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Information Theory and Entropies for Quantized Optical Waves in Complex Time-Varying Media

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1. Introduction

An important physical intuition that led to the Copenhagen interpretation of quantum mechanics is the Heisenberg uncertainty relation (HUR) which is a consequence of the noncommutativity between two conjugate observables. Our ability of observation is intrinsically limited by the HUR, quantifying an amount of inevitable and uncontrollable disturbance on measurements (Ozawa, 2004).

Though the HUR is one of the most fundamental results of the whole quantum mechanics, some drawbacks concerning its quantitative formulation are reported. As the expectation value of the commutator between two arbitrary noncommuting operators, the value of the HUR is not a fixed lower bound and varies depending on quantum state (Deutsch, 1983). Moreover, in some cases, the ordinary measure of uncertainty, i.e., the variance of canonical variables, based on the Heisenberg-type formulation is divergent (Abe et al., 2002).

These shortcommings are highly nontrivial issues in the context of information sciences. Thereby, the theory of informational entropy is proposed as an alternate optimal measure of uncertainty. The adequacy of the entropic uncertainty relations (EURs) as an uncertainty measure is owing to the fact that they only regard the probabilities of the different outcomes of a measurement, whereas the HUR the variances of the measured values themselves (Werner, 2004). According to Khinchin's axioms (Ash, 1990) for the requirements of common information measures, information measures should be dependent exclusively on a probability distribution (Pennini & Plastino, 2007). Thank to active research and technological progress associated with quantum information theory (Nielsen & Chuang, 2000; Choi et al., 2011), the entropic uncertainty band now became a new concept in quantum physics.

Information theory proposed by Shannon (Shannon, 1948a; Shannon, 1948b) is important as information-theoretic uncertainty measures in quantum physics but even in other areas such as signal and/or image processing. Essential unity of overall statistical information for a system can be demonstrated from the Shannon information, enabling us to know how information could be quantified with absolute precision. Another good measure of uncertainty or randomness is the Fisher information (Fisher, 1925) which appears as the basic ingredient in bounding entropy production. The Fisher information is a measure of accuracy in statistical theory and useful to estimate ultimate limits of quantum measurements.

Recently, quantum information theory besides the fundamental quantum optics has aroused great interest due to its potential applicability in three sub-regions which are quantum computation, quantum communication, and quantum cryptography. Information theory has contributed to the development of the modern quantum computation (Nielsen & Chuang, 2000) and became a cornerstone in quantum mechanics. A remarkable ability of quantum computers is that they can carry out certain computational tasks exponentially faster than classical computers utilizing the entanglement and superposition principle.

Stimulated by these recent trends, this chapter is devoted to the study of information theory for optical waves in complex time-varying media with emphasis on the quantal information measures and informational entropies. Information theoretic uncertainty relations and the information measures of Shannon and Fisher will be managed. The EUR of the system will also be treated, quantifying its physically allowed minimum value using the invariant operator theory established by Lewis and Riesenfeld (Lewis, 1967; Lewis & Riesenfeld, 1969). Invariant operator theory is crucial for studying quantum properties of complicated time-varying systems, since it, in general, gives exact quantum solutions for a system described by time-dependent Hamiltonian so far as its counterpart classical solutions are known.

2. Quantum optical waves in time-varying media

Let us consider optical waves propagating through a linear medium that has time-dependent electromagnetic parameters. Electromagnetic properties of the medium are in principle determined by three electromagnetic parameters such as electric permittivity ϵ, magnetic permeability μ, and conductivity σ. If one or more parameters among them vary with time, the medium is designated as a time-varying one. Coulomb gauge will be taken for convenience under the assumption that the medium have no net charge distributions. Then the scalar potential vanishes and, consequently, the vector potential is the only potential needed to consider when we develop quantum theory of electromagnetic wave phenomena. Regarding this fact, the quantum properties of optical waves in time-varying media are described in detail in Refs. (Choi & Yeon, 2005; Choi, 2012; Choi et al, 2012) and they will be briefly surveyed in this section as a preliminary step for the study of information theory.

According to separation of variables method, it is favorable to put vector potential in the form

$$\mathbf{A}(\mathbf{r}, t) = \sum_l \mathbf{u}_l(\mathbf{r}) q_l(t). \tag{1}$$

Then, considering the fact that the fields and current density obey the relations, $\mathbf{D} = \epsilon(t)\mathbf{E}$, $\mathbf{B} = \mu(t)\mathbf{H}$, and $\mathbf{J} = \sigma(t)\mathbf{E}$, in linear media, we derive equation of motion for q_l from Maxwell equations as (Choi, 2012; Choi, 2010a; Pedrosa & Rosas, 2009)

$$\ddot{q}_l + \{[\dot{\epsilon}(t) + \sigma(t)]/\epsilon(t)\}\dot{q}_l + \omega_l^2(t)q_l = 0. \tag{2}$$

Here, the angular frequency (natural frequency) is given by $\omega_l(t) = c(t)k_l$ where $c(t)$ is the speed of light in media and $k_l(= |\mathbf{k}_l|)$ is the wave number. Because electromagnetic parameters vary with time, $c(t)$ can be represented as a time-dependent form, i.e., $c(t) = 1/\sqrt{\mu(t)\epsilon(t)}$. However, $k_l(= |\mathbf{k}_l|)$ is constant since it does not affected by time-variance of the parameters.

The formula of mode function $\mathbf{u}_l(\mathbf{r})$ depends on the geometrical boundary condition in media (Choi & Yeon, 2005). For example, it is given by $\mathbf{u}_{l\nu}(\mathbf{r}) = V^{-1/2}\hat{\varepsilon}_{l\nu}\exp{(\pm i\mathbf{k}_l \cdot \mathbf{r})}$ ($\nu = 1, 2$) for the fields propagating under the periodic boundary condition, where V is the volume of the space, $\hat{\varepsilon}_{l\nu}$ is a unit vector in the direction of polarization designated by ν.

From Hamilton's equations of motion, $\dot{q}_l = \partial H_l/\partial p_l$ and $\dot{p}_l = -\partial H_l/\partial q_l$, the classical Hamiltonian that gives Eq. (3) can be easily established. Then, by converting canonical variables, q_l and p_l, into quantum operators, $\hat{q}_l$ and $\hat{p}_l$, from the resultant classical Hamiltonian, we have the quantum Hamiltonian such that (Choi et al., 2012)

$$\hat{H}_l(\hat{q}_l, \hat{p}_l, t) = \frac{1}{2\epsilon_0}e^{-\Lambda(t)}\hat{p}_l^2 + b(t)(\hat{q}_l\hat{p}_l + \hat{p}_l\hat{q}_l) + \frac{1}{2}\epsilon_0 e^{\Lambda(t)}\omega_l^2(t)\hat{q}_l^2, \tag{3}$$

where $\hat{p}_l = -i\hbar(\partial/\partial q_l)$, $\epsilon_0 = \epsilon(0)$, $b(t)$ is an arbitrary time function, and

$$\Lambda(t) = \int_0^t dt'[\dot{\epsilon}(t') + \sigma(t')]/\epsilon(t'), \tag{4}$$

$$\varpi_l^2(t) = \omega_l^2(t) + 2\dot{b}(t) + 2b(t)[\dot{\epsilon}(t) + \sigma(t)]/\epsilon(t) + 4b^2(t). \tag{5}$$

The complete Hamiltonian is obtained by summing all individual Hamiltonians: $\hat{H} = \sum_l \hat{H}_l(\hat{q}_l, \hat{p}_l, t)$.

From now on, let us treat the wave of a particular mode and drop the under subscript l for convenience. It is well known that quantum problems of optical waves in nonstationary media are described in terms of classical solutions of the system. Some researchers use real classical solutions (Choi, 2012; Pedrosa & Rosas, 2009) and others imaginary solutions (Angelow & Trifonov, 2010; Malkin et al., 1970). In this chapter, real solutions of classical equation of motion for q will be considered. Since Eq. (2) is a second order differential equation, there are two linearly independent classical solutions. Let us denote them as $s_1(t)$ and $s_2(t)$, respectively. Then, we can define an Wronskian of the form

$$\Omega = 2\epsilon_0 e^{\Lambda(t)}\left[s_1(t)\frac{ds_2(t)}{dt} - \frac{ds_1(t)}{dt}s_2(t)\right]. \tag{6}$$

This will be used at later time, considering only the case that $\Omega > 0$ for convenience.

When we study quantum problem of a system that is described by a time-dependent Hamiltonian such as Eq. (3), it is very convenient to introduce an invariant operator of the system. Such idea (invariant operator method) is firstly devised by Lewis and Riesenfeld (Lewis, 1967; Lewis & Riesenfeld, 1969) in a time-dependent harmonic oscillator as mentioned in the introductory part and now became one of potential tools for investigating quantum characteristics of time-dependent Hamiltonian systems. By solving the Liouville-von Neumann equation of the form

$$\frac{d\hat{K}}{dt} = \frac{\partial\hat{K}}{\partial t} + \frac{1}{i\hbar}[\hat{K}, \hat{H}] = 0, \tag{7}$$

we obtain the invariant operator of the system as (Choi, 2004)

$$\hat{K} = \hbar\Omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{8}$$

where Ω is chosen to be positive from Eq. (6) and $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators, respectively, that are given by

$$\hat{a} = \sqrt{\frac{1}{\hbar\Omega}}\left\{\left[\frac{\Omega}{2s(t)} - i\epsilon_0 e^{\Lambda(t)}\left(\frac{ds(t)}{dt} - 2b(t)s(t)\right)\right]\hat{q} + is(t)\hat{p}\right\}, \tag{9}$$

$$\hat{a}^\dagger = \sqrt{\frac{1}{\hbar\Omega}}\left\{\left[\frac{\Omega}{2s(t)} + i\epsilon_0 e^{\Lambda(t)}\left(\frac{ds(t)}{dt} - 2b(t)s(t)\right)\right]\hat{q} - is(t)\hat{p}\right\}, \tag{10}$$

with

$$s(t) = \sqrt{s_1^2(t) + s_2^2(t)}. \tag{11}$$

Since the system is somewhat complicate, let us develop our theory with $b(t) = 0$ from now on. Then, Eq. (5) just reduces to $\varpi_l(t) = \omega_l(t)$. Since the formula of Eq. (8) is very similar to the familiar Hamiltonian of the simple harmonic oscillator, we can solve its eigenvalue equation via well known conventional method. The zero-point eigenstate $\phi_0(q, t)$ of $\hat{K}$ is obtained from $\hat{a}\phi_0(q, t) = 0$. Once $\phi_0(q, t)$ is obtained, nth eigenstates are also derived by acting $\hat{a}^\dagger$ on $\phi_0(q, t)$ n times. Hence we finally have (Choi, 2012)

$$\phi_n(q, t) = \sqrt[4]{\frac{\delta(t)}{\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\delta(t)}q\right)$$
$$\times \exp\left\{-\frac{\delta(t)}{2}\left[1 - i\frac{2\epsilon_0 e^{\Lambda(t)}s(t)}{\Omega}\frac{ds(t)}{dt}\right]q^2\right\}, \tag{12}$$

where $\delta(t) = \Omega/[2\hbar s^2(t)]$ and H_n are Hermite polynomials.

According to the theory of Lewis-Riesenfeld invariant, the wave functions that satisfy the Schrödinger equation are given in terms of $\phi_n(q,t)$:

$$\psi_n(q,t) = \phi_n(q,t)\exp[i\theta_n(t)], \tag{13}$$

where $\theta_n(t)$ are time-dependent phases of the wave functions. By substituting Eq. (13) with Eq. (3) into the Schrödinger equation, we derive the phases to be $\theta_n(t) = -(n+1/2)\eta(t)$ where (Choi, 2012)

$$\eta(t) = \frac{\Omega}{2\epsilon_0}\int_0^t \frac{dt'}{s^2(t')e^{\Lambda(t')}} + \eta(0). \tag{14}$$

The probability densities in both q and p spaces are given by the square of wave functions, i.e., $\rho_n(q) = |\psi_n(q,t)|^2$ and $\tilde{\rho}_n(p) = |\tilde{\psi}_n(p,t)|^2$, respectively. From Eq. (13) and its Fourier component, we see that

$$\rho_n(q) = \sqrt{\frac{\delta(t)}{\pi}}\frac{1}{2^n n!}\left\{H_n[\sqrt{\delta(t)}q]\right\}^2 e^{-\delta(t)q^2}, \tag{15}$$

$$\tilde{\rho}_n(p) = \sqrt{\frac{\delta'(t)}{\pi}}\frac{1}{2^n n!}\left\{H_n[\sqrt{\delta'(t)}p]\right\}^2 e^{-\delta'(t)p^2}, \tag{16}$$

where

$$\delta'(t) = \frac{2\Omega}{\hbar\left[\frac{\Omega^2}{s^2(t)} + 4\epsilon_0^2 e^{2\Lambda(t)}\left(\frac{ds(t)}{dt}\right)^2\right]}. \tag{17}$$

The wave functions and the probability densities derived here will be used in subsequent sections in order to develop the information theory of the system.

3. Information measures for thermalized quantum optical fields

Informations of a physical system can be obtained from the statistical analysis of results of a measurement performed on it. There are two important information measures. One is the Shannon information and the other is the Fisher information. The Shannon information is also called as the Wehrl entropy in some literatures (Wehrl, 1979; Pennini & Plastino, 2004) and suitable for measuring uncertainties relevant to both quantum and thermal effects whereas quantum effect is overlooked in the concept of ordinary entropy. The Fisher information which is also well known in the field of information theory provides the extreme physical information through a potential variational principle. To manage these informations, we start from the establishment of density operator for the electromagnetic field equilibrated with its environment of temperature T. Density operator of the system obeys the Liouville-von Neumann equation such that (Choi et al., 2011)

$$\frac{\partial\hat{\varrho}(t)}{\partial t} + \frac{1}{i\hbar}[\hat{\varrho}(t),\hat{H}] = 0. \tag{18}$$

Considering the fact that invariant operator given in Eq. (8) is also established via the Liouville-von Neumann equation, we can easily construct density operator as a function of the invariant operator. This implies that the Hamiltonian $\hat{H}$ in the density operator of the simple harmonic oscillator should be replaced by a function of the invariant operator $y(0)\hat{K}$, where $y(0)\{= [2\epsilon_0 e^{\Lambda(0)} s^2(0)]^{-1}\}$ is inserted for the purpose of dimensional consideration. Thus we have the density operator in the form

$$\hat{\varrho}(t) = \frac{1}{Z} e^{-\beta\hbar W(\hat{a}^\dagger\hat{a}+1/2)}, \tag{19}$$

where $W = y(0)\Omega$, Z is a partition function, $\beta = k_b T$, and k_b is Boltzmann's constant. If we consider Fock state expression, the above equation can be expand to be

$$\varrho(t) = \frac{1}{Z} \sum_{n=0}^{\infty} |\phi_n(t)\rangle e^{-\beta\hbar W(n+1/2)} \langle\phi_n(t)|, \tag{20}$$

while the partition function becomes

$$Z = \sum_{n=0}^{\infty} \langle\phi_n(t)| e^{-\beta\hbar W(\hat{a}^\dagger\hat{a}+1/2)} |\phi_n(t)\rangle. \tag{21}$$

If we consider that the coherent state is the most classical-like quantum state, a semiclassical distribution function associated with the coherent state may be useful for the description of information measures. As is well known, the coherent state $|\alpha\rangle$ is obtained by solving the eigenvalue equation of $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{22}$$

Now we introduce the semiclassical distribution function $\mu_\varrho(\alpha)$ related with the density operator via the equation (Anderson & Halliwell, 1993)

$$\mu_\varrho(\alpha) = \langle\alpha|\varrho(t)|\alpha\rangle. \tag{23}$$

This is sometimes referred to as the Husimi distribution function (Husimi, 1940) and appears frequently in the study relevant to the Wigner distribution function for thermalized quantum systems. The Wigner distribution function is regarded as a qusi-distribution function because some parts of it are not positive but negative. In spite of the ambiguity in interpreting this negative value as a distribution function, the Wigner distribution function meets all requirements of both quantum and statistical mechanics, i.e., it gives correct expectation values of quantum mechanical observables. In fact, the Husimi distribution function can also be constructed from the Wigner distribution function through a mathematical procedure known as "Gaussian smearing" (Anderson & Halliwell, 1993). Since this smearing washes out the negative part, the negativity problem is resolved by the Hisimi's work. But it is interesting to note that new drawbacks are raised in that case, which state that the probabilities of

different samplings of q and p, relevant to the Husimi distribution function, cannot be represented by mutually exclusive states due to the lack of orthogonality of coherent states used for example in Eq. (23) (Anderson & Halliwell, 1993; Nasiri, 2005). This weak point is however almost fully negligible in the actual study of the system, allowing us to use the Husimi distribution function as a powerful means in the realm of semiclassical statistical physics.

Notice that coherent state can be rewritten in the form

$$|\alpha\rangle = \hat{D}(\alpha)|\phi_0(t)\rangle, \tag{24}$$

where $\hat{D}(\alpha)$ is the displacement operator of the form $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$. A little algebra leads to

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_n \frac{\alpha^n}{\sqrt{n!}} |\phi_n(t)\rangle. \tag{25}$$

Hence, the coherent state is expanded in terms of Fock state wave functions. Now using Eqs. (20) and (25), we can evaluate Eq. (23) to be

$$\begin{aligned}
\mu_\varrho(\alpha) &= \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\beta\hbar W(n+1/2)} |\langle \phi_n(t)|\alpha\rangle|^2 \\
&= \frac{1 - e^{-\beta\hbar W}}{\exp[(1 - e^{-\beta\hbar W})|\alpha|^2]}.
\end{aligned} \tag{26}$$

Here, we used a well known relation in photon statistics, which is

$$|\langle \phi_n(t)|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}. \tag{27}$$

As you can see, the Husimi distribution function is strongly related to the coherent state and it provides necessary concepts for establishment of both the Shannon and the Fisher informations. If we consider Eqs. (9) and (22), α (with $b(t) = 0$) can be written as

$$\alpha = \sqrt{\frac{1}{\hbar\Omega}} \left\{ \left[\frac{\Omega}{2s(t)} + i\epsilon_0 e^{\Lambda(t)} \frac{ds(t)}{dt} \right] q + is(t)p \right\}. \tag{28}$$

Hence there are innumerable number of α-samples that correspond to different pair of (q,p), which need to be examined for measurement.

A natural measure of uncertainty in information theory is the Shannon information as mentioned earlier. The Shannon information is defined as (Anderson & Halliwell, 1993)

$$I_S = -\int \frac{d^2\alpha}{\pi} \mu_\varrho(\alpha) \ln \mu_\varrho(\alpha), \tag{29}$$

where $d^2\alpha = d\text{Re}(\alpha)\, d\text{Im}(\alpha)$. With the use of Eq. (26), we easily derive it:

$$I_S = 1 + \ln \frac{1}{1 - e^{-\beta\hbar W}}. \tag{30}$$

This is independent of time and, in the limiting case of fields propagating in time-*in*dependent media that have no conductivity, W becomes natural frequency of light, leading this formula to correspond to that of the simple harmonic oscillator (Pennini & Plastino, 2004). This approaches to $I_S \simeq \ln[k_b T / (\hbar W)]$ for sufficiently high temperature, yielding the dominance of the thermal fluctuation and, consequently, permitting the quantum fluctuation to be neglected. On the other hand, as T decreases toward absolute zero, the Shannon information is always larger than unity ($I_S \geq 1$). This condition is known as the Lieb-Wehrl condition because it is conjectured by Wehrl (Wehrl, 1979) and proved by Lieb (Lieb, 1978). From this we can see that I_S has a lower bound which is connected with pure quantum effects. Therefore, while usual entropy is suitable for a measure of uncertainty originated only from thermal fluctuation, I_S plays more universal uncertainty measure covering both thermal and quantum regimes (Anderson & Halliwell, 1993).

Other potential measures of information are the Fisher informations which enable us to assess intrinsic accuracy in the statistical estimation theory. Let us consider a system described by the stochastic variable $\alpha = \alpha(x)$ with a physical parameter x. When we describe a measurement of α in order to infer x from the measurement, it is useful to introduce the coherent-state-related Fisher information that is expressed in the form

$$I_{F,x} = \int \frac{d^2\alpha}{\pi} f(\alpha(x); x) \left(\frac{\partial \ln f(\alpha(x); x)}{\partial x} \right)^2. \tag{31}$$

In fact, there are many different scenarios of this information depending on the choice of x. For a more general definition of the Fisher information, you can refer to Ref. (Pennini & Plastino, 2004).

If we take $f(\alpha(x); x) = \mu_\varrho(\alpha)$ and $x = \beta$, the Fisher's information measure can be written as (Pennini & Plastino, 2004)

$$I_{F,\beta} = \int \frac{d^2\alpha}{\pi} \mu_\varrho(\alpha) \left(\frac{\partial \ln \mu_\varrho(\alpha)}{\partial \beta} \right)^2. \tag{32}$$

Since β is the parameter to be estimated here, I_β reflects the change of μ_ϱ according to the variation of temperature. A straightforward calculation yields

$$I_{F,\beta} = \left(\frac{\hbar W}{e^{\beta\hbar W} - 1} \right)^2. \tag{33}$$

This is independent of time and of course agree, in the limit of the simple harmonic oscillator case, to the well known formula of Pennini and Plastino (Pennini & Plastino, 2004). Hence,

the change of electromagnetic parameters with time does not affect to the value of β. $I_{F,\beta}$ reduces to zero at absolute zero-temperature ($T \rightarrow 0$), leading to agreement with the third law of thermodynamics (Pennini & Plastino, 2007).

Another typical form of the Fisher informations worth to be concerned is the one obtained with the choice of $f(\alpha(x); x) = \mu_\varrho(\alpha)$ and $x = \{q, p\}$ (Pennini et al, 1998):

$$I_{F,\{q,p\}} = \int \frac{d^2\alpha}{\pi} \mu_\varrho(\alpha) \left[\sigma_{qq,\alpha} \left(\frac{\partial \ln \mu_\varrho(\alpha)}{\partial q} \right)^2 + \sigma_{pp,\alpha} \left(\frac{\partial \ln \mu_\varrho(\alpha)}{\partial p} \right)^2 \right], \tag{34}$$

where $\sigma_{qq,\alpha}$ and $\sigma_{pp,\alpha}$ are variances of q and p in the Glauber coherent state, respectively. Notice that $\sigma_{qq,\alpha}$ and $\sigma_{pp,\alpha}$ are inserted here in order to consider the weight of two independent terms in Eq. (34). As you can see, this information is jointly determined by means of canonical variables q and p. To evaluate this, we need

$$\sigma_{qq,\alpha} = \langle \alpha | \hat{q}^2 | \alpha \rangle - \langle \alpha | \hat{q} | \alpha \rangle^2, \tag{35}$$

$$\sigma_{pp,\alpha} = \langle \alpha | \hat{p}^2 | \alpha \rangle - \langle \alpha | \hat{p} | \alpha \rangle^2. \tag{36}$$

It may be favorable to represent $\hat{q}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ at this stage. They are easily obtained form the inverse representation of Eqs. (9) and (10) to be

$$\hat{q} = \sqrt{\hbar/\Omega}s(t)[\hat{a} + \hat{a}^\dagger], \tag{37}$$

$$\hat{p} = \sqrt{\hbar} \left[\left(\frac{\epsilon_0 e^{\Lambda(t)}}{\sqrt{\Omega}} \frac{ds(t)}{dt} - i\frac{\sqrt{\Omega}}{2s(t)} \right) \hat{a} + \left(\frac{\epsilon_0 e^{\Lambda(t)}}{\sqrt{\Omega}} \frac{ds(t)}{dt} + i\frac{\sqrt{\Omega}}{2s(t)} \right) \hat{a}^\dagger \right]. \tag{38}$$

Thus with the use of these, Eqs. (35) and (36) become

$$\sigma_{qq,\alpha} = \frac{\hbar s^2(t)}{\Omega}, \tag{39}$$

$$\sigma_{pp,\alpha} = \frac{\hbar\Omega}{4s^2(t)} \left[1 + 4\epsilon_0^2 e^{2\Lambda(t)} \frac{s^2(t)}{\Omega^2} \left(\frac{ds(t)}{dt} \right)^2 \right]. \tag{40}$$

A little evaluation after substituting these quantities into Eq. (34) leads to

$$I_{F,\{q,p\}} = \left[1 + 4\epsilon_0^2 e^{2\Lambda(t)} \frac{s^2(t)}{\Omega^2} \left(\frac{ds(t)}{dt} \right)^2 \right] (1 - e^{-\beta\hbar W}). \tag{41}$$

Notice that this varies depending on time. In case that the time dependence of every electromagnetic parameters vanishes and $\sigma \rightarrow 0$, this reduces to that of the simple harmonic oscillator limit, $I_{F,\{q,p\}} = 1 - e^{-\beta\hbar\omega}$, where natural frequency ω is constant, which exactly agrees with the result of Pennini and Plastino (Pennini & Plastino, 2004).

4. Husimi uncertainties and uncertainty relations

Uncertainty principle is one of intrinsic features of quantum mechanics, which distinguishes it from classical mechanics. Aside form conventional procedure to obtain uncertainty relation, it may be instructive to compute a somewhat different uncertainty relation for optical waves through a complete mathematical description of the Husimi distribution function. Bearing in mind this, let us see the uncertainty of canonical variables, associated with information measures, and their corresponding uncertainty relation. The definition of uncertainties suitable for this purpose are

$$\sigma_{\mu,qq}(t) = \langle \hat{q}^2 \rangle_\mu - \langle \hat{q} \rangle_\mu^2, \tag{42}$$

$$\sigma_{\mu,pp}(t) = \langle \hat{p}^2 \rangle_\mu - \langle \hat{p} \rangle_\mu^2, \tag{43}$$

$$\sigma_{\mu,qp}(t) = \langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle_\mu / 2 - \langle \hat{q} \rangle_\mu \langle \hat{p} \rangle_\mu, \tag{44}$$

where $\langle \hat{O}^l \rangle_\mu$ ($l = 1, 2$) with an arbitrary operator $\hat{O}$ is the expectation value relevant to the Husimi distribution function and can be evaluated from

$$\langle \hat{O}^l \rangle_\mu = \int \frac{d^2\alpha}{\pi} O^l \mu_\varrho(\alpha). \tag{45}$$

While $\langle \hat{q} \rangle_\mu = 0$ and $\langle \hat{p} \rangle_\mu = 0$ for $l = 1$, the rigorous algebra for higher orders give

$$\langle \hat{q}^2 \rangle_\mu = \frac{2\hbar s^2(t)}{\Omega} R(\beta), \tag{46}$$

$$\langle \hat{p}^2 \rangle_\mu = \frac{\hbar\Omega}{2s^2(t)} \left[1 + 4\epsilon_0^2 e^{2\Lambda(t)} \frac{s^2(t)}{\Omega^2} \left(\frac{ds(t)}{dt} \right)^2 \right] R(\beta), \tag{47}$$

$$\langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle_\mu = 4\hbar\epsilon_0 e^{\Lambda(t)} \frac{s(t)}{\Omega} \frac{ds(t)}{dt} R(\beta), \tag{48}$$

where

$$R(\beta) = \frac{1}{1 - e^{-\beta\hbar W}} + \frac{1}{2}. \tag{49}$$

Thus we readily have

$$\sigma_{\mu,qq} = \frac{2\hbar s^2(t)}{\Omega} R(\beta), \tag{50}$$

$$\sigma_{\mu,pp} = \frac{\hbar\Omega}{2s^2(t)} \left[1 + 4\epsilon_0^2 e^{2\Lambda(t)} \frac{s^2(t)}{\Omega^2} \left(\frac{ds(t)}{dt} \right)^2 \right] R(\beta). \tag{51}$$

Like other types of uncertainties in physics, the relationship between $\sigma_{\mu,qq}$ and $\sigma_{\mu,pp}$ is rather unique, i.e., if one of them become large the other become small, and there is nothing whatever one can do about it.

We can represent the uncertainty product σ_μ and the generalized uncertainty product Σ_μ in the form

$$\sigma_\mu = [\sigma_{\mu,qq}(t)\sigma_{\mu,pp}(t)]^{1/2}, \tag{52}$$

$$\Sigma_\mu = [\sigma_{\mu,qq}(t)\sigma_{\mu,pp}(t) - \sigma_{\mu,qp}{}^2(t)]^{1/2}. \tag{53}$$

Through the use of Eqs. (50) and (51), we get

$$\sigma_\mu = \hbar \left[1 + 4\epsilon_0^2 e^{2\Lambda(t)} \frac{s^2(t)}{\Omega^2} \left(\frac{ds(t)}{dt} \right)^2 \right]^{1/2} R(\beta), \tag{54}$$

$$\Sigma_\mu = \hbar R(\beta). \tag{55}$$

Notice that σ_μ varies depending on time, while Σ_μ does not and is more simple form. The relationship between σ_μ and usual thermal uncertainty relations σ obtained using the method of thermofield dynamics (Choi, 2010b; Leplae et al., 1974) are given by $\sigma_\mu = r(\beta)\sigma$ where $r(\beta) = (3e^{\beta\hbar W} - 1)/(e^{\beta\hbar W} + 1)$.

5. Entropies and entropic uncertainty relations

The HUR is employed in many statistical and physical analyses of optical data measured from experiments. This is a mathematical outcome of the nonlocal Fourier analysis (Bohr, 1928) and we can simply represent it by multiplying standard deviations of q and p together. From measurements, simultaneous prediction of q and p with high precision for both beyond certain limits levied by quantum mechanics is impossible according to the Heisenberg uncertainty principle. It is plausible to use the HUR as a measure of the spread when the curve of distribution involves only a simple hump such as the case of Gaussian type. However, the HUR is known to be inadequate when the distribution of the statistical data is somewhat complicated or reveals two or more humps (Bialynicki-Birula, 1984; Majernik & Richterek, 1997).

For this reason, the EUR is suggested as an alternative to the HUR by Bialynicki-Birula and Mycielski (Biatynicki-Birula & Mycielski, 1975). To study the EUR, we start from entropies of q and p associated with the Shannon's information theory:

$$S(\rho_n) = -\int \rho_n(q) \ln \rho_n(q) dq, \tag{56}$$

$$S(\tilde\rho_n) = -\int \tilde\rho_n(p) \ln \tilde\rho_n(p) dp. \tag{57}$$

By executing some algebra after inserting Eqs. (15) and (16) into the above equations, we get

$$S(\rho_n) = -\frac{1}{2} \ln \frac{\Omega}{2\hbar s^2(t)} + \ln(2^n n! \sqrt\pi) + n + \frac{1}{2} - \frac{1}{2^n n! \sqrt\pi} E(H_n), \tag{58}$$

$$S(\tilde{\rho}_n) = \frac{1}{2} \ln \left\{ \frac{\hbar}{2\Omega} \left[\frac{\Omega^2}{s^2(t)} + 4\epsilon_0^2 e^{2\Lambda(t)} \left(\frac{ds(t)}{dt} \right)^2 \right] \right\} + \ln(2^n n! \sqrt{\pi}) + n + \frac{1}{2}$$
$$- \frac{1}{2^n n! \sqrt{\pi}} E(H_n), \tag{59}$$

where $E(H_n)$ are entropies of Hermite polynomials of the form (Dehesa et al, 2001)

$$E(H_n) = \int_{-\infty}^{\infty} [H_n(y)]^2 e^{-y^2} \ln([H_n(y)]^2) dy. \tag{60}$$

By adding Eqs. (58) and (59) together,

$$U_E = S(\rho_n) + S(\tilde{\rho}_n), \tag{61}$$

we obtain the alternative uncertainty relation, so-called the EUR such that

$$U_E = \frac{1}{2} \ln \left\{ \hbar^2 \left[1 + 4\epsilon_0^2 e^{2\Lambda(t)} \frac{s^2(t)}{\Omega^2} \left(\frac{ds(t)}{dt} \right)^2 \right] \right\} + 2 \ln(2^n n! \sqrt{\pi})$$
$$+ 2n + 1 - \frac{2}{2^n n! \sqrt{\pi}} E(H_n). \tag{62}$$

The EUR is always larger than or at least equal to a minimum value known as the BBM (Bialynicki-Birula and Mycielski) inequality: $U_E \geq 1 + \ln \pi \simeq 2.14473$ (Haldar & Chakrabarti, 2012). Of course, Eq. (62) also satisfy this inequality. The BBM inequality tells us a lower bound of the uncertainty relation and the equality holds for the case of the simple harmonic oscillation of fields with $n = 0$.

The EUR with evolution in time, as well as information entropy itself, is a potential tool to demonstrate the effects of time dependence of electromagnetic parameters on the evolution of the system and, consequently, it deserves a special interest. The general form of the EUR can also be extended to not only other pairs of observables such as photon number and phase but also more higher dimensional systems even up to infinite dimensions.

6. Application to a special system

The application of the theory developed in the previous sections to a particular system may provide a better understanding of information theory for the system to us. Let us see the case that $\epsilon(t) = \epsilon_0$, $\sigma(t) = 0$, and

$$\mu(t) = \mu_0 \frac{1 + h}{1 + h \cos(\omega_0 t)}, \tag{63}$$

where $\mu_0[=\mu(0)]$, h, and ω_0 are real constants and $|h| \ll 1$. Then, the classical solutions of Eq. (2) are given by

$$s_1(t) = s_0 Ce_\nu(\omega_0 t/2, -\nu h/2), \tag{64}$$

$$s_2(t) = s_0 Se_\nu(\omega_0 t/2, -\nu h/2), \tag{65}$$

where s_0 is a real constant, Ce_ν and Se_ν are Mathieu functions of the cosine and the sine elliptics, respectively, and $\nu = 4k^2/[\epsilon_0 \mu_0 \omega_0^2(1+h)]$. Figure 1 is information measures for this system, plotted as a function of time. Whereas I_S and $I_{F,\beta}$ do not vary with time, $I_{F,\{q,p\}}$ oscillates as time goes by.
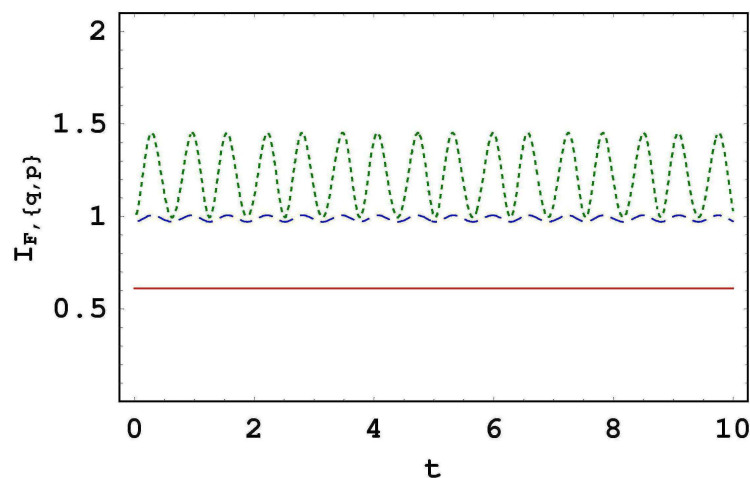

Figure 1. The time evolution of $I_{F,\{q,p\}}$. The values of (k, h) used here are (1, 0.1) for solid red line, (3, 0.1) for long dashed blue line, and (3, 0.2) for short dashed green line. Other parameters are taken to be $\epsilon_0 = 1$, $\mu_0 = 1$, $\beta = 1$, $\hbar = 1$, $\omega_0 = 5$, and $s_0 = 1$.

In case of $h \to 0$, the natural frequency in Eq. (2) become constant and $W \to \omega$. Then, Eqs. (64) and (65) become $s_1 = s_0 \cos \omega t$ and $s_2 = s_0 \sin \omega t$, respectively. We can confirm in this situation that our principal results, Eqs. (30), (33), (41), (54), and (62) reduce to those of the wave described by the simple harmonic oscillator as expected.

7. Summary and conclusion

Information theories of optical waves traveling through arbitrary time-varying media are studied on the basis of invariant operator theory. The time-dependent Hamiltonian that gives classical equation of motion for the time function $q(t)$ of vector potential is constructed. The quadratic invariant operator is obtained from the Liouville-von Neumann equation given in Eq. (7) and it is used as a basic tool for developing information theory of the system. The eigenstates $\phi_n(q, t)$ of the invariant operator are identified using the annihilation and the creation operators. From these eigenstates, we are possible to obtain the Schrödinger solutions, i.e., the wave functions $\psi_n(q, t)$, since $\psi_n(q, t)$ is merely given in terms of $\phi_n(q, t)$.
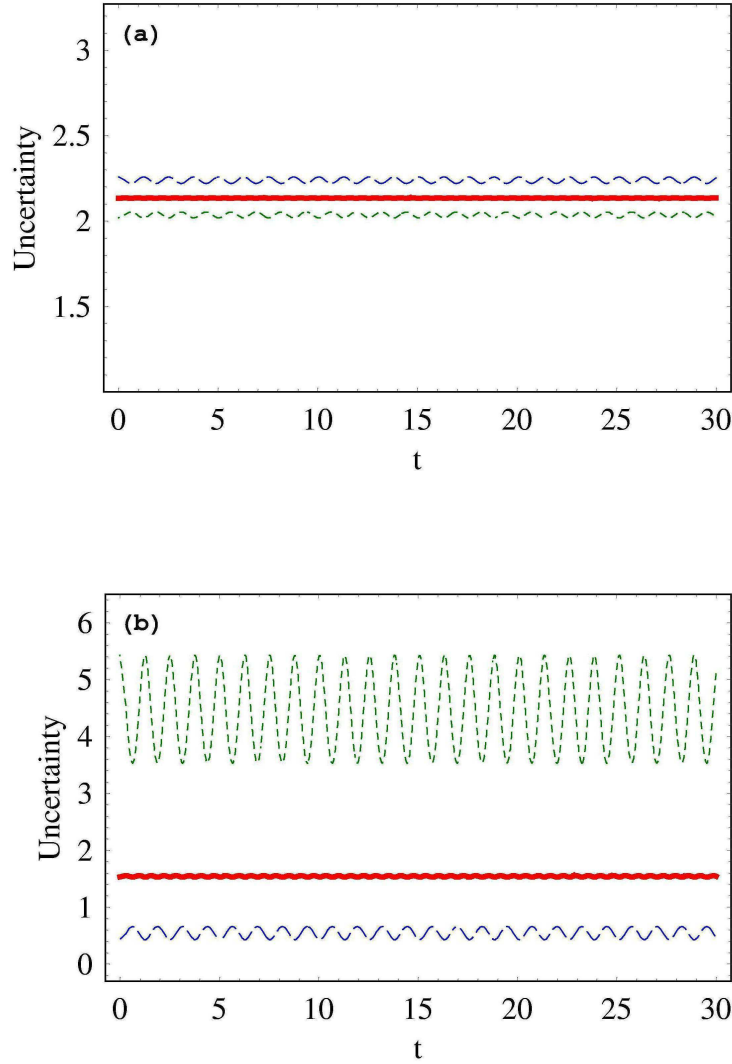
Figure 2. The Uncertainty product σ_μ (thick solid red line) together with $\sigma_{\mu,qq}$ (long dashed blue line) and $\sigma_{\mu,pp}$ (short dashed green line). The values of (k, h) used here are (1, 0.1) for (a) and (3, 0.1) for (b). Other parameters are taken to be $\epsilon_0 = 1$, $\mu_0 = 1$, $\beta = 1$, $\hbar = 1$, $\omega_0 = 5$, and $s_0 = 1$.

The semiclassical distribution function $\mu_\varrho(\alpha)$ is the expectation value of $\hat{\varrho}(t)$ in the coherent state which is the very classical-like quantum state. From Eq. (30), we see that the Shannon information does not vary with time. However, Eq. (41) shows that the Fisher information $I_{F,\{q,p\}}$ varies depending on time. It is known that the localization of the density is determined in accordance with the Fisher information (Romera et al, 2005). For this reason, the Fisher measure is regarded as a local measure while the Shannon information is a global information measure of the spreading of density. Local information measures vary depending on various derivatives of the probability density whereas global information measures follow the Kinchin's axiom for information theory (Pennini & Plastino, 2007; Plastino & Casas, 2011).
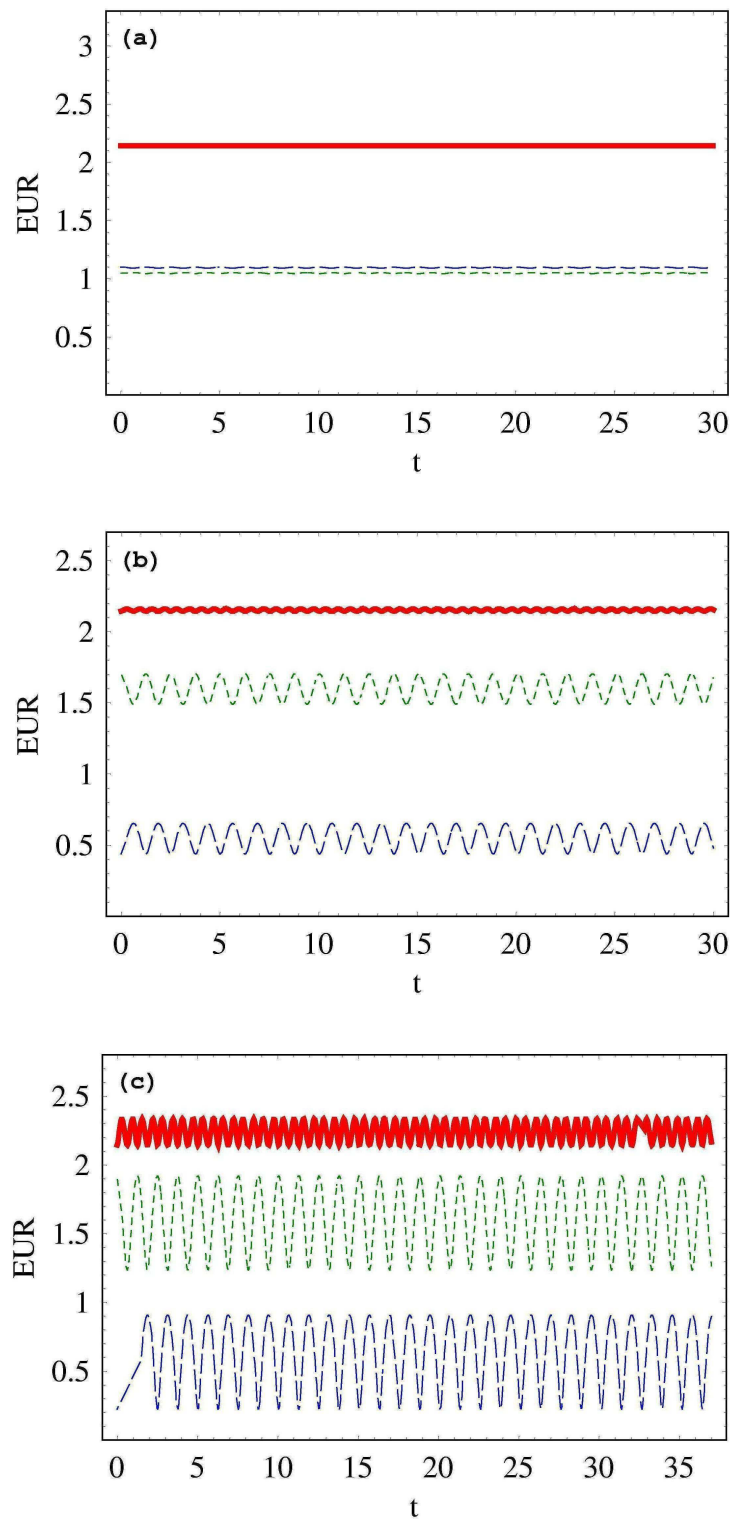
Figure 3. The EUR U_{E} (thick solid red line) together with $S(\rho_n)$ (long dashed blue line) and $S(\tilde{\rho}_n)$ (short dashed green line). The values of (k, h) used here are (1, 0.1) for (a), (3, 0.1) for (b), and (3, 0.2) for (c). Other parameters are taken to be $\epsilon_0 = 1$, $\mu_0 = 1$, $\hbar = 1$, $\omega_0 = 5$, $n = 0$, and $s_0 = 1$.

Two kinds of uncertainty products relevant to the Husimi distribution function are considered: one is the usual uncertainty product σ_μ and the other is the more generalized

product Σ_μ defined in Eq. (53). While σ_μ varies as time goes by Σ_μ is constant and both have particular relations with those in standard thermal state.

Fock state representation of the Shannon entropies in q- and p-spaces are derived and given in Eqs. (58) and (59), respectively. The EUR which is an alternative uncertainty relation is obtained by adding these two entropies. The EUR is more advantageous than the HUR in the context of information theory. The information theory is not only important in modern technology of quantum computing, cryptography, and communication, its area is now extended to a wide range of emerging fields that require rigorous data analysis like neural systems and human brain. Further developments of theoretical and physical backgrounds for analyzing statistical data obtained from a measurement beyond standard formulation are necessary in order to promote the advance of such relevant sciences and technologies.

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