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# Path Integral Methods in Generalized Uncertainty Principle

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## 1. Introduction

As we know the modern physics is based on the two fundamental pillars of physics. The first is the general relativity theory, discovered by Albert Einstein, which gave us a detailed explanation of the macro-dimension world; for example, planets, stars, galaxies and clusters of galaxies and even the extra-universe, that explains the force exerted by the gravitational field of a massive object on any body within the vicinity of its surface. It mainly uses the Riemannian geometry as a mathematical formalism. The second perspective is the quantum mechanics, that describes the micro-dimensions, such as; molecules, atoms and even the smallest components of the latter, like the electrons and quarks, which explains the three principal forces in the micro-world, (like the weak force, electromagnetic force and strong force). It uses the operator theory acting on a Hilbert space algebra (von Neumann algebras). After the mid-twentieth century a new theory in physics has been emerged called the Non-commutative (NC) geometry. It came to unify the four fundamental forces, And its roots go back to the inability of classical physics to explain certain macroscopic phenomena. Mathematically described by a Poisson manifold  $M$ , and denoted by  $F(M)$  algebra (commutative) regular functions on  $M$ , called observable. In this case, it is important to quantify these Poisson varieties (quantum mechanics) in order to obtain results more "precise" than classical mechanics. Many studies have focused on the possibility of quantification of such varieties and the idea of using the theory of algebraic deformations, called "deformation of quantization" is due to (Bayen et al, 1978). And as has been creativity in this mathematical aspect, through a group of researchers ( for example, (Bordemann et al., (2005); Makhoul, 2007)). The motivations to the occurrence of this deformation theory are multiple, in a string theory, (see for example (Veneziano, (1986); Amati et al, (1987); Konishi et al, (1990); Kato, (1990) and Guida et al, (1991) also Gross et al, 1988) in a quantum gravity, (Garay, 1995) in a non-commutative geometry, (Capozziello et al, 2000) and in a black hole physics (Scardigli, (1999); Scardigli & Casadio, (2003)).

In the same context, the innovations physicists were prominent by inserting this study algebra on several applications in physics. The first of these applications is the papers of (Kempf et al, 1995), which is based on introducing a parameter of deformation  $\beta$  in the Heisenberg uncertainty principle, given by:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \beta (\Delta p)^2 + \beta \langle p \rangle^2 \right], \quad (1)$$

Where the commutation between the position and the momentum operators in one dimension, can be written as:

$$[\hat{x}, \hat{p}] = i\hbar (1 + \beta p^2), \quad (2)$$

This example, we have found several applications in the non-relativistic quantum mechanics; such as, the harmonic oscillator of arbitrary dimensions (cf., e.g., Refs (Chang et al, (2002); Hinrichsen & Kempf, (1996); Kempf, (1994), Kempf et al, (1995) and Kempf, (1997)), the problem of the cosmological constant has been studied (Pet & Polchinski, (1999)), the effect of the minimal length on the 3-D Coulomb potential has also been studied in (Brau, (1999) and Akhoury, (2003)), the one-dimensional box (Nozari & Azizi, (2006)), the study of the dynamics of a non-relativistic particle with mass variable  $m(t)$  (cf., e.g., Ref Merad & Falek, (2009)), and also in the relativistic extension of this problem has some limited attempts, among them we mention: the Dirac equation in the presence of a minimum length in (Nouicer, (2006)), where the Dirac oscillator in one dimension has been solved exactly, the generalized Dirac equation was recently studied by Nozari (Nozari & Karami, (2005)), the bosonic oscillator DKP (spin 0 and 1)-dimensional and three-dimensional that were treated respectively in (Falek & Merad, (2009); (2010)).

On the other hand, the path integral is an alternative technical of Heisenberg and Schrödinger methods. This approach is based on the Lagrangian form, which offers an alternative view of quantum mechanics, that has quickly established itself in theoretical physics, with its extension on quantum field theory and gauge theories. The extension of this technique within the framework of deformed algebra was applied to the relativistic and non-relativistic quantum mechanics. For example, the harmonic oscillator in one dimension (Nouicer, 2006) and in D dimensions (Chergui et al, 2010) and the (1+1)-dimensional Klein–Gordon equation with mixed vector-scalar linear potentials (Merad et al, 2010) but recently it is shown that the problem concerning the choice of point discretization in the path integral is not yet resolved and this arbitrariness is fixed by comparing the discrete action in its infinitesimal form with the corresponding wave equation by which judicious choice of the discretization parameter is indicated by the order of operators ( cf., e.g., Refs Benzair et al, (2012); (2014)). This resembles the case of curved spaces in which the mid-point (i.e.  $\bar{x} = (x_j + x_{j-1}) / 2$ ) was privileged to have correct quantum correction due to the curvature. Similar arguments in the case of space-time transformations (cf., e.g., Ref (Khandekar et al, 1993)) are presented. In ( Ref (Kleinert, 1990)), an outcome considers all points of the interval in an equivalent manner but unfortunately with minimal length deformation. The problem is raised and we will say that it is more like that of the quantization with constraints (see, e.g., Ref (Lecheheb et al, 2007)).

In this chapter we propose to construct the path integral formalism in the momentum space representation to adapt this type of deformation, defined in Eqs. (1) and (2). Then, we

describe in detail the method of calculating the quantum corrections according to Feynman approach (cf., e.g., Ref (Khandekar et al, 1993)). As it is shown in (Benzair et al, (2012); (2014)), different methods gave different results, where the quantum correction  $C_T$  depends on the  $\alpha$ -point discretization interval and there are specific options for the choice of the discretization  $\alpha$ -parameter, which coincides with the equation method, and this leads to the vanishing of the term  $C_T$  and this corresponds to  $\alpha = 0$  and  $\alpha = 1/2$  within the method of (Kleinert, 1990) and to  $\alpha = \frac{1}{2} \left(1 \pm 1/\sqrt{2}\right)$  within the standard method of (Khandekar et al, 1993).

## 2. Brief review of a minimal length relation

As it seems that in the Kempf's work (cf., e.g., Ref (Kempf et al, 1995)), there is a minimal value of  $(\Delta x)_{\min}$  different zero which is given by:

$$(\Delta x)_{\min}(\langle p \rangle) = \hbar \sqrt{\beta} \sqrt{1 + \beta \langle p \rangle^2}. \quad (3)$$

$$= \hbar \sqrt{\beta} \text{ corresponds to } \langle p \rangle = 0. \quad (4)$$

The operators  $(\hat{x}, \hat{p})$  that verifies the commutation relation amended (2) may be considered as the functions of  $q$  and  $p$  operators, satisfying the relationship of canonical commutation:  $[\hat{q}, \hat{p}] = i\hbar$ , as follow

$$\hat{x} = i\hbar \left(1 + \beta p^2\right) \hat{q}, \quad \hat{p} = p. \quad (5)$$

In the momentum space representation, we define the expressions of  $\hat{x}$  and  $\hat{p}$  act on the functions  $\Psi(p)$  defined by:

$$\hat{p}.\Psi(p) = p\Psi(p), \quad \hat{x}\Psi(p) = i\hbar \left(1 + \beta p^2\right) \frac{\partial}{\partial p} \Psi(p). \quad (6)$$

The most important condition to be satisfied by the representation (2), is the preservation of the operators symmetry  $x$  and  $p$ , where their values are real. Despite the fact that  $p$  is not modified, then its symmetry is obvious; it is not the case for the  $x$  operator. Indeed, the symmetry condition is written

$$(\langle \Psi | \hat{x} | \Phi \rangle) = \langle \Psi | (\hat{x} | \Phi \rangle). \quad (7)$$

The scalar product should be defined as

$$\langle \Psi | \Phi \rangle = \int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2} \Psi^*(p) \Phi(p). \quad (8)$$

The modification of this product implies a new closure relation, which is written as

$$\int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2} |p\rangle \langle p| = 1. \quad (9)$$

Inserting the latter relation in the scalar product of two momentum eigenvectors operator, we get:

$$\langle p | p' \rangle = (1 + \beta p^2) \delta(p - p'), \quad (10)$$

also is given by

$$\langle p | p' \rangle = \delta \left( \frac{1}{\sqrt{\beta}} \arctan \sqrt{\beta} p - \frac{1}{\sqrt{\beta}} \arctan \sqrt{\beta} p' \right). \quad (11)$$

In this case the Schrödinger equation for the particle in the harmonic oscillator of momentum space representation in one dimension, can be written as

$$\hat{H} = \left( \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2} \hat{x}^2 \right) = \left[ \frac{p^2}{2m} + \frac{m\omega^2}{2} \left( i\hbar (1 + \beta p^2) \frac{\partial}{\partial p} \right)^2 \right]. \quad (12)$$

Exact solutions of spectrum energy and the normalized eigenfunctions of the bound states are defined in (Chang et al, (2002):

$$E_n = \hbar\omega \left[ \left( n + \frac{1}{2} \right) \sqrt{1 + \left( \frac{\beta\hbar m\omega}{2} \right)^2} + (n^2 + n + 1) \left( \frac{\beta\hbar m\omega}{2} \right) \right]. \quad (13)$$

and

$$\Psi_n(p) = \sqrt{\frac{2^{2\lambda-1}(\lambda+n)n!\sqrt{\beta}}{\pi\Gamma(2\lambda+n)[\Gamma(\lambda)]^{-2}}} \left[ \frac{1}{\sqrt{1+\beta p^2}} \right]^\lambda C_n^\lambda \left( \frac{\sqrt{\beta} p}{\sqrt{1+\beta p^2}} \right), \quad (14)$$

with  $C_n^\lambda$  are Gegenbauer polynomials.

### 3. Construction propagators with generalized Heisenberg principle

The purpose of this section is to discuss the propagators and the quantum corrections via the standard Feynman approach, for a non-relativistic quantum mechanics in the context of the deformed and non-deformed space at  $\alpha$ -point discretization. Then we will circulate this study on the relativistic problems through the two applications chosen below.

#### 3.1. Ordinary quantum mechanics case

In this subsection, we will illustrate the spatio-temporal technique and the method of calculating the quantum corrections according to the standard Feynman approach in the

ordinary quantum mechanics. So, in one dimension, we consider the propagator expression of ordinary non-relativistic quantum mechanics to the discontinuous form path integral

$$K_N = A_N \int \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} S_n \right\} \prod_{n=1}^N dz_n, \quad (15)$$

with  $A_N = \left( \sqrt{\frac{m}{2\pi i \hbar \varepsilon}} \right)^{N+1}$  and  $S_n$  is the discrete action into intervals  $[n-1, n]$ , takes the form follows:

$$S_n = \frac{m}{2\varepsilon} (z_n - z_{n-1})^2 - \varepsilon V(x_n). \quad (16)$$

According to the standard method of Feynman (cf., e.g., Ref (Khandekar et al, 1993)), we will apply the spatio-temporal method of processing  $\bar{z}_n^{(\alpha)} = f(\bar{q}_j^{(\alpha)})$  to  $\alpha$ -point discretization defined as:

$$\bar{z}_n^{(\alpha)} = \alpha z_n + (1 - \alpha) z_{n-1}. \quad (17)$$

Two terms of quantum corrections have appeared ( $C_{act}$ ,  $C_{mes}$ ). Let's start by calculating the correction to the  $C_{act}$  action. Developing  $\Delta z_n$  to  $\alpha$ -point discretization, we have:

$$\Delta z_n = \Delta q_n \bar{f}_n^{(\alpha)'} \left( 1 + \frac{(1-2\alpha)}{2!} \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \Delta q_n + \frac{(1-\alpha)^3 + \alpha^3}{3!} \frac{\bar{f}_n^{(\alpha)'''}}{\bar{f}_n^{(\alpha)'}} \Delta q_n^2 \right), \quad (18)$$

where the prime on the function  $\bar{f}_j^{(\alpha)}$  indicates the derivative  $\bar{f}_j^{(\alpha)}$  over  $\bar{q}_j^{(\alpha)}$ . So, the kinetic energy term in the action is:

$$\frac{\Delta z_n^2}{4\varepsilon} = \frac{\Delta q_n^2}{4\varepsilon} \left( \bar{f}_n^{(\alpha)'} \right)^2 \left( 1 + (1-2\alpha) \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \Delta q_n + \left[ \frac{(1-2\alpha)^2}{4} \left( \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \right)^2 + \frac{(1-\alpha^3) + \alpha^3}{3} \frac{\bar{f}_n^{(\alpha)'''}}{\bar{f}_n^{(\alpha)'}} \right] \Delta q_n^2 \right). \quad (19)$$

The potential energy takes a simple form:

$$\varepsilon V(z_n) = \varepsilon V(\bar{q}_n^{(\alpha)}) + O(\varepsilon^2) = \varepsilon V(\bar{q}_n^{(\alpha)}). \quad (20)$$

In addition, we note that the transformation  $z = f(q)$  made the path integral rather complicated, where the mass parameter is transformed into  $m \bar{f}_n^{(\alpha)'}$ . At this point, we would apply the transformation over time parameter in order to overcome this difficulty:

$$\varepsilon = \sigma_n f'(q_n) f'(q_{n-1}), \text{ where } \sigma_n = s_n - s_{n-1}. \quad (21)$$

Development  $f'(q_n)$  and  $f'(q_{n-1})$  at  $\alpha$ -point discretization in order two of  $\Delta q_n$ , is written as:

$$\varepsilon = \sigma_n \left( \bar{f}_n^{(\alpha)'} \right)^2 \left( 1 + (1-2\alpha) \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \Delta q_n + \left( \frac{(1-\alpha)^2 + \alpha^2}{2} \frac{\bar{f}_n^{(\alpha)'''}{\bar{f}_n^{(\alpha)'}} - \alpha(1-\alpha) \left( \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \right)^2 \right) \Delta q_n^2 \right). \quad (22)$$

From expressions (19) and (22), we can deduce the quantum correction from the action:

$$\exp \left[ \frac{i}{\hbar} \frac{m}{2\varepsilon} (\Delta z_n)^2 \right] = \exp \left[ \frac{i}{\hbar} \frac{m}{2\sigma_n} (\Delta q_n)^2 \right] (1 + C_{act}), \quad (23)$$

with

$$C_{act} = \frac{i}{\hbar} \frac{m}{8\sigma_n} \Delta q_n^4 \left[ \left( -16\alpha^2 + 16\alpha - 3 \right) \left( \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \right)^2 - \frac{2}{3} \frac{\bar{f}_n^{(\alpha)'''}{\bar{f}_n^{(\alpha)'}} \right]. \quad (24)$$

Turning now to calculate the second correction  $C_{mes}$ , we have:

$$A_N \prod_{n=1}^N dz_n = \left( \sqrt{\frac{m}{2\pi i \hbar \varepsilon}} \right)^{N+1} \prod_{n=1}^N dq_n f'(q_n). \quad (25)$$

This can be achieved by rewriting:

$$A_N \prod_{n=1}^N dz_n = [f'(q_b) f'(q_a)]^{-1/2} \prod_{n=1}^{N+1} \left( \sqrt{\frac{m f'(q_n) f'(q_{n-1})}{2\pi i \hbar \varepsilon}} \right) \prod_{n=1}^N dq_n. \quad (26)$$

Then, we develop  $f'(q_n)$  and  $f'(q_{n-1})$  to the second order of  $\Delta q_n$  as follows:

$$[f'(q_n) f'(q_{n-1})]^{1/2} = \bar{f}_n^{(\alpha)'} \left( 1 + \frac{(1-2\alpha)}{2} \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \Delta q_n + \left( \frac{(1-\alpha)^2 + \alpha^2}{4} \frac{\bar{f}_n^{(\alpha)'''}{\bar{f}_n^{(\alpha)'}} - \frac{\alpha(1-\alpha)}{2} \left( \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \right)^2 \right) \Delta q_n^2 \right) \quad (27)$$

From the expressions (26), (27) and (22) we can deduce  $C_{mes}$ ,

$$A_N \prod_{n=1}^N dz_n = \prod_{n=1}^{N+1} \left( \sqrt{\frac{m}{2\pi i \hbar \sigma_n}} \right) (1 + C_{mes}) \prod_{n=1}^N dq_n, \quad (28)$$

to the form

$$C_{mes} = \frac{(1-2\alpha)^2}{4} \left( \frac{\bar{f}_j^{(\alpha)''}}{\bar{f}_j^{(\alpha)'}} \right)^2 \Delta q_n^2. \quad (29)$$

To calculate the total quantum corrections  $C_T$ , we use the following expression

$$\langle (\Delta q)^{2n} \rangle = \left( \frac{i\hbar\sigma}{m} \right)^n (2n-1)!!, \quad (30)$$

The result is

$$\langle (\Delta q)^2 \rangle = \left( \frac{i\hbar\sigma}{m} \right), \quad \langle (\Delta q)^4 \rangle = \left( \frac{i\hbar\sigma}{m} \right)^2 (3)!! = -3 \left( \frac{\hbar\sigma}{m} \right)^2. \quad (31)$$

By combining all these results, we arrive at:

$$C_T = V_{eff} = -\sigma_n \frac{i\hbar^2}{4m} \left[ \left( \frac{11}{2} - 28\alpha^2 + 28\alpha \right) \left( \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \right)^2 - \frac{\bar{f}_n^{(\alpha)'''}{\bar{f}_n^{(\alpha)'}} \right]. \quad (32)$$

When we used the standard formalism of Feynman (cf., e.g., Ref (Khandekar et al, 1993)) to  $\alpha$ -point discretization, a single case of  $\alpha$ -point gave the same result of the method equation, this value is  $\alpha = 1/2$ , where the effective potential is

$$V_{eff} = \sigma_n \frac{i\hbar^2}{4m} \left[ \frac{3}{2} \left( \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \right)^2 - \frac{\bar{f}_n^{(\alpha)'''}{\bar{f}_n^{(\alpha)'}} \right]. \quad (33)$$

But in the presence  $\beta$  parameter deformation, the value of  $\alpha$ -point discretization is different according to what has been explained by the Feynman approach.

### 3.2. The non-relativistic QM with minimal length case

We illustrate the use of the path integral formalism of the transition amplitude in the momentum space representation for a quantum time-independent quadratic systems with the presence of nonzero minimum position uncertainty. We start with the propagator expressed as:

$$K^{(\beta)}(p_b, t_b, p_a, t_a) = \lim_{N \rightarrow \infty} \left\langle p_b \left| \prod_{j=1}^N \exp\left(\frac{i\varepsilon}{\hbar} \hat{H}\right) \right| p_a \right\rangle. \quad (34)$$

Inserting the closure relation for the momentum states (9) between each pair of infinitesimal evolution operators ( $U(j, j-1) = \exp(\frac{i\varepsilon}{\hbar} \hat{H})$ ), we obtain

$$K^{(\beta)}(p_b, t_b, p_a, t_a) = \lim_{N \rightarrow \infty} \prod_{j=1}^N \int \frac{dp_j}{(1+\beta p_j^2)} \prod_{j=1}^{N+1} \left[ 1 + \frac{i\varepsilon}{\hbar} \hat{H} \right] \langle p_j | p_{j-1} \rangle, \quad (35)$$

where the projection relation ( $\langle p_j | p_{j-1} \rangle$ ) is defined in eq.(10). Then we inject the Hamiltonian operator of a particle with nonzero minimum position uncertainty on the projection relation for any systems studied. It is clear that there are only few cases where

it is exactly solvable; namely, the case of a linear potential ( $V(x) = gx$ ) and the case of a harmonic potential ( $V(x) = \frac{m\omega^2}{2}x^2$ ). In this chapter, we will study only the form quadratic, for example, the harmonic oscillator potential in one dimension and the spinorial relativistic particle.

The construction of momentum space path integral representation of the transition amplitude for a particle moving in the potential of the harmonic oscillator in one dimension with nonzero minimum position uncertainty. Following the well-known steps to construct a quantum propagator  $K^{(\beta)}$ , we write:

$$K^{(\beta)}(p_b, t_b; p_a, t_a) = \lim_{N \rightarrow \infty} \prod_{j=1}^N \int \frac{dp_j}{(1+\beta p_j^2)} \prod_{j=1}^{N+1} \int \frac{dq_j}{2\pi\hbar} (1+\beta p_j^2) \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N+1} \left[ q_j \Delta p_j - \varepsilon \frac{p_j^2}{2m} - \varepsilon \frac{m\omega^2}{2} \left( (1+\beta p_j^2)^2 q_j^2 - 6i\hbar\beta p_j q_j (1+\beta p_j^2) - 2\hbar^2\beta (1+3\beta p_j^2) \right) \right] \right\}. \quad (36)$$

The form of expression (36) shows that the path integral over the variables  $q_j$  is Gaussian, so the result is simply written as:

$$K^{(\beta)}(p_b, t_b, p_a, t_a) = \lim_{N \rightarrow \infty} \prod_{j=1}^N \int \frac{dp_j}{(1+\beta p_j^2)} \prod_{j=1}^{N+1} \frac{1}{\sqrt{2\pi i \hbar m \omega^2 \varepsilon}} \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N+1} \left[ \frac{(\Delta p_j)^2}{2m\omega^2\varepsilon(1+\beta p_j^2)^2} + \frac{3i\hbar\beta p_j \Delta p_j}{(1+\beta p_j^2)} - \varepsilon \frac{p_j^2}{2m} - \varepsilon \frac{m\omega^2\hbar^2\beta}{2} (-2 + 3\beta p_j^2) \right] \right\}. \quad (37)$$

This latter expression shows that the kinetic term is similar a system of space dependent mass and can be removed by an  $\alpha$ -point coordinate transformation method (see, Ref (Khandekar et al, 1993)), where the  $\alpha$ -point discretization interval defined by

$$\bar{p}_j^{(\alpha)} = \alpha p_j + (1-\alpha) p_{j-1}. \quad (38)$$

So, we will introduce the function  $f(p)$ , where the first derivative of  $f(\bar{p}_j^{(\alpha)})$  is equal to  $1/(1+\beta\bar{p}_j^{(\alpha)2})$ . Thus, there are three quantum corrections obtained in expression (37).

- The first related to the action  $C_{act}^{(1)}$
- the second correction related to the measure  $C_{mes}^{(1)}$
- and the third correction related to the f-factor  $C_f^T$

Expanding the exponential  $\left( \frac{i}{\hbar} \frac{(\Delta p_j)^2}{2m\omega^2\varepsilon(1+\beta p_j^2)^2} \right)$  of the  $\alpha$ -point discretization interval, we find

$$\exp \left[ \frac{i}{\hbar} \sum_{j=1}^{N+1} \left( \frac{(\Delta p_j)^2}{2m\omega^2\epsilon(1+\beta p_j^2)} \right) \right] = \exp \left[ \frac{i}{\hbar} \sum_{j=1}^{N+1} \left( \frac{(\bar{f}_j^{(\alpha)'})^2 (\Delta p_j)^2}{2m\omega^2\epsilon} \right) \right] (1 + C_{act}^{(1)}), \quad (39)$$

where

$$C_{act}^{(1)} = \frac{i}{2\hbar m\omega^2\epsilon} \left[ \frac{2(1-\alpha)\bar{f}_j^{(\alpha)''} (\bar{f}_j^{(\alpha)'})^2}{\bar{f}_j^{(\alpha)'}} (\Delta p_j)^3 + (1-\alpha)^2 \left[ \left( \frac{\bar{f}_j^{(\alpha)''}}{\bar{f}_j^{(\alpha)'}} \right)^2 + \frac{\bar{f}_j^{(\alpha)''''}}{\bar{f}_j^{(\alpha)'}} \right] (\bar{f}_j^{(\alpha)'})^2 (\Delta p_j)^4 \right] - \frac{2(1-\alpha)^2}{(2\hbar m\omega^2\epsilon)^2} \left( \frac{\bar{f}_j^{(\alpha)''}}{\bar{f}_j^{(\alpha)'}} \right)^2 (\bar{f}_j^{(\alpha)'})^4 (\Delta p_j)^6, \quad (40)$$

and the measure term will be developed as.

$$\begin{aligned} \prod_{j=1}^N \int \frac{dp_j}{1+\beta p_j^2} &= \sqrt{(1+\beta p_b^2)(1+\beta p_a^2)} \prod_{j=1}^N \int dp_n \prod_{j=1}^{N+1} \frac{1}{\sqrt{(1+\beta p_j^2)(1+\beta p_{j-1}^2)}} \\ &= \left[ \frac{1}{f_{b,a}'} \right] \prod_{n=1}^N \int dp_j \prod_{n=1}^{N+1} \bar{f}_j^{(\alpha)'} (1 + C_m^{(1)}), \end{aligned} \quad (41)$$

where

$$C_m^{(1)} = \frac{(1-2\alpha)}{2} \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \Delta p_n + \left( \frac{(1-\alpha)^2 + \alpha^2}{4} \frac{\bar{f}_n^{(\alpha)''''}}{\bar{f}_n^{(\alpha)'}} - \frac{\alpha(1-\alpha)}{2} \left( \frac{\bar{f}_n^{(\alpha)''}}{\bar{f}_n^{(\alpha)'}} \right)^2 \right) \Delta p_n^2. \quad (42)$$

and the  $f$ -factor term will be developed as

$$\exp \left( -\frac{3\beta p_j \Delta p_j}{1+\beta p_j^2} \right) = 1 + C_f^T, \quad (43)$$

where

$$C_f^T = \frac{3}{2} \left( \frac{\bar{f}_j^{(\alpha)''}}{\bar{f}_j^{(\alpha)'}} \right) \Delta p_j + \frac{9}{8} \left( \frac{\bar{f}_j^{(\alpha)''}}{\bar{f}_j^{(\alpha)'}} \right)^2 (\Delta p_j)^2 + \frac{3}{2} (1-\alpha) \left( \frac{\bar{f}_j^{(\alpha)''''}}{\bar{f}_j^{(\alpha)'}} - \left( \frac{\bar{f}_j^{(\alpha)''}}{\bar{f}_j^{(\alpha)'}} \right)^2 \right) (\Delta p_j)^2, \quad (44)$$

Now, in order to convert this expression to the usual form of Feynman path integral, let us bring the kinetic term to the conventional one, with a constant mass term by using the following coordinate transformation  $p_j = g(k_j)$ , this transformation generates two corrections:

- the first related to the action  $C_{act}^{(2)}$

- and the other correction related to the measure  $C_m^{(2)}$

The  $\alpha$ -point expansion of  $\Delta p_j$  reads at each ( $j$ )

$$\Delta p_j = g(k_j) - g(k_{j-1}) = \Delta k_j \bar{g}_j^{(\alpha)'} \left( 1 + \frac{(1-2\alpha)}{2!} \frac{\bar{g}_j^{(\alpha)''}}{\bar{g}_j^{(\alpha)'}} \Delta k_j + \frac{(1-\alpha)^3 + \alpha^3}{3!} \frac{\bar{g}_j^{(\alpha)'''}}{\bar{g}_j^{(\alpha)'}} \Delta k_j^2 \right). \quad (45)$$

The choice of  $g$  is fixed by the following condition  $((\partial g / \partial k) = (\partial f / \partial p)^{-1})$ , that makes the transformation  $g(k) = \frac{\tan \sqrt{\beta} k}{\sqrt{\beta}}$  where the region  $p \in [-\infty, +\infty]$  is mapped to  $k \in [-\pi/2\sqrt{\beta}, \pi/2\sqrt{\beta}]$ . Thereafter, we develop the exponential of the kinetic term as

$$\exp \left[ \frac{i}{\hbar} \sum_{j=1}^{N+1} \left( \frac{(\Delta p_j)^2}{2m\omega^2\varepsilon(1+\beta p_j^2)} \right) \right] = \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N+1} \left[ \frac{\Delta k_j^2}{2m\omega^2\varepsilon} \right] \right\} [1 + C_{act}^{(1)}] [1 + C_{act}^{(2)}], \quad (46)$$

where  $C_{act}^{(2)}$  is given by

$$\begin{aligned} C_{act}^{(2)} = & \left\{ \frac{i}{2\hbar m\omega^2\varepsilon} \left[ (1-2\alpha) \frac{\bar{g}_j^{(\alpha)''}}{\bar{g}_j^{(\alpha)'}} \left( \bar{g}_j^{(\alpha)'} \right)^2 \left( \bar{f}_j^{(\alpha)'} \right)^2 \Delta k_j^3 \right. \right. \\ & + \left[ \frac{(1-2\alpha)^2}{4} \left( \frac{\bar{g}_j^{(\alpha)''}}{\bar{g}_j^{(\alpha)'}} \right)^2 + \frac{(1-\alpha)^3 + \alpha^3}{3} \frac{\bar{g}_j^{(\alpha)'''}}{\bar{g}_j^{(\alpha)'}} \right] \left( \bar{f}_j^{(\alpha)'} \right)^2 \left( \bar{g}_j^{(\alpha)'} \right)^2 \Delta k_j^4 \\ & \left. \left. - \frac{(1-2\alpha)^2}{2(2\hbar m\omega^2\varepsilon)^2} \left( \frac{\bar{g}_j^{(\alpha)''}}{\bar{g}_j^{(\alpha)'}} \right)^2 \left( \bar{g}_j^{(\alpha)'} \right)^4 \left( \bar{f}_j^{(\alpha)'} \right)^4 \Delta k_j^6 + \dots \right] \right\}. \quad (47) \end{aligned}$$

The measure induce also a correction

$$\begin{aligned} \prod_{j=1}^N \int \frac{dp_j}{1+\beta p_j^2} &= \sqrt{\frac{1}{f_b' f_a' g_b' g_a'}} \prod_{n=1}^N \int dk_j \prod_{j=1}^{N+1} \bar{f}_j^{(\alpha)'} \bar{g}_j^{(\alpha)'} \left( 1 + C_m^{(2)} \right) \left( 1 + C_m^{(1)} \right) \\ &= \prod_{j=1}^N \int dk_j \prod_{j=1}^{N+1} \left( 1 + C_m^{(1)} \right) \left( 1 + C_m^{(2)} \right), \quad (48) \end{aligned}$$

where  $C_m^{(2)}$  is given by

$$C_m^{(2)} = \frac{(1-2\alpha)}{2} \frac{\bar{g}_j^{(\alpha)''}}{\bar{g}_j^{(\alpha)'}} \Delta k_j + \left[ -\alpha \frac{(1-\alpha)}{2} \left( \frac{\bar{g}_j^{(\alpha)''}}{\bar{g}_j^{(\alpha)'}} \right)^2 + \frac{(1-\alpha)^2 + \alpha^2}{4} \frac{\bar{g}_j^{(\alpha)'''}}{\bar{g}_j^{(\alpha)'}} \right] \Delta k_j^2. \quad (49)$$

By combining all these corrections as follows:

$$1 + C_T = \left(1 + C_{act}^{(1)}\right) \left(1 + C_{act}^{(2)}\right) \left(1 + C_m^{(1)}\right) \left(1 + C_m^{(2)}\right) \left(1 + C_f^T\right). \quad (50)$$

where the corrections terms are evaluated perturbatively, using the expectation values

$$\langle (\Delta k)^{2\ell} \rangle = \left(i\hbar m \omega^2 \varepsilon\right)^\ell (2\ell - 1)!! \quad (51)$$

So through all of these corrections, one can conclude the correction total  $C_T$  depending on the  $\alpha$ -point discretization can be obtained as

$$C_T = \frac{i}{\hbar} \hbar^2 m \omega^2 \varepsilon \beta \left[ -1 + \left(8\alpha^2 - 8\alpha + \frac{5}{2}\right) \tan^2 \sqrt{\beta} k_j \right] \quad (52)$$

Furthermore, a judicious choice of the discretization parameter is indicated by the order of operators present in the wave equation method. The result coincides with the path integral approach when  $C_T$  equals

$$C_T = i\hbar \varepsilon m \omega^2 \beta \left[ -1 + \frac{3}{2} \tan^2 \sqrt{\beta} k \right], \quad (53)$$

We note that the different  $\alpha$ -point discretization value (i.e.  $\alpha = \frac{1}{2} (1 \pm 1/\sqrt{2})$ ) are obtained compared with the ordinary quantum mechanics case ( $\alpha = 1/2$ ). So the propagator (37) becomes,

$$K^{(\beta)}(p_b, t_b; p_a, t_a) = \lim_{N \rightarrow \infty} \prod_{n=1}^N \int dk_n \prod_{n=1}^{N+1} \frac{1}{\sqrt{2\pi i \hbar m \omega^2 \varepsilon}} \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} \left[ \frac{(\Delta k_n)^2}{2m\omega^2 \varepsilon} - \varepsilon \frac{\tan^2(\sqrt{\beta} k_n)}{2m\beta} \right] \right\}, \quad (54)$$

This expression is exactly the path integral representation of the transition amplitude of a particle, moving in the symmetric Pöschl-Teller potential (cf., e.g., Ref (Grouche & Steiner, 1998)):

$$K^{(\beta)} = \lim_{N \rightarrow \infty} \prod_{n=1}^N \int dk_n \prod_{n=1}^{N+1} \frac{1}{\sqrt{2\pi i \hbar m \omega^2 \varepsilon}} \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} \left[ \frac{(\Delta k_n)^2}{2m\omega^2 \varepsilon} - \varepsilon \frac{\beta \hbar^2 m \omega^2}{2} \lambda (\lambda - 1) \tan^2 \left( \sqrt{\beta} k_n \right) \right] \right\} \quad (55)$$

with  $(\lambda = (1 + (1 + (2/\beta \hbar m \omega)^2)^{1/2}) / 2)$ . The solution of this path integral is given by:

$$K^{(\beta)}(p_b, t_b; p_a, t_a) = \sum_{n=0}^{\infty} \frac{2^{2\lambda-1}(\lambda+n)n!\sqrt{\beta}}{\pi\Gamma(2\lambda+n)[\Gamma(\lambda)]^{-2}} \exp\left[-\frac{i}{\hbar} \frac{\beta\hbar^2 m\omega^2(t_b-t_a)}{2} (n^2 + 2(n+1)\lambda)\right] \\ \cos^\lambda(\sqrt{\beta}k_b) \cos^\lambda(\sqrt{\beta}k_a) C_n^\lambda(\sin(\sqrt{\beta}k_b)) C_n^\lambda(\sin(\sqrt{\beta}k_a)). \quad (56)$$

We finally obtain the spectral decomposition of the transition amplitude for the one-dimensional harmonic oscillator with nonzero minimum position uncertainty

$$K^{(\beta)}(p_b, t_b; p_a, t_a) = \sum_{n=0}^{\infty} \Psi_n(p_b) \Psi_n^*(p_a) e^{-\frac{i}{\hbar} E_n(t_b-t_a)}. \quad (57)$$

The energy spectrum is obtained from the poles of the Green function (57):

$$E_n = \frac{\beta\hbar^2 m\omega^2}{2} (n^2 + 2(n+1)\lambda). \quad (58)$$

Using the expression of  $\lambda$ , one finds:

$$E_n = \hbar\omega \left[ \left(n + \frac{1}{2}\right) \sqrt{1 + \left(\frac{\beta\hbar m\omega}{2}\right)^2} + (n^2 + n + 1) \left(\frac{\beta\hbar m\omega}{2}\right) \right]. \quad (59)$$

Also, the normalized eigenfunctions of the bound states can be easily deduced

$$\Psi_n(p) = \sqrt{\frac{2^{2\lambda-1}(\lambda+n)n!\sqrt{\beta}}{\pi\Gamma(2\lambda+n)[\Gamma(\lambda)]^{-2}}} \left[ \frac{1}{\sqrt{1+\beta p^2}} \right]^\lambda C_n^\lambda\left(\frac{\sqrt{\beta}p}{\sqrt{1+\beta p^2}}\right). \quad (60)$$

We note that equations (59) and (60) coincide exactly with those obtained in (Chang et al, 2002). Also we can verify these results when  $\beta \rightarrow 0$ , which transform to these results:

$$E_n \underset{\beta \rightarrow 0}{=} \hbar\omega \left(n + \frac{1}{2}\right), \quad \Psi_n(p) \underset{\beta \rightarrow 0}{=} \left[ \frac{1}{2^n n! \sqrt{\pi}} \right]^{1/2} \left( \frac{1}{m\hbar\omega} \right)^{1/4} e^{-\frac{p^2}{2m\hbar\omega}} H_n \left( \sqrt{\frac{1}{m\hbar\omega}} p \right). \quad (61)$$

For the one dimensional harmonic oscillator in the framework non-commutative geometry represented by Eqs. (1) and (2), the quantum corrections from the viewpoint of Feynman (cf., e.g., Ref (Khandekar et al, 1993)) at the  $\alpha$ -point discretization interval, we found only two points discretization ( $\alpha = \frac{1}{2} (1 \pm 1/\sqrt{2})$ ) consistent with differential equation (see, Ref (Chang, et al, 2002)) which gives different value of ordinary quantum mechanics.

So, in the following subsection we aim to expand this type deformation for relativistic systems.

### 3.3. The relativistic QM with minimal length

Our interest in the following section is to construct the propagator for two applications relativistic quantum mechanics in the presence of a minimal length, the first is (1+1)-dimensional Dirac oscillator, where the momentum component is shifting  $p$  by  $p - im\omega\gamma_0 x$  (cf., e.g., Ref (Szmytkowski & Gruchowski, 2001), and the second is a spinorial relativistic particle under the action of a Lorentz potential  $(V(x), \vec{A} = 0)$  plus a scalar potential  $S(x)$ , described by the (1+1)-dimensional Dirac equation:

$$(\gamma^\mu \hat{\Pi}_\mu - m + i\epsilon) \hat{S}^{(\beta)} = I, \quad (62)$$

where  $\mu = 0, 1$ ,  $\gamma_\mu$  are the Dirac matrices in the 2-dimensional Minkowski space. So, via the same procedure in the our previous work (Benzair et al, 2012 and 2014), we can obtain the standard propagator result for both systems, where there are two types of propagation, one with positive energy  $(+E_n^{(\beta)})$  propagation and the other with negative energy  $(-E_n^{(\beta)})$  propagation

$$S^{(\beta)}(p_b, p_a, t_b - t_a) = - \sum_{n=0}^{\infty} \left[ \Theta(t_b - t_a) \Psi_n^{(\beta)+}(p_b) \bar{\Psi}_n^{(\beta)+}(p_a) e^{-iE_n^{(\beta)}(t_b - t_a)} + \Theta(-(t_b - t_a)) \Psi_n^{(\beta)-}(p_b) \bar{\Psi}_n^{(\beta)-}(p_a) e^{iE_n^{(\beta)}(t_b - t_a)} \right], \quad (63)$$

For one-dimensional Dirac oscillator in the momentum space representation with the presence of minimal length uncertainty, can be expressed the energy spectrum as follows

$$E_{n,\pm}^{(\beta)} = \pm \sqrt{m^2 + \beta(m\omega)^2 n^2 + 2n(m\omega)}. \quad (64)$$

and the corresponding wave functions

$$\Psi_n^{(\beta)+}(p) = \begin{pmatrix} f_n^{(\beta)+}(p) \\ g_n^{(\beta)+}(p) \end{pmatrix}, \text{ and } \Psi_n^{(\beta)-}(p) = \begin{pmatrix} f_n^{(\beta)-}(p) \\ g_n^{(\beta)-}(p) \end{pmatrix}, \quad (65)$$

where the components of the wave functions  $f_n^{(\beta)\pm}(p)$  and  $g_n^{(\beta)\pm}(p)$  are respectively

$$f_n^{(\beta)+}(p) = \sqrt{\Gamma(\eta)^2 \frac{2^{2\eta-1}(n+1)!(n+\eta)\sqrt{\beta}(E_n^{(\beta)}+m)}{\pi\Gamma(n+2\eta)2E_n^{(\beta)}}} \left(\frac{1}{1+\beta p^2}\right)^\eta C_n^\eta \left(\frac{\sqrt{\beta}p}{1+\beta p^2}\right). \quad (66)$$

$$g_n^{(\beta)+}(p) = \frac{2i}{\sqrt{\beta}} \sqrt{\Gamma(\eta)^2 \frac{2^{2\eta-1}n!(n+\eta)\sqrt{\beta}}{\pi\Gamma(n+2\eta)2E_n^{(\beta)}(E_n^{(\beta)}+m)}} \left(\frac{1}{1+\beta p^2}\right)^{\eta+1} C_{n-1}^{\eta+1} \left(\frac{\sqrt{\beta}p}{1+\beta p^2}\right). \quad (67)$$

and

$$f_n^{(\beta)-}(p) = \sqrt{\Gamma(\eta)^2 \frac{2^{2\eta-1} n! (n+\eta) \sqrt{\beta} (E_n^{(\beta)} - m)}{\pi \Gamma(n+2\eta) 2E_n^{(\beta)}}} \left( \frac{1}{1+\beta p^2} \right)^\eta C_n^\eta \left( \frac{\sqrt{\beta} p}{1+\beta p^2} \right). \quad (68)$$

$$g_n^{(\beta)-}(p) = \frac{2i}{\sqrt{\beta}} \sqrt{\Gamma(\eta)^2 \frac{2^{2\eta-1} n! (n+\eta) \sqrt{\beta}}{\pi \Gamma(n+2\eta) 2E_n^{(\beta)} (E_n^{(\beta)} - m)}} \left( \frac{1}{1+\beta p^2} \right)^{\eta+1} C_{n-1}^{\eta+1} \left( \frac{\sqrt{\beta} p}{1+\beta p^2} \right). \quad (69)$$

and in the second application we can express these results; the energy spectrum are

$$E_n^{(\beta)\pm} = -\frac{m_0 V_0}{S_0} \pm \omega_n^{(\beta)}; T = t_b - t_a. \quad (70)$$

with

$$\omega_n^{(\beta)} = \frac{(S_0^2 - V_0^2)}{S_0} \sqrt{\beta \left( n^2 + \frac{2n}{\beta \sqrt{S_0^2 - V_0^2}} \right)}. \quad (71)$$

and the wave functions appropriate to the energy spectrum  $E_n^{(\beta)\pm}$  :

$$\Psi^{(\beta)\pm}(k) = \exp \left( i \frac{(E_n^{(\beta)\pm} V_0 + m_0 S_0)}{(S_0^2 - V_0^2)} k \right) \times \left( \begin{array}{l} \sqrt{\frac{N_n (S_0^2 - V_0^2)}{4S_0 (E_n^{(\beta)\pm} S_0 + m_0 V_0)}} \left[ \sqrt{\frac{(E_n^{(\beta)\pm} S_0 + m_0 V_0)}{(S_0 + V_0)}} v^\eta C_n^\eta(u) + \frac{2i}{\sqrt{\beta}} \sqrt{\frac{(S_0 - V_0)}{(E_n^{(\beta)\pm} S_0 + m_0 V_0)}} v^{\eta+1} C_{n-1}^{\eta+1}(u) \right] \\ \sqrt{\frac{N_n (S_0^2 - V_0^2)}{4S_0 (E_n^{(\beta)\pm} S_0 + m_0 V_0)}} \left[ -\sqrt{\frac{(E_n^{(\beta)\pm} S_0 + m_0 V_0)}{(S_0 - V_0)}} v^\eta C_n^\eta(u) + \frac{2i}{\sqrt{\beta}} \sqrt{\frac{(S_0 + V_0)}{(E_n^{(\beta)\pm} S_0 + m_0 V_0)}} v^{\eta+1} C_{n-1}^{\eta+1}(u) \right] \end{array} \right) \quad (72)$$

To use the old variables, we need the following relations

$$u = \frac{p\sqrt{\beta}}{\sqrt{1+\beta p^2}}, v = \frac{1}{\sqrt{1+\beta p^2}} \text{ and } k = \frac{\arctan}{\sqrt{\beta}} \left( \sqrt{\beta} p \right). \quad (73)$$

In the end, in order to separate the  $\beta$  dependent contribution, let us consider a very small  $\beta$ . The form of (71) can easily expand to first-order in  $\beta$ , be written as

$$\omega_n^{\ll \beta} = \sqrt{2n} \frac{(S_0^2 - V_0^2)^{3/4}}{S_0} + \beta \frac{(S_0^2 - V_0^2)^{5/4} (n^2)}{2S_0 \sqrt{2n}} + O(\beta^2). \quad (74)$$

Then we get

$$E_n^{(\beta)} = -\frac{m_0 V_0}{S_0} \pm \sqrt{2n} \frac{(S_0^2 - V_0^2)^{3/4}}{S_0} \pm \beta \frac{(S_0^2 - V_0^2)^{5/4} (n^2)}{2S_0 \sqrt{2n}} + O(\beta^2), \quad (75)$$

The first term in (75) is the energy spectrum of the ordinary Dirac equation in the presence of electromagnetic field and the second term represents the correction due to the presence of the minimal length.

### 3.4. Resolution of (1+1)-dimensional Dirac equation in position space representation

In this subsection, we'll examine the same above second system in the position space representation, and using the properties of the Hermite polynomial. We can calculate the corrections in the values of spectrum energy and this will be seen in this regard. This system is described by the (1+1)-dimensional Dirac equation

$$\{\sigma_2 \hat{p} + \sigma_3 (m_0 + S(\hat{x})) - (i\partial_t - V(\hat{x}))\} \Psi(x, t) = 0, \quad (76)$$

where  $\sigma_2$  and  $\sigma_3$  are the standard Pauli matrices

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (77)$$

We note that in (1+1) dimensions, the Dirac algebra is represented by the Pauli matrices. These reflect the invariant character of the parity of the Dirac equation. In fact in this dimension, there are no spin properties. This looks meaningless. However, in second quantization, we are obliged to use anticommutation relations to take into account the statistics of particles and to have a stable theory. At this level and even in (1+1) dimensions, the spin is an intrinsic characteristic of the particles in connection with the Wigner representation of relativistic particles.

As the potentials are time-independent, we have then to find the stationary states of this equation. Accordingly, let us choose for  $\Psi(x, t)$  the form  $\exp(-iEt) \Phi(x)$ ; we then get the following eigenvalue equation

$$\{\sigma_2 \hat{p} + \sigma_3 (m_0 + S(\hat{x})) - (E - V(\hat{x}))\} \Phi(x) = 0. \quad (78)$$

In the position space acts as

$$\hat{x} = x, \quad \hat{p} = -i\partial_x \left(1 - \frac{\beta}{3}\partial_x^2\right). \quad (79)$$

Then, a modified Dirac equation can be written as,

$$\left[-i\sigma_2 \partial_x \left(1 - \frac{\beta}{3}\partial_x^2\right) + \sigma_3 (m_0 + S(x)) - (E - V(x))\right] \Phi(x) = 0. \quad (80)$$

By using the following ansatz

$$\Phi = \left\{ -i\sigma_2 \partial_x \left( 1 - \frac{\beta}{3} \partial_x^2 \right) + \sigma_3 (m_0 + S(x)) + (E - V(x)) \right\} \chi, \quad (81)$$

where  $\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$  is a two-component function spinor, Eq.(80) becomes a differential equation of fourth order whose solution is very complicated in the presence of potentials.

Now, by suggesting that the system is subjected to the action of linear vector plus scalar potentials,

$$V(x) = V_0 x, \quad S(x) = S_0 x, \quad (82)$$

with  $S_0$  and  $V_0$  being arbitrary constants, we find that the Dirac spinor satisfies:

$$\left\{ -\frac{2}{3}\beta \partial_x^4 + \partial_x^2 + (E - V_0 x)^2 - (m_0 + S_0 x)^2 - \left( 1 - \beta \partial_x^2 \right) (S_0 \sigma_1 + iV_0 \sigma_2) \right\} \chi = 0, \quad (83)$$

We note that the linear potential, such a uniform external electromagnetic field plays a significant role in various domains of physics. For example, in particle physics, it can be regarded as a model to describe quark confinement (cf., e.g., Ref (Ferreira et al, 1971)). Further, the linear potential well has potential applications in electronics (in semiconductor devices), where the electrons are confined in almost linear quantum wells (cf., e.g., Ref (Singh, 1997)). Quantum Mechanics-Fundamentals and Applications to Technology).

Now in order to decouple the system (83), we introduce the following canonical transformation

$$\chi(x) = U \xi(x), \quad (84)$$

where  $U$  is given by

$$U = \begin{bmatrix} \frac{(V_0 + S_0)}{\sqrt{(S_0^2 - V_0^2)}} & \frac{(V_0 + S_0)}{\sqrt{(S_0^2 - V_0^2)}} \\ \sqrt{(S_0^2 - V_0^2)} & -\sqrt{(S_0^2 - V_0^2)} \end{bmatrix}. \quad (85)$$

Then the function  $\xi(x)$  satisfies the following equation

$$\left[ -\frac{2\beta}{3\alpha} \partial_x^4 + \frac{(1+\varepsilon\beta\alpha)}{\alpha} \partial_x^2 - \alpha \left[ x^2 + 2 \frac{(m_0 S_0 + E V_0)}{\alpha^2} x \right] + \frac{(E^2 - m_0^2)}{\alpha} - \varepsilon \right] \xi_\varepsilon(x) = 0, \quad (86)$$

with  $\alpha = \sqrt{(S_0^2 - V_0^2)}$  and  $\varepsilon = \pm 1$ .

As it has been mentioned previously that the solution is complicated, we try to find via the usual perturbation method of quantum mechanics the first energy correction at order 1 in  $\beta$  and point out how the introduction of the modified Heisenberg algebra affects the physical results. To do this, let us first suppose in this case that  $\alpha^2 > 0$  so as to avoid

complex eigenvalues and arrange equation (86) as a sum of two terms, one of which being the perturbative term, as follows,

$$\left[ H^0(z, \partial_z) + H^{pert}(\partial_z) \right] \xi_\varepsilon(z) = 0, \quad (87)$$

by setting

$$z = (S_0^2 - V_0^2)^{1/4} \left( x + \frac{(m_0 S_0 + E V_0)}{(S_0^2 - V_0^2)} \right). \quad (88)$$

and

$$\begin{aligned} H^0 &= \partial_z^2 - z^2 + z_1, \\ H^{pert} &= -\frac{2}{3} \beta \alpha \partial_z^4 + \varepsilon \beta \alpha \partial_z^2, \end{aligned} \quad (89)$$

where

$$z_1 = \frac{(m_0 S_0 + E V_0)^2}{(S_0^2 - V_0^2)^{3/2}} + \frac{(E^2 - m_0^2)}{\sqrt{(S_0^2 - V_0^2)}} - \varepsilon. \quad (90)$$

Now, in case where  $H^{pert}(\partial_z)$  vanishes, (i.e. when  $\beta \rightarrow 0$ ), equation (87) becomes that of the harmonic oscillator whose solution is known,

$$\xi_\varepsilon^{\beta=0}(z) = C_{n'} \exp\left(-\frac{1}{2}z^2\right) H_{n'}(z), \quad n' = n + \frac{1}{2} + \frac{\varepsilon}{2}, \quad (91)$$

with  $z_1$  verifying

$$z_1 = 2n + 1. \quad n = 0, 1, 2, \dots, \quad (92)$$

where  $n' = (n + 1, n)$ . Hence from (90) and (92), we obtain the following energy levels for our Dirac equation:

$$E_{n,\pm}^{\beta=0} = -\frac{m_0 V_0}{S_0} \pm \sqrt{2n} \frac{(S_0^2 - V_0^2)^{3/4}}{S_0}. \quad (93)$$

We note the existence of the two signs in (93) which is a characteristic property of energies in relativistic quantum mechanics. Now, to find the first correction in the energy levels, we take the expectation value of the perturbation operator by using eigenfunctions (91)

$$\Delta z_{n1} = \frac{\langle \xi_\varepsilon^{\beta=0}(z) | H^{pert} | \xi_\varepsilon^{\beta=0}(z) \rangle}{\langle \xi_\varepsilon^{\beta=0}(z) | \xi_\varepsilon^{\beta=0}(z) \rangle}. \quad (94)$$

With the help the properties of Hermite polynomial (Gradshteyn & Ryzhik, 2000), we obtain this result:

$$\Delta z_{n1} = \frac{\beta \alpha \int \exp\left(-\frac{1}{2}z^2\right) H_n(z) \left[-\frac{2}{3}\partial_z^4 + \varepsilon \partial_z^2\right] \exp\left(-\frac{1}{2}z^2\right) H_n(z) dz}{\int \exp\left(-\frac{1}{2}z^2\right) H_n(z) \exp\left(-\frac{1}{2}z^2\right) H_n(z) dz} = -\frac{\beta \alpha}{2} \left(n^2\right), \quad (95)$$

From the relation (90), we derive the expression of  $\Delta E_{n1}$  as a function of  $\Delta z_{n1}$ , and we write,

$$\Delta E_n^1 = \frac{(S_0^2 - V_0^2)^{3/2} \Delta z_{n1}}{2S_0(E_{n,\pm}^{\beta=0} S_0 + m_0 V_0)}. \quad (96)$$

Then, by substituting (95) and (93) in (96), we find,

$$\Delta E_n^1 = \pm \beta \frac{(S_0^2 - V_0^2)^2 (n^2)}{2S_0 \sqrt{(2n)} (S_0^2 - V_0^2)^{3/4}}. \quad (97)$$

The energy spectrum of this study at order 1 in  $\beta$  can be rewritten as

$$E_n(\beta) = E_{n,\pm}^{\beta=0} + \Delta E_n^1 + O(\beta^2), \quad (98)$$

which is equal to

$$E_n(\beta) = -\frac{m_0 V_0}{S_0} \pm \sqrt{(2n)} \frac{(S_0^2 - V_0^2)^{3/4}}{S_0} \pm \beta \frac{(S_0^2 - V_0^2)^{5/4} (n^2)}{2S_0 \sqrt{(2n)}} + O(\beta^2). \quad (99)$$

We note that the same correction spectrum energy obtained where using the momentum space representation defined in eq. (75).

## 4. Conclusion

We have discussed in this chapter the path integral formalism in the case of the appearance of the parameter of deformation  $\beta$  in the generalized Heisenberg principle (1), where we calculated the quantum corrections according to the Feynman approach (cf., e.g., Ref (Khandekar et al, 1993)) for Harmonic oscillator particle in one dimension. And we have shown that the different  $\alpha$  values obtained for the ordinary quantum mechanics, that make us wonder about these results. In addition, we have generalized the study of relativistic particles which have one half (1/2) spin for example Dirac oscillator and relativistic spinning particle subjected to the action of combined vector and scalar linear potentials, with a deformed commutation relation for the Heisenberg principle. We have obtained the same  $\alpha$  values for Harmonic oscillator. This has been explained in our previous works (Benzair et al, 2012; 2014). Using the residue theorem, the energy spectrum and corresponding eigenfunctions expressed in terms of Gegenbauer polynomials are then deduced as a function of the deformation parameter  $\beta$ . It has been noted above the energy spectrum of the relativistic

spinorial particle is dependent on term quadratic in  $n$  that is similar to the energy levels of a particle confined in a potential well. In addition, we studied in the second relativistic application, the energy spectrum of the Dirac equation for a spin 1/2 subjected to the action of combined vector and scalar linear potentials, with a deformed commutation relation for the Heisenberg principle, where we have used the old variable  $p$  in the position space representation, we have obtained a differential equation of fourth order whose analytic solution is complicated. We have calculated the energy correction in first order for  $\beta$  by using a usual approximation technique of quantum mechanics. We note that the two methods gave the same results for the first energy correction at order 1 in  $\beta$ . In this study, we did not take the case  $S_0^2 - V_0^2 < 0$ , so as to avoid the complex eigenvalues. But when  $S_0^2 - V_0^2 = 0$ , the calculation is very simple and we can obtain physical results.

Finally, let us signal that the problems of choosing  $\alpha$ -point discretization in the case of deformed space are under consideration.

## Author details

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