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Chapter 13

Non-Classical Memory Kernels in Linear Viscoelasticity

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Additional information is available at the end of the chapter

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Abstract

In linear viscoelasticity, a large variety of regular kernels have been classically employed, depending on the mechanical properties of the materials to be modeled. Nevertheless, new viscoelastic materials, such as viscoelastic gels, have been recently discovered and their mechanical behavior requires convolution integral with singular kernels to be described. On the other hand, when the natural/artificial aging of the viscoelastic material has to be taken into account, time-dependent kernels are needed. The aim of this chapter is to present a collection of nonstandard viscoelastic kernels, with special emphasis on singular and time-dependent kernels, and discuss their ability to reproduce experimental behavior when applied to real materials. As an application, we study some magneto-rheological elastomers, where viscoelastic and magnetic effects are coupled.

Keywords: Materials with memory, Viscoelasticity, Unbounded memory kernels, Existence of solutions, Asymptotic solutions', Behavior

1. Introduction

The stress-strain relation in linear viscoelasticity involves a convolution integral with a memory kernel. The fading memory principle requires that the memory kernel decays quickly as the elapsed time goes to infinity, but no limitation is imposed to its behavior near zero. So, a wide range of kernels may be used depending on the nature of the materials to be modeled. Starting from the rheological model of a standard viscoelastic solid, whose kernel involves a single exponential, a large variety of regular kernels have been classically employed: discrete and continuous Prony series, completely monotonic functions, etc. Recently, new viscoelastic

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materials, such as viscoelastic gels, have been described by virtue of convolution integral with singular kernels: for instance, fractional and hypergeometric kernels [1]. On the other hand, when the natural/artificial aging of the viscoelastic material has to be taken into account, time-dependent kernels are needed. Furthermore, the behavior of some new materials, for instance, ferrogel and magneto-rheological elastomers, can be determined by coupling viscoelastic and magnetic effects.

The material of this chapter is organized as follows. First we present the model of a viscoelastic body which represents the basis for our study. It is assumed to be homogeneous and isotropic, and its crucial feature is that the stress response at time t linearly depends on the whole *past history* of the strain up to t. Then, we look for the modeling of *aging isothermal viscoelasticity*, assuming that the viscoelastic structural parameters are time dependent while the material is subject to chemical or physical agents at constant temperature. Finally, singular kernel problems are addressed to, at first, in the case of a viscoelastic body and, later, when the viscoelastic behavior is coupled with magnetization. In particular, the case of magneto-viscoelastic bodies is considered. Indeed, the idea of coupling the viscoelastic behavior with magnetic effects is suggested by new materials which are obtained by inserting magnetic defects into a solid body to have the opportunity to influence the mechanical properties of the body when a magnetic field is applied.

2. Preliminary notions and notations

This section is devoted to provide the key notions concerning the model of isothermal viscoelastic body with memory. For sake of simplicity, the body is supposed homogeneous. In order to briefly introduce the subject, at the beginning, we restrict our attention to one-dimensional processes. Let ε denote the uniaxial strain and σ the corresponding tensile stress at every point x of the reference configuration of the sample. According to Boltzmann's formulation of hereditary elasticity [2], a linear viscoelastic solid may be described by a stress-strain relation in the Riemann-Stieltjes integral form

$$\sigma(x,t) = \int_{-\infty}^{t} G(t-s)\mathrm{d}_s\varepsilon(x,s) \tag{1}$$

where G is named *Boltzmann function* (or *memory kernel*) and $\varepsilon(\cdot)$ is a fading strain history, namely

$$\lim_{s\to-\infty}\varepsilon(x,s) = 0, \qquad \varepsilon(x,s) = \int_{-\infty}^{s}\mathrm{d}_s\varepsilon(x,\zeta). \tag{2}$$

In particular, when the strain history vanishes from $-\infty$ to 0, then Eq. (1) reduces to

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$$\sigma(x,t) = \int_0^t G(t-s)\mathrm{d}_s\varepsilon(x,s). \tag{3}$$

A peculiar behavior of viscoelastic solid materials is named *relaxation property*: if the solid is held at a constant strain starting from a given time $t_0 \geq 0$, the stress tends (as $t \to \infty$) to a constant value which is "proportional" to the applied constant strain. Indeed, if $\varepsilon(x, \cdot)$ is continuous on $(-\infty, t_0]$ and

$$\varepsilon(x,t) = \varepsilon(x,t_0) = \varepsilon_0(x), \quad \forall t \geq t_0,$$

it follows that

$$\lim_{t\to\infty}\sigma(x,t) = \lim_{t\to\infty} G_\infty\varepsilon(x,t) + \lim_{t\to\infty}\int_{-\infty}^{t_0}[G(t-s)-G_\infty]\mathrm{d}_s\varepsilon(x,s) = G_\infty\varepsilon_0(x),$$

where the *relaxation modulus*

$$G_\infty = \lim_{\tau\to\infty} G(\tau)$$

is assumed to be positive. Then, using Eq. (2) and letting

$$\hat{G}(\tau) = G(\tau) - G_\infty \tag{4}$$

the stress-strain relation (1) may be rewritten as

$$\sigma(x,t) = G_\infty\varepsilon(x,t) + \int_{-\infty}^{t}\hat{G}(t-s)\mathrm{d}_s\varepsilon(x,s). \tag{5}$$

Of course, the choice of G is required to satisfy some basic principles, like the *fading memory principle* and the *dissipation principle*, a thermostatic version of the second law of thermodynamics (see [3], for instance). In general, these conditions allow the memory kernel to be unbounded at the origin.

In the terminology of Dautray and Lions [4], hereditary effects with *long memory range* are represented by a convolution integral, where

$$G \in L^1(0,T) \cap C^2(0,T), \qquad \forall T > 0 \tag{6}$$

whereas a *short memory range* is related to singular kernels of the Dirac delta type. In the latter case, letting $\hat{G} = \Gamma\delta_0$, where δ_0 denotes the Dirac mass at 0^+, from Eq. (1) it follows:

$$\sigma(x,t) = G_\infty \varepsilon(x,t) + \Gamma \partial_t \varepsilon(x,t) \tag{7}$$

where ∂_t denotes partial derviation with respect to 't'.

which is named the *Kelvin-Voigt model*. On the other hand, assumption (6) may be strengthened by letting G be bounded along with its derivatives

$$G \in L^\infty(0,T) \cap C_b^2(0,T), \qquad \forall T > 0. \tag{8}$$

If this is the case,

$$G_0 = \lim_{\tau \to 0^+} G(\tau)$$

and an integration by parts changes Eq. (1) into the alternate forms

$$\sigma(x,t) = G_0 \varepsilon(t) + \int_{-\infty}^{t} G'(t-s)\varepsilon(x,s)\mathrm{d}s = G_0 \varepsilon(x,t) + \int_0^\infty G'(\tau)\varepsilon(x,t-\tau)d\tau, \tag{9}$$

where the *relaxation function* $G'(\tau)$ is the derivative with respect to τ of the Boltzmann function G. This constitutive stress-strain relation is based on the Lebesgue representation of linear functionals in the history space theory devised by Volterra [5]. Provided that Eq. (2) holds true, the Boltzmann and the Volterra constitutive relations are equivalent. The latter approach, however, can be applied to a wilder class of strain histories (uniformly bounded, for instance), in which Eq. (2) is no longer needed.

In the three-dimensional case, all fields depend on the space-time pair $(\boldsymbol{x}, t) \in \Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^3$ is the reference configuration. The displacement vector $\boldsymbol{u}(\boldsymbol{x}, t)$ is given by

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{\mu}(\boldsymbol{x},t) - \boldsymbol{x},$$

where $\boldsymbol{\mu}(\boldsymbol{x}, \cdot)$ is the motion of $\boldsymbol{x}$, and

$$\boldsymbol{E} = \frac{1}{2}\left[\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right],$$

is the infinitesimal strain tensor. Borrowing from Eq. (9), the viscoelastic Cauchy stress tensor $\boldsymbol{T}$ is given by

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$$\boldsymbol{T}(\boldsymbol{x},t) = \mathbb{G}_0\boldsymbol{E}(\boldsymbol{x},t) - \int_{-\infty}^{t}\mathbb{G}'(t-s)\boldsymbol{E}(\boldsymbol{x},s)\mathrm{d}s = \mathbb{G}_0\boldsymbol{E}(\boldsymbol{x},t) + \int_0^{\infty}\mathbb{G}'(\tau)\boldsymbol{E}(\boldsymbol{x},t-\tau)\mathrm{d}\tau. \tag{10}$$

where $\mathbb{G}:\mathbb{R}^+ \to \mathrm{Lin(Sym)}$ stands for the relaxation function and

$$\mathbb{G}_0 = \mathbb{G}(0), \qquad \mathbb{G}_\infty = \lim_{\tau\to\infty}\mathbb{G}(\tau), \qquad \mathbb{G}'(\tau) = \partial_\tau\mathbb{G}(\tau).$$

A simple manipulation of Eq. (10) yields an alternate stress-strain relation:

$$\boldsymbol{T}(\boldsymbol{x},t) = \mathbb{G}_\infty\boldsymbol{E}(\boldsymbol{x},t) - \int_0^{\infty}\mathbb{G}'(\tau)[\boldsymbol{E}(\boldsymbol{x},t) - \boldsymbol{E}(\boldsymbol{x},t-\tau)]\mathrm{d}\tau. \tag{11}$$

For fading strain histories obeying Eq. (2), an integration by parts allows Eq. (10) to be rewritten as

$$\boldsymbol{T}(\boldsymbol{x},t) = \mathbb{G}_\infty\boldsymbol{E}(\boldsymbol{x},t) + \int_{-\infty}^{t}\hat{\mathbb{G}}(t-s)\mathrm{d}_s\boldsymbol{E}(\boldsymbol{x},s) = \mathbb{G}_\infty\boldsymbol{E}(\boldsymbol{x},t) - \int_0^{\infty}\hat{\mathbb{G}}(\tau)\mathrm{d}_\tau\boldsymbol{E}(\boldsymbol{x},t-\tau), \tag{12}$$

where $\hat{\mathbb{G}}$ is defined as $\hat{G}$ in Eq. (4). The material is said to enjoy the *fading memory principle* when, for every $\varepsilon > 0$ there exists a positive time shift $S_0(\varepsilon)$, possibly dependent on the strain history, such that

$$\left|\int_0^{\infty}\mathbb{G}'(\tau+s)\boldsymbol{E}(\boldsymbol{x},t-\tau)\,\mathrm{d}\tau\right| = \left|\int_0^{\infty}\hat{\mathbb{G}}(\tau+s)\mathrm{d}_\tau\boldsymbol{E}(\boldsymbol{x},t-\tau)\right| < \varepsilon, \qquad \forall s > s_0. \tag{13}$$

Note that Eq. (13) does work even if G(t) is allowed to be singular and non-integrable at the origin. Indeed, the fading memory property requires that the memory kernel decays quickly as the *elapsed time* τ go to infinity, but no limitation is imposed to its behavior near zero.

3. Aging models in linear viscoelasticity

Aging is a gradual process in which the properties of a material change, over time or with use, due to chemical or physical agents. Corrosion, obsolescence, and weathering are examples of aging. In metallurgical processes, aging may be induced by a heat treatment (age hardening). Consequences of aging are of various types. For instance, the damages caused by melting or time-deteriorating processes are examples for decreasing stiffness in elastic springs. Instead, solidification of concrete is an irreversible transition process where the system increases its stiffness and releases a large amount of energy per volume. As pointed out in the sequel, the former type of aging is compatible with thermodynamics under isothermal conditions, while

the latter involves a latent heat and then requires a non-isothermal framework. For definiteness, in this section, we investigate viscoelastic solids and assume that the viscoelastic model holds while the material is subject to chemical or physical agents at constant temperature. It is then understood that we look for the modeling of *aging isothermal viscoelasticity*.

In modeling aging effects, we might think that in Eq. (1), the dependence of the Boltzmann function G on t and s is not merely through the difference $t - s$ but involves t and s separately. It is a central problem to understand how to model G and we would like to argue as far as possible on physical grounds. The recourse to physical arguments to model aging properties is not new in the literature (see, e.g., [6, 7]). Quite naturally one may refer to the classical rheological models with regular kernels (8) and hence to express the aging properties in terms of time-dependent elasticity and viscosity coefficients.

To this purpose, we first address attention to rheological models and, in particular, we consider the standard solid and the Wiechert-Maxwell model [8, 9]. Hence, we establish the functional providing the stress in terms of the strain. This procedure has the advantage of showing how the dependence on the present value and that on the history of ε are influenced by the rheological parameters. Next we generalize the model and look for the corresponding three-dimensional version. For a generic time-dependent relaxation function, a free energy is found to hold for the stress functional as a suitable Graffi-Volterra functional [3, 5, 10]. As a consequence, the stress functional is found to be compatible with thermodynamics subject to weak restrictions on the relaxation function.

3.1. Insights from a rheological model

To get some insights about the modeling of aging viscoelastic solids, we start from the classical standard linear solid where a Maxwell unit, consisting of a spring and a dashpot connected in series, is set in parallel with a lone spring. While we have in mind the behavior of the model in terms of elongation and forces, we extend the formulae to the continuum framework by the standard analogies stress-force and strain-elongation. It is understood that the model is framed

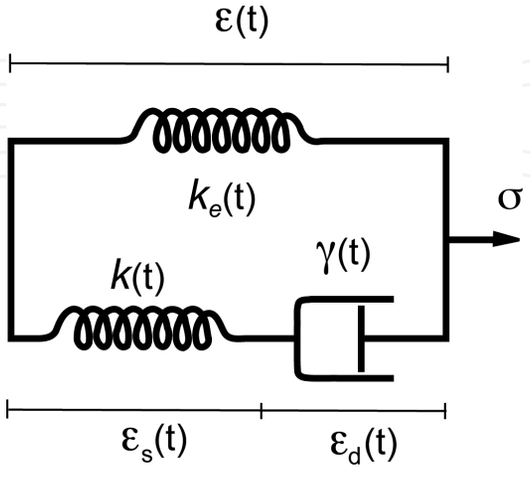


Figure 1. Mechanical scheme of an aging standard viscoelastic solid.

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within a one-dimensional picture, so that both strains and stresses are scalar fields depending on *(x,t)*. Since the elastic and Maxwell elements are in parallel, the strain is the same for every element and the applied stress is the sum of the stress in each element (see **Figure 1**).

Hereafter, the dependence on x of all the fields involved is understood and not written. For the Maxwell element, let ε_s and ε_d be the strain of the spring and that of the dashpot. Hence, denoting by ε the common strain we have

$$\varepsilon = \varepsilon_{\mathrm{s}} + \varepsilon_{\mathrm{d}}, \tag{14}$$

Let σ_e be the stress on the isolated spring while σ_m the stress on the Maxwell element. Then, the total applied stress is given by

$$\sigma = \sigma_{\mathrm{e}} + \sigma_{\mathrm{m}}.$$

Moreover let k and k_e be the elastic modulus (or rigidity) of the spring of the Maxwell element and of the spring in parallel, respectively, and γ the viscosity of the dashpot (**Figure 1**). It is the essential feature of the aging effect that k, k_e and γ are positive functions of the time t. In the Maxwell unit, the spring and the dashpot are in series and hence they are subject to the same stress so that, according to the Hook's law,

$$\sigma_{\mathrm{e}} = k_{\mathrm{e}}\varepsilon, \qquad \sigma_{\mathrm{m}} = k\varepsilon_{\mathrm{s}} = \gamma\partial_t\varepsilon_{\mathrm{d}}, \tag{15}$$

where ∂_t denotes partial differentiation with respect to time t. Using the last equality, from Eq. (14) we have

$$\frac{1}{\alpha}\partial_t\varepsilon_{\mathrm{d}} + \varepsilon_{\mathrm{d}} = \varepsilon, \qquad \alpha = \frac{k}{\gamma} \tag{16}$$

Incidentally, if k_e and k are time independent then time differentiation of Eq. (14) and use of Eq. (15) give

$$\partial_t\varepsilon = \frac{1}{k}(\partial_t\sigma - k_{\mathrm{e}}\partial_t\varepsilon) + \frac{1}{\gamma}(\sigma - k_{\mathrm{e}}\varepsilon),$$

which holds for any viscoelastic standard element. Letting $g_0 = k_e + k$ and $g_\infty = k_e$, this differential equation is equivalent to

$$\partial_t\sigma = g_0\partial_t\varepsilon - \alpha(\sigma - g_\infty\varepsilon),$$

which is commonly used in the literature. So far it is only assumed that

(A1) $k_e, k, \gamma \in \mathbb{C}^1(\mathbb{R})$ and $k_e(t), k(t) \geq 0, \ \gamma(t) \geq \gamma_0 > 0$ for every $t \in \mathbb{R}$.

(A2) $\int_{-\infty}^{t} \alpha(\xi)\mathrm{d}\xi = \infty$

The last condition is fulfilled when $\alpha = k/\gamma$ is a constant function, for instance.

We may regard Eq. (16) as a differential equation in the unknown $\varepsilon_d(t)$. Then, integration over $[t_0,t]$ yields

$$\varepsilon_d(t) = \varepsilon_d(t_0)\exp\left(-\int_{t_0}^{t}\alpha(y)\mathrm{d}y\right) + \int_{t_0}^{t}\exp\left(-\int_{s}^{t}\alpha(y)\mathrm{d}y\right)\alpha(s)\varepsilon(s)\mathrm{d}s.$$

It is convenient to let $t_0 \to -\infty$. By assuming that ε_d is uniformly bounded on $(-\infty,t]$, assumption (A2) allows us to take

$$\lim_{t_0\to-\infty}\varepsilon_d(t_0)\exp\left(-\int_{t_0}^{t}\alpha(s)\mathrm{d}s\right) = 0.$$

Hence, we have

$$\varepsilon_d(t) = \int_{-\infty}^{t}\exp\left(-\int_{s}^{t}\alpha(y)\mathrm{d}y\right)\alpha(s)\varepsilon(s)\mathrm{d}s,$$

and from the representation

$$\sigma = k_e\varepsilon + k\varepsilon_s = [k_e + k]\varepsilon - k\varepsilon_d,$$

we obtain the stress-strain relation

$$\sigma(t) = [k_e(t) + k(t)]\varepsilon(t) - \int_{-\infty}^{t}k(t)\exp\left(-\int_{s}^{t}\alpha(y)\mathrm{d}y\right)\alpha(s)\varepsilon(s)ds. \tag{17}$$

which involves both the present value $\varepsilon(t)$ and the past history $\varepsilon(s), s \in -\infty, t$. Since

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$$\exp\left(-\int_s^t \alpha(y)\mathrm{d}y\right)\alpha(s) = \partial_s \exp\left(-\int_s^t \alpha(y)\mathrm{d}y\right),$$

an integration by parts allows (17) to be rewritten as

$$\sigma(t) = k_e(t)\varepsilon(t) + \int_{-\infty}^t k(t)\exp\left(-\int_s^t \alpha(y)\mathrm{d}y\right)\partial_s\varepsilon(s)ds. \tag{18}$$

provided that ε is uniformly bounded on $[-\infty,t)$. A change of variables $\tau = t-s$ within Eq. (17) leads to the alternate form

$$\sigma(t) = [k_e(t) + k(t)]\varepsilon(t) - \int_0^\infty k(t)\exp\left(-\int_0^\tau \alpha(t-\xi)\mathrm{d}\xi\right)\alpha(t-\tau)\varepsilon(t-\tau)d\tau. \tag{19}$$

Finally, after introducing the so-called *relative history*,

$$\eta^t(\tau) = \varepsilon(t) - \varepsilon(t-\tau),$$

the stress-strain relation may be rewritten as

$$\sigma(t) = k_e(t)\varepsilon(t) + \int_0^\infty k(t)\exp\left(-\int_0^\tau \alpha(t-\xi)\mathrm{d}\xi\right)\alpha(t-\tau)\eta^t(\tau)d\tau. \tag{20}$$

3.2. Some remarks on the aging effect

To give some evidence to the aging effects, we fix a time $t_0 < t$ and we let

$$k_e = k_e(t_0), \qquad k = k(t_0), \qquad \gamma = \gamma(t_0).$$

This statement holds even if $t_0 = -\infty$ provided that we identify the constant values with the limits as $t \to -\infty$. If no aging affects the material, then

$$k_e(t) = k_e, \qquad k(t) = k, \qquad \gamma(t) = \gamma \qquad \forall t \in \mathbb{R}.$$

Otherwise, remembering that $\alpha = k/\gamma$, we introduce the functions

$$\kappa(t) = k_e(t)/k_e, \qquad \varkappa(t) = k(t)/k, \qquad w(y) = \alpha(y)/\alpha.$$

In particular, κ, $\varkappa$ and w equal unity for non-aging materials. This approach leads to identify κ and w with the aging factors of the elastic and the Maxwell elements, respectively. Moreover, Eq. (18) becomes

$$\sigma(t) = k_e\kappa(t)\varepsilon(t) - \int_{-\infty}^{t} k\exp[-\alpha(t-s)]H(t,s)\partial_s\varepsilon(s)ds \tag{21}$$

where

$$H(t,s) = \varkappa(t)\exp\left[\alpha\int_s^t [1-w(y)]\mathrm{d}y\right].$$

This suggests that aging effects can be modeled by means of two functions: κ and H. In our notation, the present value $\varepsilon(t)$ is affected by the factor $\kappa(t)$, whereas the history of ε is affected by the function $H(t,s)$. Letting

$$J(\tau) = k_e + k\exp[-\alpha\tau], \qquad J_\infty = \lim_{\tau\to\infty} J(\tau) = k_e, \qquad \hat{J}(\tau) = J(\tau) - J_\infty$$

the stress-strain relation (21) may be rewritten as

$$\sigma(t) = J_\infty\kappa(t)\varepsilon(t) + \int_{-\infty}^{t} \hat{J}(t-s)H(t,s)\partial_s\varepsilon(s)ds \tag{22}$$

For non-aging materials, $\kappa(t) = H(t,s) \equiv 1$, and this relation reduces to Eq. (5).

We end by observing that in [11] fatigue effects are modeled by using the convolution form (3) modified by the occurrence of a reduced time t_r in place of time t, that is

$$\sigma(t_r) = \int_0^{t_r} G(t_r - s)\partial_s\varepsilon(s)\mathrm{d}s = G_\infty\varepsilon(t_r) + \int_0^{t_r} \hat{G}(t_r - s)\partial_s\varepsilon(s)ds, \tag{23}$$

where

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$$t_{\mathrm{r}} = \int_0^t \frac{1}{a_{\mathrm{T}}(\tau)} \mathrm{d}\tau,$$

a_{T} being named the *time-temperature shift factor*. A similar approach may equally well model other aging effects. If we denote by *f(t)* the function associated with the aging process applied to the body, then we may introduce a *rescaled time* t_{r} which is given by

$$t_{\mathrm{r}} = \int_0^t f(\tau) \mathrm{d}\tau.$$

To our mind the use of a rescaled time t_{r} is an operative way of accounting for aging effects. Hereafter, we show that Eq. (18) may be represented as a linear convolution integral after introducing a suitable rescaled time. We start by letting

$$\exp\left(-\int_s^t \alpha(y) \mathrm{d}y \right) = \exp\big(-[A(t) - A(s)]\big)$$

where, for every fixed t,

$$A(s) = \int_0^s \alpha(y) \mathrm{d}y, \qquad s \le t,$$

is positive and nondecreasing because of (A1). Moreover, from (A2) $\lim_{s \to -\infty} A(s) = -\infty$, so that $t_{\mathrm{r}} = A(t)$ plays the role of a rescaled time. Letting $\hat{\varepsilon}(A(s)) = \varepsilon(s)$, we have

$$\partial_{A(s)} \hat{\varepsilon}(A(s)) \mathrm{d}A(s) = \partial_{\mathrm{s}} \hat{\varepsilon}(A(s)) \mathrm{d}s = \partial_s \varepsilon(s) \mathrm{d}s$$

and the stress-strain relation (18) may be rewritten as

$$\sigma(t) = k_{\mathrm{e}}(t)\varepsilon(t) + k(t) \int_{-\infty}^{A(t)} \exp\big(-[A(t) - A(s)]\big) \partial_{A(s)} \hat{\varepsilon}(A(s)) \mathrm{d}A(s).$$

This expression suggests that aging effects may be partly represented by a suitable change of the time scale within the memory integral. Indeed,

$$\hat{\sigma}(t_{\mathrm{r}}) = k_{\mathrm{e}}(t)\hat{\varepsilon}(t_{\mathrm{r}}) + k(t) \int_{-\infty}^{t_{\mathrm{r}}} \exp\big[-(t_{\mathrm{r}} - s_{\mathrm{r}})\big] \partial_{s_{\mathrm{r}}} \hat{\varepsilon}(s_{\mathrm{r}}) ds_{\mathrm{r}}, \tag{24}$$

where $t_r=A(t)$, $s_r=A(s)$, and $\hat{\sigma}(t_r)=\sigma(t)$. This expression completely matches with Eq. (23) only if k_e and k are constants. For non-aging materials, the scaling turns out to be linear, $t_r=A(t)=\alpha t$, and Eq. (24) becomes

$$\sigma(t) = k_e\varepsilon(t) + \int_{-\infty}^{t} k\exp\left[-\alpha(t-s)\right]\partial_s\varepsilon(s)\mathrm{d}s.$$

3.3. From long to short memory: a possible aging effect

Instead of (A1) and (A2), we assume here

(B1) k_e, $\gamma > 0$ are constants.

(B2) $k \in \mathbb{C}^1(\mathbb{R})$ is positive, nondecreasing and such that

$$\lim_{t\to-\infty} k(t) = \beta > 0, \qquad \lim_{t\to\infty} k(t) = \infty.$$

From (B1), the viscosity of the damper and the rigidity of the lone spring are constants, whereas (B2) translates the fact that the spring in the Maxwell element becomes completely rigid in the longtime. Under the additional very mild assumption[1]

(B3) $\lim_{t\to\infty} \frac{k'(t)}{[k(t)]^2} = 0,$

we can prove that the Kelvin-Voigt viscoelastic model (7) is recovered when $t \to \infty$. Namely, letting

$$k_t(s) = k(t)\exp\left[-\frac{1}{\gamma}\int_s^t k(y)\mathrm{d}y\right],$$

within (B1)-(B3) the distributional convergence

$$k_t \to \gamma\delta_0 \tag{25}$$

occurs as $t \to \infty$, so that Eq. (18) collapses into the Kelvin-Voigt stress-strain relation

$$\sigma_{KV} = k_e\varepsilon + \gamma\partial_t\varepsilon.$$

[1] It is easily seen that (B3) always holds, for instance, when k is eventually concave down as $t \to \infty$.

The rigorous proof can be found in [12]. Since the function $k_t(\cdot)$ is nonnegative for every t, Eq. (25) follows by showing that, for every fixed $v \geq 0$,

$$\lim_{t\to\infty}\int_v^\infty k_t(s)\mathrm{d}s = \begin{cases}\gamma & \text{if } v = 0,\\ 0 & \text{if } v > 0.\end{cases}$$

Assumptions (B1)-(B3) comply with the dissipation principle, as proved by Example 2 in Section 3.7.

3.4. The Wiechert-Maxwell model with aging

The Wiechert-Maxwell model (or Generalized Maxwell model) is composed by a bunch of (say N) Maxwell elements, assembled in parallel, and a further spring in parallel with the whole array. Since all elements are in parallel the strain is the same for every element and the applied stress is the sum of the stress in each element.

ε denotes the common strain and σ_e denotes the stress on the isolated spring, while $\sigma_1,\ldots,\sigma_N$ are the stresses on the Maxwell pairs. Moreover, let k_e, $k_1,\ldots, k_N$ be the elastic modulus (or rigidity) of the $N + 1$ springs and $\gamma_1,\ldots,\gamma_N$ the viscosity coefficients of the dashpots. It is the essential feature of the aging effect that k_e, $k_1,\ldots, k_N$ and $\gamma_1,\ldots,\gamma_N$ are functions of the time t (see **Figure 2**).

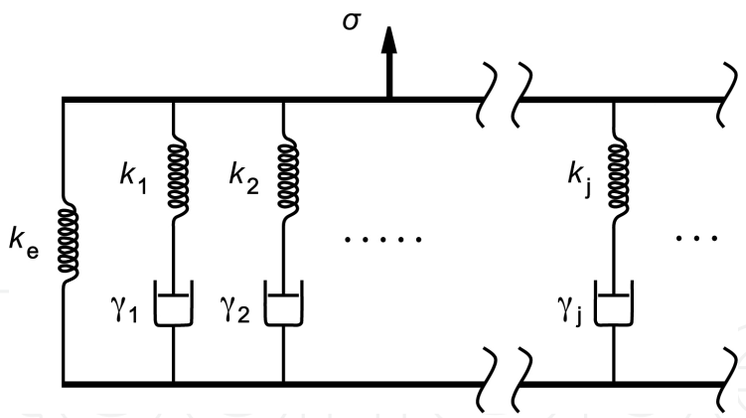


Figure 2. Mechanical scheme of a Wiechert-Maxwell model.

The dependence of k and γ on time requires that we review the elementary arguments to determine the relations among σ_e, $\sigma_1,\ldots,\sigma_N$ and ε. For each jth Maxwell element, let ε_{sj} and ε_{dj} be the strain of the spring and that of the dashpot. Hence, we have

$$\varepsilon = \varepsilon_{sj} + \varepsilon_{dj}, \qquad \sigma_e = k_e\varepsilon, \qquad \sigma_j = k_j\varepsilon_{sj} = \gamma_j\partial_t\varepsilon_{dj}.$$

As a consequence, $\gamma_j\partial_t\varepsilon_{dj} + k_j\varepsilon_{dj} = k_j\varepsilon$, and then

$$\partial_t \varepsilon_{dj} + \alpha_j \varepsilon_{dj} = \alpha_j \varepsilon, \qquad \alpha_j = k_j / \gamma_j. \tag{26}$$

Previous assumptions are generalized for any $j = 1,\ldots,N$, as follows:

(C1) $k_e, k_j, \gamma_j \in \mathbb{C}^1(\mathbb{R})$ and $k_e(t) \geq 0$, $k_j(t) \geq 0$, $\gamma_j(t) \geq \gamma 0 > 0$, for every $t \in \mathbb{R}$,

(C2) $\int_{-\infty}^{t} \alpha_j(\xi)\mathrm{d}\xi = \infty$, for every $t \in \mathbb{R}$.

By regarding Eq. (26) as a differential equation in the unknown $\varepsilon_{dj}(t)$, for the jth Maxwell element we have

$$\sigma_j(t) = k_j(t)\varepsilon(t) - k_j(t)\int_{-\infty}^{t} \exp\left(-\int_s^t \alpha_j(y)\mathrm{d}y\right)\alpha_j(s)\varepsilon(s)\mathrm{d}s.$$

The whole stress on the Wiechert-Maxwell model is then given by

$$\sigma(t) = \left[k_e(t) + \sum_{j=1}^{N} k_j(t)\right]\varepsilon(t) - \sum_{j=1}^{N} k_j(t)\int_{-\infty}^{t} \exp\left(-\int_s^t \alpha_j(y)\mathrm{d}y\right)\alpha_j(s)\varepsilon(s)ds. \tag{27}$$

To give some evidence to the aging effects as in Section 3.2, we assume that all springs and dashpots within the Maxwell elements have common aging factors.

(C3) There exist a time $t_0 \in \mathbb{R}$ (possibly, $t_0 = -\infty$) and two functions $\varkappa, w: \mathbb{R} \to \mathbb{R}$ such that $\varkappa(t_0) = w(t_0) = 1$ and for every $j = 1,\ldots,N$

$$k_j(t) = k_j \varkappa(t), \quad \alpha_j(y) = \alpha_j - \bar{\alpha}[1 - w(y)], \text{ where } \bar{\alpha} = \frac{1}{N}\sum_{j=1}^{N} \alpha_j$$

In particular, $\sum_{j=1}^{N} \alpha_j(y) = \sum_{j=1}^{N} \alpha_j w(y)$. Defining the aging factors as follows:

$$\kappa(t) = k_e(t) / k_e(t_0), \qquad H(t,s) = \varkappa(t)\exp\left[\bar{\alpha}\int_s^t [1 - w(y)]\mathrm{d}y\right],$$

the stress-strain relation (27) may be rewritten in the form (22) by letting

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$$J_\infty = k_e(t_0), \qquad \hat{J}(\tau) = \sum_{j=1}^{N} k_j \exp[-\alpha_j \tau].$$

As in the standard solid model, the present value $\varepsilon(t)$ is affected by the factor $\kappa(t)$ only, whereas the history of ε is affected by the function $H(t, s)$.

3.5. Time-dependent linear viscoelasticity

Borrowing from the Wiechert-Maxwell solid developed above, we now state the uniaxial stress-strain constitutive equation that allows for time-dependent properties. If we introduce the function

$$G(t,s) = k_e(t) + \sum_{j=1}^{N} k_j(t) \exp\left(-\int_s^t \alpha_j(y)\mathrm{d}y \right)$$

which is defined on the half plane $\mathcal{D} = \{(t, s) \in \mathbb{R}^2 : s \le t\}$, the Wiechert-Maxwell constitutive law (27) may be rewritten into the general form

$$\sigma(x,t) = G_0(t)\varepsilon(x,t) - \int_{-\infty}^{t} \partial_s G(t,s)\varepsilon(x,s)\mathrm{d}s, \tag{28}$$

where ∂_s denotes partial differentiation with respect to the variable s and

$$G_0(t) := G(t,t) = k_e(t) + \sum_{j=1}^{N} k_j(t),$$

for all $t \in \mathbb{R}$. In addition, from (C1)–(C2) we have

$$G_\infty(t) := \lim_{s \to -\infty} G(t,s) = k_e(t) > 0, \qquad G_0(t) - G_\infty(t) = \sum_{j=1}^{N} k_j(t) \ge 0.$$

For further convenience, we define $\breve{G}$: $\mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ as $\breve{G}(t, \tau) = G(t, t - \tau)$ so that

$$G_0(t) = \breve{G}(t,0) > 0, \quad G_\infty(t) = \lim_{\tau \to \infty} \breve{G}(t,\tau) > 0, \quad \partial_{\boldsymbol{\tau}} G(t, t-\boldsymbol{\tau}) = \partial_{\boldsymbol{\tau}} \breve{G}(t,\tau).$$

Finally, remembering that $G_0(t) = G(t, t)$, an integration by part of (28) yields

$$\sigma(x,t) = \int_{-\infty}^{t} G(t,s)\partial_s \varepsilon(x,s)ds = G_\infty(t)\varepsilon(t) + \int_{-\infty}^{t} \hat{G}(t,s)\partial_s \varepsilon(x,s)ds \tag{29}$$

provided that Eq. (2) holds and

$$\hat{G}(t,s) = G(t,s) - G_\infty(t).$$

The classical expressions (5) and (9) are recovered from Eqs. (29) and (28), respectively, by simply assuming that $G(t, s) = G(t\text{-}s)$, $s \le t$. If this is the case, G_0 and G_∞ turn out to be constants.

We now look for a general, though linear, time-dependent three-dimensional model. According to Eq. (28), the Cauchy stress tensor $\boldsymbol{T}$ is given by

$$\boldsymbol{T}(\boldsymbol{x},t) = \mathbb{G}_0(t)\boldsymbol{E}(\boldsymbol{x},t) - \int_{-\infty}^{t} \partial_s \mathbb{G}(t,s)\boldsymbol{E}(\boldsymbol{x},s)\mathrm{d}s, \tag{30}$$

where $\mathbb{G}$ stands for the t-dependent relaxation function and

$$\mathbb{G} : \mathcal{D} \to \mathrm{Lin(Sym)}, \qquad \mathbb{G}_0(t) := \mathbb{G}(t,t).$$

Letting $\breve{\mathbb{G}}$: $\mathbb{R} \times \mathbb{R}^+ \to$ Lin(Sym) such that

$$\breve{\mathbb{G}}(t,\tau) = \mathbb{G}(t, t-\tau),$$

a change of the integration variable into Eq. (30) yields an alternate stress-strain relation

$$\boldsymbol{T}(\boldsymbol{x},t) = \mathbb{G}_0(t)\boldsymbol{E}(\boldsymbol{x},t) + \int_0^{\infty} \partial_\tau \breve{\mathbb{G}}(t,\tau)\boldsymbol{E}(\boldsymbol{x},t-\tau)\mathrm{d}\tau, \tag{31}$$

or equivalently

$$\boldsymbol{T}(\boldsymbol{x},t) = \mathbb{G}_\infty(t)\boldsymbol{E}(\boldsymbol{x},t) - \int_0^{\infty} \partial_\tau \breve{\mathbb{G}}(t,\tau)[\boldsymbol{E}(\boldsymbol{x},t) - \boldsymbol{E}(\boldsymbol{x},t-\tau)]\mathrm{d}\tau. \tag{32}$$

Moreover,

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$$\mathbb{G}_0(t) = \breve{\mathbb{G}}(t,0), \qquad \mathbb{G}_\infty(t) = \lim_{\tau\to\infty}\breve{\mathbb{G}}(t,\tau), \qquad \partial_\tau\breve{\mathbb{G}}(t,\tau) = \partial_\tau\mathbb{G}(t,t-\tau).$$

For non-aging materials, $\mathbb{G}(t, s)$ and $\breve{\mathbb{G}}(t, \tau)$ reduce to $\mathbb{G}(t\text{-}s)$ and $\mathbb{G}(\tau)$, respectively.

Hereafter, for ease in writing, we introduce the function

$$\boldsymbol{\mathcal{G}}(t,\tau) = -\partial_\tau\breve{\mathbb{G}}(t,\tau), \qquad (t,\tau)\in\mathbb{R}\times\mathbb{R}^+, \tag{33}$$

which is assumed to satisfy the following properties.

(M1) $(t,\tau)\mapsto\boldsymbol{\mathcal{G}}(t,\tau)\in L^\infty(\mathcal{C})$ for every compact set $\mathcal{C}\subset\mathbb{R}\times\mathbb{R}^+$.

(M2) For every fixed $t\in\mathbb{R}$, the map $\tau\mapsto\boldsymbol{\mathcal{G}}(t,\tau)$ is positive semi-definite, absolutely continuous and summable on $\mathbb{R}^+$. Then, for every $t\in\mathbb{R}$

$$\int_0^\infty\boldsymbol{\mathcal{G}}(t,\tau)d\tau = \mathbb{G}_0(t)-\mathbb{G}_\infty(t)\geq 0.$$

Besides, it is differentiable for all $\boldsymbol{\tau}\in\mathbb{R}^+$ and

$$(t,\tau)\mapsto\partial_\tau\boldsymbol{\mathcal{G}}(t,\tau)\in L^\infty(\mathcal{C})$$

for every compact set $\mathcal{C}\subset\mathbb{R}\times\mathbb{R}^+$.

(M3) For every fixed $\boldsymbol{\tau}>0$, the map $t\mapsto\boldsymbol{\mathcal{G}}(t,\tau)$ is differentiable for all $t\in\mathbb{R}$. Besides,

$$(t,\tau)\mapsto\partial_\tau\boldsymbol{\mathcal{G}}(t,\tau)\in L^\infty(\mathcal{C})$$

for every compact set $\mathcal{C}\subset\mathbb{R}\times\mathbb{R}^+$.

(M4) There exists a nonnegative scalar function M: $\mathbb{R}\to\mathbb{R}^+$, bounded on bounded intervals, such that

$$\partial_t\boldsymbol{\mathcal{G}}(t,\tau)+\partial_\tau\boldsymbol{\mathcal{G}}(t,\tau)\leq -M(t)\boldsymbol{\mathcal{G}}(t,\tau)$$

for every $(t, \tau)\in\mathbb{R}\times\mathbb{R}^+$.

According to (M2), the t-dependent relaxation function $\breve{\mathbb{G}}$ may be represented as

$$\breve{\mathbb{G}}(t,\tau) = \mathbb{G}_0(t) - \int_0^\tau \boldsymbol{\mathcal{G}}(t,\sigma)d\sigma.$$

Borrowing from the scalar case, $\mathbb{G}_\infty(t)$ is assumed to be positive definite for every $t \in \mathbb{R}$, namely

$$\mathbb{G}_\infty(t)\boldsymbol{E}\cdot\boldsymbol{E} > 0 \quad \forall \boldsymbol{E} \in \mathrm{Sym}.$$

Finally, an integration by part of Eq. (30) yields

$$\boldsymbol{T}(\boldsymbol{x},t) = \int_{-\infty}^{t} \mathbb{G}(t,s)\partial_s \boldsymbol{E}(\boldsymbol{x},s)\mathrm{d}s = \mathbb{G}_\infty(t)\boldsymbol{E}(\boldsymbol{x},t) + \int_{-\infty}^{t} \hat{\mathbb{G}}(t,s)\partial_s \boldsymbol{E}(\boldsymbol{x},s)\mathrm{d}s \tag{34}$$

provided that Eq. (2) holds for $\boldsymbol{E}$ and

$$\hat{\mathbb{G}}(t,s) = \mathbb{G}(t,s) - \mathbb{G}_\infty(t).$$

As an advantage, within Eq. (34), $\mathbb{G}$ may be unbounded at the origin.

In order to stress the aging effects, we might assume the following *factorization of the memory kernel G*.

(M5) There exist three functions, $\kappa : \mathbb{R} \to \mathbb{R}^+$, $\mathbb{H} : \mathcal{D} \to \mathrm{Lin}^+(\mathrm{Sym})$ and $\mathbb{J} : \mathbb{R}^+ \to \mathrm{Lin}(\mathrm{Sym})$, such that $\mathbb{H}$ is uniformly bounded, $\lim_{\tau\to\infty} \mathbb{J}(\tau) = \mathbb{J}_\infty$ and, for every $t \in \mathbb{R}$ and $s < t$,

$$\hat{\mathbb{G}}(t,s) = \mathbb{H}(t,s)[\mathbb{J}(t-s) - \mathbb{J}_\infty], \qquad \mathbb{G}_\infty(t) = \mathbb{J}_\infty \kappa(t).$$

Accordingly, the stress-strain relation (34) may be rewritten into the form (22). The aging factors κ and $\mathbb{H}$ reduces to unit when non-aging materials are considered.

So far, we restrict our attention to scrutinize stress-strain relations in the form (30). In particular, for isotropic materials $\breve{\mathbb{G}}$ takes the special form

$$\breve{\mathbb{G}}(t,\tau) = \lambda(t,\tau)\mathbf{1}\otimes\mathbf{1} + 2\mu(t,\tau)\mathbb{I},$$

where $\mathbf{1}$ is the unit second-order tensor, $\mathbb{I}$ is the symmetric fourth-order identity tensor, and λ, μ: $\mathbb{R} \times \mathbb{R}^+$ are named *Lamé relaxation functions*. Accordingly,

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$$\boldsymbol{\mathcal{G}}(t,\tau) = -\partial_\tau \lambda(t,\tau)\mathbf{1}\otimes\mathbf{1} - 2\partial_\tau \mu(t,\tau)\mathbb{I},$$

3.6. A Wiechert-type three-dimensional model

In the sequel, we scrutinize the special isotropic vector-valued kernel $\boldsymbol{\mathcal{G}} = \mathcal{G}_1\mathbf{1}\otimes\mathbf{1} + \mathcal{G}_2\mathbb{I}$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are given by

$$\mathcal{G}_i(t,\tau) = \sum_{j=1}^{N} k_{ji}(t)\alpha_{ji}(t-\tau)\exp\left(-\int_0^\tau \alpha_{ji}(t-y)\mathrm{d}y\right) \qquad i=1,2, \tag{35}$$

as in the rheological Wiechert-Maxwell model devised in Section 3.1. We first prove that properties (M1)-(M4) hold provided that some additional restrictions are imposed on the material functions k_{ji} and α_{ji}. Finally, we give some examples of these functions that fulfill these conditions.

- (M1) Starting from (C1), it is quite trivial to prove this property.
- (M2) By virtue of Eq. (35) and (C1), G_i, i=1,2, are positive and continuously differentiable with respect to t and τ. Moreover,

$$\int_0^\infty \boldsymbol{\mathcal{G}}(t,\tau)\mathrm{d}\tau = \int_0^\infty \mathcal{G}_1(t,\tau)\mathrm{d}\tau\,\mathbf{1}\otimes\mathbf{1} + \int_0^\infty \mathcal{G}_2(t,\tau)\mathrm{d}\tau\,\mathbb{I} = \sum_{j=1}^{N} k_{j1}(t)\,\mathbf{1}\otimes\mathbf{1} + \sum_{j=1}^{N} k_{j2}(t)\,\mathbb{I}. \tag{36}$$

Hence, $\boldsymbol{\mathcal{G}}(t)$, is summable and vanishing at infinity for every $t \in \mathbb{R}$. In addition, we have

$$\partial_\tau \mathcal{G}_i(t,\tau) = \sum_{j=1}^{N}[\alpha'_{ji}(t-\tau) + \alpha_{ji}^2(t-\tau)]\exp\left(-\int_0^\tau \alpha_{ji}(t-y)\mathrm{d}y\right), \quad i=1,2,$$

Hence (M2) is fulfilled.

- (M3) It is obviously true as $\alpha_{ji} \in \mathbb{C}^1(\mathbb{R})$ by virtue of (C1)-(C2). In particular,

$$\partial_t \mathcal{G}_i(t,\tau) = \sum_{j=1}^{N} [k'_{ji}(t)\alpha_{ji}(t-s) + k_{ji}(t)\alpha'_{ji}(t-\tau)$$
$$- k_{ji}(t)\alpha_{ji}(t-\tau)[\alpha_{ji}(t) - \alpha_{ji}(t-\tau)]] \exp\left(-\int_0^{\tau} \alpha_{ji}(t-y)\mathrm{d}y\right)$$
$$= \sum_{j=1}^{N} [k'_{ji}(t) - k_{ji}(t)\alpha_{ji}(t)]\alpha_{ji}(t-\tau) \exp\left(-\int_0^{\tau} \alpha_{ji}(t-y)\mathrm{d}y\right) - \partial_\tau \mathcal{G}_i(t,\tau).$$

- (M4) In order to prove this property we need more restrictive conditions. Since

$$\partial_t \boldsymbol{\mathcal{G}} + \partial_\tau \boldsymbol{\mathcal{G}} = (\partial_t \mathcal{G}_1 + \partial_\tau \mathcal{G}_1)\mathbf{1} \otimes \mathbf{1} + (\partial_t \mathcal{G}_2 + \partial_\tau \mathcal{G}_2)\mathbb{I}$$

  a sufficient condition to ensure (M4) is given by

$$\partial_t \mathcal{G}_i + \partial_\tau \mathcal{G}_i \le k_{ji}^2(t)\mathcal{G}_i, \qquad i = 1,2. \tag{37}$$

In order to prove these inequalities, we now assume

$$k'_{ji}(t)\gamma_{ji}(t) \le k_{ji}^2(t), \quad \forall t \in \mathbb{R}. \tag{38}$$

and for every $t \in \mathbb{R}$ we let

$$M(t) = \min_{i=1,2} \min_{j=1,..,N} \left[\alpha_{ji}(t) - \frac{k'_{ji}(t)}{k_{ji}(t)}\right].$$

It is apparent that $M(t) \ge 0$ and then from (35) it follows:

$$\partial_t \mathcal{G}_i(t,\tau) + \partial_\tau \mathcal{G}_i(t,\tau) = -\sum_{j=1}^{N} \left[\alpha_{ji}(t) - \frac{k'_{ji}(t)}{k_{ji}(t)}\right] k_{ji}(t)\alpha_{ji}(t-\tau) \exp\left(-\int_0^{\tau} \alpha_{ji}(t-y)\mathrm{d}y\right)$$
$$\le -M(t)\sum_{j=1}^{N} k_{ji}(t)\alpha_{ji}(t-\tau) \exp\left(-\int_0^{\tau} \alpha_{ji}(t-y)\mathrm{d}y\right) = -M(t)\mathcal{G}_i(t,\tau)$$

When non-aging material parameters are involved, Eq. (37) reduces to $\partial_\tau \mathcal{G}_i + M\mathcal{G}_i \le 0$, $i = 1, 2$, which implies the exponential decay of the kernels.

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3.7. Some examples

We present here some special expressions of material functions k_{ji} and α_{ji}, $j = 1, 2,\ldots, N$, $i = 1, 2$, which fulfill properties (M1)-(M4).

- *Example 1.*

  For simplicity, we restrict our attention to a single Maxwell element. Letting $j = 1$. and $i = 1, 2$, we choose

$$k_{1i}(t) = \kappa_i, \quad \gamma_{1i}(t) = \frac{\eta_i}{e^{\beta_i t} + 1}, \qquad \beta_i, \kappa_i, \eta_i > 0$$

  and then

$$\alpha_{1i}(t) = \frac{\kappa_i}{\eta_i}(e^{\beta_i t} + 1),$$

  so that (A1) and (A2) hold true. Condition (38) is fulfilled for all $t \in \mathbb{R}$ and for every choice of the parameters, so that

$$M(t) = \min_{i=1,2}\left[\frac{\kappa_i}{\eta_i}(e^{\beta_i t} + 1)\right] > 0 \qquad \forall t \in \mathbb{R}.$$

- *Example 2.*

  Otherwise, for $j = 1$. and $i = 1,2$, we can choose

$$k_{1i}(t) = \kappa_i(e^{\omega_i t} + 1), \quad \gamma_{1i}(t) = \eta_i, \qquad \forall t \in \mathbb{R}, \quad i = 1,2,$$

  where ω_i, k_i, $\eta_i > 0$. Accordingly,

$$\alpha_{1i}(t) = \frac{\kappa_i}{\eta_i}(e^{\omega_i t} + 1), \qquad \forall t \in \mathbb{R}, \quad i = 1,2,$$

  so that (A1) and (A2) hold true. On the other hand, condition (38) reduces to

$$\omega_i \eta_i e^{\omega_i t} \le \kappa_i(e^{\omega_i t} + 1)^2, \quad i = 1,2.$$

  which is equivalent to

$$\frac{\omega_i \eta_i - 2\kappa_i}{\kappa_i} \le e^{\omega_i t} + e^{-\omega_i t} = \cosh \omega_i t, \quad i = 1,2,$$

and is fulfilled for all $t \in \mathbb{R}$ provided that $\omega_i \le 3\, K_i / \eta_i$. If this is the case,

$$M(t) = \min_{i=1,2} \left[\frac{\kappa_i (e^{\omega_i t} + 1)^2 - \omega_i \eta_i e^{\omega_i t}}{\eta_i (e^{\omega_i t} + 1)} \right] > 0 \qquad \forall t \in \mathbb{R}.$$

3.8. Motion, free energies, and thermodynamics

We now derive the motion equation related to the time-dependent viscoelastic stress-strain relation (32) and we examine its compatibility with thermodynamics. The displacement field $\boldsymbol{u}$: $\Omega \times \mathbb{R} \to \mathbb{R}^3$, relative to the reference configuration $\Omega \subset \mathbb{R}^3$, is subject to the *equation of motion*

$$\rho \partial_{tt} \boldsymbol{u} = \nabla \cdot \boldsymbol{T} + \boldsymbol{f},$$

where $\boldsymbol{f}$ is the body force, per unit volume. Hence, from Eqs. (32)–(33), we obtain

$$\rho \partial_{tt} \boldsymbol{u}(\boldsymbol{x},t) - \nabla \cdot \mathbb{G}_\infty(t) \nabla \boldsymbol{u}(\boldsymbol{x},t) - \nabla \cdot \int_0^\infty \boldsymbol{\mathcal{G}}(t,s) \nabla [\boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t-s)] \mathrm{d}s = \boldsymbol{f}(\boldsymbol{x},t). \tag{39}$$

In order to introduce the initial boundary value problem for this equation, we have to take in mind that it is not invariant under time shift.

Consistent with linear viscoelasticity, we restrict attention to isothermal processes, namely those where the temperature is constant and uniform. Hence, the local form of the second law inequality reduces to the *dissipation inequality*

$$-\rho \frac{\mathrm{d}}{\mathrm{d}t} \psi + \boldsymbol{T} \cdot \boldsymbol{D} \ge 0,$$

where ρ is the mass density, ψ is the Helmholtz free energy density per unit volume, and $\boldsymbol{D}$ is the stretching tensor. Again for consistency with the linearity of the model, we let the mass density ρ be constant and take the approximation

$$\boldsymbol{D} \simeq \partial_t \boldsymbol{E} = \frac{1}{2} \left[\nabla \partial_t \boldsymbol{u} + \nabla \partial_t \boldsymbol{u}^T \right].$$

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Accordingly, we take the dissipation inequality in the form

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\psi \le \boldsymbol{T} \cdot \nabla \partial_t \boldsymbol{u}. \tag{40}$$

In materials with memory, the motion equations are required to rule both the displacement instantaneous value $\boldsymbol{u}$(t) and its history up to t. Letting $t_0 \in \mathbb{R}$ be arbitrarily fixed, we define the *relative displacement history* $\zeta^t(\boldsymbol{x},s)$, with $(t,s) \in [t_0,T] \times \mathbb{R}^+$, by

$$\zeta^t(\boldsymbol{x},s) = \begin{cases} \boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t-s), & s \le t - t_0, \\ \zeta_{t_0}(\boldsymbol{x},s-t+t_0) + \boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t_0), & s > t - t_0, \end{cases} \tag{41}$$

where ζ_{t_0} is the prescribed initial (relative) past history of $\boldsymbol{u}$ up to t_0,

$$\zeta_{t_0}(\boldsymbol{x},s) = \boldsymbol{u}(\boldsymbol{x},t_0) - \boldsymbol{u}(\boldsymbol{x},t_0 - s) \qquad s \in [0,+\infty).$$

Accordingly, $\zeta^t(\boldsymbol{x},0) = 0$ and the motion equation (39) becomes a system

$$\begin{cases} \rho \partial_{tt} \boldsymbol{u}(\boldsymbol{x},t) - \nabla \cdot \mathsf{G}_\infty(t) \nabla \boldsymbol{u}(\boldsymbol{x},t) - \nabla \cdot \int_0^\infty \boldsymbol{\mathcal{G}}(t,s) \nabla \zeta^t(\boldsymbol{x},s) \mathrm{d}s = \boldsymbol{f}(\boldsymbol{x},t), \\ \partial_t \zeta^t(\boldsymbol{x},s) = \partial_t \boldsymbol{u}(\boldsymbol{x},t) - \partial_s \zeta^t(\boldsymbol{x},s) \end{cases} \tag{42}$$

where $\boldsymbol{u} : \Omega \times [t_0, +\infty)$ and $\zeta^t : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$, $T \in [t_0, +\infty)$ are the unknown variables. Their initial conditions are prescribed at $t_0 \in \mathbb{R}$ as follows

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x},t_0) = \boldsymbol{u}_{t_0}(\boldsymbol{x}), \\ \partial_t \boldsymbol{u}(\boldsymbol{x},t_0) = \boldsymbol{v}_{t_0}(\boldsymbol{x}), \\ \zeta^{t_0}(\boldsymbol{x},s) = \zeta_{t_0}(\boldsymbol{x},s), \quad s \in [0,+\infty). \end{cases} \tag{43}$$

Let $H^0 = [\mathrm{L}^2(\Omega)]^3$ and $H^1 = [\mathrm{H}^1{}_0(\Omega)]^3$, and let $\langle \cdot, \cdot \rangle_j$ denote the usual inner product in H^j, $j = 0,1$. For every $t \ge t_0$, we introduce the family of memory spaces

$$\mathcal{M}_t = L^2_{\mathcal{G}}(\mathbb{R}^+; H^1), \qquad \langle \boldsymbol{\zeta}, \boldsymbol{\xi} \rangle_{\mathcal{M}_t} = \int_0^\infty \langle \boldsymbol{\mathcal{G}}(t,s)\boldsymbol{\zeta}(s), \boldsymbol{\xi}(s) \rangle_1 \mathrm{d}s,$$

where $\langle \cdot, \cdot \rangle_{\mathcal{M}_t}$ denotes the t-dependent weighted L^2 inner product equipping each $\mathcal{M}_t$. In this functional framework, the motion equation admits a unique regular solution. The proof of this result can be found in [12, Th. 4.5].

Theorem 1. *Let* $\mathcal{H}_t = H^1 \times H^0 \times \mathcal{M}_t$ and $\boldsymbol{f} \in H^0$. Under assumptions (M1)-(M4), for every $T > t_0$ and every initial datum $\boldsymbol{z}_{t_0} = (\boldsymbol{u}_{t_0}, \boldsymbol{v}_{t_0}, \boldsymbol{\zeta}_{t_0}) \in \mathcal{H}_{t_0}$, problem *(42)-(43)* admits a unique solution $\boldsymbol{z}(t) = (\boldsymbol{u}(t), \partial_t \boldsymbol{u}(t), \boldsymbol{\zeta}^t)$ on the interval $[t_0, T]$ *such that*

$$\boldsymbol{u} \in \mathcal{C}([t_0, T], H^1) \cap \mathcal{C}^1([t_0, T], H^0), \qquad \boldsymbol{\zeta}^t \in \mathcal{M}_t, \quad \forall t \in [t_0, T],$$

and

$$\sup_{t \in [t_0, T]} \|\boldsymbol{z}(t)\|_{\mathcal{H}_t} < C,$$

for some $C > 0$ *depending only on* T, t_0 *and the size of the initial datum* $\|z\|_{\mathcal{H}_{t_0}}$.

Now, we introduce a time-dependent free energy density borrowing its expression from the Graffi's single-integral quadratic form (see [13] and references therein). Let

$$\psi(\nabla \boldsymbol{u}(t), \nabla\, \boldsymbol{\zeta}^t, t) = \frac{1}{2}\mathbb{G}_\infty(t)\nabla \boldsymbol{u}(t) \cdot \nabla \boldsymbol{u}(t) + \frac{1}{2}\int_0^\infty \boldsymbol{\mathcal{G}}(t,s)\nabla\, \boldsymbol{\zeta}^t(s) \cdot \nabla\, \boldsymbol{\zeta}^t(s) \mathrm{d}s.$$

For ease in writing, hereafter the dependence on x is understood and not written. In addition, we assume $\rho = 1$. After integrating over Ω, we end up with the total free energy functional

$$\Psi_t(\boldsymbol{u}(t), \boldsymbol{\zeta}^t) = \int_\Omega \psi(\nabla \boldsymbol{u}(t), \nabla\, \boldsymbol{\zeta}^t, t) \mathrm{d}v = \frac{1}{2}\langle \mathbb{G}_\infty(t)\boldsymbol{u}(t), \boldsymbol{u}(t) \rangle_1 + \frac{1}{2}\| \boldsymbol{\zeta}^t \|^2_{\mathcal{M}_t}.$$

Theorem 2. For an aging viscoelastic material, the dissipation inequality (40) is fulfilled provided that (M4) holds and

$$\mathbb{G}'_\infty(t) \le 0, \quad \forall t \in \mathbb{R}. \tag{44}$$

Proof. First we observe that

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$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbb{G}_\infty(t)\boldsymbol{u}(t),\boldsymbol{u}(t)\rangle_1 = 2\langle \mathbb{G}_\infty(t)\boldsymbol{u}(t),\partial_t\boldsymbol{u}(t)\rangle_1 + \langle \mathbb{G}'_\infty(t)\boldsymbol{u}(t),\boldsymbol{u}(t)\rangle_1$$

Then, by virtue of (42), and some integration by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\zeta}^t\|^2_{\mathcal{M}_t} = \int_0^\infty \langle [\partial_t\boldsymbol{\mathcal{G}}(t,s)+\partial_s\boldsymbol{\mathcal{G}}(t,s)]\,\boldsymbol{\zeta}^t(s),\boldsymbol{\zeta}^t(s)\rangle_1\,\mathrm{d}s + 2\langle\,\boldsymbol{\zeta}^t,\partial_t\boldsymbol{u}(t)\rangle_{\mathcal{M}_t},$$

and, taking into account (41),

$$\begin{aligned}\langle \boldsymbol{T}(t),\nabla\partial_t\boldsymbol{u}(t)\rangle_0 &= \langle \mathbb{G}_\infty(t)\boldsymbol{u}(t)+\int_0^\infty \boldsymbol{\mathcal{G}}(t,s)\boldsymbol{\zeta}^t(s)\mathrm{d}s,\partial_t\boldsymbol{u}(t)\rangle_1\\ &= \mathbb{G}_\infty(t)\langle \boldsymbol{u}(t),\partial_t\boldsymbol{u}(t)\rangle_1 + \langle\boldsymbol{\zeta}^t,\partial_t\boldsymbol{u}(t)\rangle_{\mathcal{M}_t}\end{aligned}$$

In summary, we end up with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(\boldsymbol{u}(t),\boldsymbol{\zeta}^t) &= \langle \boldsymbol{T}(t),\nabla\partial_t\boldsymbol{u}(t)\rangle_0 + \frac{1}{2}\langle \mathbb{G}'_\infty(t)\boldsymbol{u}(t),\boldsymbol{u}(t)\rangle_1\\ &\quad + \frac{1}{2}\int_0^\infty \langle [\partial_t\boldsymbol{\mathcal{G}}(t,s)+\partial_s\boldsymbol{\mathcal{G}}(t,s)]\,\boldsymbol{\zeta}^t(s),\boldsymbol{\zeta}^t(s)\rangle_1\,\mathrm{d}s\end{aligned}$$

Owing to (M4), this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(\boldsymbol{u}(t),\boldsymbol{\zeta}^t) \le \langle \boldsymbol{T}(t),\nabla\partial_t\boldsymbol{u}(t)\rangle_0 - \frac{1}{2}M(t)\|\nabla\,\boldsymbol{\zeta}^t\|^2_{\mathcal{M}_t} + \frac{1}{2}\langle \mathbb{G}'_\infty(t)\nabla\boldsymbol{u}(t),\nabla\boldsymbol{u}(t)\rangle$$

where $M(t) \ge 0$, and (44) finally implies the dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(\boldsymbol{u}(t),\boldsymbol{\zeta}^t) \le \langle \boldsymbol{T}(t),\nabla\partial_t\boldsymbol{u}(t)\rangle_0.$$

4. Singular kernel models in linear viscoelasticity

The study of singular kernel problems is motivated by the modeling of new materials and, in particular, of the mechanical behavior of some new viscoelastic polymers and bio-inspired materials. As noticed in [14], the appropriate way to handle the response of certain time-dependent systems exhibiting long tail memories is to account for power laws, both for creep and relaxation, leading to the occurrence of fractional hereditariness. Another example encountered in natural materials is mineralized tissues as bones, ligaments, and tendons. They exhibit a marked power-law time-dependent behavior under applied loads (see e.g. [15]), since

the high stiffness of the crystals in such tissues is combined with the exceptional hereditariness of the collagen protein-based matrix. In all these cases, we are forced to abandon the regularity assumptions (8) and assume the memory kernels obey Eq. (6) and are unbounded at the origin.

The idea of singular kernels to model particular cases of viscoelastic behaviors was introduced by Boltzmann [2] in the nineteenth century. The fast growth of polymer science motivated further developments of viscoelasticity in the middle of the twentieth century [16, 17], but a Volterra-type integro-differential equation with a regular kernel (typically, a finite sum of exponentials) was preferred to the Boltzmann approach in the modeling of the mechanical response [5, 18]. Later, however, many authors addressed their interest to singular kernel problems, both under the analytical as well as the model point of view [19–24], and their thermodynamical admissibility was analyzed in [25]. In modern viscoelasticity, it is a central problem to understand how to model the memory kernels, and it should be argued as far as possible on physical grounds. So, the first question to answer to is why do we consider singular kernel models. More recently, new viscoelastic materials, such as viscoelastic gels, have been discovered and their mechanical properties are well described by virtue of convolution integral with singular kernels: for instance, fractional and hypergeometric kernels [1]. This applicative interest gave rise to a wide research activity concerning singular kernel problems, both in rigid thermodynamics with memory as well as in viscoelasticity (see, for instance, [26–31], and especially concerning applications of fractional calculus to the theory of viscoelasticity and the study of new bio-inspired materials [15, 32–35]. A recent book [36] provides an overview on this subject. In this framework, Fabrizio [37] analyzes the connection between Volterra and fractional derivatives models and shows how experimental results motivate us to adopt, as in this present article, less restrictive functional requirements on the kernel representing the relaxation modulus.

4.1. Singular isothermal viscoelastic body with memory

To start with, the one-dimensional classical viscoelasticity problem is recalled. It reads

$$u_{tt} = G(0)u_{xx} + \int_0^t G'(t-\tau)u_{xx}(\tau)d\tau + f \tag{45}$$

$$u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1 \ \text{ in } \ \Omega;\ u = 0 \ \text{ on } \ \Sigma = \partial\Omega \times (0,T) \tag{46}$$

where $\Omega = (0,1)$. When, to model the physical behavior of new materials or polymers, the regularity assumptions on the relaxation modulus are relaxed, G is assumed to satisfy the following functional requirements

$$G \in L^1(0,T) \cap C^2(0,T), \qquad G' \notin L^1(0,T),\ \forall T \in \mathbb{R} \tag{47}$$

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that is, now, the relaxation function $G(t)$ is not required to be finite at $t = 0$ and then Eq. (45) loses its meaning and, hence, needs to be replaced by a different one. The method to overcome this difficulty, devised in [28], consists in the introduction of a suitable sequence of regular problems, depending on a *small parameter* $0 < \varepsilon \ll 1$ which, in the limit $\varepsilon \to 0$ reduce to the singular problem under investigation. The key steps of the approximation strategy can be sketched as follows.

- Let K, termed *integrated relaxation function*, denote

$$K(\xi) := \int_0^{\xi} G(\tau) d\tau \ , \ K(0) = 0 \, ; \tag{48}$$

  it is well defined, since $G \in L^1(0,T), \forall T \in \mathbb{R}^+$.

- Then, introduce the regular problems:

$$P^{\varepsilon} : \ u_{tt}^{\varepsilon} = G^{\varepsilon}(0) u_{xx}^{\varepsilon} + \int_0^t G'^{\varepsilon}(t-\tau) u_{xx}^{\varepsilon}(\tau) d\tau + f \quad \text{where} \quad G^{\varepsilon}(\cdot) := G(\varepsilon + \cdot) \tag{49}$$

together with the initial and boundary conditions

$$u^{\varepsilon}|_{t=0} = u_0(x), \quad u_t^{\varepsilon}|_{t=0} = u_1(x), \quad u^{\varepsilon}|_{\partial\Omega\times(0,T)} = 0 \ , \ t < T. \tag{50}$$

- For each ε, the problem P^{ε} is a *regular* approximated problem since $G^{\varepsilon}(0)$ is finite and, therefore, the initial boundary value problem (49)-(50) admits a unique solution:
- then, find approximated solutions u^{ε}, $0 < \varepsilon \ll 1$,
- show the existence of the limit solution $u := \lim_{\varepsilon \to 0} u^{\varepsilon}$
- prove the uniqueness of the limit solution u which represents a weak solution admitted by the *singular* problem.

Note that, corresponding to each value of ε, the problem P^{ε} is equivalent to the *integral* equation:

$$P^{\varepsilon} : \ u^{\varepsilon}(t) = \int_0^t K^{\varepsilon}(t-\tau) u_{xx}^{\varepsilon}(\tau) \mathrm{d}\tau + u_1 t + u_0 + \int_0^t \mathrm{d}\tau \int_0^{\tau} f(\xi) \mathrm{d}\xi, \tag{51}$$

Partial derivation w.r. to t, twice, of Eq. (51) delivers Eq. (49) together with initial and boundary conditions (50). Furthermore, when $\varepsilon = 0$, we obtain the well-defined problem

$$P : \ u(t) = \int_0^t K(t-\tau) u_{xx}(\tau) \mathrm{d}\tau + u_1 t + u_0 + \int_0^t \mathrm{d}\tau \int_0^{\tau} f(\xi) \mathrm{d}\xi \tag{52}$$

where the superscripts, in the case $\varepsilon = 0$, are omitted for notational simplicity. Hence, the following theorems can be proved. Here only the outlines of the proofs are given; the details are comprised in [28] when homogeneous Dirichlet b.c.s (50) are imposed and in [27] when homogeneous Neumann b.c.s are considered.

Theorem 1*Given u^ε solution to the integral problem P^ε (51), then*

$$\exists u(t) = \lim_{\varepsilon \to 0} u^\varepsilon(t) \ \ in \ \ L^2(Q), Q = \Omega \times (0,T). \tag{53}$$

Proof's outline:

- weak formulation, on introduction of test functions $\varphi \in H^1(\Omega \times (0,T)$ s.t. $\varphi_x = 0$, on $\partial\Omega$,
- consider separately the terms without ε,
- the terms with u^ε and K^ε,
- prove convergence via Lebesgue's theorem.

Furthermore, the weak solution, as stated in the following theorem, is unique.

Theorem 2*The integral problem (52) admits a unique weak solution.*

Proof's outline: The result is proved by contradiction, see [28] for details, assuming there are two different solution and, then, showing that such an assumption leads to a contradiction.

As a final remark, we wish to emphasize that, since the isothermal rigid viscoelasticity model exhibits remarkable analogies, under the analytical point of view [38], with rigid thermodynamics with memory, then, analogous results can be obtained also in the study of singular kernel problems in such a framework [29].

4.2. Magneto-viscoelasticity problems

This section is concerned about a problem in magneto-viscoelasticity, again under the assumption of a memory kernel singular at the origin. The interest in magneto-viscoelastic material finds its motivation in the growing interest in new materials such as magneto-rheological elastomers or, in general, magneto-sensitive polymeric composites (see [39–41] and references therein). The model adopted here to describe the magneto-elastic interaction is introduced in [42]. Evolution problems in magneto-elasticity are studied in [43] and, later magneto-viscoelasticity problems are considered in [44, 45]. Notably, under the analytical viewpoint, when the coupling with magnetization is considered, the problem to study is modeled via a nonlinear integro-differential system while the purely viscoelastic problem is linear.

To understand the model equations, a brief introduction on the model magnetization here adopted, based on [46], who revisited the Gilbert magnetization model. Accordingly, when $\Omega \subset \mathbb{R}^3$ denotes the body configuration, the related magnetization changes according to the

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Landau Lifshitz equation, which, in Gilbert form, where m represents the magnetization vector reads

$$\gamma^{-1}\boldsymbol{m}_t - \boldsymbol{m}\times\left(a\Delta\boldsymbol{m} - \boldsymbol{m}_t\right) = 0 \ \ ,|\boldsymbol{m}| = 1, \gamma, a \in \mathbb{R}^+. \tag{54}$$

The quantities of interest, in the general three-dimensional case, are the following ones:

$\boldsymbol{u} := \boldsymbol{u}(\boldsymbol{x}, t)$ *displacement vector*
$\boldsymbol{m} := \boldsymbol{m}(\boldsymbol{x}, t)$ *magnetization vector*
$\mathbb{G}(s) = \{G_{klmn}(s)\},\ s \in [0, T]$ *visco-elasticity tensor*
$\mathbb{L} = \{\lambda_{klmn}\}$ *magneto-elasticity tensor*
$\mathbb{E} = \{\epsilon_{lm}\}$ *strain tensor*
$\boldsymbol{\epsilon}(\boldsymbol{u}) := \{\epsilon_{lm}(\boldsymbol{u})\} = \frac{1}{2}\{\boldsymbol{u}_{l,m} + \boldsymbol{u}_{m,l}\}$ *deformation tensor*
$\mathbb{G}\,\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} = G_{klmn}\epsilon_{kl}(\boldsymbol{u})\epsilon_{mn}(\boldsymbol{v})$
$\mathbb{L}\boldsymbol{m}\otimes\boldsymbol{m} = \{\lambda_{klmn}m_k m_l\}$
$\mathbb{L}\boldsymbol{m}\otimes\nabla\boldsymbol{u} = \{\lambda_{klmn}m_k\epsilon_{lm}(\boldsymbol{u})\}$
$\mathbb{L}\boldsymbol{m}\otimes\boldsymbol{m}\cdot\nabla\boldsymbol{u} = \lambda_{klmn}m_k m_l\epsilon_{mn}(\boldsymbol{u})$

where the coefficients λ_{klmn} are subject to the condition

$$\lambda_{ijkl} = \lambda_1\delta_{ijkl} + \lambda_2\delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{55}$$

Then, the following constitutive assumptions are assumed. Thus, the exchange magnetization energy is given by

$$E_{ex}(\mathbf{m}) = \frac{1}{2}\int_\Omega a_{ij}m_{k,i}m_{k,j}d\Omega \tag{56}$$

where

- $a_{ij} = a_{ji}$ symmetric positive definite matrix
- $a_{ij} = a\,\delta_{ji},\ a \in \mathbb{R}^+$ diagonal matrix (most materials).

Then, the magneto-elastic energy is given by

$$E_{em}(\mathbf{m}, H) = \frac{1}{2}\int_\Omega \boldsymbol{\lambda}_{ijkl}\, m_i\, m_j\, \varepsilon_{kl}(H)\, d\Omega \tag{57}$$

- $\lambda_{ijkl} = \lambda_1\delta_{ijkl} + \lambda_2\delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$,
- $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise,
- $\lambda_1, \lambda_2, \lambda_3$ $\mathbb{R}_{\text{constants}}$.

The viscoelastic energy is given by

$$E_{ve}(\boldsymbol{u}) = \frac{1}{2}\int_\Omega G_{klmn}(0)\varepsilon_{kl}\varepsilon_{mn}\mathrm{d}\Omega + \frac{1}{2}\int_0^t d\tau\left(\int_\Omega G'_{klmn}(t-\tau)\varepsilon_{kl}(\tau)\varepsilon_{mn}(\tau)d\Omega\right) \tag{58}$$

where the tensor's entries of $\mathbb{G}$ satisfy

- $G_{klmn} = G_{mnkl} = G_{lkmn}$
- $G_{klmn}e_{kl}e_{mn} \geq \beta e_{kl}e_{kl}, \qquad \beta > 0, e_{kl} = e_{lk}$
- $G'_{klmn}e_{kl}e_{mn} \leq 0$
- $G''_{klmn}e_{kl}e_{mn} \geq 0$

Then, the total energy of the system is given by

$$E(\mathbf{m}, u) = E_{\text{ex}}(\mathbf{m}) + E_{em}(\mathbf{m}, u) + E_{ve}(u), \tag{59}$$

taking into account, further to the single magnetic and viscoelastic contribution, of the exchange energy.

4.2.1. A regular magneto-viscoelasticity problem

The problem we are concerned about is the behavior of a viscoelastic body subject also to the presence of a magnetic field; in the one-dimensional case, it is modeled by the nonlinear system

$$\begin{cases} u_{tt} - G(0)\ u_{xx} - \int_0^t G'(t-\tau)\ u_{xx}(\tau)\ \mathrm{d}\tau - \dfrac{\lambda}{2}(\Lambda(\mathbf{m})\cdot\mathbf{m})_x = f, \\ \\ \mathbf{m}_t + \mathbf{m}\dfrac{|\mathbf{m}|^2 - 1}{\varepsilon} + \lambda\Lambda(\mathbf{m})u_x - \mathbf{m}_{xx} = 0, \end{cases} \quad \text{in } \mathcal{Q} \tag{60}$$

where $\Omega = (0,1)$, $\mathcal{Q} := \Omega \times (0, T)$ and $\mathcal{M} \equiv (0,\boldsymbol{m}$, where $\boldsymbol{m} = (m_1, m_2)$, denotes the magnetization vector, orthogonal to the conductor, since $\boldsymbol{u} \equiv (u, 0, 0)$, when both quantities are written in $\mathbb{R}^3$

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in addition, v is the outer unit normal at the boundary $\partial\Omega$, Λ is a linear operator defined by $\Lambda(\boldsymbol{m}) = (m_1, m_2)$ the scalar function u is the displacement in the direction of the conductor itself, here identified with the x axis and λ is a positive parameter. In addition, the term f represents an external force which also includes the deformation history.

In [44], the existence and uniqueness of the solution to the problem given by (60), together with the following initial and boundary conditions, is proved

$$u(\cdot,0) = u_0 = 0,\ \mathbf{m}(\cdot,0) = \boldsymbol{m}_0, \quad |\boldsymbol{m}_0| = 1 \quad \text{in } \Omega, \tag{61}$$

$$u = 0, \qquad \frac{\partial \boldsymbol{m}}{\partial \nu} = 0 \qquad \text{on} \quad \Sigma = \partial\Omega \times (0,T), \tag{62}$$

under the assumptions

$$\begin{cases} u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega), \quad \mathrm{m}_0 \in \boldsymbol{H}^1(\Omega), \\ f \in L^2(\Omega \times (0,T)), \quad G(t) \in C^2(0,T), \end{cases} \tag{63}$$

Then, the following existence and uniqueness result [44] holds.

Theorem 3 Given the problem (60)-(63), it admits a unique solution for any given T > 0 and ε small enough (i.e., $\varepsilon\lambda^{-2}GT$), s.t.

- $u \in C^0([0,T];\ H_0^1(\Omega)) \cap C^1([0,T];\ L^2(\Omega))$,
- $\boldsymbol{m} \in \mathbf{C}^0([0,T]; H^1(\Omega)) \cap \mathbf{L}^2(0,T; H^2(\Omega))$,
- $\boldsymbol{m}_t \in \mathbf{L}^2(0,T; L^2(\Omega))$.

The proof, is based on the a priori estimate on the viscoelastic term:

$$\frac{1}{2}\int_\Omega |\varphi_x|^2\, dx + \frac{1}{2}\int_\Omega |\varphi_t|^2\, dx \le \alpha e^{T} C(f, \varphi_0, \varphi_1), \quad \alpha, C \in \mathbb{R}^+ \tag{64}$$

A result of existence, in a three-dimensional regular magneto-viscoelasticity problem, is given in [45].

4.2.2. A singular magneto-viscoelasticity problem

Now, as in the purely viscoelastic case, when the requirement $G' \in L^1(0,T)$ is removed, the magneto-viscoelasticity problem cannot be written under the form (60); however, since $G \in L^1(0,T)$ via integration with respect to time of the integro-differential equation, it can be formulated in the following equivalent form

$$\begin{cases} u_t(t) - \int_0^t G(t-\tau)u_{xx}(\tau)d\tau - u_1 - \int_0^t \frac{\lambda}{2}(\Lambda(\mathbf{m})\cdot\mathbf{m})_x d\tau = \int_0^t f(\tau)d\tau \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{Q} \\ \mathbf{m}_t + \mathbf{m}\frac{|\mathbf{m}|^2 - 1}{\delta} + \lambda\Lambda(\mathbf{m})u_x - \mathbf{m}_{xx} = 0\,, \end{cases} \tag{65}$$

The strategy to prove the existence result [47], relies on the fact that the classical problem (60) as soon as the initial time is $t_0 = \varepsilon$, for any arbitrary $\varepsilon > 0$, the relaxation modulus satisfies the classical regularity requirements, namely, as in subSection 4.0.1, $G^\varepsilon(\cdot) := G(\varepsilon + \cdot)$ implies that $G^\varepsilon \in C^2[0,\ T]$ Hence, each *time-translated* approximated problems

$$P^\varepsilon : \begin{cases} u^\varepsilon_{tt} - G^\varepsilon(0)u^\varepsilon_{xx} - \int_0^t G'^\varepsilon(t-\tau)u^\varepsilon_{xx}(\tau)d\tau - \frac{\lambda}{2}(\Lambda(\mathbf{m}^\varepsilon)\cdot\mathbf{m}^\varepsilon)_x = f \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{Q} \\ \mathbf{m}^\varepsilon_t + \mathbf{m}^\varepsilon\frac{|\mathbf{m}^\varepsilon|^2 - 1}{\delta} + \lambda\Lambda(\mathbf{m}^\varepsilon)u^\varepsilon_x - \mathbf{m}^\varepsilon_{xx} = 0, \end{cases} \tag{66}$$

with the assigned initial and boundary conditions

$$u^\varepsilon(\cdot,0) = u_0 = 0,\ u^\varepsilon_t(\cdot,0) = u_1,\quad \mathbf{m}^\varepsilon(\cdot,0) = \mathbf{m}_0,\quad \text{in } \Omega, \tag{67}$$

$$u^\varepsilon = 0,\qquad \frac{\partial \mathbf{m}^\varepsilon}{\partial \nu} = 0 \qquad \text{on} \quad \Sigma = \partial\Omega\times(0,T), \tag{68}$$

is regular. Then, according to [44], the problem P^ε admits a unique strong solution. According to [47], where all the needed proofs are given, the following existence result can be stated.

Theorem 3.1*For all* $T > 0$, *there exists a weak solution* $(u, \boldsymbol{m})$ *to the problem* (65)-(61)-(62), *that is a vector function* $(u, \boldsymbol{m})$ *s.t.*

- $u \in L^{\infty}(0,T;H_0^1(\Omega))$;
- $u_t \in L^{\infty}(0,T;L^2(\Omega))$;
- $\mathrm{m} \in L^{\infty}(0,T;H^1(\Omega))$;
- $\mathrm{m}_t \in L^2(\mathcal{Q})$.

which satisfies

$$
\begin{aligned}
&-\int_{\mathcal{Q}} \phi_t\, u^{\varepsilon}\,(t)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{Q}}\int_0^t G^{\varepsilon}\,(t-\tau)\,u_x^{\varepsilon}\,(\tau)\,\phi_x\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{Q}}\int_0^t\frac{\lambda}{2}\,\Lambda(\mathbf{m}^{\varepsilon})\cdot\mathbf{m}^{\varepsilon}\,\phi_t\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{\mathcal{Q}}\left[u_1+\int_0^t f(\tau)\,\mathrm{d}\tau\right]\phi\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{Q}}\boldsymbol{\psi}_t\cdot\mathbf{m}^{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t+\\
&\int_{\mathcal{Q}}\mathbf{m}_0\cdot\boldsymbol{\psi}\,(\cdot,0)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{Q}}\left(\frac{|\,\mathbf{m}^{\varepsilon}\,|^2-1}{\delta}\right)\boldsymbol{\psi}\cdot\mathbf{m}^{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{\mathcal{Q}}\lambda\,u_x^{\varepsilon}\,\Lambda\,(\mathbf{m}^{\varepsilon})\cdot\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t-\int_{\mathcal{Q}}\mathbf{m}_x^{\varepsilon}\cdot\boldsymbol{\psi}_x\,\mathrm{d}x\,\mathrm{d}t=0\,.
\end{aligned}
\tag{69}
$$

$\forall\phi$ smooth s.t. $\phi(0,t)=\phi(1,t)=0,\ \phi(\cdot,T)=0,$ and $\forall\boldsymbol{\psi}\equiv(\psi_1,\psi_2)$ s.t. $\boldsymbol{\psi}(x,T)=0,$ where ϕ and $\boldsymbol{\psi}$ arbitrarily chosen test functions in $\mathcal{Q}$.

The proof, not included here, is provided in [47].

Proof's Outline:

- consider the viscoelastic energy associated to the problem to obtain a suitable *a priori estimate*
- consider the energy connected to interaction between magnetic and viscoelastic effects to obtain further suitable estimates
- consider the total energy together with smooth enough initial data to estimate the energy at the generic time *t*
- introduce an appropiate weak formulation and suitable test functions
- consider separately the limit process when $\varepsilon \to 0$

As a closing remark, we can note that, under the applicative point of view as well as under the analytical one, the free energy associated to the model plays a crucial role. Indeed, the proof relies on estimates which are based on the free energies connected to the model here adopted. Specifically, the viscoelastic energy allows [47], also in the magneto-viscoelastic case, to prove an a priori estimate on which the subsequent results are based. This is not surprising since the connection relating free energies and evolution problems is well known; see for instance [48] and references therein.

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