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Chapter 2

Oscillation Criteria for Second-Order Neutral Damped Differential Equations with Delay Argument

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Additional information is available at the end of the chapter

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Abstract

The chapter is devoted to study the oscillation of all solutions to second-order nonlinear neutral damped differential equations with delay argument. New oscillation criteria are obtained by employing a refinement of the generalized Riccati transformations and integral averaging techniques.

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1. Introduction

In the chapter, we are mainly concerned with the oscillatory behavior of solutions to second-order nonlinear neutral damped differential equations with delay argument of the form

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + p(t)\left(z'(t)\right)^{\alpha} + q(t)f\left(x(\sigma(t))\right) = 0, \quad t \geq t_0, \tag{1}$$

where $\alpha \geq 1$ is a quotient of positive odd integers and

$$z(t) = x(t) + a(t)x(\tau(t)). \tag{2}$$

Throughout, we suppose that the following hypotheses hold:

i. $r,\ p,\ q \in C(\mathscr{I}, \mathbb{R}^+)$, where $\mathscr{I} = [t_0, \infty)$ and $\mathbb{R}^+ = (0,\infty)$;

ii. $a \in C(\mathscr{I},\ \mathbb{R})$, $0 \leq a\ (t) \leq 1$;

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iii. $\tau \in C(\mathcal{I}, \mathbb{R})$, $\tau(t) \leq t$, $\tau(t) \to \infty$ as $t \to \infty$;

iv. $\sigma \in C^1(\mathcal{I}, \mathbb{R})$, $\sigma(t) \leq t$, $\sigma'(t) \geq 0$, $\sigma(t) \to \infty$ as $t \to \infty$;

v. $f \in C(\mathbb{R}, \mathbb{R})$, such that $xf(x) > 0$ and $f(x)/x^\beta \geq k > 0$ for $x \neq 0$, where k is a constant and β is the ratio of odd positive integers.

By a solution of Eq. (1), we mean a nontrivial real-valued function $x(t)$, which has the property $z(t) \in C^1([T_x, \infty))$, $r(t)\left(z'(t)\right)^\alpha \in C^1([T_x, \infty))$, $T_x \geq t_0$, and satisfies Eq. (1) on $[T_x, \infty)$. In the sequel, we will restrict our attention to those solutions $x(t)$ of Eq. (1) that satisfy the condition

$$\sup\{|x(t)| : T \leq t < \infty\} > 0 \quad \text{for} \quad T \geq T_x. \tag{3}$$

We make the standing hypothesis that Eq. (1) admits such a solution. As is customary, a solution of Eq. (1) is said to be oscillatory if it is neither eventually positive nor eventually negative on $[T_x, \infty)$ and otherwise, it is termed nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Remark 1. All the functional inequalities considered in the sequel are assumed to hold eventually, that is, they are satisfied for all t large enough.

Oscillation theory was created in 1836 with a paper of Jacques Charles François Sturm published in *Journal des Mathematiqués Pures et Appliqueés*. His long and detailed memoir [1] was one of the first contributions in Liouville's newly founded journal and initiated a whole new research into the qualitative analysis of differential equations. Heretofore, the theory of differential equations was primarily about finding solutions of a given equation and so was very limited. Contrarily, the main idea of Sturm was to obtain geometric properties of solutions (such as sign changes, zeros, boundaries, and oscillation) directly from the differential equation, without benefit of solutions themselves.

Henceforth, the oscillation theory for ordinary differential equations has undergone a significant development. Nowadays, it is considered as coherent, self-contained domain in the qualitative theory of differential equations that is turning mainly toward the study of solution properties of functional differential equations (FDEs).

The problem of obtaining sufficient conditions for asymptotic and oscillatory properties of different classes of FDEs has experienced long-term interest of many researchers. This is caused by the fact that differential equations, especially those with deviating argument, are deemed to be adequate in modeling of the countless processes in all areas of science. For a summary of the most significant efforts and recent findings in the oscillation theory of FDEs and vast bibliography therein, we refer the reader to the excellent monographs [2–6].

In a neutral delay differential equation the highest-order derivative of the unknown function appears both with and without delay. The study of qualitative properties of solutions of such equations has, besides its theoretical interest, significant practical importance. This is due to the fact that neutral differential equations arise in various phenomena including problems concerning electric networks containing lossless transmission lines (as in high-speed computers where such lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar or in the solution of variational problems with time delays. We refer the reader to the monograph [7] for further applications in science and technology.

So far, most of the results obtained in the literature has centered around the special *undamped* form of Eq. (1), i.e., when $p(t) = 0$ (for example, see Refs. [8–18]). For instance, in one of the pioneering works on the subject, Grammatikopoulos et al. [8] studied the second-order neutral differential equation with constant delay of the form

$$(x(t) + a(t)x(t-\tau))'' + q(t)x(t-\tau) = 0 \tag{4}$$

and proved that Eq. (4) is oscillatory if

$$\int_{t_0}^{\infty} q(s)\Big(1-a(s-\tau)\Big)\mathrm{d}s = \infty. \tag{5}$$

Later on, Grace and Lalli [9] extended the results from [8] to the more general equation

$$\Big(r(t)(x(t) + a(t)x(t-\tau)')\Big)' + q(t)f\Big(x(t-\tau)\Big) = 0, \tag{6}$$

with

$$\frac{f(x)}{x} \geq k, \quad k > 0 \quad \text{and} \quad \int_{t_0}^{\infty} \frac{\mathrm{d}s}{r(s)} = \infty \tag{7}$$

and showed that Eq. (6) is oscillatory if there exists a continuously differentiable function $\rho(t)$ such that

$$\int_{t_0}^{\infty}\left(\rho(s)q(s)(1-a(s-\tau)) - \frac{\left(\rho'(s)\right)^2 r(s-\tau)}{4k\rho(s)}\right)\mathrm{d}s = \infty. \tag{8}$$

In Ref. [10], Dong has involved to study the oscillation problem for a half-linear case of Eq. (1) and by defining a sequence of continuous functions has obtained various kinds of better results. Afterward, his approach has been further developed by several authors, see, e.g., [11–14]. However, it appears that very little is known regarding the oscillation of Eq. (1) with $p(t) \neq 0$ and $\alpha \neq \beta$. Motivated by the results of Ref. [10], this chapter presents some new oscillation criteria, which are applicable on Eq. (1).

On the other hand, Eq. (1) can be considered as a natural generalization of the second-order delay differential equation of the form

$$\left(r(t)\left(x'(t)\right)^{\alpha}\right) + p(t)\left(x'(t)\right)^{\alpha} + q(t)f\Big(x(\sigma(t))\Big) = 0. \tag{9}$$

Very recently, the authors of [19] studied the oscillation problem of Eq. (9) with $p(t) = 0$ and $\alpha = \beta$. Their ideas, which are based on careful investigation of classical techniques covering

Riccati transformations and integral averages, will be extended to the more general equation (1).

2. Main results

For the simplicity and without further mention, we use the following notations:

$$A(t) = \exp\left(-\int_{t_0}^{t} \frac{p(s)}{r(s)} \mathrm{d}s\right), \quad Q(t) = kq(t)\left(1-a(\sigma(t))\right)^{\beta}, \tag{10}$$

$$R(t) = \int_{t}^{\infty} \left(\frac{A(s)}{r(s)}\right)^{\frac{1}{\alpha}} \mathrm{d}s, \quad \tilde{Q}(t) = q(t)\left(1-a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\beta}, \tag{11}$$

$$P(t) = \frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}, \quad \tilde{q}(t) = Q(t) + \frac{p(t)A(t)}{r(t)} \int_{t}^{\infty} \frac{Q(s)}{A(s)} \mathrm{d}s, \tag{12}$$

where $\phi(t) \in C^1(\mathscr{I}, \mathbb{R})$ is a given function and will be specified later.

The organization of this chapter is as follows. Before stating our main results, we present two lemmas that ensure that any solution $x(t)$ of Eq. (1) satisfies the condition

$$z(t) > 0, \quad z'(t) > 0, \quad \left(r(t)\left(z'(t)\right)^{\alpha}\right)' < 0, \tag{13}$$

for t sufficiently large. Next, we get our main oscillation results for Eq. (1) by employing the generalized Riccati transformations and integral averaging techniques. We base our arguments on the assumption that the function $P(t)$ is positive or negative.

Lemma 1. Assume that

$$\int_{t_0}^{\infty} \left(\frac{A(s)}{r(s)}\right)^{\frac{1}{\alpha}} \mathrm{d}s = \infty \tag{14}$$

holds and Eq. (1) has a positive solution $x(t)$ on $\mathscr{I}$. Then there exists a $T \in \mathscr{I}$, sufficiently large, such that

$$z(t) > 0, \quad z'(t) > 0, \quad \left(r(t)\left(z'(t)\right)^{\alpha}\right)' < 0, \tag{15}$$

on $[T, \infty)$.

Proof. Since, $x(t)$ is a positive solution of Eq. (1) on $\mathscr{I}$, then, by the assumptions (iii) and (iv), there exists a $t_1 \in \mathscr{I}$ such that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ on $[t_1, \infty)$. Define the function $z(t)$ as in Eq. (2). Then it is easy to see that $z(t) \geq x(t) > 0$, for $t \geq t_1$, and at the same time, from Eq. (1), we get

$$\left(r(t)\left(z^{'}(t)\right)^{\alpha}\right)^{'}+p(t)\left(z^{'}(t)\right)^{\alpha}\quad=-q(t)f\left(x(\sigma(t))\right)<0. \tag{16}$$

We assert that $\frac{r(t)}{A(t)}\left(z^{'}(t)\right)^{\alpha}$ is decreasing. Clearly, by writing the left-hand side of Eq. (16) in the form

$$\left(r(t)\left(z^{'}(t)\right)^{\alpha}\right)^{'}+\frac{p(t)}{r(t)}r(t)\left(z^{'}(t)\right)^{\alpha}<0, \tag{17}$$

we get

$$\left(\frac{r(t)}{A(t)}\left(z^{'}(t)\right)^{\alpha}\right)^{'}=-\frac{q(t)}{A(t)}f(x(\sigma(t)))<0 \tag{18}$$

and so the assertion is proved.

Now, we claim that $z^{'}(t)>0$ on $[t_1,\infty)$. If not, then there exists $t_2\in[t_1,\infty)$ such that $z^{'}(t_2)<0$. Using the fact that $\frac{r(t)}{A(t)}\left(z^{'}(t)\right)^{\alpha}$ is decreasing, we obtain, for $t\geq t_2$,

$$\frac{r(t)}{A(t)}\left(z^{'}(t)\right)^{\alpha}<c:=\frac{r(t_2)}{A(t_2)}\left(z^{'}(t_2)\right)^{\alpha}<0. \tag{19}$$

Integrating the above inequality from t_2 to t, we find that

$$z(t)<z(t_2)+c^{\frac{1}{\alpha}}\int_{t_2}^{t}\left(\frac{A(s)}{r(s)}\right)^{\frac{1}{\alpha}}\mathrm{d}s \tag{20}$$

for $t\geq t_2$. By condition (14), $z(t)$ approaches to $-\infty$ as $t\to\infty$, which contradicts the fact that $z(t)$ is eventually positive. Therefore, $z^{'}(t)>0$ and from Eq. (1), we have that $\left(r(t)\left(z^{'}(t)\right)^{\alpha}\right)^{'}<0$. The proof is complete.

Lemma 2. Assume that

$$\int_{t_0}^{\infty}\left(\frac{A(u)}{r(u)}\int_{t_0}^{u}\frac{\tilde{Q}(s)R^{\beta}(\sigma(s))}{A(s)}\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}u=\infty, \tag{21}$$

holds and Eq. (1) has a positive solution $x(t)$ on $\mathcal{I}$. Then there exists $T\in\mathcal{I}$, sufficiently large, such that

$$z(t)>0,\quad z^{'}(t)>0,\quad \left(r(t)\left(z^{'}(t)\right)^{\alpha}\right)^{'}<0, \tag{22}$$

on $[T,\infty)$.

Proof. Similarly to the proof of Lemma 1, we assume that there exists $t_2 \in \mathscr{I}$ such that $z'(t) < 0$ on $[t_2, \infty)$. Taking Eq. (18) into account, we have

$$z'(s) \leq \left(\frac{r(t)}{A(t)}\frac{A(s)}{r(s)}\right)^{\frac{1}{\alpha}} z'(t), \tag{23}$$

for $s \geq t \geq t_2$. Integrating the above inequality from t to t', $t' \geq t \geq t_2$, we get

$$z(t') \leq z(t) + \left(\frac{r(t)}{A(t)}\right)^{\frac{1}{\alpha}} z'(t) \int_t^{t'} \left(\frac{r(s)}{A(s)}\right)^{-\frac{1}{\alpha}} \mathrm{d}s. \tag{24}$$

Letting $t' \to \infty$, we have

$$z(t) \geq -R(t)\left(\frac{r(t)}{A(t)}\right)^{\frac{1}{\alpha}} z'(t), \tag{25}$$

which yields

$$\left(\frac{z(t)}{R(t)}\right) \geq 0 \tag{26}$$

and hence we see that $\frac{z(t)}{R(t)}$ is nondecreasing. By Eq. (2) and (iii), we have

$$\begin{aligned} x(t) &= z(t) - a(t)x(\tau(t)) \\ &\geq z(t) - a(t)z(\tau(t)) \\ &\geq \left(1 - a(t)\frac{R(\tau(t))}{R(t)}\right) z(t), \end{aligned} \tag{27}$$

which together with Eq. (1) and the assumption (v) yields

$$\begin{aligned} \left(r(t)\left(z'(t)\right)^{\alpha}\right)' + p(t)\left(z'(t)\right)^{\alpha} &\leq -kq(t)\left(1 - a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\beta} z^{\beta}(\sigma(t)) \\ &= -k\tilde{Q}(t) z^{\beta}(\sigma(t)). \end{aligned} \tag{28}$$

On the other hand, from Eq. (23), we have

$$\frac{r(t)\left(z'(t)\right)^{\alpha}}{A(t)} \leq \frac{r(t_2)\left(z'(t_2)\right)^{\alpha}}{A(t_2)}, \tag{29}$$

that is,

$$\frac{r(t)}{A(t)}\left(-z'(t)\right)^{\alpha} \geq \frac{r(t_2)}{A(t_2)}\left(-z'(t_2)\right)^{\alpha} := \gamma^{\alpha} \tag{30}$$

for some positive constant γ. Setting Eq. (30) into Eq. (25), we obtain

$$z(t) \geq \gamma R(t) \tag{31}$$

and so, Eq. (28) becomes

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + p(t)\left(z'(t)\right)^{\alpha} \leq -\tilde{\gamma}\tilde{Q}(t)R^{\beta}(\sigma(t)), \tag{32}$$

where $\tilde{\gamma} := k\gamma^{\beta}$. Now, if we define the function

$$U(t) = r(t)\left(-z'(t)\right)^{\alpha} > 0, \tag{33}$$

then

$$U'(t) + \frac{p(t)}{r(t)}U(t) \geq \tilde{\gamma}\tilde{Q}(t)R^{\beta}(\sigma(t)), \tag{34}$$

or, equally

$$\left(\frac{U(t)}{A(t)}\right)' \geq \tilde{\gamma}\frac{\tilde{Q}(t)R^{\beta}(\sigma(t))}{A(t)}. \tag{35}$$

Integrating the above inequality from t_2 to t, we get

$$U(t) \geq \tilde{\gamma}A(t)\int_{t_2}^{t}\frac{\tilde{Q}(s)R^{\beta}(\sigma(s))}{A(s)}\mathrm{d}s \tag{36}$$

or

$$r(t)\left(-z'(t)\right)^{\alpha} \geq \tilde{\gamma}A(t)\int_{t_2}^{t}\frac{\tilde{Q}(s)R^{\beta}(\sigma(s))}{A(s)}\mathrm{d}s. \tag{37}$$

It follows from this last inequality that

$$0 < z(t) \leq z(t_2) - \tilde{\gamma}\int_{t_2}^{t}\left(\frac{A(u)}{r(u)}\int_{t_2}^{u}\frac{\tilde{Q}(s)R^{\beta}(\sigma(s))}{A(s)}\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}u \tag{38}$$

for $t \geq t_2$. As $t \to \infty$, then by condition Eq. (21), $z(t)$ approaches to $-\infty$, which contradicts the fact that $z(t)$ is eventually positive. Therefore, $z'(t) > 0$ and from Eq. (1), we have $\left(r(t)\left(z'(t)\right)^{\alpha}\right)' < 0$. The proof is complete.

Lemma 3. Assume that

$$\int_{t_0}^{\infty}\left(\frac{A(u)}{r(u)}\int_{t_0}^{u}\frac{\tilde{Q}(s)}{A(s)}\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}u = \infty, \tag{39}$$

holds and Eq. (1) has a positive solution $x(t)$ on $\mathcal{I}$. Then there exists $T \in \mathcal{I}$, sufficiently large, such that either

$$z(t) > 0, \quad z^{'}(t) > 0, \quad \left(r(t)\left(z^{'}(t)\right)^{\alpha}\right)^{'} < 0, \tag{40}$$

on $[T, \infty)$ or $\lim_{t\to\infty} x(t) = 0$.

Proof. As in the proof of Lemma 1, we assume that there exists $t_2 \in \mathcal{I}$ such that $z^{'}(t) < 0$ on $[t_2, \infty)$. So, $z(t)$ is decreasing and

$$\lim_{t\to\infty} z(t) =: b \geq 0 \tag{41}$$

exists. Therefore, there exists $t_3 \in [t_2, \infty)$ such that

$$z(\sigma(t)) > z(t) \geq b > 0. \tag{42}$$

As in the proof of Lemma 2, we obtain Eq. (27), i.e.,

$$\begin{aligned} x(\sigma(t)) &\geq \left(1 - a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right) z(\sigma(t)) \\ &\geq b\left(1 - a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right), \quad \text{for} \quad t \geq t_3. \end{aligned} \tag{43}$$

Thus,

$$\begin{aligned} \left(r(t)\left(z^{'}(t)\right)^{\alpha}\right)^{'} + p(t)\left(z^{'}(t)\right)^{\alpha} &\leq -\tilde{b}\, q(t)\left(1 - a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\beta} \\ &= -\tilde{b}\tilde{Q}(t), \end{aligned} \tag{44}$$

where $\tilde{b} := kb^{\beta}$.

Define the function $U(t)$ as in Eq. (103). Then Eq. (44) becomes

$$\left(\frac{U(t)}{A(t)}\right)^{'} \geq \tilde{b}\,\frac{\tilde{Q}(t)}{A(t)}. \tag{45}$$

Integrating the above inequality twice from t_3 to t, one gets

$$0 < z(t) \leq z(t_3) - \tilde{b}\int_{t_3}^{t}\left(\frac{A(u)}{r(u)}\int_{t_3}^{u}\frac{\tilde{Q}(s)}{A(s)}\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}u, \tag{46}$$

for $t \geq t_3$. As $t \to \infty$, then by condition (39), $z(t)$ approaches to $-\infty$, which contradicts the fact that $z(t)$ is eventually positive. Thus, $b = 0$ and from $0 \leq x(t) \leq z(t)$, we see that $\lim_{t\to\infty} x(t) = 0$. The proof is complete.

Using results of Lemmas 1 and 2, we can obtain the following oscillation criteria for Eq. (1).

Theorem 1. Let conditions (i)–(v) and one of the conditions (14) or (21) hold. Furthermore, assume that there exists a positive continuously differentiable function $\phi(t)$ such that, for all sufficiently large, T, $T_1 \geq T$,

$$P(t) \geq 0 \tag{47}$$

on $[T, \infty)$ and

$$\limsup_{t\to\infty}\left\{\phi(t)A(t)\int_t^{\infty}\frac{Q(s)}{A(s)}\mathrm{d}s \quad +\int_{T_1}^{t}\left[\phi(s)Q(s)-\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\phi(s)r(\sigma(s))\left(P(s)\right)^{\alpha+1}}{\left(\beta\sigma'(s)\psi(s)\right)^{\alpha}}\right]\mathrm{d}s\right\}=\infty, \tag{48}$$

where

$$\psi(t)=\begin{cases} c_1, & c_1 \text{ is some positive constant if } \beta>\alpha \\ 1, & \text{if } \beta=\alpha \\ c_2\left(\int_T^t r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^{\frac{\beta-\alpha}{\alpha}}, & c_2 \text{ is some positive constant if } \beta<\alpha. \end{cases} \tag{49}$$

Then, Eq. (1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (1). Then, without loss of generality, we may assume that there exists $T\in\mathcal{I}$ large enough, so that $x(t)$ satisfies the conclusions of Lemma 1 or 2 on $[T, \infty)$ with

$$x(t)>0, \quad x(\tau(t))>0, \quad x(\sigma(t))>0 \tag{50}$$

on $[T, \infty)$. In particular, we have

$$z(t)>0, \quad z'(t)>0, \quad \left(r(t)\left(z'(t)\right)^{\alpha}\right)'<0, \quad \text{for} \quad t\geq T. \tag{51}$$

By Eq. (2) and the assumption (iii), we get

$$\begin{aligned} x(t) &= z(t)-a(t)x(\tau(t)) \\ &\geq z(t)-a(t)z(\tau(t)) \\ &\geq (1-a(t))z(t), \end{aligned} \tag{52}$$

which together with Eq. (1) implies

$$\begin{aligned} \left(r(t)\left(z'(t)\right)^{\alpha}\right)'+\frac{p(t)}{r(t)}\left(z'(t)\right)^{\alpha} &\leq -kq(t)\left(1-a(\sigma(t))\right)^{\beta}z^{\beta}(\sigma(t)) \\ &= -Q(t)z^{\beta}(\sigma(t)). \end{aligned} \tag{53}$$

We consider the generalized Riccati substitution

$$w(t)=\phi(t)\frac{r(t)\left(z'(t)\right)^{\alpha}}{z^{\beta}(\sigma(t))}>0, \quad \text{for} \quad t\geq T. \tag{54}$$

As in the proof of Lemma 1, we get Eq. (18), which in view of the assumption (v) yields

$$\left(\frac{r(t)}{A(t)}\left(z'(t)\right)^{\alpha}\right)' \leq -\frac{Q(t)}{A(t)} z^{\beta}(\sigma(t)). \tag{55}$$

Integrating Eq. (55) from t to ∞ and using the fact that $z(t)$ is increasing, we have

$$\begin{aligned} \frac{r(t)}{A(t)}\left(z'(t)\right)^{\alpha} &\geq \int_t^{\infty} \frac{Q(s)}{A(s)} z^{\beta}(\sigma(s))\mathrm{d}s \\ &\geq z^{\beta}(\sigma(t)) \int_t^{\infty} \frac{Q(s)}{A(s)}\mathrm{d}s. \end{aligned} \tag{56}$$

So it follows from Eq. (56) and the definition (54) of $w(t)$ that

$$w(t) = \phi(t)\frac{r(t)\left(z'(t)\right)^{\alpha}}{z^{\beta}(\sigma(t))} \geq \phi(t)A(t)\int_t^{\infty} \frac{Q(s)}{A(s)}\mathrm{d}s. \tag{57}$$

By Eq. (53) we can easily prove that

$$\begin{aligned} w'(t) &= \left(r(t)\left(z'(t)\right)^{\alpha}\right)' \frac{\phi(t)}{z^{\beta}(\sigma(t))} + \left(\frac{\phi(t)}{z^{\beta}(\sigma(t))}\right)' r(t)\left(z'(t)\right)^{\alpha} \\ &\leq -\frac{\phi(t)}{z^{\beta}(\sigma(t))}\left(p(t)\left(z'(t)\right)^{\beta} + Q(t)z^{\beta}(\sigma(t))\right) \\ &\quad + r(t)\left(z'(t)\right)^{\alpha}\left(\frac{\phi'(t)}{z^{\beta}(\sigma(t))} - \frac{\phi(t)\left(z^{\beta}(\sigma(t))\right)}{z^{\beta+1}(\sigma(t))}\right) \\ &\leq -\phi(t)Q(t) + w(t)\left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right) \\ &\quad -\beta\phi(t)\frac{r(t)(z'(t)^{\beta}z'(\sigma(t)\sigma'(t)}{z^{\beta+1}(\sigma(t))}. \end{aligned} \tag{58}$$

On the other hand, since $r(t)\left(z'(t)\right)^{\alpha}$ is decreasing, we have

$$\frac{z'(\sigma(t))}{z'(t)} \geq \left(\frac{r(t)}{r(\sigma(t))}\right)^{\frac{1}{\alpha}} \tag{59}$$

and thus Eq. (58) becomes

$$\begin{aligned} w'(t) \leq &\; -\phi(t)Q(t) + P(t)w(t) \\ &\; -\frac{\beta\phi(t)\sigma'(t)}{r^{\frac{1}{\alpha}}(\sigma(t))}\left(\frac{w(t)}{\phi(t)}\right)^{\frac{\alpha+1}{\alpha}} z^{\frac{\beta-\alpha}{\alpha}}(\sigma(t)). \end{aligned} \tag{60}$$

Now, we consider the following three cases:

Case I: $\beta > \alpha$.

In this case, since $z'(t) > 0$ for $t \geq T$, then there exists $T_1 \geq T$ such that $z(\sigma(t)) \geq c$ for $t \geq T_1$. This implies that

$$z^{\frac{\beta-\alpha}{\alpha}}(\sigma(t)) \geq c^{\frac{\beta-\alpha}{\alpha}} := c_1 \tag{61}$$

Case II: $\beta = \alpha$.

In this case, we see that $z^{\frac{\beta-\alpha}{\alpha}}(\sigma(t)) = 1$.

Case III: $\beta < \alpha$.

Since $r(t)\left(z'(t)\right)^{\alpha}$ is decreasing, there exists a constant d such that

$$r(t)\left(z'(t)\right)^{\alpha} \leq d \tag{62}$$

for $t \geq T$. Integrating the above inequality from T to t, we have

$$z(t) \leq z(T) + \int_T^t \left(\frac{d}{r(s)}\right)^{\frac{1}{\alpha}} \mathrm{d}s. \tag{63}$$

Hence, there exists $T_1 \geq T$ and a constant d_1 depending on d such that

$$z(t) \leq d_1 \int_T^t r^{-\frac{1}{\alpha}}(s)\mathrm{d}s, \quad \text{for} \quad t \geq T_1 \tag{64}$$

and thus

$$z^{\frac{\beta-\alpha}{\alpha}}(\sigma(t)) \geq d_1^{\frac{\beta-\alpha}{\alpha}} \left(\int_T^t r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^{\frac{\beta-\alpha}{\alpha}} = d_2 \left(\int_T^t r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^{\frac{\beta-\alpha}{\alpha}} \tag{65}$$

for some positive constant d_2.

Using these three cases and the definition of $\psi(t)$, we get

$$w'(t) \leq -\phi(t)Q(t) + P(t)w(t) - \frac{\beta\sigma'(t)\psi(t)}{\left(\phi(t)r(\sigma(t))\right)^{\frac{1}{\alpha}}} w^{\frac{1+\alpha}{\alpha}}(t) \tag{66}$$

for $t \geq T_1 \geq T$. Setting

$$A := P(t), \tag{67}$$

$$B := \frac{\beta\sigma'(t)\psi(t)}{\left(\phi(t)r(\sigma(t))\right)^{\frac{1}{\alpha}}}, \tag{68}$$

and using the inequality

$$Au - Bu^{\frac{1+\alpha}{\alpha}} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A^{\alpha+1}}{B^\alpha}, \tag{69}$$

we obtain

$$w'(t) \le -\phi(t)Q(t) + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\phi(t)r(\sigma(t))\left(P(t)\right)^{\alpha+1}}{\left(\beta\sigma'(t)\psi(t)\right)^\alpha}. \tag{70}$$

Integrating the above inequality from T_1 to t, we have

$$w(t) \le w(T_1) - \int_{T_1}^{t} \left(\phi(s)Q(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\phi(s)r(\sigma(s))\left(P(s)\right)^{\alpha+1}}{\left(\beta\sigma'(s)\psi(s)\right)^\alpha}\right) \mathrm{d}s. \tag{71}$$

Taking Eq. (57) into account, we get

$$\begin{aligned} w(T_1) \quad & \ge \phi(t)A(t)\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s \\ & + \int_{T_1}^{t} \left(\phi(s)Q(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\phi(s)r(\sigma(s))\left(P(s)\right)^{\alpha+1}}{\left(\beta\sigma'(s)\psi(s)\right)^\alpha}\right) \mathrm{d}s. \end{aligned} \tag{72}$$

Taking the lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction to the condition (48). This completes the proof.

Remark 2. Note that the presence of the term $\phi(t)A(t)\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s$ in Eq. (57) improves a number of related results in, e.g., [9, 13–18, 20].

Setting $\phi(t) = t$ in Eq. (57), then the following corollary becomes immediate.

Corollary 1. Let conditions (i)–(v) and one of the conditions (14) or (21) hold. Assume that, for all sufficiently large, T, $T_1 \ge T$,

$$tp(t) \le r(t) \tag{73}$$

on $[T, \infty)$ and

$$\limsup_{t\to\infty} \left\{ tA(t)\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s \quad + \int_{T_1}^{t} \left[sQ(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{sr(\sigma(s))\left(\frac{1}{s} - \frac{p(s)}{r(s)}\right)^{\alpha+1}}{\left(\beta\sigma'(s)\psi(s)\right)^\alpha} \right] \mathrm{d}s \right\} = \infty, \tag{74}$$

where $\psi(t)$ is as in Theorem 1. Then Eq. (1) is oscillatory.

Corollary 2. Assume that the conditions (39) and (74) hold. Then Eq. (1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Next, we present some complementary oscillation results for Eq. (1) by using an integral averaging technique due to Philos. We need the class of functions F. Let

$$\mathcal{D}_0 = \{(t, s) : t > s \geq t_0\} \quad \text{and} \quad \mathcal{D} = \{(t, s) : t > s \geq t_0\} \tag{75}$$

The function $H(t, s) \in C(\mathcal{D}, \mathbb{R})$ is said to belong to a class F if

(a) $H(t, t) = 0$ for $t \geq T$, $H(t, s) > 0$ for $(t, s) \in \mathcal{D}_0$

(b) $H(t, s)$ has a continuous and nonpositive partial derivative on $\mathcal{D}_0$ with respect to the second variable such that

$$\frac{\partial}{\partial s}\Big(H(t, s)\phi(s)\Big) - H(t, s)\frac{\phi(s)p(s)}{r(s)} = -h(t, s)\Big(H(t, s)\phi(s)\Big)^{\frac{\alpha}{\alpha+1}} \tag{76}$$

for all $(t, s) \in \mathcal{D}_0$.

Theorem 2. Let conditions (i)–(v) and one of the conditions (14) or (21) hold. Furthermore, assume that there exist functions $H(t, s)$, $h(t, s) \in F$ such that, for all sufficiently large, T, for $T_1 \geq T$,

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{H(t, T_1)}\int_{T_1}^{t} & \Big(H(t, s)\Big(\phi(s)Q(s) + \rho(s)\phi(s)p(s)\Big) \\ & - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{h^{\alpha+1}(t, s)r(\sigma(s))}{\beta^\alpha\Big(\sigma'(s)\psi(s)\Big)^\alpha}\Big)ds = \infty\end{aligned} \tag{77}$$

where $\phi(t)$ and $\rho(t)$ are continuously differentiable functions and $\psi(t)$ is as in Theorem 1. Then Eq. (1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (1). Then, without loss of generality, we may assume that there exists $T \in \mathcal{I}$ large enough, so that $x(t)$ satisfies the conclusions of Lemma 1 or 2 on $[T, \infty)$ with

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \tag{78}$$

on $[T, \infty)$. In particular, we have

$$z(t) > 0, \quad z'(t) > 0, \quad \Big(r(t)\Big(z'(t)\Big)^\alpha\Big)' < 0, \quad \text{for} \quad t \geq T. \tag{79}$$

Define the function $w(t)$ as

$$w(t) = \phi(t)r(t)\left(\frac{\Big(z'(t)\Big)^\alpha}{z^\beta(\sigma(t))} + \rho(t)\right) \geq \phi(t)r(t)\rho(t), \tag{80}$$

where $\rho(t) \in C^1(\mathcal{I}, \mathbb{R})$. Similarly to the proof of Theorem 1, we obtain the inequality

$$\begin{aligned} w'(t) \le \; & -\phi(t)Q(t) + \phi(t)\Big(r(t)\rho(t)\Big)' + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)w(t) \\ & - \frac{\beta\sigma'(t)\psi(t)}{\Big(\phi(t)r(\sigma(t))\Big)^{\frac{1}{\alpha}}}\Big(w(t) - \phi(t)r(t)\rho(t)\Big)^{\frac{1+\alpha}{\alpha}}. \end{aligned} \tag{81}$$

Multiplying Eq. (81) by $H(t, s)$, integrating with respect to s from T_1 to t for $t \ge T_1 \ge T$, and using (a) and (b), we find that

$$\begin{aligned} & \int_{T_1}^{t} H(t, s)\phi(s)\Big(Q(s) - \Big(r(s)\rho(s)\Big)\Big)' \mathrm{d}s \\ & \le -\int_{T}^{t} H(t, s)w'(s)\mathrm{d}s + \int_{T_1}^{t} H(t, s)\left(\frac{\phi'(s)}{\phi(s)} - \frac{p(s)}{r(s)}\right)w(s)\mathrm{d}s \\ & - \int_{T_1}^{t} \frac{\beta H(t, s)\sigma'(s)\psi(s)}{\Big(\phi(s)r(\sigma(s))\Big)^{\frac{1}{\alpha}}}\Big(w(s) - \phi(s)r(s)\rho(s)\Big)^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \\ & = H(t, s)w(s)\Big|^{T_1 t} + \int_{T_1}^{t}\left(\frac{\partial}{\partial s}H(t, s) + H(t, s)\left(\frac{\phi'(s)}{\phi(s)} - \frac{p(s)}{r(s)}\right)\right)w(s)\mathrm{d}s \\ & - \int_{T_1}^{t} \frac{\beta H(t, s)\sigma'(s)\psi(s)}{\Big(\phi(s)r(\sigma(s))\Big)^{\frac{1}{\alpha}}}\Big(w(s) - \phi(s)r(s)\rho(s)\Big)^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \\ & = H(t, T_1)w(T_1) + \int_{T_1}^{t} -\frac{h(t, s)}{\phi(s)}\Big(H(t, s)\phi(s)\Big)^{\frac{\alpha}{\alpha+1}}w(s)\mathrm{d}s \\ & - \int_{T_1}^{t} \frac{\beta H(t, s)\sigma'(s)\psi(s)}{\Big(\phi(s)r(\sigma(s))\Big)^{\frac{1}{\alpha}}}\Big(w(s) - \phi(s)r(s)\rho(s)\Big)^{\frac{1+\alpha}{\alpha}}\mathrm{d}s \end{aligned} \tag{82}$$

Setting

$$A := -\frac{h(t, s)}{\phi(s)}[H(t, s)\phi(s)]^{\frac{\alpha}{\alpha+1}}, \quad B := \frac{\beta H(t, s)\sigma'(s)\psi(s)}{\Big(\phi(s)r(\sigma(s))\Big)^{\frac{1}{\alpha}}} \tag{83}$$

and

$$C := \phi(s)r(s)\rho(s) \tag{84}$$

and using the inequality

$$Au - B(u - C)^{\frac{1+\alpha}{\alpha}} \le AC + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{A^{\alpha+1}}{B^{\alpha}}, \tag{85}$$

we obtain

$$\begin{aligned} & \int_{T_1}^{t} H(t, s)\phi(s)\Big(Q(s) - \Big(r(s)\rho(s)\Big)'\Big)\mathrm{d}s \\ & \le H(t, T_1)w(T_1) + \int_{T_1}^{t} -h(t, s)r(s)\rho(s)[H(t, s)\phi(s)]^{\frac{\alpha}{\alpha+1}}\mathrm{d}s \\ & + \int_{T_1}^{t} \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{h^{\alpha+1}(t, s)r(\sigma(s))}{\beta^{\alpha}\Big(\sigma'(s)\psi(s)\Big)^{\alpha}}\mathrm{d}s \end{aligned} \tag{86}$$

Thus,

$$\begin{aligned} H(t,\,T_1)w(T_1) \quad &\geq \int_{T_1}^{t} H(t,\,s)\phi(s)\Big(Q(s)-\big(r(s)\rho(s)\big)'\Big)\mathrm{d}s \\ &\quad + \int_{T_1}^{t} h(t,\,s)r(s)\rho(s)[H(t,\,s)\phi(s)]^{\frac{\alpha}{\alpha+1}}\mathrm{d}s \\ &\quad -\int_{T_1}^{t} \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{h^{\alpha+1}(t,\,s)r(\sigma(s))}{\beta^\alpha\big(\sigma'(s)\psi(s)\big)^\alpha}\mathrm{d}s. \end{aligned} \tag{87}$$

That is,

$$\begin{aligned} &H(t,\,T_1)w(T_1) \\ &\geq \int_{T_1}^{t} H(t,\,s)\phi(s)\Big(Q(s)-\big(r(s)\rho(s)\big)'\Big)\mathrm{d}s \\ &+ \int_{T_1}^{t} -r(s)\rho(s)\left(\frac{\partial}{\partial s}\big(H(t,\,s)\phi(s)\big)-H(t,\,s)\frac{\phi(s)p(s)}{r(s)}\right)\mathrm{d}s \\ &-\int_{T_1}^{t} \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{h^{\alpha+1}(t,\,s)r(\sigma(s))}{\beta^\alpha\big(\sigma'(s)\psi(s)\big)^\alpha}\mathrm{d}s \\ &= \int_{T_1}^{t} H(t,\,s)\Big(\phi(s)Q(s)+\rho(s)\phi(s)p(s)\Big)\mathrm{d}s \\ &-H(t,\,s)\phi(s)r(s)\rho(s)\Big|_{T_1}^{t} - \int_{T_1}^{t} \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{h^{\alpha+1}(t,\,s)r(\sigma(s))}{\beta^\alpha\big(\sigma'(s)\psi(s)\big)^\alpha}\mathrm{d}s \end{aligned} \tag{88}$$

It follows that

$$\begin{aligned} &\int_{T_1}^{t} H(t,\,s)\Big(\phi(s)Q(s)+\rho(s)\phi(s)p(s)\Big)\mathrm{d}s \\ -\quad &\int_{T_1}^{t} \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{h^{\alpha+1}(t,\,s)r(\sigma(s))}{\beta^\alpha\big(\sigma'(s)\psi(s)\big)^\alpha}\mathrm{d}s \\ &\leq H(t,\,T_1)\Big(w(T_1)-\phi(T_1)r(T_1)\rho(T_1)\Big), \end{aligned} \tag{89}$$

which is a contradiction to Eq. (77). The proof is complete.

Remark 3. Authors in [15, 20] studied a partial case of Eq. (1) by employing the generalized Riccati substitution (80). Note that the function $\rho(t)$ used in the generalized Riccati substitution (80) finally becomes unimportant. Thus, we can put $\rho(t)=0$ and obtain similar results to those from [15, 20].

In the next part, we provide several oscillation results for Eq. (1) under the assumption that the function $P(t)$ is nonpositive. These results generalize those from [10] for Eq. (1) in such sense that $\alpha\neq\beta$ and $p(t)\neq 0$.

Theorem 3. Let conditions (i)–(v) and one of the conditions (14) or (21) hold. Furthermore, assume that there exists a continuously differentiable function $\phi(t)$ such that, for all sufficiently large, T, $T_1\geq T$,

$$P(t) \le 0 \tag{90}$$

on $[T, \ \infty)$ and

$$\limsup_{t\to\infty}\left[\phi(t)A(t)\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s + \int_{T_1}^t \phi(s)\left(Q(s) - A(s)P(s)\int_s^\infty \frac{Q(u)}{A(u)}\mathrm{d}u\right)\mathrm{d}s\right] = \infty. \tag{91}$$

Then Eq. (1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (1). Then, without loss of generality, we may assume that there exists $T \in \mathcal{I}$ large enough, so that $x(t)$ satisfies the conclusions of Lemma 1 or 2 on $[T, \infty)$ with

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \tag{92}$$

on $[T, \infty)$. In particular, we have

$$z(t) > 0, \quad z'(t) > 0, \quad \left(r(t)\left(z'(t)\right)^\alpha\right)' < 0, \quad \text{for} \quad t \ge T. \tag{93}$$

Proceeding as in the proof of Theorem 1, we obtain the inequality (66), i.e.,

$$w'(t) \le \quad -\phi(t)Q(t) + P(t)w(t) - \frac{\beta\sigma'(t)\psi(t)}{\left(\phi(t)r(\sigma(t))\right)^{\frac{1}{\alpha}}} w^{\frac{1+\alpha}{\alpha}}(t) \tag{94}$$

for $t \ge T_1 \ge T$. Using Eq. (90), and setting Eq. (57) in Eq. (94), we get

$$\begin{aligned} w'(t) \quad &\le -\phi(t)Q(t) + \phi(t)A(t)P(t)\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s \\ &\quad - \frac{\beta\sigma'(t)\psi(t)}{\left(\phi(t)r(\sigma(t))\right)^{\frac{1}{\alpha}}} w^{\frac{1+\alpha}{\alpha}}(t) \\ &\le -\phi(t)Q(t) + \phi(t)A(t)P(t)\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s, \end{aligned} \tag{95}$$

that is,

$$w'(t) + \phi(t)Q(t) - \phi(t)A(t)P(t)\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s \le 0. \tag{96}$$

Integrating the above inequality from T_1 to t, we have

$$\begin{aligned} w(T_1) \quad &\ge w(t) + \int_{T_1}^t \left(\phi(s)Q(s) - \phi(s)A(s)P(s)\int_s^\infty \frac{Q(u)}{A(u)}\mathrm{d}u\right)\mathrm{d}s \\ &\ge \phi(t)A(t)\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s + \int_{T_1}^t \left(\phi(s)Q(s) - \phi(s)A(s)P(s)\int_s^\infty \frac{Q(u)}{A(u)}\mathrm{d}u\right)\mathrm{d}s \end{aligned} \tag{97}$$

Taking the lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition Eq. (91). This completes the proof.

Setting $\phi(t) = 1$, we have the following consequence.

Corollary 3. Let conditions (i)–(v) and one of the conditions (14) or (21) hold. Assume that

$$\limsup_{t\to\infty}\left[A(t)\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s + \int_{T_1}^t \tilde{q}(s)\mathrm{d}s\right] = \infty, \tag{98}$$

for all sufficiently large T, for $T_1 \geq T$. Then Eq. (1) is oscillatory.

Define a sequence of functions $\{y_n(t)\}_{n=0}^\infty$ as

$$y_0(t) = \int_t^\infty \tilde{q}(s)\mathrm{d}s, \qquad t\geq T \tag{99}$$

$$y_n(t) = \int_t^\infty \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))}\left(y_{n-1}(s)\right)^{\frac{1+\alpha}{\alpha}}\mathrm{d}s + y_0(t), \qquad t\geq T, \qquad n = 1, 2, 3, \ldots, \tag{100}$$

for $T \geq t_0$ sufficiently large.

By induction, we can see that $y_n \leq y_{n+1}$, $n = 1, 2, 3, \ldots$.

Lemma 4. Let conditions (i)–(v) and one of the conditions (14) or (21) hold. Assume that $x(t)$ is a positive solution of Eq. (1) on $\mathscr{I}$. Then there exists $T \in \mathscr{I}$, sufficiently large, such that

$$w(t) \geq y_n(t), \tag{101}$$

where $w(t)$ and $y_n(t)$ are defined as Eqs. (54) and (100), respectively. Furthermore, there exists a positive function $y(t)$ on $[T_1, \infty)$, $T_1 \geq T$, such that

$$\lim_{n\to\infty} y_n(t) = y(t) \tag{102}$$

and

$$y(t) = \int_t^\infty \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))}\left(y(s)\right)^{\frac{1+\alpha}{\alpha}}\mathrm{d}s + y_0(t). \tag{103}$$

Proof. Similarly to the proof of Theorem 3, we obtain Eq. (95). Setting $\phi(t) = 1$ in Eq. (95), we get

$$w'(t) + Q(t) + \frac{p(t)A(t)}{r(t)}\int_t^\infty \frac{Q(s)}{A(s)}\mathrm{d}s + \frac{\beta\sigma'(t)\psi(t)}{r^{\frac{1}{\alpha}}(\sigma(t))}w^{\frac{1+\alpha}{\alpha}}(t) \leq 0 \tag{104}$$

for $t \geq T_1 \geq T$. Integrating Eq. (104) from t to t', we get

$$w(t') - w(t) + \int_t^{t'} \tilde{q}(s)\mathrm{d}s + \int_t^{t'} \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))}w^{\frac{1+\alpha}{\alpha}}(s)\mathrm{d}s \leq 0 \tag{105}$$

or

$$w(t') - w(t) + \int_t^{t'} \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))}w^{\frac{1+\alpha}{\alpha}}(s)\mathrm{d}s \leq 0. \tag{106}$$

We assert that

$$\int_t^\infty \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} w^{\frac{1+\alpha}{\alpha}}(s)\mathrm{d}s < \infty. \tag{107}$$

If not, then

$$w(t') \le w(t) - \int_t^{t'} \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} w^{\frac{1+\alpha}{\alpha}}(s)\mathrm{d}s \to -\infty \tag{108}$$

as $t' \to \infty$, which contradicts to the positivity of $w(t)$ and thus the assertion is proved. By Eq. (104), we see that $w(t)$ is decreasing that means

$$\lim_{t\to\infty} w(t) = k, \quad k \ge 0. \tag{109}$$

By virtue of Eq. (107), we have $k = 0$. Thus, letting $t' \to \infty$ in Eq. (105), we get

$$\begin{aligned} w(t) \quad &\ge \int_t^\infty \tilde{q}(s)\mathrm{d}s + \int_t^\infty \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} w^{\frac{1+\alpha}{\alpha}}(s)\mathrm{d}s \\ &= y_0(t) + \int_t^\infty \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} w^{\frac{1+\alpha}{\alpha}}(s)\mathrm{d}s, \end{aligned} \tag{110}$$

that is,

$$w(t) \ge \int_t^\infty \tilde{q}(s)\mathrm{d}s = y_0(t). \tag{111}$$

Moreover, by induction, we have that

$$w(t) \ge y_n(t), \quad \text{for} \quad t \ge T_1, \quad n = 1, 2, 3, \dots. \tag{112}$$

Thus, since the sequence $\{y_n(t)\}_{n=0}^\infty$ is monotone increasing and bounded above, it converges to $y(t)$. Letting $n \to \infty$ and using Lebesgue monotone convergence theorem in Eq. (100), we get Eq. (103). The proof is complete.

Theorem 4. Let conditions (i)–(v) and one of the conditions (14) or (21) hold. If

$$\lim_{t\to\infty} inf \left(\frac{1}{y_0(t)} \int_t^\infty \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} \left(y_0(s)\right)^{\frac{1+\alpha}{\alpha}} \mathrm{d}s \right) > \frac{\alpha}{(\alpha+1)^{\frac{1+\alpha}{\alpha}}}, \tag{113}$$

where $\psi(t)$ is as in Theorem 1, then Eq. (1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (1). Then, without loss of generality, we may assume that there exists $T \in \mathcal{I}$ large enough, so that $x(t)$ satisfies the conclusions of Lemma 1 or 2 on $[T, \infty)$ with

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \tag{114}$$

on $[T, \infty)$. In particular, we have

$$z(t) > 0, \quad z^{'}(t) > 0, \quad \left(r(t)\left(z^{'}(t)\right)^{\alpha}\right)^{'} < 0, \quad \text{for} \quad t \geq T. \tag{115}$$

By Eq. (113), there exists a constant $\gamma > \frac{\alpha}{(\alpha+1)^{\frac{1+\alpha}{\alpha}}}$ such that

$$\lim_{t\to\infty} inf \frac{1}{y_0(t)} \int_t^{\infty} \frac{\beta\sigma^{'}(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} \left(y_0(s)\right)^{\frac{1+\alpha}{\alpha}} \mathrm{d}s > \gamma. \tag{116}$$

Proceeding as in the proof of Lemma 4, we obtain Eq. (110) and from that, we have

$$\frac{w(t)}{y_0(t)} \geq 1 + \frac{1}{y_0(t)} \int_t^{\infty} \frac{\beta\sigma^{'}(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} \left(y_0(s)\right)^{\frac{1+\alpha}{\alpha}} \left(\frac{w(s)}{y_0(s)}\right)^{\frac{1+\alpha}{\alpha}} \mathrm{d}s \tag{117}$$

Let

$$\lambda = \inf_{t \geq t_1} \frac{w(t)}{y_0(t)}. \tag{118}$$

Then it is easy to see that $\lambda \geq 1$ and

$$\lambda \geq 1 + \lambda^{\frac{1+\alpha}{\alpha}} \gamma, \tag{119}$$

which contradicts the admissible value of λ and γ, and thus completes the proof.

Theorem 5. Let conditions (i)–(v), one of the conditions (14) or (21) hold, and $y_n(t)$ be defined as in Eq. (100). If there exists some $y_n(t)$ such that, for T sufficiently large,

$$\lim_{t\to\infty} sup \; y_n(t) \left(\int_T^{\sigma(t)} r^{-\frac{1}{\alpha}}(s) \mathrm{d}s\right)^{\alpha} > \frac{1}{\psi(t)}, \tag{120}$$

where $\psi(t)$ is as in Theorem 1, then Eq. (1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (1). Then, without loss of generality, we may assume that there exists $T \in \mathcal{I}$ large enough, so that $x(t)$ satisfies the conclusions of Lemma 1 or 2 on $[T, \infty)$ with

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \tag{121}$$

on $[T, \infty)$. In particular, we have

$$z(t) > 0, \quad z^{'}(t) > 0, \quad \left(r(t)\left(z^{'}(t)\right)^{\alpha}\right)^{'} < 0, \quad \text{for} \quad t \geq T. \tag{122}$$

Proceeding as in the proof of Theorem 3 and using defining $w(t)$ as in Eq. (54), for $T_1 \geq T$, we get

$$\begin{aligned}\frac{1}{w(t)} &= \frac{z^{\beta}(\sigma(t))}{r(t)\left(z'(t)\right)^{\alpha}} \\ &\geq \frac{\psi(t)}{r(t)}\left(\frac{z(\sigma(t))}{z'(t)}\right)^{\alpha} \\ &= \frac{\psi(t)}{r(t)}\left(\frac{z(T_1) + \int_{T_1}^{\sigma(t)} r^{-\frac{1}{\alpha}}(s) r^{\frac{1}{\alpha}}(s) z'(s)\mathrm{d}s}{z'(t)}\right)^{\alpha} \\ &\geq \psi(t)\left(\int_{T_1}^{\sigma(t)} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^{\alpha}\end{aligned} \tag{123}$$

Thus,

$$w(t)\left(\int_{T}^{\sigma(t)} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^{\alpha} \leq \frac{1}{\psi(t)}\left(\frac{\int_{T}^{\sigma(t)} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}{\int_{T_1}^{\sigma(t)} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)^{\alpha} \tag{124}$$

And therefore,

$$\limsup_{t\to\infty}\ w(t)\left(\int_{T}^{\sigma(t)} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^{\alpha} \leq \frac{1}{\psi(t)}, \tag{125}$$

which contradicts Eq. (120). The proof is complete.

Theorem 6. Let conditions (i)–(v), one of the conditions (14) or (21) hold, and $y_n(t)$ be defined as in Eq. (100). If there exists some $y_n(t)$ such that

$$\int_{T_1}^{\infty} \tilde{q}(t)\exp\left(\int_{T_1}^{t} \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} y_n^{\frac{1}{\alpha}}(s)\mathrm{d}s\right)\mathrm{d}t = \infty \tag{126}$$

or

$$\int_{T_1}^{\infty} \frac{\beta\sigma'(t)\psi(t) y_n^{\frac{1}{\alpha}}(t) y_0(t)}{r^{\frac{1}{\alpha}}(\sigma(t))}\exp\left(\int_{T_1}^{t} \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} y_n^{\frac{1}{\alpha}}(s)\mathrm{d}s\right)\mathrm{d}t = \infty, \tag{127}$$

for T sufficiently large and $T_1 \geq T$, where $\psi(t)$ is as in Theorem 1, then Eq. (1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (1). Then, without loss of generality, we may assume that there exists $T \in \mathscr{I}$ large enough, so that $x(t)$ satisfies the conclusions of Lemma 1 or 2 on $[T, \infty)$ with

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \tag{128}$$

on $[T, \infty)$. In particular, we have

$$z(t) > 0, \quad z'(t) > 0, \quad \left(r(t)\left(z'(t)\right)^{\alpha}\right)' < 0, \quad \text{for} \quad t \geq T. \tag{129}$$

From Eq. (103), we have

$$y'(t) = -\frac{\beta \sigma'(t)\psi(t)}{r^{\frac{1}{\alpha}}(\sigma(t))}\left(y(t)\right)^{\frac{1+\alpha}{\alpha}} - \tilde{q}(t), \tag{130}$$

for all $t \geq T_1 \geq T$. Since $y(t) \geq y_n(t)$, Eq. (130) yields

$$y'(t) \leq -\frac{\beta \sigma'(t)\psi(t)}{r^{\frac{1}{\alpha}}(\sigma(t))} y_n^{\frac{1}{\alpha}}(t) y(t) - \tilde{q}(t). \tag{131}$$

Multiplying the above inequality by the integration factor

$$\exp\left(\int_{T_1}^{t} \frac{\beta \sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} y_n^{\frac{1}{\alpha}}(s) \mathrm{d}s\right), \tag{132}$$

one gets

$$\begin{aligned} y(t) \leq \quad & \exp\left(-\int_{T_1}^{t} \frac{\beta \sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} y_n^{\frac{1}{\alpha}}(s) \mathrm{d}s\right) \\ & \left(y(t_1) - \int_{T_1}^{t} \tilde{q}(s) \exp\left(\int_{T_1}^{s} \frac{\beta \sigma'(u)\psi(u)}{r^{\frac{1}{\alpha}}(\sigma(u))} y_n^{\frac{1}{\alpha}}(u) \mathrm{d}u\right) \mathrm{d}s\right), \end{aligned} \tag{133}$$

from which we have that

$$\int_{T_1}^{t} \tilde{q}(s) \exp\left(\int_{T_1}^{s} \frac{\beta \sigma'(u)\psi(u)}{r^{\frac{1}{\alpha}}(\sigma(u))} y_n^{\frac{1}{\alpha}}(u) \mathrm{d}u\right) \mathrm{d}s \leq y(T_1) < \infty. \tag{134}$$

This is a contradiction with Eq. (126).

Now denote

$$u(t) = \int_{t}^{\infty} \frac{\beta \sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))}\left(y(s)\right)^{\frac{1+\alpha}{\alpha}} \mathrm{d}s \tag{135}$$

Taking the derivative of $u(t)$, one gets

$$\begin{aligned} u'(t) &= -\frac{\beta\sigma'(t)\psi(t)}{r^{\frac{1}{\alpha}}(\sigma(t))}\Big(y(t)\Big)^{\frac{1+\alpha}{\alpha}} \\ &\leq -\frac{\beta\sigma'(t)\psi(t)}{r^{\frac{1}{\alpha}}(\sigma(t))} y_n^{\frac{1}{\alpha}}(t)y(t) \\ &= \frac{\beta\sigma'(t)\psi(t)}{r^{\frac{1}{\alpha}}(\sigma(t))} y_n^{\frac{1}{\alpha}}(t)\Big(u(t)+y_0(t)\Big) \end{aligned} \tag{136}$$

Proceeding in a similar manner to that above, we conclude that

$$\int_{T_1}^{\infty} \frac{\beta\sigma'(t)\psi(t)}{r^{\frac{1}{\alpha}}(\sigma(t))} y_n^{\frac{1}{\alpha}}(t)y_0(t)\exp\left(\int_{T_1}^{t} \frac{\beta\sigma'(s)\psi(s)}{r^{\frac{1}{\alpha}}(\sigma(s))} y_n^{\frac{1}{\alpha}}(s)\mathrm{d}s\right)\mathrm{d}t < \infty, \tag{137}$$

which contradicts to Eq. (127). The proof is complete.

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