Chapter 4

Quantum Communication Processes and Their Complexity

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1. Introduction

The complex systems and their dynamics are treated various way. Ohya looked for synthesizing method to treat complex systems. He established Information Dynamics [36] which is a new concept unifying the dynamics of a state and the complexity of the system itself. By applying Information Dynamics, one can discuss in a unified frame the problems such as in mathematics, physics, biology, information science. Information Dynamics is growing as one of the research fields, for instance, the international journal named "Open Systems and Information Dynamics" in 1992 has appeared. In ID, there are two types of complexity, that is, (a) a complexity of state describing system itself and (b) a transmitted complexity between two systems. Entropies of classical and quantum information theory are the example of the complexities of (a) and (b).

Shannon [52] found that the entropy, introduced in physical systems by Clausius and Boltzmann, can be used to express the amount of information by means of communication processes, and he proposed the so-called information communication theory at the middle part of the 20th century. In his information theory, the entropy and the mutual entropy (information) are most important concepts. The entropy relates to the complexity of ID measuring the amount of information of the state of system. The mutual entropy (information) corresponds to the transmitted complexity of ID representing the amount of information correctly transmitted from the initial system to the final system through a channel, and it was extended to the mutual entropy on the continuous probability space by Gelfand– Kolmogorov - Yaglom [17,23], which was defined by using the relative entropy of two states by Kullback-Leibler [26].

Laser is often used in the current communication. A formulation of information theory being able to treat quantum effects is necessary, which is the so-called quantum information theory. The quantum information theory is important in both mathematics and engineering. It has been developed with quantum entropy theory and quantum probability. In quantum information

theory, one of the important problems is to investigate how much information is exactly transmitted to the output system from the input system through a quantum channel. The amount of information of the quantum input system is described by the quantum entropy defined by von Neumann [29] in 1932. The C*-entropy was introduced in [33,35] and its property is discussed in [28,21]. The quantum relative entropy was introduced by Umegaki [55] and it is extended to general quantum system by Araki [4,5], Uhlmann [54] and Donald [14]. Furthermore, it had been required to extend the Shannon's mutual entropy (information) of classical information theory to that in the quantum one. The classical mutual entropy is defined by using the joint probability expressing a correlation between the input system and the output system. However, it was shown by Urbanik [56] that in quantum system there does not generally exists a joint probability distribution. The semi-classical mutual entropy was introduced by Holevo, Livitin, Ingarden [18,20] for classical input and output passing through a possible quantum channel. By introducing a new notion, the so-called compound state, in 1983 Ohya formulated the mutual entropy [31,32] in a complete quantum mechanical system (i.e., input state, output state and channel are all quantum mechanical), which is called the Ohya mutual entropy. It was generalized to C*-algebra in [Oent84]. The quantum capacity [40] is defined by taking the supremum for the Ohya mutual entropy. By using the Ohya quantum mutual entropy, one can discuss the efficiency of the information transmission in quantum systems [28,27,44,34,35], which allows the detailed analysis of optical communication processes. Concerning quantum communication processes, several studies have been done in [31,32,35,40,41]. Recently, several mutual entropy type measures (Lindblad - Nielsen entropy [10] and Coherent entropy [6]) were defined by using the entropy exchange. One can classify these mutual entropy type measures by calculating their measures for the quantum channel. These entropy type complexities are explained in [39,43].

The entangled state is an important concept for quantum theory and it has been studied recently by several authors. One of the remarkable formulations to search the entanglement state is the Jamiolkowski's isomorphism [22]. It might be related to the construction of the compound state in quantum communication processes. One can discuss the entangled state generated by the beam splitting and the squeezed state.

The classical dynamical (or Kolmogorov-Sinai) entropy S(T) [23] for a measure preserving transformation T was defined on a message space through finite partitions of the measurable space. The classical coding theorems of Shannon are important tools to analyze communication processes which have been formulated by the mean dynamical entropy and the mean dynamical mutual entropy. The mean dynamical entropy represents the amount of information per one letter of a signal sequence sent from the input source, and the mean dynamical mutual entropy does the amount of information per one letter of the signal received in the output system. In this chapter, we will discuss the complexity of the quantum dynamical system to calculate the mean mutual entropy with respect to the modulated initial states and the attenuation channel for the quantum dynamical systems [59].

The quantum dynamical entropy (QDE) was studied by Connes-Størmer [13], Emch [15], Connes-Narnhofer-Thirring [12], Alicki-Fannes [3], and others [9,48,19,57,11]. Their dynamical entropies were defined in the observable spaces. Recently, the quantum dynamical entropy and the quantum dynamical mutual entropy were studied by the present authors [34,35]: (1) the dynamical entropy is defined in the state spaces through the complexity of Information

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Dynamics [36]. (2) It is defined through the quantum Markov chain (QMC) was done in [2]. (3) The dynamical entropy for a completely positive (CP) map was defined in [25]. In this chapter, we will discuss the complexity of the quantum dynamical process to calculate the generalized AOW entropy given by KOW entropy for the noisy optical channel [58].

2. Quantum channels

The signal of the input quantum system is transmitted through a physical device, which is called a quantum channel. The concept of channel has been performed an important role in the progress of the quantum information communication theory. The mathematical representation of the quantum channel is a mapping from the input state space to the output state space. In particular, the attenuation channel [31] and the noisy optical channel [44] are remarkable examples of the quantum channels describing the quantum optical communication processes. These channels are related to the mathematical desctiption of the beam splitter.

Here we review the definition of the quantum channels.

Let $\left(B(\mathcal{H}_1), \mathfrak{S}(\mathcal{H}_1)\right)$ and $\left(B(\mathcal{H}_2), \mathfrak{S}(\mathcal{H}_2)\right)$ be input and output systems, respectively, where $B(\mathcal{H}_k)$ is the set of all bounded linear operators on a separable Hilbert space $\mathcal{H}_k$ and $\mathfrak{S}(\mathcal{H}_k)$ is the set of all density operators on $\mathcal{H}_k$ ($k=1, 2$). Quantum channel Λ^* is a mapping from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$.

1. Λ^* is called a linear channel if Λ^* satisfies $\Lambda^*(\lambda\rho_1 + (1-\lambda)\rho_2) = \lambda\Lambda^*(\rho_1) + (1-\lambda)\Lambda^*(\rho_2)$ for any $\rho_1, \rho_2 \in \mathfrak{S}(\mathcal{H}_1)$ and any $\lambda \in [0, 1]$.

2. Λ^* is called a completely positive (CP) channel if Λ^* is linear and its dual map Λ from $B(\mathcal{H}_2)$ to $B(\mathcal{H}_1)$ holds

$$\sum_{i,j=1}^{n} A_i^* \Lambda(B_i^* B_j) A_j \geq 0$$

for any $n \in N$, any $B_j \in B(\mathcal{H}_2)$ and any $A_j \in B(\mathcal{H}_1)$, where the dual map Λ of Λ^* is defined by $tr\Lambda^*(\rho)B = tr\rho\Lambda(B)$ for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$ and any $B \in B(\mathcal{H}_2)$. Almost all physical transformations can be described by the CP channel [30,39,21,46, 43].

3. Quantum communication processes

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Hilbert spaces expressing noise and loss systems, respectively. Quantum communication process including the influence of noise and loss is denoted by the following scheme [31]: Let ρ be an input state in $\mathfrak{S}(\mathcal{H}_1)$, ζ be a noise state in $\mathfrak{S}(\mathcal{K}_1)$.

$$\begin{array}{ccc} \mathfrak{S}(\mathcal{H}_1) & \xrightarrow{\Lambda^*} & \mathfrak{S}(\mathcal{H}_2) \\ \gamma^* \downarrow & & \uparrow a^* \\ \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1) & \xrightarrow{\Pi^*} & \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2) \end{array}$$

The above maps γ^*, a^* are given as

$$\gamma^*(\rho) = \rho \otimes \xi, \quad \rho \in \mathfrak{S}(\mathcal{H}_1),$$
$$a^*(\sigma) = tr_{\mathcal{K}_2}\sigma, \quad \sigma \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2).$$

The map Π^* is a CP channel from $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$ to $\mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ given by physical properties of the device transmitting signals. Hence the channel for the above process is given as

$$\Lambda^*(\rho) \equiv tr_{\mathcal{K}_2}\Pi^*(\rho \otimes \zeta) = (a^* \circ \Pi^* \circ \gamma^*)(\rho)$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$. Based on this scheme, the noisy optical channel is constructed as follows:

4. Noisy optical channel

Noisy optical channel Λ^* with a noise state ζ was defined by Ohya and NW [44] such as

$$\Lambda^*(\rho) \equiv tr_{\mathcal{K}_2}\Pi^*(\rho \otimes \zeta) = tr_{\mathcal{K}_2}V(\rho \otimes \zeta)V^*,$$

where $\zeta = |m_1\rangle\langle m_1|$ is the m_1 photon number state in $\mathfrak{S}(\mathcal{K}_1)$ and V is a mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ denoted by

$$V(|n_1\rangle \otimes |m_1\rangle) = \sum_{j=0}^{n_1+m_1} C_j^{n_1,m_1} |j\rangle \otimes |n_1 + m_1 - j\rangle,$$

where

$$C_j^{n_1,m_1} = \sum_{r=L}^{K} (-1)^{n_1+j-r} \frac{\sqrt{n_1!\, m_1!\, j!\, (n_1+m_1-j)!}}{r!\,(n_1-j)!\,(j-r)!\,(m_1-j+r)!} \alpha^{m_1-j+2r} (-\bar{\beta})^{n_1+j-2r},$$

and $|n_1\rangle$ is the n_1 photon number state vector in $\mathcal{H}_1$, and α, β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. K and L are constants given by $K = \min\{n_1, j\}$, $L = \max\{m_1 - j, 0\}$. We have the following theorem.

Theorem The noisy optical channel Λ^* with noise state $|m\rangle\langle m|$ is described by

$$\Lambda^*(\rho) = \sum_{i=0}^{\infty} O_i V Q^{(m)} \rho Q^{(m)*} V^* O_i^*,$$

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where $Q^{(m)} \equiv \sum_{l=0}^{\infty} (|y_l\rangle \otimes |m\rangle)\langle y_l|$, $O_i \equiv \sum_{k=0}^{\infty} |z_k\rangle(\langle z_k| \otimes \langle i|)$, $\{|y_l\rangle\}$ is a CONS in $\mathcal{H}_1$, $\{|z_k\rangle\}$ is a CONS in $\mathcal{H}_2$ and $\{|i\rangle\}$ is the set of number states in $\mathcal{K}_2$.

In particular for the coherent input states

$$\rho = |\xi\rangle\langle\xi| \otimes |\kappa\rangle\langle\kappa| \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1),$$

the output state of Π^* is obtained by

$$\Pi^*(|\xi\rangle\langle\xi| \otimes |\kappa\rangle\langle\kappa|) = |\alpha\xi + \beta\kappa\rangle\langle\alpha\xi + \beta\kappa| \otimes |-\bar{\beta}\xi + \bar{\alpha}\kappa\rangle\langle-\bar{\beta}\xi + \bar{\alpha}\kappa|.$$

$$|\xi\rangle\langle\xi| \rightarrow \begin{array}{c} |\kappa\rangle\langle\kappa| \\ \downarrow \\ \searrow \\ \downarrow \\ |-\bar{\beta}\xi + \bar{\alpha}\kappa\rangle\langle-\bar{\beta}\xi + \bar{\alpha}\kappa| \end{array} \rightarrow |\alpha\xi + \beta\kappa\rangle\langle\alpha\xi + \beta\kappa|$$

5. Attenuation channel

The noisy optical channel with a vacuum noise is called the attenuation channel introduced in [31] by

$$\Lambda_0^*(\rho) \equiv tr_{\mathcal{K}_2}\Pi_0^*(\rho \otimes \zeta_0) = tr_{\mathcal{K}_2}V_0(\rho \otimes |0\rangle\langle 0|)V_0^*,$$

where $|0\rangle\langle 0|$ is the vacuum state in $\mathfrak{S}(\mathcal{K}_1)$ and V_0 is a mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ given by

$$V_0(|n_1\rangle \otimes |0\rangle) = \sum_j^{n_1} C_j^{n_1} |j\rangle \otimes |n_1 - j\rangle,$$

$$C_j^{n_1} = \sqrt{\frac{n_1!}{j!(n_1-j)!}}\alpha^j(-\bar{\beta})^{n_1-j}$$

In particular, for the coherent input state

$$\rho = |\xi\rangle\langle\xi| \otimes |0\rangle\langle 0| \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1),$$

one can obtain the output state

$$\Pi_0^*(|\xi\rangle\langle\xi| \otimes |0\rangle\langle 0|) = |\alpha\xi\rangle\langle\alpha\xi| \otimes |-\bar{\beta}\xi\rangle\langle-\bar{\beta}\xi|.$$

Lifting $\mathcal{E}_0^*$ from $\mathfrak{S}(\mathcal{H})$ to $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ in the sense of Accardi and Ohya [1] is denoted by

$$\mathcal{E}_0^*(|\xi\rangle\langle\xi|) = |\alpha\xi\rangle\langle\alpha\xi| \otimes |\beta\xi\rangle\langle\beta\xi|.$$

$\mathcal{E}_0^*$ (or Π_0^*) is called a beam splitting. Based on liftings, the beam splitting was studied by Accardi - Ohya [1] and Fichtner - Freudenberg - Libsher [16].

6. Information dynamics

We are interested to study the dynamics of state change or the complexity of state for several systems. Information dynamics (ID) is a new concept introduced by Ohya [36] to construct a theory under the ID's framework by synthesizing these investigating schemes. In ID, a complexity of state describing system itself and a transmitted complexity between two systems are used. The examples of these complexities are the Shannon's entropy and the mutual entropy (information) in classical entropy theory. In quantum entropy theory, it was known that the von Neumann entropy and the Ohya mutual entropy relate to these complexities. Recently, several mutual entropy type measures (the Lindblad - Nielsen entropy [10] and the Coherent entropy [6]) were proposed by means of the entropy exchange for an input state and a channel.

7. Concept of information dynamics

Ohya introduced Information Dynamics (ID) synthesizing dynamics of state change and complexity of state. Based on ID, one can study various problems of physics and other fields. Channel and two complexities are key concepts of ID. Two kinds of complexities $C^{\mathcal{S}}(\rho)$, $T^{\mathcal{S}}(\rho;\Lambda^*)$ are used in ID. $C^{\mathcal{S}}(\rho)$is a complexity of a state ρ measured from a subset $\mathcal{S}$ and $T^{\mathcal{S}}(\rho;\Lambda^*)$ is a transmitted complexity according to the state change from ρ to $\Lambda^*\rho$. Let $\mathcal{S}$, $\bar{\mathcal{S}}$, $\mathcal{S}_t$ be subsets of $\mathfrak{S}(\mathcal{H}_1)$, $\mathfrak{S}(\mathcal{H}_2)$, $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, respectively. These complexities should fulfill the following conditions as follows:

8. Complexity of system

1. For any $\rho \in \mathcal{S}$, $C^{\mathcal{S}}(\rho)$ is nonnegative (i.e., $C^{\mathcal{S}}(\rho) \geq 0$)

2. For a bijection j from $ex\mathfrak{S}(\mathcal{H}_1)$ to $ex\mathfrak{S}(\mathcal{H}_1)$,

$C^{\mathcal{S}}(\rho) = C^{\mathcal{S}}(j(\rho))$

is hold, where $ex\mathfrak{S}(\mathcal{H}_1)$ is the set of all extremal points of $\mathfrak{S}(\mathcal{H}_1)$.

3. For $\rho \otimes \sigma \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $\rho \in \mathfrak{S}(\mathcal{H}_1)$, $\sigma \in \mathfrak{S}(\mathcal{H}_2)$,

$C^{\mathcal{S}_t}(\rho \otimes \sigma) = C^{\mathcal{S}}(\rho) + C^{\bar{\mathcal{S}}}(\sigma)$

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It means that the complexity of the state $\rho \otimes \sigma$ of totally independent systems are given by the sum of the complexities of the states ρ and σ.

9. Transmitted complexity

(1′) For any $\rho \in \mathcal{S}$ and a channel Λ^*, $T^{\mathcal{S}}(\rho;\Lambda^*)$ is nonnegative (i.e., $T^{\mathcal{S}}(\rho;\Lambda^*) \geq 0$)

(4) $C^{\mathcal{S}}(\rho)$ and $T^{\mathcal{S}}(\rho;\Lambda^*)$ satisfy the following inequality $0 \leq T^{\mathcal{S}}(\rho;\Lambda^*) \leq C^{\mathcal{S}}(\rho)$.

(5) If the channel Λ^* is given by the identity map id, then $T^{\mathcal{S}}(\rho;id) = C^{\mathcal{S}}(\rho)$ is hold.

The examples of the above complexities are the Shannon entropy $S(p)$ for $C^{\mathcal{S}}(p)$ and the classical mutual entropy $I(p;\Lambda^*)$ for $T^{\mathcal{S}}(p;\Lambda^*)$, respectively Let us consider these complexities for quantum communication processes.

10. Quantum entropy

Since the present optical communication is using the optical signal including quantum effect, it is necessary to construct new information theory dealing with those quantum phenomena in order to discuss the efficiency of information transmission of optical communication processes rigorously. The quantum information theory is important in both mathematics and engineering, and it contains several topics, for instance, quantum entropy theory, quantum communication theory, quantum teleportation, quantum entanglement, quantum algorithm, quantum coding theory and so on. It has been developed with quantum entropy theory and quantum probability. In quantum information theory, one of the important problems is to investigate how much information is exactly transmitted to the output system from the input system through a quantum channel. The amount of information of the quantum communication system is described by the quantum mutual entropy defined by Ohya [31], based on the quantum entropy by von Neumann [29], and the quantum relative entropy by Umegaki [55], Araki [4] and Uhlmann [54]. The quantum information theory directly relates to quantum communication theory, for instance, [40,41,45]. One of the most important communication processes is quantum teleportation, whose new treatment was studied in [24]. It is important to classify quantum states. One of such classifications is to study entanglement and separability of states (see [7,8]). There have been lots of trials in finite dimensional Hilbert spaces. Quantum mechanics should be basically discussed in infinite dimensional Hilbert spaces. We have to study such a classification in infinite dimensional Hilbert spaces.

10.1. Von Neumann entropy

The study of the entropy in quantum system was begun by von Neumann [29] in 1932. For any state given by the density operator ρ, the von Neumann entropy is defined by

$$S(\rho) = -tr\rho\log\rho, \qquad \forall\, \rho \in \mathfrak{S}(\mathcal{H}).$$

Since the von Neumann entropy satisfies the conditions (1),(2),(3) of the complexity of state of ID, it seems to be considered as an example of the complexity of state $C(\rho)=S(\rho)$.

10.2. Entropy for general systems

Here we briefly explain Let us comment general entropies of states in C*-dynamical systems. The C*-entropy ($\mathcal{S}$-mixing entropy) was introduced by Ohya in [33,35] and its property is discussed in [28,21].

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha(G))$ be a C*-dynamical system and $\mathcal{S}$ be a weak* compact and convex subset of $\mathfrak{S}(\mathcal{A})$. For example, $\mathcal{S}$ is given by $\mathfrak{S}(\mathcal{A})$ (the set of all states on $\mathcal{A}$), $I(\alpha)$ (the set of all invariant states for α), $K(\alpha)$ (the set of all KMS states),and so on. Every state $\varphi \in \mathcal{S}$ has a maximal measure μ pseudosupported on $ex\mathcal{S}$ such that

$$\varphi = \int_{\mathcal{S}} \omega d\mu,$$

where $ex\mathcal{S}$ is the set of all extreme points of $\mathcal{S}$. The measure μ giving the above decomposition is not unique unless $\mathcal{S}$ is a Choquet simplex. The set of all such measures is denoted by $M_\varphi(\mathcal{S})$ and $D_\varphi(\mathcal{S})$is the subset of $M_\varphi(\mathcal{S})$ constituted by

$$D(\mathcal{S}) = \Big\{ M_\varphi(\mathcal{S}); \quad \exists \mu_k \subset \mathbb{R}^+ \text{ and } \{\varphi_k\} \subset exS \\ s.t. \quad \sum_k \mu_k = 1, \quad \mu = \sum_k \mu_k \delta(\varphi_k) \Big\}$$

where $\delta(\varphi)$ is the Dirac measure concentrated on an initial state φ. For a measure $\mu \in D_\varphi(\mathcal{S})$, the entropy type functional $H(\mu)$ is given by

$$H(\mu) = -\sum_k \mu_k \log \mu_k.$$

For a state $\varphi \in \mathcal{S}$ with respect to $\mathcal{S}$, Ohya introduced the C*-entropy ($\mathcal{S}$-mixing entropy) [33,35] defined by

$$S^{\mathcal{S}}(\varphi) = \begin{cases} \inf\{H(\mu); \quad \mu \in D_\varphi(\mathcal{S})\} \\ +\infty \qquad \text{if } D_\varphi(\mathcal{S}) = \varnothing. \end{cases}$$

It describes the amount of information of the state φ measured from the subsystem $\mathcal{S}$. If $\mathcal{S} = \mathfrak{S}(\mathcal{A})$, then $S^{\mathfrak{S}(\mathcal{A})}(\varphi)$ is denoted by $S(\varphi)$. This entropy is an extension of the von Neumann entropy mentioned above.

10.3. Quantum relative entropy

The classical relative entropy in continuous probability space was defined by Kullback-Leibler [26]. It was developed in noncommutative probability space. The quantum relative entropy was first defined by Umegaki [55] for σ-finite von Neumann algebras, which denotes a certain difference between two states. It was extended by Araki [4] and Uhlmann [54] for general von Neumann algebras and *-algebras, respectively.

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10.4. Umegakirelative entropy

The relative entropy of two states was introduced by Umegaki in [55] for σ-finite and semi-finite von Neumann algebras. Corresponding to the classical relative entropy, for two density operators ρ and σ, it is defined as

$$S(\rho,\ \sigma)=\begin{cases} tr\rho(\log\rho-\log\sigma) & (s(\rho)\ll s(\sigma)), \\ \infty & (\text{else}), \end{cases}$$

where $s(\rho)\ll s(\sigma)$ means the support projection $s(\sigma)$ of σ is greater than the support projection $s(\rho)$ of ρ. It means a certain difference between two quantum states ρ and σ. The Umegaki's relative entropy satisfies (1) positivity, (2) joint convexity, (3) symmetry, (4) additivity, (5) lower semicontinuity, (6) monotonicity. Araki [4] and Uhlmann [54] extended this relative entropy for more general quantum systems.

10.5. Relative entropy for general systems

The relative entropy for two general states was introduced by Araki [4,5] in von Neumann algebra and Uhlmann [54] in *-algebra. The above properties are held for these relative entropies.

10.5.1. Araki's relative entropy[4,5]

Let $\mathcal{N}$ be a σ-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and φ, ψ be normal states on $\mathcal{N}$ given by $\varphi(\cdot)=\langle x,\ \cdot\ x\rangle$ and $\psi(\cdot)=\langle y,\ \cdot\ y\rangle$ with $x,\ y\in\mathcal{K}$ (i.e., $\mathcal{K}$ is a positive natural cone) $\subset\mathcal{H}$. On the domain $\mathcal{N}y+(I-s^{\mathcal{N}'}(y))\mathcal{H}$, the operator $S_{x,y}$ is defined by

$$S_{x,y}(Ay+z)=s^{\mathcal{N}}(y)A^{*}x,\quad A\in\mathcal{N}\quad \left(z\in\mathcal{H}\text{ is satisfying } s^{\mathcal{N}'}(y)z=0\right),$$

where $s^{\mathcal{N}}(y)$ (the $\mathcal{N}$-support of y) is the projection from $\mathcal{H}$ to $\{\mathcal{N}'y\}^{-}$. Using this $S_{x,y}$, the relative modular operator $\Delta_{x,y}$ is defined as $\Delta_{x,y}=(S_{x,y})^{*}\overline{S_{x,y}}$, whose spectral decomposition is denoted by $\int_0^\infty \lambda de_{x,y}(\lambda)$ ($\overline{S_{x,y}}$ is the closure of $S_{x,y}$). Then the Araki's relative entropy is given by

Definition The Araki's relative entropy of φ and ψ is defined by

$$S(\psi,\ \varphi)=\begin{cases} \int_0^\infty \log\lambda d\langle y,\ e_{x,y}(\lambda)y\rangle & (\psi\ll\varphi), \\ \infty & (\text{otherwise}), \end{cases}$$

where $\psi\ll\varphi$ means that $\varphi(A^{*}A)=0$ implies $\psi(A^{*}A)=0$ for $A\in\mathcal{N}$.

10.5.2. Uhlmann's relative entropy[54]

Let $\mathcal{L}$ be a complex linear space and $p,\ q$ be two semi-norms on $\mathcal{L}$. $H(\mathcal{L}(p,\ q))$ is the set of all positive Hermitian forms α on $\mathcal{L}$ satisfying $|\alpha(x,\ y)|\le p(x)q(y)$ for all $x,\ y\in\mathcal{L}$. For $x\in\mathcal{L}$, the quadratical mean $QM(p,\ q)$ of p and q is defined by

$$QM(p, q)(x)=\sup\{\alpha(x, x)^{1/2};\ \alpha\in H(\mathcal{L}(p, q))\}.$$

For each $x\in\mathcal{L}$, there exists a family of semi-norms $p_t(x)$ of $t\in[0, 1]$, which is called the quadratical interpolation from p to q, satisfying the following conditions:

1. For any $x\in\mathcal{L}$, $p_t(x)$ is continuous in t,
2. $p_{1/2}=QM(p, q)$
3. $p_{t/2}=QM(p, p_t)\quad(\forall\, t\in[0, 1])$
4. $p_{(t+1)/2}=QM(p_t, q)\quad(\forall\, t\in[0, 1])$

This semi-norm p_t is denoted by $QI_t(p, q)$. It is shown that for any positive Hermitianforms α, β, there exists a unique function $QF_t(\alpha, \beta)$ of $t\in[0, 1]$ with values in the set $H(\mathcal{L}(p, q))$ such that $QF_t(\alpha, \beta)(x, x)^{1/2}$ is the quadratical interpolation from $\alpha(x, x)^{1/2}$ to $\beta(x, x)^{1/2}$. For $x\in\mathcal{L}$, the relative entropy functional $S(\alpha, \beta)(x)$ of α and β is defined as

$$S(\alpha, \beta)(x)=-\liminf_{t\to+0}\frac{1}{t}\{QF_t(\alpha, \beta)(x, x)-\alpha(x, x)\}.$$

Let $\mathcal{L}$ be a *-algebra $\mathcal{A}$. For positive linear functional φ, ψ on $\mathcal{A}$, two Hermitian forms φ^L , ψ^R are given by $\varphi^L(A, B)=\varphi(A^*B)$ and $\psi^R(A, B)=\psi(BA^*)$.

Definition The Uhlmann's relative entropy of φ and ψ is defined by

$$S(\psi, \varphi)=S(\psi^R, \varphi^L)(I).$$

10.5.3. Ohya mutual entropy [31]

The Ohya mutual entropy [31] with respect to the initial state ρ and a quantum channel Λ^* is described by

$$I(\rho;\Lambda^*)\equiv\sup\left\{\sum_n S(\Lambda^*E_n, \Lambda^*\rho),\ \rho=\sum_n\lambda_nE_n\right\},$$

where $S(\cdot, \cdot)$ is the Umegaki's relative entropy and $\rho=\sum_n\lambda_nE_n$ represents a Schatten-von Neumann (one dimensional orthogonal) decomposition [49] of ρ. Since the Schatten-von Neumann decomposion of a state ρ is not unique unless all eigenvalues of ρ do not degenerate, the Ohya mutual entropy is defined by taking a supremum for all Schatten-von Neumann decomposion of a state ρ. Then the Ohya mutual entropy satisfies the following Shannon's type inequality [31]

$$0\le I(\rho, \Lambda^*)\le\min\{S(\rho), S(\Lambda^*\rho)\},$$

where $S(\rho)$ is the von Neumann entropy. This inequalities show that the Ohya mutual entropy represents the amount of information correctly carried from the input system to the output

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system through the quantum channel. The capacity denotes the ability of the information transmission of the communication processes, which was studied in [40,41,45].

For a certain set $\mathcal{S} \subset S(\mathcal{H}_1)$ satisfying some physical conditions, the capacity of quantum channel Λ^* [40] is defined by

$$C_q^{\mathcal{S}}(\Lambda^*) \equiv \sup\{I(\rho;\Lambda^*); \rho \in \mathcal{S}\}.$$

If $\mathcal{S} = S(\mathcal{H}_1)$ holds, then the capacity is denoted by $C_q(\Lambda^*)$. Then the following theorem for the attenuation channel was proved in [40].

Theorem For a subset $\mathcal{S}_n \equiv \{\rho \in S(\mathcal{H}_1); \dim s(\rho) = n\}$, the capacity of the attenuation channel Λ_0^* satisfies

$$C_q^{\mathcal{S}_n}(\Lambda_0^*) = \log n,$$

where $s(\rho)$ is the support projection of ρ.

10.6. Mutual entropy for general systems

Based on the classical relative entropy, the mutual entropy was discussed by Shannon to study the information transmission in classical systems and it was extended by Ohya [33,34,35] for fully general quantum systems.

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha(G))$ be a unital C^*-system and $\mathcal{S}$ be a weak* compact convex subset of $\mathfrak{S}(\mathcal{A})$. For an initial state $\varphi \in \mathcal{S}$ and a channel $\Lambda^* : \mathfrak{S}(\mathcal{A}) \to \mathfrak{S}(\mathcal{B})$, two compound states are

$$\Phi_\mu^{\mathcal{S}} = \int_{\mathcal{S}} \omega \otimes \Lambda^* \omega \, d\mu,$$

$$\Phi_0 = \varphi \otimes \Lambda^* \varphi.$$

The compound state $\Phi_\mu^{\mathcal{S}}$ expresses the correlation between the input state φ and the output state $\Lambda^*\varphi$. The mutual entropy with respect to $\mathcal{S}$ and μ is given by

$$I_\mu^{\mathcal{S}}(\varphi; \Lambda^*) = S(\Phi_\mu^{\mathcal{S}}, \Phi_0)$$

and the mutual entropy with respect to $\mathcal{S}$ is defined by Ohya [33] as

$$I^{\mathcal{S}}(\varphi; \Lambda^*) = \sup\{I_\mu^{\mathcal{S}}(\varphi; \Lambda^*); \mu \in M_\varphi(\mathcal{S})\}.$$

10.7. Mutual entropy type complexity

Shor [53] and Bennet et al [6,10] proposed the mutual type measures so-called the coherent entropy and the Lindblad-Nielson entropy by using the entropy exchange [50] defined by

$$S_e(\rho, \Lambda^*) = -trW \log W,$$

where W is a matrix $W = (W_{ij})_{i,j}$ with

$W_{ij} \equiv tr A_i^* \rho A_j$

for a state ρ concerning a Stinespring-Sudarshan-Kraus form

$\Lambda^*(\cdot) \equiv \sum_j A_j^* \cdot A_j$,

of a channel Λ^*. Then the coherent entropy $I_C(\rho; \Lambda^*)$ [53] and the Lindblad-Nielson entropy $I_L(\rho; \Lambda^*)$ [10] are given by

$I_C(\rho; \Lambda^*) \equiv S(\Lambda^* \rho) - S_e(\rho, \Lambda^*)$,

$I_L(\rho; \Lambda^*) \equiv S(\rho) + S(\Lambda^* \rho) - S_e(\rho, \Lambda^*)$.

In this section, we compare with these mutual types measures. By comparing these mutual entropies for quantum information communication processes, we have the following theorem [47]:

Theorem Let $\{A_j\}$ be a projection valued measure withdim $A_j = 1$. For arbitrary state ρ and the quantum channel $\Lambda^*(\cdot) \equiv \sum_j A_j \cdot A_j^*$, one has

1. $0 \le I(\rho; \Lambda^*) \le \min\{S(\rho), S(\Lambda^* \rho)\}$ (Ohya mutual entropy),
2. $I_C(\rho; \Lambda^*) = 0$ (coherent entropy),
3. $I_L(\rho; \Lambda^*) = S(\rho)$ (Lindblad-Nielsen entropy).

For the attenuation channel Λ_0^*, one can obtain the following theorems [47]:

Theorem For any state $\rho = \sum_n \lambda_n |n\rangle\langle n|$ and the attenuation channel Λ_0^* with $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$, one has

1. $0 \le I(\rho; \Lambda_0^*) \le \min\{S(\rho), S(\Lambda_0^* \rho)\}$ (Ohya mutual entropy),
2. $I_C(\rho; \Lambda_0^*) = 0$ (coherent entropy),
3. $I_L(\rho; \Lambda_0^*) = S(\rho)$ (Lindblad-Nielsen entropy).

Theorem For the attenuation channel Λ_0^* and the input state $\rho = \lambda |0\rangle\langle 0| + (1-\lambda) |\theta\rangle\langle\theta|$, we have

1. $0 \le I(\rho; \Lambda_0^*) \le \min\{S(\rho), S(\Lambda_0^* \rho)\}$ (Ohya mutual entropy),
2. $-S(\rho) \le I_C(\rho; \Lambda_0^*) \le S(\rho)$ (coherent entropy),
3. $0 \le I_L(\rho; \Lambda_0^*) \le 2S(\rho)$ (Lindblad-Nielsen entropy).

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The above Theorem shows that the coherent entropy $I_C(\rho; \Lambda_0^*)$ takes a minus value for $|\alpha|^2 < |\beta|^2$ and the Lindblad-Nielsen entropy $I_L(\rho; \Lambda_0^*)$ is greater than the von Neumann entropy of the input state ρ for $|\alpha|^2 > |\beta|^2$. Therefore Ohya mutual entropy is most suitable one for discussing the efficiency of information transmission in quantum processes. Since the above theorems and other results [47] we could conclude that Ohya mutual entropy might be most suitable one for discussing the efficiency of information transmission in quantum communication processes. It means that Ohya mutual entropy can be considered as the transmitted complexity for quantum communication processes.

11. Quantum dynamical entropy

The classical dynamical (or Kolmogorov-Sinai) entropy S(T) [23] for a measure preserving transformation T was defined on a message space through finite partitions of the measurable space.

The classical coding theorems of Shannon are important tools to analyse communication processes which have been formulated by the mean dynamical entropy and the mean dynamical mutual entropy. The mean dynamical entropy represents the amount of information per one letter of a signal sequence sent from an input source, and the mean dynamical mutual entropy does the amount of information per one letter of the signal received in an output system.

The quantum dynamical entropy (QDE) was studied by Connes-Størmer [13], Emch [15], Connes- Narnhofer-Thirring [12], Alicki-Fannes [3], and others [9,48,19,57,11]. Their dynamical entropies were defined in the observable spaces. Recently, the quantum dynamical entropy and the quantum dynamical mutual entropy were studied by the present authors [34,35]: (1) The dynamical entropy is defined in the state spaces through the complexity of Information Dynamics [36]. (2) It is defined through the quantum Markov chain (QMC) was done in [2]. (3) The dynamical entropy for a completely positive (CP) maps was introduced in [25].

12. Mean entropy and mean mutual entropy

The classical Shannon 's coding theorems are important subject to study communication processes which have been formulated by the mean entropy and the mean mutual entropy based on the classical dynamical entropy. The mean entropy shows the amount of information per one letter of a signal sequence of an input source, and the mean mutual entropy denotes the amount of information per one letter of the signal received in an output system. Those mean entropies were extended in general quantum systems.

In this section, we briefly explain a new formulation of quantum mean mutual entropy of K-S type given by Ohya [35,27].

In quantum information theory, a stationary information source is denoted by a C^* triple $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \theta_{\mathcal{A}})$ with a stationary state φ with respect to $\theta_{\mathcal{A}}$; that is, $\mathcal{A}$ is a unital C^*-algebra, $\mathfrak{S}(\mathcal{A})$ is the set of all states over $\mathcal{A}$, $\theta_{\mathcal{A}}$ is an automorphism of $\mathcal{A}$, and $\varphi \in \mathfrak{S}(\mathcal{A})$ is a state over $\mathcal{A}$ with $\varphi \circ \theta_{\mathcal{A}} = \varphi$.

Let an output C^*-dynamical system be the triple $(\mathcal{B}, \mathfrak{S}(\mathcal{B}), \theta_{\mathcal{B}})$, and $\Lambda^* : \mathfrak{S}(\mathcal{A}) \to \mathfrak{S}(\mathcal{B})$ be a covariant channel which is a dual of a completely positive unital map $\Lambda : \mathcal{B} \to \mathcal{A}$ such that $\Lambda \circ \theta_{\mathcal{B}} = \theta_{\mathcal{A}} \circ \Lambda$.

In this section, we explain functional $S_\mu^{\mathcal{S}}(\varphi; \alpha^M)$, $S^{\mathcal{S}}(\varphi; \alpha^M)$, $I_\mu^{\mathcal{S}}(\varphi; \alpha^M, \beta^N)$ and $I^{\mathcal{S}}(\varphi; \alpha^M, \beta^N)$ introduced in [35,27] for a pair of finite sequences of $\alpha^M = (\alpha_1, \alpha_2, \cdots, \alpha_M)$, $\beta^N = (\beta_1, \beta_2, \cdots, \beta_N)$ of completely positive unital maps $\alpha_m : \mathcal{A}_m \to \mathcal{A}$, $\beta_n : \mathcal{B}_n \to \mathcal{B}$ where $\mathcal{A}_m$ and $\mathcal{B}_n$ $(m = 1, \cdots, M, n = 1, \cdots, N)$ are finite dimensional unital C^*-algebras.

Let $\mathcal{S}$ be a weak * convex subset of $\mathfrak{S}(\mathcal{A})$ and φ be a state in $\mathcal{S}$. We denote the set of all regular Borel probability measures μ on the state space $\mathfrak{S}(\mathcal{A})$ of $\mathcal{A}$ by $M_\varphi(\mathcal{S})$, so that μ is maximal in the Choquet ordering and μ represents $\varphi = \int_{S(\mathcal{A})} \omega d\mu(\omega)$. Such measures is taken by extremal decomposition measures for φ, Using Choquet's theorem, one can be shown that there exits such measures for any state $\varphi \in \mathfrak{S}(\mathcal{A})$. For a given finite sequences of completely positive unital maps $\alpha_m : \mathcal{A}_m \to \mathcal{A}$ from finite dimensional unitalC^*-algebras $\mathcal{A}_m$ $(m = 1, \cdots, M)$ and a given extremal decomposition measure μ of φ, the compound state of $\alpha_1^*\varphi, \alpha_2^*\varphi, \cdots, \alpha_M^*\varphi$ on the tensor product algebra $\bigotimes_{m=1}^{M} \mathcal{A}_m$ is given by [35,27]

$$\Phi_\mu^{\mathcal{S}}(\alpha^M) = \int_{S(\mathcal{A})} \bigotimes_{m=1}^{M} \alpha_m^* \omega d\mu(\omega).$$

Furthermore $\Phi_\mu^{\mathcal{S}}(\alpha^M \cup \beta^N)$ is a compound state of $\Phi_\mu^{\mathcal{S}}(\alpha^M)$ and $\Phi_\mu^{\mathcal{S}}(\beta^N)$ with $\alpha^M \cup \beta^N \equiv (\alpha_1, \alpha_2, \cdots, \alpha_M, \beta_1, \beta_2, \cdots, \beta_N)$ constructed as

$$\Phi_\mu^{S}(\alpha^M \cup \beta^N) = \int_{S(A)} \left(\bigotimes_{m=1}^{M} \alpha_m^* \omega \right) \otimes \left(\bigotimes_{n=1}^{N} \beta_n^* \omega \right) d\mu$$

For any pair (α^M, β^N) of finite sequences $\alpha^M = (\alpha_1, \cdots, \alpha_M)$ and $\beta^N = (\beta_1, \cdots, \beta_N)$ of completely positive unital maps $\alpha_m : \mathcal{A}_m \to \mathcal{A}$, $\beta_n : \mathcal{B}_n \to \mathcal{A}$ from finite dimensional unital C^* -algebras and any extremal decomposition measure μ of φ, the entropy functional S_μ and the mutual entropy functional I_μ are defined in [35,27] such as

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$$S_{\mu}^{\mathcal{S}}(\varphi;\alpha^{M})=\int_{S(A)}S\left(\bigotimes_{m=1}^{M}\alpha_{m}^{*}\omega,\ \Phi_{\mu}^{\mathcal{S}}(\alpha^{M})\right)d\mu(\omega),$$

$$I_{\mu}^{\mathcal{S}}(\varphi;\alpha^{M},\beta^{N})=S\left(\Phi_{\mu}^{S}\left(\alpha^{M}\cup\beta^{N}\right),\ \Phi_{\mu}^{\mathcal{S}}(\alpha^{M})\otimes\Phi_{\mu}^{\mathcal{S}}(\beta^{N})\right),$$

where $S(\cdot,\ \cdot)$ is the relative entropy.

For a given pair of finite sequences of completely positive unitalmaps $\alpha^{M}=(\alpha_{1},\ \cdots,\ \alpha_{M})$, $\beta^{N}=(\beta_{1},\ \cdots,\ \beta_{N})$, the functional $S^{\mathcal{S}}(\varphi;\alpha^{M})$ (resp. $I^{\mathcal{S}}(\varphi;\alpha^{M},\beta^{N})$) is given in [35,27] by taking the supremum of $S_{\mu}^{\mathcal{S}}(\varphi;\alpha^{M})$ (resp. $I_{\mu}^{\mathcal{S}}(\varphi;\alpha^{M},\beta^{N})$) for all possible extremal decompositions μ's of φ:

$$S^{\mathcal{S}}(\varphi;\alpha^{M})=\sup\left\{S_{\mu}^{\mathcal{S}}(\varphi;\alpha^{M});\quad \mu\in M_{\varphi}(\mathcal{S})\right\},$$

$$I^{\mathcal{S}}(\varphi;\alpha^{M},\beta^{N})=\sup\left\{I_{\mu}^{\mathcal{S}}(\varphi;\alpha^{M},\beta^{N});\quad \mu\in M_{\varphi}(\mathcal{S})\right\}.$$

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be a unital C^{*}-algebra with a fixed automorphism $\theta_{\mathcal{A}}$ (resp. $\theta_{\mathcal{B}}$), Λ be a covariant completely positive unital map from $\mathcal{B}$ to $\mathcal{A}$, and φ be an invariant state over $\mathcal{A}$, i.e., $\varphi\circ\theta_{\mathcal{A}}=\varphi$.

$$\alpha^{N}\equiv\left(\alpha,\ \theta_{\mathcal{A}}\circ\alpha,\ \cdots,\ \theta_{\mathcal{A}}^{N-1}\circ\alpha\right),$$

$$\beta_{\Lambda}^{N}\equiv\left(\Lambda\circ\beta,\ \Lambda\circ\theta_{\mathcal{B}}\circ\beta,\ \cdots,\ \Lambda\circ\theta_{\mathcal{B}}^{N-1}\circ\beta\right).$$

For each completely positive unital map $\alpha:\mathcal{A}_{0}\to\mathcal{A}$ (resp. $\beta:\mathcal{B}_{0}\to\mathcal{B}$) from a finite dimensional unital C^{*}-algebra $\mathcal{A}_{0}$ (resp. $\mathcal{B}_{0}$) to $\mathcal{A}$ (resp. $\mathcal{B}$), $\tilde{S}^{\mathcal{S}}(\varphi;\theta_{\mathcal{A}},\alpha)$, $\tilde{I}^{\mathcal{S}}(\varphi;\Lambda^{*},\theta_{\mathcal{A}},\theta_{\mathcal{B}},\alpha,\beta)$ are given in [35,27] by

$$\tilde{S}^{\mathcal{S}}(\varphi;\theta_{\mathcal{A}},\alpha)=\liminf_{N\to\infty}\frac{1}{N}S^{\mathcal{S}}(\varphi;\alpha^{N}),$$

$$\tilde{I}^{\mathcal{S}}(\varphi;\Lambda^{*},\theta_{\mathcal{A}},\theta_{\mathcal{B}},\alpha,\beta)=\liminf_{N\to\infty}\frac{1}{N}I^{\mathcal{S}}(\varphi;\alpha^{M},\beta^{N}).$$

The functional $\tilde{S}^{\mathcal{S}}(\varphi;\theta_{\mathcal{A}})$ and $\tilde{I}^{\mathcal{S}}(\varphi;\Lambda^{*},\theta_{\mathcal{A}},\theta_{\mathcal{B}})$ are defined by taking the supremum for all possible $\mathcal{A}_{0}$'s, α's, $\mathcal{B}_{0}$'s, and β's:

$$\tilde{S}^{\mathcal{S}}(\varphi;\theta_{\mathcal{A}})=\sup_{\alpha}\tilde{S}^{\mathcal{S}}(\varphi;\theta_{\mathcal{A}},\alpha),$$

$$\tilde{I}^{\mathcal{S}}(\varphi;\Lambda^{*},\theta_{\mathcal{A}},\theta_{\mathcal{B}})=\sup_{\alpha,\beta}\tilde{I}^{\mathcal{S}}(\varphi;\Lambda^{*},\theta_{\mathcal{A}},\theta_{\mathcal{B}},\alpha,\beta).$$

Then the fundamental inequality in information theory holds for $\tilde{S}^{\mathcal{S}}(\varphi;\theta_{\mathcal{A}})$ and $\tilde{I}^{\mathcal{S}}(\varphi;\Lambda^{*},\theta_{\mathcal{A}},\theta_{\mathcal{B}})$ [35].

12.1. Proposition

$$0\le\tilde{I}^{\mathcal{S}}(\varphi;\Lambda^{*},\theta_{\mathcal{A}},\theta_{\mathcal{B}})\le\min\left\{\tilde{S}^{\mathcal{S}}(\varphi;\theta_{\mathcal{A}}),\ \tilde{S}^{\mathcal{S}}(\Lambda^{*}\varphi;\theta_{\mathcal{B}})\right\}.$$

These functional $\tilde{S}^{\mathfrak{S}}(\varphi;\theta_{\mathcal{A}})$ and $\tilde{I}^{\mathfrak{S}}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}})$ are constructed from the functional $S_{\mu}^{\mathfrak{S}}(\varphi;\alpha^N)$ and $I_{\mu}^{\mathfrak{S}}(\varphi;\alpha^N,\beta^N)$ coming from information theory and these functionals are obtained by using a channel transformation, so that those functionals contains the dynamical entropy as a special case [35,27]. Moreover these functionals contain usual K-S entropies as follows [35,27].

Proposition If $\mathcal{A}_k$, $\mathcal{A}$ are abelian C^*-algebras and each α_k is an embedding, then our functionals coincide with classical K-S entropies:

$$S_{\mu}^{\mathfrak{S}(\mathcal{A})}(\varphi;\alpha^M)=S_{\mu}^{classical}\left(\bigvee_{m=1}^{M}\tilde{A}_m\right),$$

$$I_{\mu}^{\mathfrak{S}(\mathcal{A})}(\varphi;\alpha^M,\beta_{id}^N)=I_{\mu}^{classical}\left(\bigvee_{m=1}^{M}\tilde{A}_m,\bigvee_{n=1}^{N}\tilde{B}_n\right)$$

for any finite partitions $\tilde{A}_m$, $\tilde{B}_n$ of a probability space $(\Omega,\mathfrak{F},\varphi)$.

In general quantum structure, we have the following theorems [35,27].

Theorem Let α_m be a sequence of completely positive maps $\alpha_m:\mathcal{A}_m\to\mathcal{A}$ such that there exist completely positive maps $\alpha_m^{'}:\mathcal{A}\to\mathcal{A}_m$ satisfying $\alpha_m\circ\alpha_m^{'}\to id_{\mathcal{A}}$ in the pointwise topology. Then:

$$\tilde{S}^{\mathfrak{S}}(\varphi;\theta_{\mathcal{A}})=\lim_{m\to\infty}\tilde{S}^{\mathfrak{S}}(\varphi;\theta_{\mathcal{A}},\alpha_m).$$

Theorem Let α_m and β_m be sequences of completely positive maps $\alpha_m:\mathcal{A}_m\to\mathcal{A}$ and $\beta_m:\mathcal{B}_m\to\mathcal{B}$ such that there exist completely positive maps $\alpha_m^{'}:\mathcal{A}\to\mathcal{A}_m$ and $\beta_m^{'}:\mathcal{B}\to\mathcal{B}_m$ satisfying $\alpha_m\circ\alpha_m^{'}\to id_{\mathcal{A}}$ and $\beta_m\circ\beta_m^{'}\to id_{\mathcal{B}}$ in the pointwise topology. Then one has

$$\tilde{I}^{\mathfrak{S}}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}})=\lim_{m\to\infty}\tilde{I}^{\mathfrak{S}}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}},\alpha_m,\beta_m).$$

The above theorem is a Kolmogorov-Sinai type convergence theorem for the mutual entropy [35,27,28,34].

In particular, a quantum extension of classical formulation for information transmission giving a basis of Shannon's coding theorems can be considered in the case that $A=\bigotimes_{-\infty}^{\infty}A_0$, $B=\bigotimes_{-\infty}^{\infty}B_0$, $S=\mathfrak{S}$ and θ_A, θ_B are shift operators, both denoted by θ. In this case, the channel capacity is defined as [40,41,45,46,38,39,42,43]

$$\tilde{C}(\Lambda^*)\equiv\sup\{\tilde{I}^{\mathfrak{S}}(\varphi;\Lambda^*,\theta);\varphi\in\mathfrak{S}\}.$$

Using this capacity, one can consider Shannon's coding theorems in fully quantum systems.

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13. Computations of mean entropies for modulated states

Based on the paper [59], we here explain general modulated states and briefly review some examples of modulated states (PPM, OOK, PSK).

Let $\{a_1, \cdots, a_N\}$ be an alphabet set constructing the input signals and $N \equiv \{E_1, \cdots, E_N\}$ be the set of one dimensional projections on a Hilbert space $\mathcal{H}$ satisfying

1. $E_n \perp E_m \quad (n \neq m)$

2. E_n corresponds to the alphabet a_n.

We denote the set of all density operators on $\mathcal{H}$ generated by

$$\mathfrak{S}_0 \equiv \left\{\rho_0 = \sum_{n=1}^{N} \lambda_n E_n; \rho_0 \geq 0,\ tr\rho_0 = 1\right\},$$

where an element of $\mathfrak{S}_0$ represents a state of the quantum input system. The state is transmitted from the quantum input system to the quantum modulator in order to send information effectively, whose transmitted state is called the quantum modulated state. The quantum modulated states are denoted as follows: Let M be an ideal modulator and $N \equiv \{E_1^{(M)}, \cdots, E_N^{(M)}\}$ be the set of one dimensional projections on a Hilbert space $\mathcal{H}_M$ for modulated signals satisfying

$E_n^{(M)} \perp E_m^{(M)} (n \neq m)$, and we represent the set of all density operators on $\mathcal{H}_M$ by

$$\mathfrak{S}_0^{(M)} \equiv \left\{\rho_0^{(M)} = \sum_{n=1}^{N} \mu_n E_n^{(M)};\quad \rho_0^{(M)} \geq 0,\quad tr\rho_0^{(M)} = 1\right\},$$

where an element of $\mathfrak{S}_0^{(M)}$ represents a modulated state of the quantum input system. There are many expressions for the modulations. In this section, we take the modulated states by means of the photon number states.

γ_M^* is a modulator M if $\gamma_M^*(E_n) = E_n^{(M)}$ is a map from $\mathfrak{S}_0$ to $\mathfrak{S}_0^{(M)}$ satisfying (1) γ_M is a completely positive unital map from $\mathcal{A}_0$ to $\mathcal{A}$. Moreover γ_{IM}^* is called an ideal modulator IM if (1) $\gamma_{IM}^*(E_n) = E_n^{(M)}$ is a modulator from $\mathfrak{S}_0$ to $\mathfrak{S}_0^{(M)}$, $\gamma_{IM}^*(E_n) \perp \gamma_{IM}^*(E_m)$ for any orthogonal $E_n \in \mathfrak{S}_0$. Some examples of ideal modulator are given as follows:

1. For any $E_n \in \mathfrak{S}_0$, the PPM (Pulse Position Modulator) is defined by

$$\gamma_{PPM}^*(E_n) \equiv E_n^{(PPM)}$$
$$= E_0^{PAM} \otimes \cdots \otimes E_0^{(PAM)} \otimes E_d^{(PAM)} \otimes E_0^{(PAM)} \otimes \cdots E_0^{(PAM)}$$

where $E_0^{(PAM)}$ is the vacuum state on $\mathcal{H}_{(PAM)}$.

2. For $E_1, E_2 \in \mathfrak{S}_0$, the OOK (On-Off Keying) is defined by

$$\gamma^{*}_{OOK}(E_1) \equiv E_1^{(OOK)} = |0><0|,$$
$$\gamma^{*}_{OOK}(E_2) \equiv E_2^{(OOK)} = |\kappa><\kappa|$$

where $|\kappa><\kappa|$ is the coherent state on $\mathcal{H}_{OOK}$.

3. For $E_1,\ E_2 \in S_0$, the PSK (Phase Shift Keying) is defined by

$$\gamma^{*}_{PSK}(E_1) \equiv E_1^{(PSK)} = |-\kappa><-\kappa|,$$
$$\gamma^{*}_{PSK}(E_2) \equiv E_2^{(PSK)} = |\kappa><\kappa|$$

where $|\kappa><\kappa|$, $|-\kappa><-\kappa|$ are the coherent states on $\mathfrak{H}_{PSK}$.

Now we briefly review the calculation of the mean mutual entropy of K-S type for the modulated state (PSK) by means of the coherent state. Other calculations are obtained in [59].

$\alpha^{N}_{(IM)}$, $\beta^{N}_{(IM)}$ are given by

$$\alpha^{N}_{(IM)} \equiv (\alpha \circ \tilde{\gamma}_{(IM)},\ \theta_{\mathcal{A}} \circ \alpha \circ \tilde{\gamma}_{(IM)},\ \cdots,\ \theta_{\mathcal{A}}^{N-1} \circ \alpha \circ \tilde{\gamma}_{(IM)}),$$
$$\beta^{N}_{(IM)} \equiv (\tilde{\gamma}_{(IM)} \circ \Lambda \circ \beta,\ \tilde{\gamma}_{(IM)} \circ \Lambda \circ \theta_{\mathcal{B}} \circ \beta,\ \cdots,\ \tilde{\gamma}_{(IM)} \circ \Lambda \circ \theta_{\mathcal{B}}^{N-1} \circ \beta),$$

where $\tilde{\Lambda} \equiv \overset{\infty}{\underset{i=-\infty}{\otimes}} \Lambda$ and $\tilde{\gamma}_{(IM)} \equiv \overset{\infty}{\underset{i=-\infty}{\otimes}} \gamma_{(IM)}$ are held.

PSK. For an initial state $\rho = \overset{\infty}{\underset{i=-\infty}{\otimes}} \rho_i \in \overset{\infty}{\underset{i=-\infty}{\otimes}} S_i$, let $\rho_i = \nu |-\kappa\rangle\langle -\kappa| + (1-\nu)|\kappa\rangle\langle\kappa|$ $(0 \le \nu \le 1)$. The Schatten decomposition of ρ_i is obtained as

$$\rho_i = \sum_{n_i=1}^{2} \lambda_{n_i} E_{n_i}^{(PSK)},$$

where the eigenvalues λ_{n_i} of ρ_i are

$$\lambda_1 = \frac{1}{2}\left\{1 + \sqrt{1 - 4\nu(1-\nu)\left(1 - \exp\left(-|2\kappa|^2\right)\right)}\right\},$$
$$\lambda_2 = \frac{1}{2}\left\{1 - \sqrt{1 - 4\nu(1-\nu)\left(1 - \exp\left(-|2\kappa|^2\right)\right)}\right\}.$$

Two projections $E_{n_i}^{(PSK)}$ $(n_i = 1, 2)$ and the eigenvectors $|e_{n_i}^{(PSK)}\rangle$ of λ_{n_i} $(n_i = 1, 2)$ are given by

$$E_{n_i}^{(PSK)} = |e_{n_i}^{(PSK)}\rangle\langle e_{n_i}^{(PSK)}|,$$

$$|e_{n_i}^{(PSK)}\rangle = a_{n_i}|-\kappa\rangle + b_{n_i}|\kappa\rangle, \quad (n_i = 1, 2),$$

where

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$$|b_{n_i}|^2=\frac{1}{\tau_{n_i}^2+2\exp\left(-|\kappa|^2\right)\tau_{n_i}+1},$$

$$|a_{n_i}|^2=\tau_{n_i}^2|b_{n_i}|^2,$$

$$a_{n_i}\overline{b_{n_i}}=\overline{a_{n_i}}b_{n_i}=\tau_{n_i}|b_{n_i}|^2$$

$$\tau_1=\frac{-(1-2\nu)+\sqrt{1-4\nu(1-\nu)\left(1-\exp\left(-|2\kappa|^2\right)\right)}}{2(1-\nu)\exp\left(-|\kappa|^2\right)},$$

$$\tau_2=\frac{-(1-2\nu)-\sqrt{1-4\nu(1-\nu)\left(1-\exp\left(-|2\kappa|^2\right)\right)}}{2(1-\nu)\exp\left(-|\kappa|^2\right)}.$$

For the above initial state $E_{n_i}^{(PSK)}$, one can obtain the output state for the attenuation channel Λ^* as follows:

$$\Lambda^* E_{n_i}^{(PSK)}=\sum_{n_i'=1}^{2}\hat{\lambda}_{n_i,n_i'}\overset{\frown}{E}{}^{(PSK)}_{n_i,n_i'}\quad(n_i=1,\ 2),$$

where the eigenvalues $\hat{\lambda}_{n_i,n_i'}$ of $\Lambda^* E_{n_i}^{(OOK)}$ are given by ($n_i=1,\ 2$)

$$\hat{\lambda}_{n_i,1}=\frac{1}{2}\left\{1+\sqrt{1-4\mu_{n_i}\left(1-\mu_{n_i}\right)\left(1-|\langle u_{n_i,1},\ u_{n_i,2}\rangle|^2\right)}\right\},$$

$$\hat{\lambda}_{n_i,2}=\frac{1}{2}\left\{1-\sqrt{1-4\mu_{n_i}\left(1-\mu_{n_i}\right)\left(1-|\langle u_{n_i,1},\ u_{n_i,2}\rangle|^2\right)}\right\},$$

$$\mu_{n_i}=\frac{1}{2}\left(1+\exp\left(-(1-\eta)|\kappa|^2\right)\right)\frac{\tau_{n_i}^2+2\exp\left(-|\alpha\kappa|^2\right)\tau_{n_i}+1}{\tau_{n_i}^2+2\exp\left(-|\kappa|^2\right)\tau_{n_i}+1}.$$

$$|\langle u_{n_i,n_i'},\ u_{n_i,n_i'}\rangle|^2=1,$$

$$\langle u_{n_i,1},\ u_{n_i,2}\rangle=\frac{\tau_{n_i}^2-1}{\sqrt{\left(\tau_{n_i}^2+1\right)^2-4\exp\left(-|2\alpha\kappa|^2\right)\tau_{n_i}^2}}\quad(n_i=1,\ 2).$$

$\overset{\frown}{E}{}^{(PSK)}_{n_i,n_i'}$ are the eigenstates with respect to $\hat{\lambda}_{n_i,n_i'}$. Then we have

$$\Phi_E(\alpha_{(PSK)}^N)=\bigotimes_{i=0}^{N-1}\gamma_{(PSK)}^*\circ\alpha^*\circ\theta_{\mathrm{A}}^{*i}(\rho)=\bigotimes_{i=0}^{N-1}\gamma_{(PSK)}^*(\rho_i)$$

$$=\sum_{n_0=1}^{2}\cdots\sum_{n_{N-1}=1}^{2}\left(\prod_{k=0}^{N-1}\lambda_{n_k}\right)\left(\bigotimes_{i=0}^{N-1}E_{n_i}^{(PSK)}\right)$$

When Λ^* is given by the attenuation channel, we get

$$\Phi_E(\beta^N_{\Lambda(PSK)}) = \bigotimes_{i=0}^{N-1} \beta^* \circ \theta^{*i}_{\mathsf{B}} \circ \Lambda^* \circ \gamma^*_{(PSK)}(\rho)$$
$$= \sum_{n_0=1}^{2} \cdots \sum_{n_{N-1}=1}^{2} \left(\prod_{k=0}^{N-1} \lambda_{n_k} \right) \left(\bigotimes_{i=0}^{N-1} \Lambda^* E^{(PSK)}_{n_i} \right)$$

The compound states through the attenuation channel Λ^* becomes

$$\Phi_E(\alpha^N_{(PSK)} \cup \beta^N_{\Lambda(PSK)})$$
$$= \sum_{n_0=1}^{2} \cdots \sum_{n_{N-1}=1}^{2} \left(\prod_{k=0}^{N-1} \lambda_{n_k} \right) \sum_{n'_0=1}^{2} \cdots \sum_{n'_{N-1}=1}^{2} \left(\prod_{k'=0}^{N-1} \hat{\lambda}_{n_{k'}, n'_{k'}} \right)$$
$$\times \left(\bigotimes_{i=0}^{N-1} E^{(PSK)}_{n_i} \right) \otimes \left(\bigotimes_{i'=0}^{N-1} \frown^{(PSK)} E_{n_{i'}, n'_{i'}} \right)$$
$$\Phi_E(\alpha^N_{(PSK)}) \otimes \Phi_E(\beta^N_{\Lambda(PSK)})$$
$$= \sum_{n_0=1}^{2} \cdots \sum_{n_{N-1}=1}^{2} \left(\prod_{k=0}^{N-1} \lambda_{n_k} \right) \sum_{m_0=1}^{2} \cdots \sum_{m_{N-1}=1}^{2} \left(\prod_{k'=0}^{N-1} \lambda_{m_{k'}} \right)$$
$$\times \sum_{m'_0=0}^{2} \cdots \sum_{m'_{N-1}=0}^{2} \left(\prod_{k''=0}^{N-1} \hat{\lambda}_{m_{k''}, m'_{k''}} \right)$$
$$\times \left(\bigotimes_{i=0}^{N-1} E^{(PSK)}_{n_i} \right) \otimes \left(\bigotimes_{i'=0}^{N-1} \frown^{(PSK)} E_{m_{i'}, m'_{i'}} \right)$$

Lemma For an initial state $\rho = \bigotimes_{i=-\infty}^{\infty} \rho_i \in \bigotimes_{i=-\infty}^{\infty} \mathsf{S}_i$, we have

$$I_E(\rho; \alpha^N_{(PSK)}, \beta^N_{(PSK)})$$
$$= \sum_{n'_0=1}^{2} \cdots \sum_{n'_{N-1}=1}^{2} \sum_{n_0=1}^{2} \cdots \sum_{n_{N-1}=1}^{2} \left(\prod_{k=0}^{N-1} \lambda_{n_k} \hat{\lambda}_{n_k, n'_k} \right) \log \frac{\prod_{k=0}^{N-1} \hat{\lambda}_{n_k, n'_k}}{\sum_{m_0=1}^{2} \cdots \sum_{m_{N-1}=1}^{2} \left(\prod_{k'=0}^{N-1} \lambda_{m_{k'}} \hat{\lambda}_{m_{k'}, n'_{k'}} \right)}.$$

By using the above lemma, we have the following theorem.

Theorem For an initial state $\rho = \bigotimes_{i=-\infty}^{\infty} \rho_i \in \bigotimes_{i=-\infty}^{\infty} \mathsf{S}_i$, we have

$$\tilde{S}(\rho; \theta_{\mathsf{A}'}, \alpha^N_{(PSK)}) = \lim_{N \to \infty} \frac{1}{N} S(\rho; \alpha^N_{(PSK)}) = -\sum_{n=1}^{2} \lambda_n \log \lambda_n$$

and

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$$\tilde{I}(\rho;\Lambda^*,\ \theta_{\tilde{\ }},\ \theta_{\tilde{\ }},\ \alpha^N_{(PSK)},\ \beta^N_{(PSK)})=\sum_{n'=1}^{2}\sum_{n=1}^{2}\lambda_n\hat{\lambda}_{n,n'}\log\frac{\hat{\lambda}_{n,n'}}{\sum_{m=1}^{2}\lambda_m\hat{\lambda}_{m,n'}}.$$

14. KOW dynamical entropy

In this section, we briefly explain the definition of the KOW entropy according to [25].

For a normal state ω on $B(\mathcal{K})$ and a normal, unital CP linear map Γ from $B(\mathcal{K})\otimes B(\mathcal{H})$ to $B(\mathcal{K})\otimes B(\mathcal{H})$, one can define a transition expectation $E^{\Gamma,\omega}$ from $B(\mathcal{K})\otimes B(\mathcal{H})$ to $B(\mathcal{H})$ by

$$E^{\Gamma,\omega}(\widetilde{A})=\omega(\Gamma(\widetilde{A}))=tr_{\mathcal{K}}\widetilde{\omega}\Gamma(\widetilde{A}),\quad \forall\ \widetilde{A}\in B(\mathcal{K})\otimes B(\mathcal{H})$$

in the sense of [1,25], where $\widetilde{\omega}\in\mathfrak{S}(\mathcal{K})$ is a density operator associated to ω. The dual map E is a lifting from $\mathfrak{S}(\mathcal{H})$ to $\mathfrak{S}(\mathcal{K}\otimes\mathcal{H})$ by

$$E^{*\Gamma,\omega}(\rho)=\Gamma^*(\widetilde{\omega}\otimes\rho).$$

in the sense of Accardi and Ohya [1]. For a normal, unital CP map Λ from $B(\mathcal{H})$ to $B(\mathcal{H})$ and the identity map id on $B(\mathcal{K})$, the transition expectation

$$E_{\Lambda}^{\Gamma,\omega}(\widetilde{A})=\omega((id\otimes\Lambda)\Gamma(\widetilde{A})),\quad \forall\ \widetilde{A}\in B(\mathcal{K})\otimes B(\mathcal{H})$$

and the lifting is defined by

$$E_{\Lambda}^{*\Gamma,\omega}(\rho)=\Gamma^*\big(\widetilde{\omega}\otimes\Lambda^*(\rho)\big),\quad \forall\ \rho\in\mathfrak{S}(\mathcal{H}),$$

where $id\otimes\Lambda$ is a normal, unital CP map from $B(\mathcal{K})\otimes B(\mathcal{H})$ to $B(\mathcal{K})\otimes B(\mathcal{H})$ and Λ^* is a quantum channel [30,21,31,39,44,46,43] from $\mathfrak{S}(\mathcal{H})$ to $\mathfrak{S}(\mathcal{H})$ with respect to an input signal state ρ and a noise state $\widetilde{\omega}$. Based on the following relation

$$\begin{aligned}&tr_{(\otimes_1^n\mathcal{K})\otimes\mathcal{H}}\Phi_{\Lambda,n}^{*\Gamma,\omega}(\rho)\big(A_1\otimes\cdots\otimes A_n\otimes B\big)\\&\equiv tr_{\mathcal{H}}\rho\Big(E_{\Lambda}^{\Gamma,\omega}\big(A_1\otimes E_{\Lambda}^{\Gamma,\omega}\big(A_2\otimes\cdots A_{n-1}\otimes E_{\Lambda}^{\Gamma,\omega}\big(A_n\otimes B\big)\cdots\big)\big)\Big)\end{aligned}$$

for all $A_1, A_2, \cdots, A_n\in B(\mathcal{K})$, $B\in B(\mathcal{H})$ and any $\rho\in\mathfrak{S}(\mathcal{H})$, a lifting $\Phi_{\Lambda,n}^{*\Gamma,\omega}$ from $\mathfrak{S}(\mathcal{H})$ to $\mathfrak{S}\big(\big(\otimes_1^n\mathcal{K}\big)\otimes\mathcal{H}\big)$ and marginal states are given by

$$\rho_{\Lambda,n}^{\Gamma,\omega}\equiv tr_{\mathcal{H}}\Phi_{\Lambda,n}^{*\Gamma,\omega}(\rho)\in\mathfrak{S}\big(\otimes_1^n\mathcal{K}\big)\text{ and }\bar{\rho}_{\Lambda,n}^{\Gamma,\omega}\equiv tr_{\otimes_1^n\mathcal{K}}\Phi_{\Lambda,n}^{*\Gamma,\omega}(\rho)\in\mathfrak{S}(\mathcal{H})$$

where $\Phi_{\Lambda,n}^{*\Gamma,\omega}(\rho)$ is a compound state with respect to $\bar{\rho}_{\Lambda,n}^{\Gamma,\omega}$ and $\rho_{\Lambda,n}^{\Gamma,\omega}$ in the sense of [25,31] .

Definition The quantum dynamical entropy with respect to Λ, ρ, Γ and ω is defined by

$$\widetilde{S}(\Lambda;\rho,\ \Gamma,\ \omega)\equiv\limsup_{n\to\infty}\frac{1}{n}S\big(\rho_{\Lambda,n}^{\Gamma,\omega}\big),$$

where $S\left(\rho_{\Lambda,n}^{\Gamma,\omega}\right)$ is the von Neumann entropy of $\rho_{\Lambda,n}^{\Gamma,\omega}\in\mathfrak{S}\left(\otimes_1^n\mathcal{K}\right)$ defined by $S\left(\rho_{\Lambda,n}^{\Gamma,\omega}\right)=-tr\rho_{\Lambda,n}^{\Gamma,\omega}\log\rho_{\Lambda,n}^{\Gamma,\omega}$. The dynamical entropy with respect to Λ and ρ is defined as $\tilde{S}(\Lambda;\rho)\equiv\sup\{\mathfrak{S}(\Lambda;\rho,\Gamma,\omega);\Gamma,\omega\}$.

15. Formulation of generalized AF and AOW entropies by KOW entropy

In this section, I briefly explain the generalized AF and AOW entropies based on the KOW entropy [25].

For a finite operational partition of unity $\gamma_1,\ \cdots,\ \gamma_d\in B(\mathcal{H})$, i.e., $\sum_{i=1}^{d}\gamma_i^*\gamma_i=I$, and a normal unital CP map Λ from $B(\mathcal{H})$ to $B(\mathcal{H})$, transition expectations E_Λ^γ from $M_d\otimes B(\mathcal{H})$ to $B(\mathcal{H})$ and $E_\Lambda^{\gamma(0)}$ from $M_d^0\otimes B(\mathcal{H})$ to $B(\mathcal{H})$ are defined by

$$E_\Lambda^\gamma\left(\sum_{i,j=1}^{d}E_{ij}\otimes A_{ij}\right)\equiv\sum_{i,j=1}^{d}\Lambda\left(\gamma_i^*A_{ij}\gamma_j\right),$$

$$E_\Lambda^{\gamma(0)}\left(\sum_{i,j=1}^{d}E_{ij}\otimes A_{ij}\right)\equiv\sum_{i=1}^{d}\Lambda\left(\gamma_i^*A_{ii}\gamma_i\right),$$

where $E_{ij}=|e_i\rangle\langle e_j|$ with normalized vectors $e_i\in\mathcal{H}$, $i=1,2,\ \cdots,\ d\le\dim\mathcal{H}$, M_d in $B(\mathcal{H})$ is the $d\times d$ matrix algebra and M_d^0 is a subalgebra of M_d consisting of diagonal elements of M_d. Then the quantum Markov states

$$\rho_{\Lambda,n}^\gamma=\sum_{i_1,\cdots,i_n=1}^{d}\sum_{j_1,\cdots,j_n=1}^{d}tr_{\mathcal{H}\mathcal{E}}\rho\Lambda\left(W_{j_1i_1}\left(\Lambda\left(W_{j_2i_2}\left(\cdots\Lambda\left(W_{j_ni_n}\left(I_{\mathcal{H}\mathcal{E}}\right)\right)\right)\right)\right)\right)$$
$$\times E_{i_1j_1}\otimes\cdots\otimes E_{i_nj_n}$$

and $\rho_{\Lambda,n}^{\gamma(0)}$ is obtained by

$$\rho_{\Lambda,n}^{\gamma(0)}=\sum_{i_1,\cdots,i_n=1}^{d}tr_{\mathcal{H}\mathcal{E}}\rho\Lambda\left(W_{i_1i_1}\left(\Lambda\left(W_{i_2i_2}\left(\cdots\Lambda\left(W_{i_ni_n}\left(I_{\mathcal{H}\mathcal{E}}\right)\right)\right)\right)\right)\right)E_{i_1i_1}\otimes\cdots\otimes E_{i_ni_n}$$
$$=\sum_{i_1,\cdots,i_n=1}^{d}p_{i_1,\cdots,i_n}E_{i_1i_1}\otimes\cdots\otimes E_{i_ni_n},$$

where

$$W_{ij}(A)\equiv\gamma_i^*A\gamma_j,\quad A\in B(\mathcal{H}),$$
$$W_{ij}^*(\rho)\equiv\gamma_j\rho\gamma_i^*,\quad\rho\in\mathfrak{S}(\mathcal{H}),$$
$$p_{i_1,\cdots,i_n}\equiv tr_{\mathcal{H}\mathcal{E}}\rho\Lambda\left(W_{i_1i_1}\left(\Lambda\left(W_{i_2i_2}\left(\cdots\Lambda\left(W_{i_ni_n}\left(I_{\mathcal{H}\mathcal{E}}\right)\right)\right)\right)\right)\right)$$
$$=tr_{\mathcal{H}\mathcal{E}}W_{i_ni_n}^*\left(\Lambda^*\cdots\Lambda^*\left(W_{i_2i_2}^*\left(\Lambda^*\left(W_{i_1i_1}^*\left(\Lambda^*(\rho)\right)\right)\right)\right)\right).$$

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Therefore the generalized AF entropy $\tilde{S}_{\mathcal{B}}(\Lambda;\rho)$ and the generalized AOW entropy $\tilde{S}^{(0)}_{\mathcal{B}}(\Lambda;\rho)$ of Λ and ρ with respect to a finite dimensional subalgebra $\mathcal{B}\subset B(\mathcal{H})$ are obtained by

$$\tilde{S}_{\mathcal{B}}(\Lambda;\rho)\equiv\sup_{\{\gamma_i\}\subset\mathcal{B}}\tilde{S}(\Lambda;\rho,\{\gamma_i\}),$$

$$\tilde{S}^{(0)}_{\mathcal{B}}(\Lambda;\rho)\equiv\sup^{(0)}_{\{\gamma_i\}\subset\mathcal{B}}\tilde{S}(\Lambda;\rho,\{\gamma_i\}),$$

where the dynamical entropies $\tilde{S}(\Lambda;\rho,\{\gamma_i\})$ and $\tilde{S}^{(0)}(\Lambda;\rho,\{\gamma_i\})$ are given by

$$\tilde{S}(\Lambda;\rho,\{\gamma_i\})\equiv\limsup_{n\to\infty}\frac{1}{n}S\left(\rho^{\gamma}_{\Lambda,n}\right),$$

$$\tilde{S}^{(0)}(\Lambda;\rho,\{\gamma_i\})\equiv\limsup_{n\to\infty}\frac{1}{n}S\left(\rho^{\gamma(0)}_{\Lambda,n}\right).$$

16. Computations of generalized AOW entropy for modulated states

Then we have the following theorem [25]:

16.1. Theorem

$$\tilde{S}_{\mathcal{B}}(\Lambda;\rho)\le\tilde{S}^{(0)}_{\mathcal{B}}(\Lambda;\rho).$$

$\tilde{S}^{(0)}_{\mathcal{B}}(\Lambda;\rho)$ is equal to the AOW entropy if $\{\gamma_i\}$ is PVM (projection valued measure) and Λ is given by an automorphism θ. $\tilde{S}_{\mathcal{B}}(\Lambda;\rho)$ is equal to the AF entropy if $\{\gamma_i{}^*\gamma_i\}$ is POV (positive operator valued measure) and Λ is given by an automorphism θ. For the noisy optical channel, the generalized AOW entropy can be obtained in [58] as follows.

Theorem [58] When ρ is given by $\rho=\lambda|0\rangle\langle 0|+(1-\lambda)|\xi\rangle\langle\xi|$ and Λ^* is the noisy optical channel with the cohetent noise $|\kappa\rangle\langle\kappa|$ and parameters α, β satisfying $|\alpha|^2+|\beta|^2=1$, the quantum dynamical entropy with respect to Λ, ρ and $\{\gamma_j\}$ is obtained by

$$\tilde{S}^{(0)}(\Lambda;\rho,\{\gamma_j\})=-\sum_{j,k}q_{k,j}q_j\log q_{k,j},$$

where

$$q_j=\lambda|\langle\beta\kappa,x_j\rangle|^2+(1-\lambda)|\langle\alpha\xi+\beta\kappa,x_j\rangle|^2$$

$$q_{k,j}=\nu_j^+|\langle x_k,y_j^+\rangle|^2+\left(1-\nu_j^+\right)|\langle x_k,y_j^-\rangle|^2,$$

$$|y_j^+\rangle=a_j^+|\beta\kappa\rangle+b_j^+|\alpha\xi+\beta\kappa\rangle,$$

$$|y_j^-\rangle=a_j^-|\beta\kappa\rangle-b_j^-|\alpha\xi+\beta\kappa\rangle,$$

$$a_j^+ = \varepsilon_j^+ a_j,$$

$$a_j^- = \varepsilon_j^- a_j,\ b_j^+ = \varepsilon_j^+ b_j,$$

$$b_j^- = \varepsilon_j^- b_j,$$

$$\varepsilon_j^+ = \sqrt{\frac{\tau_j^2 + 2\exp\left(-\frac{1}{2}|\xi|^2\right)\tau_j + 1}{\tau_j^2 + 2\exp\left(-\frac{1}{2}|\alpha\xi|^2\right)\tau_j + 1}},$$

$$\varepsilon_j^- = \sqrt{\frac{\tau_j^2 + 2\exp\left(-\frac{1}{2}|\xi|^2\right)\tau_j + 1}{\tau_j^2 - 2\exp\left(-\frac{1}{2}|\alpha\xi|^2\right)\tau_j + 1}},$$

$$\nu_j^+ = \frac{1}{2}\left(1 + \exp\left(-\frac{1}{2}(1-|\alpha|^2)|\xi|^2\right)\right)\frac{1}{\left(\varepsilon_j^+\right)^2},$$

$$\tau_j = \frac{-(1-2\lambda)}{2(1-\lambda)\exp\left(-\frac{1}{2}|\xi|^2\right)} + (-1)^j \frac{\sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-|\xi|^2\right)\right)}}{2(1-\lambda)\exp\left(-\frac{1}{2}|\xi|^2\right)},$$

$$|b_j|^2 = \frac{1}{\tau_j^2 + 2\exp\left(-\frac{1}{2}|\xi|^2\right)\tau_j + 1},$$

$$|a_j|^2 = \tau_j^2 |b_j|^2,\ \bar{a}_j b_j = a_j \bar{b}_j = \tau_j |b_j|^2$$

Theorem [58] For $n \geqq 3$, the above compound state $\rho_{\Lambda,n}^{\gamma(0)}$ is written by

$$\rho_{\Lambda,n}^{\gamma(0)} = \sum_{j_1,\cdots,j_n=1}^{2} q_{j_1,\cdots,j_n} \bigotimes_{k=1}^{n} |x_{j_k}\rangle\langle x_{j_k}|,$$

where

$$q_{j_1,\cdots,j_n} \equiv tr_{\mathcal{H}} W_{j_n j_n}^* \left(\Lambda^*\left(\cdots \Lambda^*\left(W_{j_2 j_2}^*\left(\Lambda^*\left(W_{j_1 j_1}^*\left(\Lambda^*(\rho)\right)\right)\right)\right)\cdots\right)\right),$$

$$\Lambda^*(\rho) = \lambda |\beta\kappa\rangle\langle\beta\kappa| + (1-\lambda)|\alpha\xi + \beta\kappa\rangle\langle\alpha\xi + \beta\kappa|,$$

$$W_{jj}^*\left(\Lambda^*(\rho)\right) \equiv \gamma_j^* \Lambda^*(\rho)\gamma_j = \left(\lambda |\langle\beta\kappa, x_j\rangle|^2 + (1-\lambda)|\langle\alpha\xi + \beta\kappa, x_j\rangle|^2\right)|x_j\rangle\langle x_j|.$$

Based on [40,41,45,46], one can obtain

$$\Lambda^*\left(|x_j\rangle\langle x_j|\right) = \nu_j^+ |y_j^+\rangle\langle y_j^+| + \left(1-\nu_j^+\right)|y_j^-\rangle\langle y_j^-|.$$

Thus we have

$$q_{j_1,\cdots,j_n} = \prod_{k=2}^{n}\left(\nu_{j_{k-1}}^+ |\langle x_{j_k}, y_{j_{k-1}}^+\rangle|^2 + \left(1-\nu_{j_{k-1}}^+\right)|\langle x_{j_k}, y_{j_{k-1}}^-\rangle|^2\right)$$

$$\times\left(\lambda |\langle\beta\kappa, x_j\rangle|^2 + (1-\lambda)|\langle\alpha\xi + \beta\kappa, x_j\rangle|^2\right)$$

$$= \prod_{k=2}^{n} q_{j_k, j_{k-1}} q_{j_1}.$$

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If $_j q_{k,j} q_j = q_k$ is hold , then we get the dynamical entropy with respect to Λ, ρ and $\{\gamma_j\}$ such as $\tilde{S}^{(0)}(\Lambda; \rho, \{\gamma_j\}) = -\sum_{j,k} q_{k,j} q_j \log q_{k,j}$.

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