and

$$\tilde{\theta} = \frac{1}{\tilde{A}} \frac{d\tilde{A}}{d\tau}, \tag{2.20}$$

where $\tilde{A}$ denotes cross-sectional area of a stream tube lying in a spacelike two-dimensional tangent subspace orthogonal to both u^a and m^a. The arc-length measured along a streamline (or wordline) is denoted by τ and $\frac{d}{d\tau}$ represents the directional derivative along a streamline. On account of Eqs. (2.19) and (2.20) , Eq. (2.15) takes the form

$$\frac{d}{d\tau}\left(\mu\omega\tilde{A}\right) = 0, \tag{2.21}$$

which is a relativistic version of Helmholtz's third theorem. It is observed from Eq. (2.21) that the variation in vorticity flux is measured along a streamline and the flux inside a stream tube remains constant in proper time τ as the fluid evolves according to Euler's equation of motion. This result can be understood in a sense that the volume occupied by a stream tube is the product of cross-sectional area of a stream tube and its length measured along a vortex line (being a material line). The vorticity flux passing through such cross-sectional area remains constant in proper time τ.

On account of Eq. (2.12), Eq. (2.13) can be put in two different forms which are given as below:

$$L_u V^a = V^b_{;b} u^a - u^b_{;b} V^a \tag{2.22}$$

where L_u denotes the Lie derivative with respect to u^a.

$$\left(\frac{V^a}{n}\right)_{;b} u^b = u^a_{;b}\left(\frac{V^b}{n}\right) + u^a\left(u'_b \frac{V^b}{n}\right), \tag{2.23}$$

In the derivation of Eq. (2.23), baryon conservation law Eq. (2.2) is used.

Projecting Eq. (2.22) orthogonal to both u^a and V^a, we get

$$p^a_b L_u V^b = 0, \tag{2.24}$$

which is Greenberg's transport law [8]. Eq. (2.24) tells us that the timelike congruence generated by u^a and the spacelike congruence generated by V^a span a family of timelike 2-surfaces. Such 2-surfaces are referred to as vortex flux surfaces (or vorticity flux surfaces) [11]. Furthermore, Eq. (2.24) implies that vortex lines are material lines.

Eq. (2.23) is in a form as has been derived by Bekenstein and Oron [10] in the case of the magnetic field vector in order to show the frozen-in property of the magnetic field in a perfectly conducting magnetofluid dynamics. Similar conclusion holds for the vorticity flux vector V^a. Eq. (2.23) shows that the orthogonal connecting vector of any two fluid particles lying on a vortex line is proportional to $\left(\frac{V^a}{n}\right)$. This in turn leads to the fact that the fluid particles lying once on a vortex line will continue to remain on the same vortex line for all times as the fluid evolves.

3. Vorticity winding in axisymmetric stationary fluid configuration

This section is devoted to the study of vorticity winding caused by meridional circulation via differential rotation of a perfect fluid which is assumed to be axisymmetric and stationary. In this case, a space time admits two linearly independent commuting Killing vectors: $\xi^a_{(t)}$ generating a translational symmetry with open timelike lines as orbits and the other is a spacelike Killing vector $\xi^a_{(\emptyset)}$ generating rotations with closed orbits about a symmetry axis. There exists a family of invariant timelike 2-surfaces (which are identified with vorticity flux surfaces) generated by these two Killing vectors that correspond to ignorable coordinates $x^4 = t$ and $x^3 = \emptyset$ (so that ${}^a_{(t)} = \delta^a_t \wedge \xi^a_{(\emptyset)} = \delta^a_{\emptyset}$). The ignorable coordinates t and $\varnothing$ are usually called toroidal coordinates. We choose poloidal coordinates $x^1 = r$ and $x^2 = \theta$. All physical quantities including the metric tensor g_{ab} are independent of t and $\varnothing$.

From Eq. (2.8), we get

$$W_{t\theta} = -W_{tr}\frac{u^r}{u^\theta} \tag{3.1a}$$

$$W_{\varnothing\theta} = -W_{\varnothing r}\frac{u^r}{u^\theta} \tag{3.1b}$$

$$W_{rt}u^{t} + W_{r\varnothing}u^{\varnothing} + W_{r\theta}u^{\theta} = 0. \tag{3.1c}$$

Since W_{ab} is a curl, we have

$$W_{[ab;c]} = 0, \tag{3.2}$$

where the square bracket around indices indicates skew-symmetrization. From Eq. (3.2), we get

$$W_{tr,\theta} + W_{\theta t,r} = 0, \tag{3.3a}$$

$$W_{\theta\varnothing,r} + W_{\varnothing r,\theta} = 0. \tag{3.3b}$$

It follows from Eqs. (3.1a) and (3.3a) that

$$\frac{d}{d\tau}\left(lnW_{tr}\right) = -u^{\theta}\left(\frac{u^{r}}{u^{\theta}}\right)_{,r}. \tag{3.4a}$$

Similarly, substitution of Eq. (3.1b) in Eq. (3.3b) gives

$$\frac{d}{d\tau}\left(lnW_{\varnothing r}\right) = -u^{\theta}\left(\frac{u^{r}}{u^{\theta}}\right)_{,r}. \tag{3.4b}$$

From Eqs. (3.4a) and (3.4b), we get

$$\frac{d}{d\tau}\left(ln\frac{W_{tr}}{W_{\varnothing r}}\right) = 0 \tag{3.5}$$

which on integration along the streamline generated by u^{a} gives

$$\frac{W_{tr}}{W_{\varnothing r}} = A = \frac{W_{t\theta}}{W_{\varnothing\theta}}, \tag{3.6}$$

where A is constant along the streamline. Eq. (3.4b) can be rewritten as:

http://dx.doi.org/10.5772/65372

$$\frac{d}{d\tau}\left(lnW_{\varnothing r}\right)=-\left(u^{r}_{,r}+u^{\theta}_{,\theta}\right)+\frac{d}{d\tau}\left(lnu^{\theta}\right) \tag{3.7}$$

From Eq. (2.2), we get

$$u^{r}_{,r}+u^{\theta}_{,\theta}+\frac{d}{d\tau}\left(ln\sqrt{-g}\,n\right)=0 \tag{3.8}$$

Substitution of Eq. (3.8) in Eq. (3.7) gives

$$\frac{d}{d\tau}\left(ln\frac{W_{\varnothing r}}{n\sqrt{-g}u^{\theta}}\right)=0 \tag{3.9}$$

which on integration along the streamline gives

$$W_{\varnothing r}=Bn\sqrt{-g}u^{\theta}, \tag{3.10}$$

where B is constant along the streamline.

Substituting Eq. (3.10) in Eq. (3.1b), we get

$$W_{\varnothing\theta}=-Bn\sqrt{-g}u^{r}. \tag{3.11}$$

Substituting Eqs. (3.6) and (3.10) in Eq. (3.1c), we get

$$W_{r\theta}=Bn\sqrt{-g}u^{t}\left(A+\Omega\right), \tag{3.12}$$

where $\Omega=\dfrac{u^{\emptyset}}{u^{t}}$ is the local velocity of rotation of the fluid.

It follows from Eqs. (3.6), (3.10) and (3.11) that

$$W_{tr}=ABn\sqrt{-g}u^{\theta},\ W_{t\theta}=-ABn\sqrt{-g}u^{r}. \tag{3.13}$$

Eq. (2.9) gives

$$V^{a} = \frac{-1}{2}\eta^{abcd}u_{b}W_{cd}. \tag{3.14}$$

Substitution of Eqs. (3.10)–(3.13) in the coordinate expansion of Eq. (3.14) gives

$$V^{a} = -Bn\left[\left(\delta_{t}^{a} - A\delta_{\varnothing}^{a}\right) + \left(u_{t} - Au_{\varnothing}\right)u^{a}\right] \tag{3.15}$$

which is the required expression for the vorticity flux vector. We now use Eq. (3.15) to find conserved quantity along a streamline from the conservation of the fluid helicity. The fluid helicity vector is given by Carter [11, 24] and Bekenstein [18] as follows

$$H^{a} = .^{ab}\left(\mu u_{b}\right) = -\mu V^{a} \tag{3.16}$$

whose divergence vanishes,

$$\left(\mu V^{a}\right)_{;a} = 0. \tag{3.17}$$

This is fluid helicity conservation. Eq. (3.17) can be cast in form

$$\left(\mu\sqrt{-g}V^{a}\right)_{,a} = 0 \tag{3.18}$$

or equivalently

$$\left(ln\mu\right)_{,a}V^{a}\sqrt{-g} + \left(\sqrt{-g}V^{r}\right)_{,r} + \left(\sqrt{-g}V^{\theta}\right)_{,\theta} = 0. \tag{3.19}$$

Substituting Eq. (3.15) in Eq. (3.19), we get

$$\frac{d}{d\tau}\left[ln\mu B\left(u_{t} - Au_{\varnothing}\right)\right] = 0 \tag{3.20}$$

which on integration along a streamline gives

$$\mu\left(u_t - Au_\varnothing\right) = -C\left(say\right), \tag{3.21}$$

where the constant B is absorbed in a constant C along a streamline. The energy and angular momentum scalars in axisymmetric stationary fluid configuration are of the forms [11]

$$E = -\mu u_t, L = \mu u_\varnothing. \tag{3.22}$$

From Eqs. (3.21) and (3.22), it can be seen that

$$C = E + \mathrm{AL} \tag{3.23}$$

which shows that the linear combination of energy and angular momentum is constant along a streamline but varies from streamline to streamline. Thus, C is a streamline invariant.

We now consider in view of Eq. (3.15) as follows

$$\Omega_{,a}V^a = -Bn\Omega_{,a}\left[\left(\delta_t^a - A\delta_\varnothing^a\right) + \left(u_t - Au_\varnothing\right)u^a\right] - B\left(u_t - Au_\varnothing\right)n\Omega_{,a}u^a. \tag{3.24}$$

Setting $V^a = \mu\omega^a$ and multiplying Eq. (3.24) by μ

and using Eq. (3.21) in the resulting equation, we get

$$\mu^2\Omega_{,a}\omega^a = nBC\frac{d\Omega}{d\tau}. \tag{3.25}$$

In order to provide a meaningful interpretation of Eq. (3.25), we proceed as follows. In the absence of meridional circulation (i.e., $u^r = 0 = u^\theta$), it can be seen from Eqs. (3.1c) and (3.6) that $\frac{W_{tr}}{W_{\emptyset r}} = \mathrm{A} = -\Omega$. This shows that the streamline invariant A is the angular velocity of vorticity flux surfaces commoving with the fluid in the absence of meridional circulation. But in the presence of meridional circulation (i.e., $u^r \neq 0, u^\theta \neq 0$, on account of Eqs. (3.1c) and (3.12), we observe that $A \neq -\Omega$. This means that the streamline invariant A which is seen to represent the angular velocity of vorticity flux surfaces will no longer be equal to fluid's angular velocity Ω in the presence of meridional circulation and hence we set $A = -\Omega_V$, where Ω_V represents the angular velocity of vorticity flux surfaces. Thus, meridional circulation causes mismatch between the angular velocities of vorticity flux surfaces and of the fluid. A short calculation from Eqs. (2.12) and (3.12) gives

$$\Omega - \Omega_V = \frac{\mu(u_\varphi \omega_t - u_t \omega_\varphi)}{nBK(u^t)^2} \tag{3.26}$$

where

$$K = (g_{t\varphi}^2 - g_{tt} g_{\varphi\varphi}) > 0. \tag{3.27}$$

It is evident from Eq. (3.26) that the toroidal components ω_t and $\omega_\emptyset$ of the fluid's vorticity vector vanish in the absence of meridional circulation. Notice that $A = -\Omega_V$ is streamline invariant, i.e., $\frac{d\Omega_V}{d\tau} = 0$. Thus, it follows from Eqs. (3.25) and (3.26) that

$$\mu^2 \Omega_{,a} \omega^a = nC \frac{d}{d\tau}\left[\frac{\mu\left(u_\varphi \omega_t - u_t \omega_\varphi\right)}{nK\left(u^t\right)^2}\right]. \tag{3.28}$$

The left-hand side of Eq. (3.28) represents the differential rotation along the poloidal components of the vorticity vector, whereas the right-hand side gives the proper time rate of change of a combination of toroidal components of the vorticity vector. This implies that the differential rotation along the poloidal vorticity cannot vanish until the vanishing of toroidal vorticity. Since the presence of meridional circulation ensures the existence of toroidal vorticity, meridional circulation causes the kind of effect that mimics like vorticity winding by stretching frozen-in vortex lines via differential rotation.

Because $\omega_t = 0 = \omega_\varphi$ in the absence of meridional circulation, Eq. (3.28) reduces to

$$\Omega_{,a} \omega^a = 0 \tag{3.29}$$

which is the law of gravitational isorotation as is pointed out by Glass [26]. Thus, it seems that the meridional circulation causes vorticity winding due to the breakdown of the gravitational isorotation.

Further investigation is needed on the lines of recent work by Birkl et al. [27] in order to understand the role of meridional circulation in relation to vorticity winding via differential rotation.

4. Vorticity energy conservation in axisymmetric stationary case

This section is devoted to the derivation of vorticity energy conservation assuming that the perfect fluid configuration is axisymmetric and stationary. Substituting Eq. (2.12) in Eq. (2.13) and contracting the resulting equation with V_a, we get

$$\frac{d}{d\tau}\left(\frac{1}{2}V^2\right) = u^a_{;b}V_aV^b - V^2u^b_{;b}. \tag{4.1}$$

Eliminating $u^b_{;b}$ with the aid of baryon conservation law Eq. (2.2), we obtain from Eq. (4.1) that

$$\frac{d}{d\tau}\left(\frac{1}{2}V^2\right) = \frac{V^2}{n}\frac{dn}{d\tau} + V_{b;a}V^bu^a \tag{4.2}$$

which can be converted to the form

$$\frac{d}{d\tau}\left(V^2\right) = \frac{V^2}{n}\frac{dn}{d\tau} + \left(V_{b;a} - V_{a;b}\right)V^bu^a. \tag{4.3}$$

The last term on the right-hand side of Eq. (4.3) with the help of Eq. (3.15) can be simplified in the following form

$$\left(V_{b,a} - V_{a,b}\right)V^bu^a = nB\left(\frac{dV_t}{d\tau} - \frac{dV_\varnothing}{d\tau}\right). \tag{4.4}$$

Substitution of Eq. (4.4) in Eq. (4.3) gives

$$\frac{d}{d\tau}\left(\frac{V^2}{n}\right) + B\frac{d}{d\tau}\left(V_t - AV_\varnothing\right) = 0. \tag{4.5}$$

Since B and A are streamline invariant, we obtain from Eq. (4.5) that

$$\frac{d}{d\tau}\left[\frac{V^2}{n} + B\left(V_t - AV_\varnothing\right)\right] = 0 \tag{4.6}$$

which on integration along a streamline gives that

$$\frac{V^2}{n} + B\left(V_t - AV_{\varnothing}\right) = D\ (say). \tag{4.7}$$

Since $V^a = \mu\omega^a$, $V^2 = \mu^2\omega^2$, $V_t = \mu\omega_t$, and $V_{\varnothing} = \mu\omega_{\varnothing}$. Thus, Eq. (4.7) takes the form

$$\frac{\mu^2\omega^2}{n} + B\mu\left(\omega_t - A\omega_{\varnothing}\right) = D. \tag{4.8}$$

The first term of Eq. (4.8) represents the energy of the vorticity flux vector per baryon. The second term indicates the presence of covariant toroidal components of the vorticity vector ω^a. The quantity D is streamline invariant of the motion of frozen-in vortex lines and can be thought of as a vorticity energy conservation arising from fluid helicity conservation.

Acknowledgements

The author wishes to thank the Board of Editors for valuable suggestion regarding the revision of the manuscript.

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References

[1] P. G. Saffman, Vortex Dynamics, Cambridge University Press, Cambridge 1992.

[2] J. Serrin, Mathematical Principle of Classical Fluid Dynamics in, Encyclopedia of Physics/Handbuck der Physik vol8/1 edited by S. Flugge (Springer Verlag Berlin 1959) p. 162–163.

[3] W. Schröder and H. J. Treder, Found. Phys. 26, 417 (1996).

[4] H. Eshraghi, Phys. Plasma. 10, 3577 (2003).

http://dx.doi.org/10.5772/65372

[5] U. R. Fischer, Ann. Phys. 278, 62 (1999).

[6] U. Ben-Y'acov, J. Phys. A: Math. Gen. 27, 7165 (1994).

[7] I. B. Birula, Z. B. Birula, and C. Sliwa, Phys. Rev. A61, 032110 (2000).

[8] P. J. Greenberg, J. Math. Anal. Appl. 30, 128 (1970).

[9] J. Ehlers, Gen. Rel. Grav. 25, 1225 (1993).

[10] J. D. Bekenstein and E. Oron, Phys. Rev. D18, 1809 (1978).

[11] B. Carter, in Active Galactic Nuclei, edited by C. Hazard and S. Mitton, p. 273–279, Cambridge University Press, Cambridge, 1979.

[12] W. Kundt and M. Trümper, Z. Phys. 192, 41 (1966).

[13] H. K. Moffat, J. Fluid Mech. 35, 117 (1969).

[14] H. K. Moffat and R. L. Ricca, Proc. Roy. Soc. Lond. A439, 411 (1992).

[15] H. K. Moffat, Proc. Nat. Acad. Sci. 111, 3663 (2014).

[16] V. I. Arnold and B. A. Khesin, Topological Methods in Hydrodynamics, Springer-Verlag, NY, USA, 1998.

[17] J. Cantarella, D. De Truck, H. Gluc, and M. Twytel, J. Math. Phys. 41, 5615 (2000).

[18] J. D. Bekenstein, Astroph. J. 319, 207 (1987).

[19] M. Tsamparlis and D. P. Mason, J. Math. Phys. 24, 1577 (1983).

[20] T. Zwerger and E. Müller Astron. Astroph. 320, 209 (1997).

[21] C. D. Ott, A. Burrows, E. Livine, and R. Walder, Astroph. J. 600, 834 (2004).

[22] M. D. Duez, Y. T. Liu, S. L. Shapiro, M. Shibata, and B. C. Stephens, Phys. Rev. D73, 104015 (2006).

[23] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, Freeman, San Francisco 1973.

[24] B. Carter and I. M. Khalatnikov, Ann. Phys. 219, 243 (1992).

[25] G. Prasad, Rigidly Connected Magnetic lines: Twisting and Winding of magnetic lines, to be submitted for publication in Gen. Rel. Grav. (2016).

[26] E. N. Glass, J. Math. Phys. 18, 708 (1977).

[27] R. Birkl, N. Stergioulas, and E. Müller, Phys. Rev. D84, 023003 (2011)

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