Chapter 3

On Finite Element Vibration Analysis of Carbon Nanotubes

Ishan Ali Khan and Seyed M. Hashemi

Additional information is available at the end of the chapter

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Abstract

In this chapter, a finite element formulation is proposed to study the natural frequencies of double-walled carbon nanotubes modeled as, both, local and nonlocal Euler-Bernoulli beams, coupled with van der Waals interaction forces. The formulation uses Galerkin-weighted residual approach and employs Hermite cubic polynomial function to derive the linear eigenvalue problem. Natural frequencies are found for clamped-free, clamped-clamped and simply supported-simply supported boundary conditions. The results are in good agreement with the formulations found in the literature. The effect of nonlocal factor on the natural frequencies of the system is found out by comparing local and nonlocal results. Additionally, the universality of the proposed model is proven by application to a double-elastic Euler-Bernoulli beam. This formulation paves way for Finite Element Method (FEM) analysis of multi-walled CNTs—either locally or nonlocally.

Keywords: carbon nanotubes, Euler-Bernoulli beam, DWCNTs, finite element analysis, nonlocal continuum mechanics, vibrations

1. Introduction

Carbon nanotubes are tubules of carbon, in the dimensional range of *nanometre* (in the order of 10e−9 m), with atoms arranged in a way that gives them exceptional properties. The high aspect ratio, the ratio of length to its diameter, and chirality are the factors that contribute to the same. Carbon nanotubes are classified as single- and multi-walled depending on the number of tubes held concentrically. A considerable amount of research has been dedicated to the study of mechanical properties of both these types using experimental or theoretical methods. Theoretical methods involve the representation of C─C bonds as is, conveniently named as atomistic modelling methods, or as continuous structures such as bar, beam and shell. The latter is usually employed to study the vibrational properties as the use of atomistic models require comparatively large computational time and effort, and are often accurate for a particular time period in the loading cycle. The laws of continuum structures have been found to be at the nanometre scale as well [1].

Continuum structures, by definition, consider the model to be continuous and ignore the inter-atomic forces. Eringen and Edelen [2] and Eringen [3, 4] proposed a model that would connect the lattice mechanics with the continuum mechanics, named as nonlocal continuum mechanics. This is achieved by introducing nonlocal factors in the governing equations [5]. As will be seen in the following sections, the introduction of these nonlocal factors have shown improved description of the model. Peddieson et al. [6] extended the nonlocal continuum models to nanotechnology and hence paved the way for these theoretical models to represent the properties of nanomaterials with greater accuracy.

Finite element methods have been employed for analysis of continuum structures since 1950s, as the numerical technique is very adaptable in terms of complex geometries and boundary conditions. The method involves the division of the structure into number of 'elements' and using appropriate interpolations functions to express the approximate solutions. The eigenvalue problem hence achieved is solved for system's natural frequencies and mode shapes.

The study of vibrations of carbon nanotubes, modeled as continuum models has gained considerable attention with beams and shells being used extensively. The elastic model has been presented by Ru [7] to study the column buckling of multi-walled carbon nanotubes, where the adjacent tubes were presented to be interacting through van der Waals interaction forces. Ru [8] also studied the transverse vibrations of single- and multi-walled carbon nanotubes by considering them as Euler-Bernoulli beams. Sufficient attention was given to the effect of van der Waals interaction forces on the vibrational frequency. Yoon et al. [9] studied the non-coaxial vibrational modes of multi-walled carbon nanotubes embedded in an elastic medium and deduced that the concentric geometry of carbon nanotubes will be distorted at ultrahigh frequencies. Nonlinear free vibrations of multi-walled carbon nanotubes embedded in an elastic medium were studied by Fu et al. [10]. The relation between nonlinear free vibration frequency, stiffness of the surrounding medium and van der Waals interaction forces was discussed and amplitude curves presented. Vibrational characteristic analysis of double-walled carbon nanotube (DWCNT) with simply supported boundary condition using multiple Euler-Bernoulli beam model was investigated by Natsuki et al. [11]. It was seen that the amplitudes of inner and outer tubes are non-coaxial and in opposite direction. Natsuki also studied DWCNTs embedded in elastic medium, represented by Winkler spring model, for free supported boundary condition [12] and DWCNTs with different lengths for inner and outer tubes [13]. Xu et al. [14] studied the vibrations of DWCNTs with nonlinear van der Waals interaction forces and concluded that it does not affect the coaxial free vibrations. Elishakoff and Pentaras [15, 16] studied the natural frequencies of DWCNTs and TWCNTs using approximate methods, like Bubnov-Galerkin and Petrov-Galerkin and exact solutions. The coupling between the concentric tubes is considered as van der Waals interaction forces, in both the cases.

Nonlocal continuum mechanics approach has also been used extensively in the recent years. Constitutive relations of nonlocal elasticity to be used in analysis of carbon nanotubes for Euler-Bernoulli, Timoshenko beams and cylindrical shells were presented by Wang and Wang [17]. Extensive attention was given to the shear stress and strain relation for the Timoshenko beams. Wang [18] also studied the wave propagation in carbon nanotubes using nonlocal Euler-Bernoulli and nonlocal Timoshenko beams and concentrated on the effect of nonlocal factor on the mechanical behaviour of carbon nanotubes. Yoon et al. [19] also performed a similar study concluded that the rotary inertia and shear deformation play a significant role when the measured natural frequency is much closer to the critical frequency. One more such study was performed by Wang and Varadan [20], who developed explicit frequency solutions for both single-walled carbon nanotubes (SWCNTs) and double-walled carbon nanotube (DWCNTs) on the basis of nonlocal continuum mechanics and employed both the beam theories. The ratio of local and nonlocal frequencies is used to study the effect of nonlocal factor. Arash and Wang [21] reviewed various studies on the nonlocal continuum theory for carbon nanotubes and graphene sheets. They also compared the nonlocal models to the local equivalents. Reddy [22] reformulated governing equations, based on nonlocal theory, and presented analytical solutions for bending, buckling and vibrations of Euler-Bernoulli, Timoshenko, Reddy and Levinson beams. Further, Reddy and Pang [23] also presented the governing equations of Euler-Bernoulli and Timoshenko beams, based on nonlocal theory, specifically for carbon nanotubes and used these equations to study the vibrations, buckling and static bending. Murmu and Pradhan [24] employed the differential quadrature (DQ) method to study the thermo-mechanical vibration of SWCNT embedded in an elastic medium, represented as Winkler-type elastic foundation, using nonlocal elasticity theory. It was found that the difference between local and nonlocal frequencies is particularly high for low temperature changes. Narendar et al. [25] studied the wave propagation in SWCNTs, under magnetic field, using nonlocal Euler-Bernoulli beam theory. De Rosa and Lippiello [26] analysed the free vibration of DWCNTs, modeled as nonlocal Euler-Bernoulli beams, using cell-discretization method (CDM) and a semi-analytical Rayleigh-Schmidt method. A rare study of forced vibrations of carbon nanotubes was performed by Karaoglu and Aydogyu [27] using both local and nonlocal Euler-Bernoulli beam models, for both SWCNTs and DWCNTs. Khosrozadeh and Hajabasi [28] studied the free vibrations of DWCNTs embedded in an elastic medium with nonlinear interlayer van der Waals interaction forces using nonlocal Euler-Bernoulli beam model. Fang et al. [29] performed a similar study for DWCNTs embedded in elastic medium considering it to be Winkler model.

Timoshenko models have also been used extensively as it considers the rotational inertia and transverse shear. Wang et al. [30] considered it and studied the free vibration of MWCNTs using differential quadrature (DQ) method to solve the governing equations. It is seen that the frequencies are over-predicted by Euler-Bernoulli model when the aspect ratio is small. Yang et al. [31] studied the nonlinear free vibration of SWCNT with the von Karman geometric nonlinearity considering them as nonlocal Timoshenko beams. Nonlinear free vibration of embedded DWCNTs modeled as nonlocal Timoshenko beams was presented by Ke et al. [32] where again differential quadrature (DQ) technique was employed, and the surrounding elastic medium was represented by spring-based Winkler model. Pradhan [33] presented a finite element model using nonlocal Timoshenko beams and Galerkin weighted residual technique. The buckling of carbon nanotubes using Timoshenko model was studied by Zhang et al. [34], and various studies on buckling were extensively reviewed by Wang et al. [35] and Shima [36].

A theory similar to nonlocal theory was presented by Park and Gao [37] using modified couple stress theory where internal material length parameter is considered and a variational model is presented basing on minimum total energy potential for Euler-Bernoulli beam model. It found a relation between deflections and beam thickness. Another nonlocal theory, for nanobeams, was presented by Thai [38] in which the governing equations were derived using Hamilton's principle and analytical solutions were presented. The results were found to be comparable to Timoshenko and Reddy beam theories even though the formulation is similar to that of Euler-Bernoulli theory. A generalized nonlocal beam theory, to study properties of nanobeams, was presented by Aydogdu [39]. Various beam models were studied as a special case of this beam theory for simply supported nanobeams, and the numerical results were found to be stand valid. Lu et al. [40] compared various nonlocal beam models that are used for carbon nanotubes and made some interesting observations.

Various variational techniques have also been employed with nonlocal theories to study the vibrational behaviour of carbon nanotubes. Adali [41] employed a semi-inverse method and presented one such model. Alshorbagy et al. [42] developed a nonlocal finite element for static analysis of nanobeams. Eltaher et al. [43] analysed the vibrations of nonlocal Euler-Bernoulli nanobeams using finite element method. A similar study was performed by Phadikar and Pradhan [44], where finite element analysis of nanoplates and nanobeams was presented. All these models are found to be efficient for SWCNTs. Ansari and Hemmatnezhad [45] studied the nonlinear vibrations of MWCNTs embedded in an elastic medium using variational iteration method. A finite element analysis of embedded Timoshenko beam was also performed by Hemmatnezhad and Ansari [46], where carbon nanotubes were modeled as nonlocal beam using statically exact shape functions to achieve an eigenvalue problem. Shakouri et al. [47] studied the transverse vibrations of DWCNTs modeled as nonlocal Euler-Bernoulli beams using Galerkin method. Dinçkal [48], more recently, proposed a finite element model that uses exact dynamic shape functions (trigonometric in nature) to form the frequency-dependant dynamic stiffness matrix and hence study the free vibrations of carbon nanotubes, modeled as both nonlocal Euler-Bernoulli and Timoshenko beams. Ehteshami and Hajabasi [49] and Zhang et al. [50] performed an analytical study and presented exact solutions for finding natural frequencies of DWCNTs modeled as nonlocal Euler-Bernoulli beams. The results of both these studies have been considered as benchmark to verify the results obtained in the current study. This chapter exploits the generality of the FEM technique to obtain the natural frequencies of DWCNTs, coupled with van der Waals interaction forces. The model has been proposed and applied to both local (classical) Euler-Bernoulli and nonlocal Euler-Bernoulli beam cases. The results are compared with the values presented in the literature.

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2. Formulation

This section gives the formulation for the proposed models. Section 2.1 would present the formulation for local Euler-Bernoulli beam and Section 2.2 for nonlocal Euler-Bernoulli beam. As will be seen ahead, the formulation uses Galerkin weighted residual method and Hermite cubic polynomial functions, in both the cases in interest of generality.

2.1. DWCNTs based on local Euler-Bernoulli beam

Euler-Bernoulli beam, also known as the classical beam, is known to efficiently represent a carbon nanotube, in the case when the aspect ratio (L/d ratio) is sufficiently high. The governing equation for an Euler-Bernoulli beam is given by [51]:

$$EIw'''' + \rho A\ddot{w} = p(x) \tag{1}$$

where EI is the bending rigidity, w is the displacement, ρ is the mass density, A is the cross-sectional area, and the derivatives of transverse displacement with respect to beam length and time are represented by prime (′) and a dot (.), respectively. Ru [8] used the multiple beam model to derive the governing equations for DWCNTs. It is taken as two single-walled carbon nanotubes coupled by an interaction force, represented by van der Waal's coefficient, which is estimated as:

$$c = \frac{\pi\varepsilon R_1 R_2 \sigma^6}{a^4}\left[\frac{1001\sigma^6}{3}H^{13} - \frac{1120}{9}H^7\right] \tag{2}$$

where σ and ε are the van der Waal's radius and Lennard-Jones potential's well depth, $a = 0.142$ —the C—C bond length and R_1 and R_2 are the inner and outer radius, respectively, whereas

$$H^m = \left(R_1 + R_2\right)^{-m}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\left(1 - K\cos^2\theta\right)^{\frac{m}{2}}}\left(m = 7,13\right) \tag{3}$$

and

$$K = \frac{4R_1 R_2}{\left(R_1 + R_2\right)^2} \tag{4}$$

The governing equations for a DWCNT are [8]:

$$
\begin{aligned}
E_1 I_1 w_1'''' + \rho_1 A_1 \ddot{w}_1 &= c w_2 - c w_1 \\
E_2 I_2 w_2'''' + \rho_2 A_2 \ddot{w}_2 &= -c w_2 + c w_1
\end{aligned}
\tag{5}
$$

where the subscripts 1, 2 for E, I, w, ρ, A represent the inner and outer tubes, respectively. Simple harmonic motion is assumed, i.e., $w_j(x,t) = W_j e^{i\omega t}$ is assumed to remove time dependency from Eq. (5) and $i = 1, 2$ to represent inner and outer carbon nanotube, respectively, and ω and t represent the circular frequency and time. Using simple harmonic motion in governing equations modifies them as:

$$
\begin{aligned}
E_1 I_1 W_1'''' + \left(c - \rho_1 A_1 \omega^2\right) W_1 - c W_2 &= 0 \\
E_2 I_2 W_2'''' + \left(c - \rho_2 A_2 \omega^2\right) W_2 - c W_1 &= 0
\end{aligned}
\tag{6}
$$

Following Galerkin weighted residual method, weight functions are introduced as $\delta W_{1,2}$ and the transverse displacements continue to be represented by $W_{1,2}$. The weighted-integral form of the governing equations is given as:

$$
\begin{aligned}
\bar{W}_1^k &= \int_0^L \left(E_1 I_1 W'''' \delta W_1 + \left(c - \rho_1 A_1 \omega^2\right) W_1 \delta W_1 - c W_2 \delta W_1 \right) dx = 0 \\
\bar{W}_2^k &= \int_0^L \left(E_2 I_2 W'''' \delta W_2 + \left(c - \rho_2 A_2 \omega^2\right) W_2 \delta W_2 - c W_1 \delta W_2 \right) dx = 0
\end{aligned}
\tag{7}
$$

The weak integral form of the governing equations is eliminated by performing integration by parts, twice on Eq. (7), as given below:

$$
\begin{aligned}
\bar{W}_2^k = \int_0^L \left(E_2 I_2 W_2'' \delta W_2'' + \left(c - \rho_2 A_2 \omega^2\right) W_2 \delta W_2 - c W_1 \delta W_2 \right) dx + \cancel{\left[(E_2 I_2 W''') \delta W_2 \right]_0^L} \\
- \cancel{\left[(E_2 I_2 W'') \delta W_2' \right]_0^L} = 0
\end{aligned}
$$

$$\bar{W}_2^{\,k} = \int_0^L \left(E_2 I_2 W_2^{"} \delta W^{"}_2 + \left(c - \rho_2 A_2 \omega^2 \right) W_2 \delta W_2 - c W_1 \delta W_2 \right) dx + \cancel{[(E_2 I_2 W_2^{'''}) \delta W_2]_0^L} - \cancel{[(E_2 I_2 W^{"}) \delta W_2^{'}]_0^L} = 0 \tag{8}$$

The boundary terms in Eq. (8) which vanish with the application of system boundary conditions, represent the shear force and moment of the beams are given by $V = E_i I_i W_i'''$ and $M = E_i I_i W''$ and remain similar to those of a classical Euler-Bernoulli beam. The domain length of the DWCNT is now discretized with four nodes and two Degrees of Freedom (DOF) per node (one transverse displacement and one slope per node) such that:

$$\begin{aligned} \bar{W}^{k}_{w_1} &= \int_0^{l^k} \delta W_1^{"} \left(E_1 I_1 W_1^{"} \right) dx + \int_0^{l^k} \delta W_1 \left(c - \rho_1 A_1 \omega^2 \right) W_1 - \int_0^{l^k} \delta W_1 c W_2 = 0 \\ \bar{W}^{k}_{w_2} &= \int_0^{l^k} \delta W_2^{"} \left(E_2 I_2 W_2^{"} \right) dx + \int_0^{l^k} \delta W_2 \left(c - \rho_2 A_2 \omega^2 \right) W_2 - \int_0^{l^k} \delta W_2 c W_1 = 0 \end{aligned} \tag{9}$$

The present work uses Hermite type polynomial approximations to derive the finite element formulation such that the interpolation function is given as:

$$\bar{W}(x) = \sum_1^4 N_i(x_i) W_i \tag{10}$$

where $\langle N_i(x_i) \rangle$ is a row vector of cubic shape functions, of the beam elements given by:

$$\begin{aligned} N_1 &= \frac{1}{L^3}\left(2x^3 - 3x^2 L + L^3\right) \quad & N_2 &= \frac{1}{L^3}\left(x^3 L - 2x^2 L + xL\right) \\ N_3 &= \frac{1}{L^3}\left(-2x^3 + 3x^2 L\right) \quad & N_4 &= \frac{1}{L^3}\left(x^3 L - x^2 L^2\right) \end{aligned} \tag{11}$$

2.2. DWCNTs based on nonlocal Euler-Bernoulli beam

The concept of nonlocality, which is accounted for with the introduction of a nonlocal or a small scale factor was extended to the field of nanotechnology by Peddieson et al. [6] where the author applied it to Euler-Bernoulli beam model. The small scale effect or the nonlocal parameter is given as $\mu = e_0 a$ to consider the stress at one point of the system as the function of strain at all points of the system. This is achieved with e_0 being the material constant (to be determined for each material) and a being the length of C─C bond. Subsequently, researchers like Ehteshami and Hajabasi [49], Zhang et al. [50] and Wang and Varadan [20] developed the model for DWCNTs using nonlocal Euler-Bernoulli beam model. The derivation of governing equations requires the classic Hooke's law for uniaxial stress state to be modified, taking nonlocal elasticity into consideration, as:

$$\sigma - (e_0 a)^2 \frac{\partial^2 \sigma}{\partial x^2} = E\varepsilon \tag{12}$$

with σ and ε being axial stress and strain, respectively. The equations of motion, perpendicular to x axis, are given by:

$$\frac{\partial S}{\partial x} - \rho A \frac{\partial^2 w}{\partial t^2} = 0; V - \frac{\partial M}{\partial x} = 0 \tag{13}$$

where ρ is the mass density of material, A is the cross-sectional area, w is deflection of the beam and M and S denotes the bending moment and shear force, respectively. Using these in the moment equilibrium conditions, results in the governing equation for nonlocal Euler-Bernoulli beam, given as:

$$p = EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \mu^2 \left(\rho A \frac{\partial^4 w}{\partial x^4} - \frac{\partial^2 p}{\partial x^2} \right) \tag{14}$$

For nonlocal DWCNTs, the governing equations are derived to be [20, 49, 50]:

$$E_1 I_1 \frac{\partial^4 w_1}{\partial x^2} + \rho_1 A_1 \frac{\partial^2 w_1}{\partial t^2} - \mu^2 \rho_1 A_1 \frac{\partial^4 w_1}{\partial x^2 \partial t^2} - c(w_2 - w_1) + \mu^2 c \left(\frac{\partial^2 w_2}{\partial x^2} - \frac{\partial^2 w_1}{\partial x^2} \right) = 0$$

$$E_2 I_2 \frac{\partial^4 w_2}{\partial x^2} + \rho_2 A_2 \frac{\partial^2 w_2}{\partial t^2} - \mu^2 \rho_2 A_2 \frac{\partial^4 w_2}{\partial x^2 \partial t^2} - c(w_1 - w_2) + \mu^2 c \left(\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 w_2}{\partial x^2} \right) = 0 \tag{15}$$

The solution of the equations is in the form $w_j(x,t) = W_j e^{i\omega t}$, similar to the local Euler-Bernoulli formulation. The time dependency is removed by considering the transverse displacement with respect to time as circular frequency, ω^2:

$$\begin{aligned} &E_1 I_1 W_1'''' - m_1\omega^2 W_1 - \mu^2 m_1\omega^2 W_1'' - cW_2 + cW_1 + \mu^2 cW_2'' - \mu^2 cW_1'' = 0 \\ &E_2 I_2 W_2'''' - m_2\omega^2 W_2 - \mu^2 m_2\omega^2 W_2'' - cW_1 + cW_2 + \mu^2 cW_1'' - \mu^2 cW_2'' = 0 \end{aligned} \tag{16}$$

To maintain uniformity in mathematical modelling with the formulation proposed for local Euler-Bernoulli model, Galerkin weighted residual model is used for FEM, with δW_i taken as weighted residual. The weak integral form of governing equations takes the form:

$$\begin{aligned} &\int_0^L (E_1 I_1 W_1''''\delta W_1 - m_1\omega^2 W_1\delta W_1 - \mu^2 m_1\omega^2 W_1''\delta W_1 - cW_2\delta W_1 + cW_1\delta W_1 + \mu^2 cW_2''\delta W_1 \\ &-\mu^2 cW_1''\delta W_1)dx = 0 \\ &\int_0^L (E_2 I_2 W_2''''\delta W_2 - m_2\omega^2 W_2\delta W_2 - \mu^2 m_2\omega^2 W_2''\delta W_2 - cW_1\delta W_2 + cW_2\delta W_2 + \mu^2 cW_1''\delta W_2 \\ &-\mu^2 cW_2''\delta W_2)dx = 0 \end{aligned} \tag{17}$$

Integration by parts is performed on the above set of equations once, and the weak form vanishes with the application of system boundary conditions,

$$\begin{aligned} &\int_0^L \begin{pmatrix} \left(\delta W_1'\left(-E_1 I_1 W_1'''\right)\right) - \left(\delta W_1\left(m_1\omega^2 W_1\right)\right) - \left(\delta W_1'\left(\mu^2 m_1\omega^2 W_1'\right)\right) \\ +\left(\delta W_1\left(cW_1\right)\right) - \left(\delta W_1\left(cW_2\right)\right) - \left(\delta W_1'\left(\mu^2 cW_2'\right)\right) + \left(\delta W_1'\left(\mu^2 cW_1'\right)\right) \end{pmatrix} dx = 0 \\ &\int_0^L \begin{pmatrix} \left(\delta W_2'\left(-E_2 I_2 W_2'''\right)\right) - \left(\delta W_2\left(m_2\omega^2 W_2\right)\right) - \left(\delta W_2'\left(\mu^2 m_2\omega^2 W_2'\right)\right) \\ +\left(\delta W_2\left(cW_2\right)\right) - \left(\delta W_2\left(cW_1\right)\right) - \left(\delta W_2'\left(\mu^2 cW_1'\right)\right) + \left(\delta W_2'\left(\mu^2 cW_2'\right)\right) \end{pmatrix} dx = 0 \end{aligned} \tag{18}$$

The resulting internal shear forces for both the beams are given as:

$$\begin{aligned} V_1 &= E_1 I_1 W_1''' + \mu^2\left(m_1\omega^2 W_1' + c\left(W_2' - W_1'\right)\right) \\ V_2 &= E_2 I_2 W_2''' + \mu^2\left(m_2\omega^2 W_2' + c\left(W_1' - W_2'\right)\right) \end{aligned} \tag{19}$$

and the resultant bending moments are given as:

$$\begin{aligned} M_1 &= E_1 I_1 W_1'' - \mu^2\left(m_1\omega^2 W_1 - c\left(W_2 - W_1\right)\right) \\ M_2 &= E_2 I_2 W_2'' - \mu^2\left(m_2\omega^2 W_2 - c\left(W_1 - W_2\right)\right) \end{aligned} \tag{20}$$

Integration by parts is performed once again on the first terms in Eq. (18) and after rearranging for clarity, the following set of equations is obtained:

$$\begin{aligned} \bar{W}_{w_1}^k &= E_1 I_1 \int_0^{l^k}\left(W_1'' \delta W_1''\right)dx - m_1\omega^2\left(\int_0^{l^k}\left(W_1\delta W_1\right) + \mu^2\int_0^{l^k}\left(W_1'\delta W_1'\right)\right)dx \\ &+ c\left(\int_0^{l^k}\left(W_1\delta W_1\right) + \mu^2\int_0^{l^k}\left(W_1'\delta W_1'\right)\right)dx - c\left(\int_0^{l^k}\left(W_2\delta W_1\right) + \mu^2\int_0^{l^k}\left(W_2'\delta W_1'\right)\right)dx \\ \bar{W}_{w_2}^k &= E_2 I_2 \int_0^{l^k}\left(W_2'' \delta W_2''\right)dx - m_2\omega^2\left(\int_0^{l^k}\left(W_2\delta W_2\right) + \mu^2\int_0^{l^k}\left(W_2'\delta W_2'\right)\right)dx \\ &+ c\left(\int_0^{l^k}\left(W_2\delta W_2\right) + \mu^2\int_0^{l^k}\left(W_2'\delta W_2'\right)\right)dx - c\left(\int_0^{l^k}\left(W_1\delta W_2\right) + \mu^2\int_0^{l^k}\left(W_1'\delta W_2'\right)\right)dx \end{aligned} \tag{21}$$

Using Hermite cubic polynomial functions, given in Eq. (11), the finite element model is developed. The element mass and stiffness matrices for both the local and nonlocal scenarios are explicitly presented in the Appendix A. An FEM code in MATLAB solves the linear eigenvalue problem which is given as:

$$\det\left(K - \omega^2 M\right) = 0 \tag{22}$$

where K represents global stiffness matrix and M represents the global mass matrix, respectively, and is achieved after assembly. **Table 1** gives the element matrices K_e and M_e, which are to be assembled to form the global matrices for both local and nonlocal cases.

Local Euler-Bernoulli beam	Nonlocal Euler-Bernoulli beam
$K_e = \begin{pmatrix} k_1 + k_3 & -k_4 \\ -k_3 & k_2 + k_4 \end{pmatrix}$	$K_e = \begin{pmatrix} k_1 + k_3 + k_5 & -\left(k_4 + k_6\right) \\ -\left(k_3 + k_5\right) & k_2 + k_4 + k_6 \end{pmatrix}$
$M_e = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$	$M_e = \begin{pmatrix} m_1 + m_3 & 0 \\ 0 & m_2 + m_4 \end{pmatrix}$

Table 1. Elemental matrices for local and nonlocal DWCNTs modeled as Euler-Bernoulli beams.

3. Results and discussion

The robustness and generality of proposed FEM model is demonstrated by its application to a double-elastic Euler-Bernoulli beam and then to carbon nanotubes modeled as Euler-Bernoulli beam, both locally and nonlocally. Natural frequencies are calculated for clamped-free, clamped-clamped and simply supported-simply supported boundary conditions. The parameters of the local double elastic Euler-Bernoulli beam are taken as:

$$\rho_1 = \rho_2 = 1 kg/m^3 \quad E_1 = E_2 = 1 N/m^2 \quad A_1 = 1\ \mathrm{m}^2 \quad A_2 = 2\ \mathrm{m}^2$$
$$I_1 = 1\ \mathrm{m}^4 \quad I_2 = 2\ \mathrm{m}^4 \quad c = 10; \mu = 0.1 \quad L = 1\ \mathrm{m}$$

Boundary conditions	[14]	Present study	Error percentage
C—F	3.52	3.52	0.01
C—C	22.37	22.37	0
S—S	9.87	9.87	0

Table 2. Non-dimentionalized natural frequencies of a local double-elastic Euler-Bernoulli beam.

These parameters are so selected that the double beam replicates the geometry of carbon nanotubes, i.e., both are concentric and are coupled with an interaction forces between the walls. The natural frequencies of the double-elastic system, in this chapter, are normalized using $\bar{\omega} = \omega L^2 \sqrt{\rho A / EI}$ where $E = E_1 + E_2$ and $I = I_1 + I_2$. The values presented by the proposed model converge at about five elements, but the data tabulated, for all cases, are taken for 10 elements, unless mentioned otherwise. The non-dimentionalized natural frequencies for three classic boundary conditions are given in **Table 2**.

Boundary conditions	[50]	Present study	Error percentage
C—F	3.45	3.44	−0.54
C—C	20.23	21.10	4.38
S—S	9.42	9.42	0

Table 3. Non-dimentionalized natural frequencies of a nonlocal double-elastic Euler-Bernoulli beam.

The percentage error has been calculated by $((\text{approximate}-\text{exact})/\text{exact}) \times 100$ in this study. These values are compared to the first natural frequency obtained using the analytical solutions found in Xu et al. The values are in good agreement and hence the proposed model is found to be efficient to find the natural frequencies of local double-elastic Euler-Bernoulli beam. The applicability of the proposed nonlocal model is demonstrated by the values in **Table 3**. It is to be noted that the values are presented for a single value nonlocal factor, where $\mu = 0.1$, and it can be confirmed that $\mu = 0$ would give the same values as presented in **Table 1** as that would reduce the nonlocal formulation to local one. The values presented in **Table 1** are compared to the values obtained using the beam parameters in analytical solutions provided by Zhang et al. [50]. It can be seen the error percentages are small hence validating the proposed model.

Table 4 gives the error percentage between local and nonlocal double-elastic Euler-Bernoulli beam when $L/d = 10$. This error helps to measure the effect of nonlocal factor on the natural frequencies of double-elastic Euler-Bernoulli beams. It is seen that the values obtained using nonlocal model is less than that obtained using local double-elastic Euler-Bernoulli model. This can be attributed to the presence of a nonlocal factor that considers the strain at one point to be function of all the points in the body.

Boundary conditions	Local	Nonlocal	Error percentage
C—F	3.52	3.44	2.25
C—C	22.37	21.10	5.65
S—S	9.87	9.42	4.60

Table 4. Error percentage between natural frequencies of local and nonlocal double-elastic Euler-Bernoulli beam.

It is concluded that a novel double-elastic Euler-Bernoulli beam FEM model has been proposed, and its veracity established for both local and nonlocal cases, the same model will now be applied to carbon nanotubes with the parameters used in Xu et al. [14] and Elishakoff and Pentaras [15], given below:

$$\begin{array}{ll} \rho_1 = \rho_2 = 2.3\ \text{g/cm}^3 & E_1 = E_2 = 1\ \text{Tpa} \\ c = 71.11\ \text{GPa} & e_0 \approx 0.82; a = 0.142\ \text{nm} \\ r_{1,i}\text{(inner radius, inner tube)} = 0.18\ \text{nm} & r_{1,o}\text{(outer radius, inner tube)} = 0.52\ \text{nm} \\ r_{2,i}\text{(inner radius, outer tube)} = 0.53\ \text{nm} & r_{2,o}\text{(outer radius, outer tube)} = 0.87\ \text{nm} \end{array}$$

Boundary conditions	[14]	Present study	Error percentage
C—F	0.16	0.16	0.01
C—C	1.06	1.06	0
S—S	4.67	4.67	0

Table 5. Natural frequencies of a local DWCNTs modeled as Euler-Bernoulli beam (10^{12} rads/s).

The natural frequencies of a local DWCNT, modeled as Euler-Bernoulli beams, are presented in **Table 5**. The nonlocal factor μ is introduced as a product of material constant and characteristic length. Zhang et al. calculated the material constant as the ratio of local result to nonlocal result for axial buckling strain of SWCNT and arrived at a value $e_0 \approx 0.82$, and the characteristic length is taken to be the length of C—C bond i.e., $a = 0.142$ nm. The natural frequencies calculated using the proposed model are compared to the values obtained using the analytical formulation presented in Xu et al. [14]. The error percentages in **Table 5** confirm that the proposed formulation holds valid even at nanoscale level. Further, negative error percentage denotes that the values obtained using the present FEM model are a little higher than those presented by Ehteshami and Hajabasi [49] (see **Table 6**).

Boundary conditions	[49]	[50]	Present study	Error percentage 1	Error percentage 2
C—F	0.166	0.16	0.16	0	0.01
C—C	1.06	1.06	1.06	−0.04	0.04
S—S	0.467	0.46	0.46	−0.03	0

Table 6. Natural frequencies of a nonlocal DWCNT modeled as Euler-Bernoulli beam (10^{12} rads/s).

Boundary conditions	Local	Nonlocal	Error percentage
C—F	0.16	0.16	0.02
C—C	1.06	1.06	0.04
S—S	0.46	0.46	0.03

Table 7. Error percentage between natural frequencies of local and nonlocal DWCNTs modeled as Euler-Bernoulli beam (10^{12} rads/s).

1	L/d	Local	Nonlocal	Error percentage
C—F	20	0.41	0.41	0
C—C		2.65	2.65	0.01
S—S		1.17	1.17	0
C—F	40	0.10	0.10	0
C—C		0.66	0.66	0
S—S		0.29	0.29	0
C—F	60	0.04	0.04	0
C—C		0.29	0.29	0
S—S		0.13	0.13	0
C—F	80	0.02	0.02	0
C—C		0.16	0.16	0
S—S		0.07	0.07	0

Table 8. Error percentage between natural frequencies of local and nonlocal DWCNTs modeled as Euler-Bernoulli beam with different L/d ratios (10^{12} rads/s).

Employing the FEM model proposed in Section 1.2, the natural frequencies of nonlocal DWCNTs modeled as Euler-Bernoulli beam are given in **Table 6**. The natural frequencies thus obtained are compared to the values obtained using the analytical formulations given by Ehteshami and Hajabasi [49] Zhang et al. [50]. The error percentages, between present study and these two studies, are presented as error percentage 1 and 2, respectively. The effect of nonlocal factor on the natural frequencies is best understood when the error percentage between local results and nonlocal results are compared, presented in **Table 7**. It will be noted that the error percentage is very small compared to the error percentage reported in **Table 4**. This difference is attributed to the value of nonlocal factor—the double-elastic beam model considers it to be between 1 and 2 and DWCNTs have it in the order of $\mathrm{e}-10$, to be exact $1.1644\mathrm{e}-10$. It is also seen (**Table 8**) that as L/d ratio is increased, the natural frequency and the error percentage, between local and nonlocal results, decrease, i.e., as the aspect ratio increases, the effect of nonlocal factor decreases.

It is noted that though many tables in this chapter may have identical values in the columns that are being compared, the indicated percentage error is a nonzero number, or on the contrary, the columns being compared might not have identical values, but the indicated error percentage is zero. A justification for these values stems from the large magnitude of the numbers reported. For the sake of uniformity, we have rounded off the numbers to 2 decimal points.

4. Conclusion

This chapter proposed FEM formulations to find the natural frequencies of DWCNTs modeled as, both, local and nonlocal Euler-Bernoulli beams, coupled with van der Waals interaction forces. It uses Galerkin weighted residual approach and Hermite cubic polynomial functions to form the stiffness and mass matrices. The formulation is applied to double-elastic Euler-Bernoulli beam and to DWCNTs, to check its generality. The natural frequencies thus obtained are in agreement with those of the formulations found in the literature. It is seen that as the L/d ratio of the system increases, the difference between local and nonlocal formulations reduces. It has also been observed that difference between local and nonlocal system is dependent on the magnitude of the nonlocal factor considered. These formulations would let researchers apply FEM to multi-walled CNTs in order to better understand their behaviour.

Nomenclature

a, length of the C—C bond

A, cross-sectional area

c, van der Waal's interaction coefficient

E, Young's Modulus

I, second area moment of inertia

M, bending moment

N, shape functions

t, time

V, shear force

w, lateral displacement

W, amplitude

x, length of the beam

δW, weighting function

μ, nonlocal factor

ρ, mass density

ω, circular frequency

e_0, material constant

Appendix A. Element matrices

$$k_1 = \begin{pmatrix} \frac{12E_1I_1}{l^3} & \frac{6E_1I_1}{l^2} & \frac{-12E_1I_1}{l^3} & \frac{6E_1I_1}{l^2} \\ & \frac{4E_1I_1}{l} & \frac{-6E_1I_1}{l^2} & \frac{2E_1I_1}{l} \\ & Sym & \frac{12E_1I_1}{l^3} & \frac{-6E_1I_1}{l^2} \\ & & & \frac{4E_1I_1}{l} \end{pmatrix} \begin{pmatrix} w_{11} \\ \theta_{11} \\ w_{12} \\ \theta_{12} \end{pmatrix} k_3 = \begin{pmatrix} \frac{156c}{420} & \frac{22lc}{420} & \frac{54c}{420} & \frac{-13lc}{420} \\ & \frac{4l^2c}{420} & \frac{13lc}{420} & \frac{-3l^2c}{420} \\ & Sym & \frac{156c}{420} & \frac{-22lc}{420} \\ & & & \frac{4l^2c}{420} \end{pmatrix} \begin{pmatrix} w_{11} \\ \theta_{11} \\ w_{12} \\ \theta_{12} \end{pmatrix}$$

$$k_2 = \begin{pmatrix} \frac{12E_2I_2}{l^3} & \frac{6E_2I_2}{l^2} & \frac{-12E_2I_2}{l^3} & \frac{6E_2I_2}{l^2} \\ & \frac{4E_2I_2}{l} & \frac{-6E_2I_2}{l^2} & \frac{2E_2I_2}{l} \\ & Sym & \frac{12E_2I_2}{l^3} & \frac{-6E_2I_2}{l^2} \\ & & & \frac{4E_2I_2}{l} \end{pmatrix} \begin{pmatrix} w_{21} \\ \theta_{21} \\ w_{22} \\ \theta_{22} \end{pmatrix} k_4 = \begin{pmatrix} \frac{156c}{420} & \frac{22lc}{420} & \frac{54c}{420} & \frac{-13lc}{420} \\ & \frac{4l^2c}{420} & \frac{13lc}{420} & \frac{-3l^2c}{420} \\ & Sym & \frac{156c}{420} & \frac{-22lc}{420} \\ & & & \frac{4l^2c}{420} \end{pmatrix} \begin{pmatrix} w_{21} \\ \theta_{21} \\ w_{22} \\ \theta_{22} \end{pmatrix}$$

$$k_5 = \begin{pmatrix} \frac{6\mu^2 c}{5l} & \frac{\mu^2 c}{10} & \frac{-6\mu^2 c}{5l} & \frac{\mu^2 c}{10} \\ & \frac{2l\mu^2 c}{15} & \frac{-\mu^2 c}{10} & \frac{-\mu^2 cl}{30} \\ & Sym & \frac{6\mu^2 c}{5l} & \frac{-\mu^2 c}{10} \\ & & & \frac{2l\mu^2 c}{15} \end{pmatrix} \begin{pmatrix} w_{11} \\ \theta_{11} \\ w_{12} \\ \theta_{12} \end{pmatrix} k_6 = \begin{pmatrix} \frac{6\mu^2 c}{5l} & \frac{\mu^2 c}{10} & \frac{-6\mu^2 c}{5l} & \frac{\mu^2 c}{10} \\ & \frac{2l\mu^2 c}{15} & \frac{-\mu^2 c}{10} & \frac{-\mu^2 cl}{30} \\ & Sym & \frac{6\mu^2 c}{5l} & \frac{-\mu^2 c}{10} \\ & & & \frac{2l\mu^2 c}{15} \end{pmatrix} \begin{pmatrix} w_{21} \\ \theta_{21} \\ w_{22} \\ \theta_{22} \end{pmatrix}$$

$$m_1 = \begin{pmatrix} \frac{156m_1 l}{420} & \frac{22l^2 m_1}{420} & \frac{54m_1 l}{420} & \frac{-13l^2 m_1}{420} \\ & \frac{4l^3 m_1}{420} & \frac{13l^2 m_1}{420} & \frac{-3l^3 m_1}{420} \\ & Sym & \frac{156m_1 l}{420} & \frac{-22l^2 m_1}{420} \\ & & & \frac{4l^3 m_1}{420} \end{pmatrix} \begin{pmatrix} w_{11} \\ \theta_{11} \\ w_{12} \\ \theta_{12} \end{pmatrix} m_3 = \begin{pmatrix} \frac{6\mu^2 m_1}{5l} & \frac{\mu^2 m_1}{10} & \frac{-6\mu^2 m_1}{5l} & \frac{\mu^2 m_1}{10} \\ & \frac{2l\mu^2 m_1}{15} & \frac{-\mu^2 m_1}{10} & \frac{-\mu^2 m_1 l}{30} \\ & Sym & \frac{6\mu^2 m_1}{5l} & \frac{-\mu^2 m_1}{10} \\ & & & \frac{2l\mu^2 m_1}{15} \end{pmatrix} \begin{pmatrix} w_{11} \\ \theta_{11} \\ w_{12} \\ \theta_{12} \end{pmatrix}$$

$$m_2 = \begin{pmatrix} \frac{156m_2 l}{420} & \frac{22l^2 m_2}{420} & \frac{54m_2 l}{420} & \frac{-13l^2 m_2}{420} \\ & \frac{4l^3 m_2}{420} & \frac{13l^2 m_2}{420} & \frac{-3l^3 m_2}{420} \\ & Sym & \frac{156m_2 l}{420} & \frac{-22l^2 m_2}{420} \\ & & & \frac{4l^3 m_2}{420} \end{pmatrix} \begin{pmatrix} w_{21} \\ \theta_{21} \\ w_{22} \\ \theta_{22} \end{pmatrix} m_4 = \begin{pmatrix} \frac{6\mu^2 m_2}{5l} & \frac{\mu^2 m_2}{10} & \frac{-6\mu^2 m_2}{5l} & \frac{\mu^2 m_2}{10} \\ & \frac{2l\mu^2 m_2}{15} & \frac{-\mu^2 m_2}{10} & \frac{-\mu^2 m_2 l}{30} \\ & Sym & \frac{6\mu^2 m_2}{5l} & \frac{-\mu^2 m_2}{10} \\ & & & \frac{2l\mu^2 m_2}{15} \end{pmatrix} \begin{pmatrix} w_{21} \\ \theta_{21} \\ w_{22} \\ \theta_{22} \end{pmatrix}$$

Author details

Ishan Ali Khan and Seyed M. Hashemi*

*Address all correspondence to: smhashem@ryerson.ca

Department of Aerospace Engineering, Ryerson University, Toronto, Canada

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