Chapter 5

Schrödinger Equation as a Hamiltonian System, Essential Nonlinearity, Dynamical Scalar Product and some Ideas of Decoherence

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1. Introduction

Our main idea is to suggest some new model of nonlinearity in quantum mechanics. The nonlinearity we discuss is non-perturbative and geometrically motivated, in any case it is not an auxiliary correction to the linear background. It has a group-theoretic motivation based on the assumption of the "large" symmetry group. In a sense, it develops further our ideas suggested earlier in [1–4].

It is well known that quantum mechanics is still plagued by some paradoxes concerning decoherence, measurement process and the reduction of the state vector. In spite of certain optimistic opinions, the problem is still unsolved, although many interesting ideas have been formulated, like that about subsystems of a large (infinite) quantum system or stochastic quantum Markov processes with the spontaneous reduction of state vectors. There is still an opinion that the main problem is the linearity of the Schrödinger equation, which seems to be drastically incompatible with the mentioned problems [5, 6]. But at the same time, that linearity works beautifully when describing the unobserved unitary quantum evolution, finding the energy levels and in all statistical predictions. It seems that either we are faced here with some completely new type of science, roughly speaking, based on some kind of solipsism with the irreducible role of human being in phenomena, or perhaps we deal with a very sophisticated and delicate nonlinearity which becomes active and remarkable just in the process of interaction between quantum systems and "large" classical objects.

The main idea is to analyse the Schrödinger equation and corresponding relativistic linear wave equations as usual self-adjoint equations of mathematical physics, thus ones derivable from variational principles. It is easy to construct their Lagrangians. Some problems appear when trying to formulate Hamiltonian formalism, because Lagrangians for the Schrödinger or Dirac equations are highly degenerate and the corresponding Legendre transformation is uninvertible and leads to constraints in the phase space. Nevertheless, using the Dirac formalism for such Lagrangians, one can find the corresponding Hamiltonian formalism. Incidentally, it turns out that introducing the second-order time derivatives to dynamical equations, even as small corrections, one can obtain the regular Legendre transformation. In non-relativistic quantum mechanics there are certain hints suggesting just such a modification in the nano-scale physics [1, 7, 8]. One can also show that in $SU(2,2)$-invariant gauge models, i.e., roughly-speaking, in conformal theory, it is more natural to begin with the four-component Klein-Gordon amplitude and then to derive the Dirac behavior as an unexpected aspect of the Klein-Gordon theory [4, 9]. This leads us to certain interesting statements concerning the pairing of fundamental quarks and leptons in electroweak interactions.

We begin just like in [1] from the first- and second-order (in time) Schrödinger equations for a finite-level system, i.e., for the finite "configuration space". We construct the "direct nonlinearity" as a non-quadratic term in Lagrangian, but further on we concentrate on our main idea. It consists in that we follow the conceptual transition from the special to general relativity. Namely, just like in the passing to the theory of general relativity, the metric tensor loses its status of the absolute geometric object and becomes included into degrees of freedom (gravitational field), so in our treatment the Hilbert-space scalar product becomes a dynamical quantity which satisfies together with the state vector the system of differential equations. The main idea is that there is no fixed scalar product metric and the dynamical term of Lagrangian, describing the self-interaction of the metric, is invariant under the total group $GL(n,\mathbb{C})$. But this invariance is possible only for models non-quadratic in the metric, just like in certain problems of the dynamics of "affinely-rigid" body [2, 3]. There is a natural metric of this kind and it introduces to the theory a very strong nonlinearity which induces also the effective nonlinearity of the wave equation, even if there is no "direct nonlinearity" in it. The structure of Lagrangian and equations of motion is very beautiful, as usual in high-symmetry problems. Nevertheless, the very strong nonlinearity prevents us to find a rigorous solution. Nevertheless, there are some partial results, namely, if we fix the behaviour of wave function to some simple form and provide an academic discussion of the resulting behaviour of the scalar product, then it turns out that there are rigorous exponential solutions, including ones infinitely growing and ones exponentially decaying in future. This makes some hope for describing, e.g., some decay/reduction phenomena. Obviously, the full answer will be possible only when we will be able to find a rigorous solution for the total system. We are going to repeat the same discussion for the more realistic infinite-level system, when the wave function is defined somehow on the total configuration space like, e.g., the arithmetic space. As usual when passing from the finite to infinite dimensions, some essentially new features appear then, nevertheless, one can hope that the finite-dimensional results may be to some extent applicable. This will be done both in the usual non-relativistic Schrödinger wave equation and for the relativistic Klein-Gordon and Dirac equations. In any of those cases we are dealing with two kinds of degrees of freedom, i.e., dynamical variables: wave function and scalar product. They are mutually interacting.

All said above concerns the self-adjoint model of the Schrödinger equation, derivable from variational principle. However, one can also ask what would result if we admitted "dissipative" models, where the Schrödinger equation does possess some "friction-like" term. As yet we have not a ready answer, nevertheless, the question is well formulated and we will try to check what might appear in a consequence of such a generalization. Maybe some quantum model of dissipation, i.e., of the open system, but at a moment we are unable to answer the question.

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2. Nonlinear Schrödinger equation as a self-adjoint equation of mathematical physics

It is known, although not often noticed and declared, that the Schrödinger equation and other equations of quantum mechanics, including relativistic ones, are self-adjoint, i.e., derivable from variational principles. Therefore, they may be expressed in Hamiltonian terms, i.e., quantum mechanics becomes a kind of analytical mechanics, usually with an infinite number of degrees of freedom (excepting finite-level systems). So, as far as one deals with the unobserved quantum system, its evolution may be described within the classical mathematical framework of Hamiltonian mechanics and canonical transformations. Of course, this breaks down when quantum springs, jumps, occur, i.e., when one is faced with phenomena like macroscopic observation, measurements and decoherence. This happens when a small/quantum system interacts with a large/classical object showing some characteristic instability. There were various ways of explaining those catastrophic phenomena and their statistical rules. It is very interesting that those rules are based on the Hilbert space geometry, but in addition some unpredictable, statistical phenomena appear. There were many attempts of explanation, based either on the extension to larger, infinite systems or on the idea of spontaneous stochastic reduction. But one of the permanent motives is the hypothesis of nonlinearity, especially one which is "silent" in the evolution of the unobserved quantum system, but becomes essential in the process of interaction with the large and unstable classical system. This is also our line in this paper. Mathematical methods of nonlinear analytical mechanics just seem to suggest some attempts of solution.

To explain the main ideas we start from the simple finite-level system, i.e., one with a finite-dimensional unitary space of states. Let us denote this complex linear space by W and put $\dim_{\mathbb{C}} W = n$. The dual, antidual and complex-conjugate spaces will be denoted respectively by W^*, $\overline{W}^* = \overline{W^*}$, and $\overline{W}$. As usual, W^* is the space of $\mathbb{C}$-linear functions on W. Having the same finite dimension, the spaces W, W^* are isomorphic, however in a non-canonical way until we introduce some unitary structure to W. But some comments are necessary concerning the complex conjugate spaces $\overline{W}^* = \overline{W^*}$, $\overline{W}$. It must be stressed that in general nothing like the complex structure is defined in W. It is a structure-less space and the half-linear (semi-linear) bar-operations are defined pointwisely. Therefore, for any $f \in W^*$ and for any $u \in W$ the corresponding $\overline{f} \in \overline{W}^*$, $\overline{u} \in \overline{W}$ are given by

$$\left(\overline{f}\right)(w) = \overline{f(w)}, \qquad \overline{u}(g) = \overline{u(g)}, \tag{1}$$

where w, g are arbitrary elements of W and W^*. Therefore, under the bar-operation $\overline{W}$ is canonically anti-isomorphic with W and $\overline{W}^*$ with W^*. Nevertheless, the bar-operation acts between different linear spaces and this is often essential. The spaces W and $\overline{W}$, and similarly W^* and $\overline{W}^*$ may be mutually identified only in important, nevertheless mathematically exceptional, situations when by the very definition W is a linear subspace of the space of $\mathbb{C}$-valued functions on a given "configuration space" Q. Then we simply define pointwisely

$$\overline{\psi}(q) := \overline{\psi(q)} \tag{2}$$

and so W becomes identical with $\overline{W}$. In general this is impossible. Let us mention of course that for the n-level system, Q is an n-element set.

Let us quote a few analytical formulas. We choose a pair of mutually dual bases $(e_1, \ldots, e_n)$, $\left(e^1, \ldots, e^n\right)$ in W, W^* and induced pair of dual bases $\left(\overline{e}_{\overline{1}}, \ldots, \overline{e}_{\overline{1n}}\right)$, $\left(\overline{e}^{\overline{1}}, \ldots, \overline{e}^{\overline{1n}}\right)$ in the complex-conjugate spaces $\overline{W}$ and $\overline{W}^* \simeq \overline{W^*}$. Then the complex conjugates of vectors

$$u = u^a e_a \in W, \qquad f = f_a e^a \in W^* \tag{3}$$

are analytically expressed as

$$\overline{u} = \overline{u}^{\overline{a}} \overline{e}_{\overline{a}}, \qquad \overline{f} = \overline{f}_{\overline{a}} \overline{e}^{\overline{a}}, \tag{4}$$

where, obviously, $\overline{u}^{\overline{a}}$, $\overline{f}_{\overline{a}}$ are the usual complex conjugates of numbers u^a, f_a.

In quantum mechanics one uses often sesquilinear forms, usually Hermitian ones. Usually our sesquilinear forms are antilinear (half-linear) in the first argument and linear in the second, therefore,

$$F(au + bw, v) = \overline{a}F(u, v) + \overline{b}F(w, v), \tag{5}$$

i.e., analytically

$$F(u, w) = F_{\overline{a}b} \overline{u}^{\overline{a}} w^b. \tag{6}$$

So, they are elements of $\overline{W}^* \otimes W^*$. For Hermitian forms we have

$$F(u, w) = \overline{F(w, u)}, \qquad F_{\overline{a}b} = \overline{F}_{b\overline{a}}. \tag{7}$$

If F is non-degenerate,

$$\det\left[F_{\overline{a}b}\right] \neq 0, \tag{8}$$

then the inverse form $F^{-1} \in W \otimes \overline{W}$ does exist with coefficients $F^{a\overline{b}}$ such that

$$F^{a\overline{c}} F_{\overline{c}b} = \delta^a{}_b, \qquad F_{\overline{a}c} F^{c\overline{b}} = \delta_{\overline{a}}{}^{\overline{b}}. \tag{9}$$

For any quantum system there are two Hermitian forms: a) the scalar product $\Gamma \in \overline{W}^* \otimes W^*$ and b) the Hamiltonian form ${}_\Gamma H$ obtained by the Γ-lowering of the first index of the Hamilton operator $H \in L(W) \simeq W \otimes W^*$. The Hamilton operator H is Γ-Hermitian, i.e.,

$$\Gamma(H\psi, \varphi) = \Gamma(\psi, H\varphi). \tag{10}$$

Analytically the sesquilinear form ${}_\Gamma H$,

$${}_\Gamma H_{\overline{a}b} = \Gamma_{\overline{a}c} H^c{}_b, \tag{11}$$

is simply Hermitian without any relationship to Γ, and from the Langrangian point of view it is more fundamental than H itself. The finite-level Schrödinger equation

$$i\hbar \frac{d\psi^a}{dt} = H^a{}_b \psi^b \tag{12}$$

is derivable from the Lagrangian

$$L(1) = \frac{i\hbar}{2}\Gamma_{\overline{a}b}\left(\overline{\psi}^{\overline{a}}\dot{\psi}^b - \dot{\overline{\psi}}^{\overline{a}}\psi^b\right) - {}_\Gamma H_{\overline{a}b}\overline{\psi}^{\overline{a}}\psi^b. \tag{13}$$

Having in view some kind of "generality" it may be convenient to admit some general constant coefficients α, γ:

$$L(1) = i\alpha\Gamma_{\overline{a}b}\left(\overline{\psi}^{\overline{a}}\dot{\psi}^b - \dot{\overline{\psi}}^{\overline{a}}\psi^b\right) - \gamma_\Gamma H_{\overline{a}b}\overline{\psi}^{\overline{a}}\psi^b. \tag{14}$$

It is seen that unlike in the Schrödinger equation, from the variational point of view ${}_\Gamma H$ is more fundamental. It should be denoted rather as $\chi_{\overline{a}b}$, and $H^a{}_b$ with the convention of the Γ-raised first index of χ, as

$$H^a{}_b = \left({}^\Gamma\chi\right)^a{}_b = \Gamma^{a\overline{c}}\chi_{\overline{c}b}. \tag{15}$$

One does not do so because of the prevailing role of Schrödinger equation over its variational interpretation. The Hermitian structure of Γ and ${}_\Gamma H$ imply that $L(1)$ is real. The descriptor (1) refers to the first-order polynomial dependence of $L(1)$ on the time derivatives of ψ. Obviously, the corresponding Legendre transformation leads to phase-space constraints and to the Dirac procedure in canonical formalism. It is interesting to admit some regularization by allowing L to contain the terms quadratic in generalized velocities, just in the spirit of analytical mechanics. The corresponding Lagrangian will have the following form:

$$L(1,2) = i\alpha\Gamma_{\overline{a}b}\left(\overline{\psi}^{\overline{a}}\dot{\psi}^b - \dot{\overline{\psi}}^{\overline{a}}\psi^b\right) + \beta\Gamma_{\overline{a}b}\dot{\overline{\psi}}^{\overline{a}}\dot{\psi}^b - \gamma_\Gamma H_{\overline{a}b}\overline{\psi}^{\overline{a}}\psi^b. \tag{16}$$

To be more precise, in the term quadratic in velocities one can admit some more general Hermitian form, not necessarily the one proportional to $\Gamma_{\overline{a}b}$. However, we do not do things like those in this paper. Let us stress that α, β, γ are real constants.

One circumstance must be stressed: we use as "independent" components ψ^a, $\overline{\psi}^{\overline{a}}$. The procedure is not new. The same is done in variational principles of field theory [10]. Lagrangians are real, based on Hermitian forms, therefore in variational procedure it is sufficient to subject, e.g., only $\overline{\psi}^{\overline{a}}$ to the modification $\overline{\psi}^{\overline{a}} \mapsto \overline{\psi}^{\overline{a}} + \delta\overline{\psi}^{\overline{a}}$. Then, e.g., for the action functional

$$I(1,2) = \int L(1,2)dt \tag{17}$$

one obtains

$$\frac{\delta I(1,2)}{\delta\overline{\psi}^{\overline{a}}(t)} = 2i\alpha\Gamma_{\overline{a}b}\frac{d\psi^b}{dt} - \beta\Gamma_{\overline{a}b}\frac{d^2\psi^b}{dt^2} - \gamma_\Gamma H_{\overline{a}b}\psi^b \tag{18}$$

and the resulting Schrödinger equation:

$$2i\alpha \frac{d\psi^a}{dt} - \beta \frac{d^2\psi^a}{dt^2} = \gamma H^a{}_b \psi^b. \tag{19}$$

And this is all, because the variation with respect to ψ^a leads to the complex-conjugate equation. This is a convenient and commonly used procedure.

The language of analytical mechanics, in this case finite-dimensional one, opens some possibility of introducing nonlinearity to quantum-mechanical equations. The simplest way is to believe in Schrödinger equation but reinterpreting it in terms of Hamiltonian mechanics, to introduce some naturally looking nonlinear perturbations to it. The simplest way is to introduce to L some non-quadratic potential term $\mathcal{V}\left(\psi, \overline{\psi}\right)$ and the corresponding action term to I:

$$I(\mathcal{V}) = \int \mathcal{V} dt. \tag{20}$$

For example, the simplest possibility is to use the term like

$$\mathcal{V}\left(\psi, \overline{\psi}\right) = f\left(\Gamma_{\overline{a}b} \overline{\psi}^{\overline{a}} \psi^b\right), \tag{21}$$

with some model function $f : \mathbb{R} \to \mathbb{R}$. In various physical applications one uses often the quartic term:

$$f(y) = \frac{\varkappa}{2}(y-b)^2. \tag{22}$$

When using the $\mathcal{V}$-term, one obtains after the variational procedure the following nonlinear Schrödinger equation:

$$2i\alpha \frac{d\psi^a}{dt} - \beta \frac{d^2\psi^a}{dt^2} = \gamma H^a{}_b \psi^b + f' \psi^a, \tag{23}$$

where f' denotes the usual first-order derivative of f. This is the simplest model containing the superposition of first- and second-order time derivatives of ψ. It is very simple because of being a finite-level system and because of the direct introduction of nonlinearity as a perturbation of the primarily linear model. Nevertheless, it demonstrates some interesting features of nonlinearity and of the mixing of derivatives order.

The problem of the order of derivatives is strongly related to the structure of Hamiltonian mechanics of our systems. It occurs also in corresponding problems of field theory. Let us mention some elementary facts. As usual, it is convenient to use the doubled number of degrees of freedom ψ^a, $\overline{\psi}^{\overline{a}}$ and the corresponding canonical momenta π_a, $\overline{\pi}_{\overline{a}}$. The symplectic form is given by

$$\omega = d\pi_a \wedge d\psi^a + d\overline{\pi}_{\overline{a}} \wedge d\overline{\psi}^{\overline{a}}, \tag{24}$$

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and the resulting Poisson bracket is expressed as follows:

$$\{F,G\} = \frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \pi_a} + \frac{\partial F}{\partial \overline{\psi}^{\overline{a}}}\frac{\partial G}{\partial \overline{\pi}_{\overline{a}}} - \frac{\partial F}{\partial \pi_a}\frac{\partial G}{\partial \psi^a} - \frac{\partial F}{\partial \overline{\pi}_{\overline{a}}}\frac{\partial G}{\partial \overline{\psi}^{\overline{a}}}. \tag{25}$$

The Hamiltonian vector field is given by

$$X_F = \frac{\partial F}{\partial \pi_a}\frac{\partial}{\partial \psi^a} + \frac{\partial F}{\partial \overline{\pi}_{\overline{a}}}\frac{\partial}{\partial \overline{\psi}^{\overline{a}}} - \frac{\partial F}{\partial \psi^a}\frac{\partial}{\partial \pi_a} - \frac{\partial F}{\partial \overline{\psi}^{\overline{a}}}\frac{\partial}{\partial \overline{\pi}_{\overline{a}}}. \tag{26}$$

It must be stressed that all dynamical quantities in this formalism are considered as independent on their complex conjugates:

$$\left\langle d\psi^a, \frac{\partial}{\partial \psi^b}\right\rangle = \delta^a{}_b, \qquad \left\langle d\overline{\psi}^{\overline{a}}, \frac{\partial}{\partial \overline{\psi}^{\overline{b}}}\right\rangle = \delta^{\overline{a}}{}_{\overline{b}}, \tag{27}$$

but

$$\left\langle d\psi^a, \frac{\partial}{\partial \overline{\psi}^{\overline{b}}}\right\rangle = 0, \qquad \left\langle d\overline{\psi}^{\overline{a}}, \frac{\partial}{\partial \psi^b}\right\rangle = 0, \tag{28}$$

and similarly,

$$\left\langle d\pi_a, \frac{\partial}{\partial \pi_b}\right\rangle = \delta_a{}^b, \qquad \left\langle d\overline{\pi}_{\overline{a}}, \frac{\partial}{\partial \overline{\pi}_{\overline{b}}}\right\rangle = \delta_{\overline{a}}{}^{\overline{b}}, \tag{29}$$

$$\left\langle d\pi_a, \frac{\partial}{\partial \overline{\pi}_{\overline{b}}}\right\rangle = 0, \qquad \left\langle d\overline{\pi}_{\overline{a}}, \frac{\partial}{\partial \pi_b}\right\rangle = 0. \tag{30}$$

All the remaining basic evaluations are vanishing, in particular those for $d\psi^a$, $d\overline{\psi}^{\overline{a}}$ with $\partial/\partial\pi_b$, $\partial/\partial\overline{\pi}_{\overline{b}}$, and similarly, for $d\pi_a$, $d\overline{\pi}_{\overline{a}}$ with $\partial/\partial\psi^b$, $\partial/\partial\overline{\psi}^{\overline{b}}$.

Let us write down the Hamilton equations of motion. Their form depends strongly on the occurrence of second time derivatives in the "Schrödinger equation". For simplicity let us begin with the assumption that $\beta \neq 0$ and our equation is second-order in time derivatives. Then the Legendre transformation is given by the formulas:

$$\pi_a = i\alpha\overline{\psi}^{\overline{b}}\Gamma_{\overline{b}a} + \beta\dot{\overline{\psi}}^{\overline{b}}\Gamma_{\overline{b}a}, \qquad \overline{\pi}_{\overline{a}} = -i\alpha\Gamma_{\overline{a}b}\psi^b + \beta\Gamma_{\overline{a}b}\dot{\psi}^b. \tag{31}$$

They are invertible and

$$\dot{\psi}^a = \frac{1}{\beta}\Gamma^{a\overline{b}}\overline{\pi}_{\overline{b}} + \frac{i\alpha}{\beta}\psi^a, \qquad \dot{\overline{\psi}}^{\overline{a}} = \frac{1}{\beta}\pi_b\Gamma^{b\overline{a}} - \frac{i\alpha}{\beta}\overline{\psi}^{\overline{a}}. \tag{32}$$

The Lagrangian "energy" function is given by

$$\mathcal{E} = \beta \Gamma_{\overline{a}b} \dot{\overline{\psi}}^{\overline{a}} \dot{\psi}^{b} + \gamma_{\Gamma} H_{\overline{a}b} \overline{\psi}^{\overline{a}} \psi^{b} + \mathcal{V}\left(\psi, \overline{\psi}\right), \tag{33}$$

and substituting here the above inverse formula we obtain the "Hamilton function" in the sense of analytical mechanics:

$$\mathcal{H} = \frac{1}{\beta}\left(\Gamma^{a\overline{b}} \pi_a \overline{\pi}_{\overline{b}} + i\alpha \left[\pi_a \psi^a - \overline{\pi}_{\overline{a}} \overline{\psi}^{\overline{a}}\right]\right) + \left(\frac{\alpha^2}{\beta}\Gamma_{\overline{a}b} + \gamma_{\Gamma} H_{\overline{a}b}\right) \overline{\psi}^{\overline{a}} \psi^{b} + \mathcal{V}\left(\psi, \overline{\psi}\right). \tag{34}$$

It is clear that the "energy" function is always globally defined in the tangent bundle, but the "Hamiltonian" $\mathcal{H}$ does exist as a function on the cotangent bundle only if β does not vanish. The special case $\beta = 0$ is essentially singular. Let us mention that it is a general rule that differential equations are catastrophically sensitive to the vanishing of coefficients at highest-order derivatives. In any case the Schrödinger equation modified by terms with second derivatives is essentially different than the usual, first-order equation. The problem has to do both with some doubts concerning the occurrence of second derivatives but also with certain hopes and new physical ideas. Obviously, if $\beta \neq 0$, the second-order Schrödinger equation is equivalent to the following canonical Hamilton equations:

$$\frac{d\psi^a}{dt} = \{\psi^a, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial \pi_a}, \qquad \frac{d\pi_a}{dt} = \{\pi_a, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial \psi^a}. \tag{35}$$

Let us mention that there are various arguments for the second-order differential equations as fundamental ones for quantum theory. In a sense, in conformal SU$(2,2)$-ruled geometrodynamics, some kind of Dirac behaviour is a byproduct of the quadruplet of the gauge Klein-Gordon equation [4, 9]. Besides, in nano-physics there are also some other arguments for the mixing of first- and second-order Schrödinger equations [1, 8]. In the SU$(2,2)$-gauge theory there are also some interesting consequences of this mixing within the framework of the standard model.

Nevertheless, it is also convenient to discuss separately the degenerate Schrödinger (Schrödinger-Dirac?) model based on the first derivatives. Our Legendre transformation becomes then

$$\pi_a = i\alpha \overline{\psi}^{\overline{b}} \Gamma_{\overline{b}a}, \qquad \overline{\pi}_{\overline{a}} = -i\alpha \Gamma_{\overline{a}b} \psi^{b}. \tag{36}$$

It does not depend on velocities at all. The same concerns the energy function:

$$\mathcal{E} = \gamma_{\Gamma} H_{\overline{a}b} \overline{\psi}^{\overline{a}} \psi^{b} = \gamma\Gamma\left(\psi, \widehat{H}\psi\right). \tag{37}$$

Strictly speaking, Hamiltonian is defined only on the manifold of "primary constraints" in the sense of Dirac, $M = \mathcal{L}\left(W \times \overline{W} \times W \times \overline{W}\right) \subset W \times \overline{W} \times W^{*} \times \overline{W}^{*}$, where $\mathcal{L}$ is just the

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above Legendre transformation. Some authors, including Dirac himself, define Hamiltonian $\mathcal{H}$ all over the phase space, however, it is then non-unique, and namely

$$\mathcal{H} = \mathcal{H}_0 + \lambda^a \left(\pi_a - i\alpha\overline{\psi}^{\overline{b}}\Gamma_{\overline{b}a}\right) + \overline{\lambda}^{\overline{a}} \left(\overline{\pi}_{\overline{a}} + i\alpha\Gamma_{\overline{a}b}\psi^b\right), \tag{38}$$

where λ^a, $\overline{\lambda}^{\overline{a}}$ are Lagrange multipliers and

$$\mathcal{H}_0 = \gamma_\Gamma H_{\overline{a}b}\overline{\psi}^{\overline{a}}\psi^b + \mathcal{V}(\psi, \overline{\psi}). \tag{39}$$

One can easily show that the Dirac secondary constraints coincide with the primary ones, $M_S = M$, and the multipliers are given by

$$\lambda^a = -\frac{i\gamma}{2\alpha}H^a{}_b\psi^b - \frac{i}{2\alpha}\Gamma^{a\overline{c}}\frac{\partial\mathcal{V}}{\partial\overline{\psi}^{\overline{c}}}, \tag{40}$$

$$\overline{\lambda}^{\overline{a}} = \frac{i\gamma}{2\alpha}\overline{\psi}^{\overline{b}}H_{\overline{b}}{}^{\overline{a}} + \frac{i}{2\alpha}\frac{\partial\mathcal{V}}{\partial\psi^c}\Gamma^{c\overline{a}}, \tag{41}$$

where operations on indices of H are meant in the sense of the metric tensor Γ.

It is clear that $M = M_S$ has the complex dimension n, but its real dimension is $2n$, always even, as it should be with symplectic manifolds. The following quantities, π-s doubled in a consequence of this "complex-real",

$$\Pi_a = 2i\alpha\overline{\psi}^{\overline{b}}\Gamma_{\overline{b}a}, \qquad \overline{\Pi}_{\overline{a}} = -2i\alpha\Gamma_{\overline{a}b}\psi^b, \tag{42}$$

may be used to represent the canonical conjugate momenta. It follows in particular, that on the constraints submanifold M we have the following Poisson brackets:

$$\{\psi^a, \psi^b\}_M = 0, \qquad \{\overline{\psi}^{\overline{a}}, \overline{\psi}^{\overline{b}}\}_M = 0, \qquad \{\psi^a, \overline{\psi}^{\overline{b}}\}_M = \frac{1}{2i\alpha}\Gamma^{a\overline{b}}. \tag{43}$$

Therefore, it is seen that up to normalization the complex conjugates $\overline{\psi}^{\overline{a}}$ coincide with canonical momenta conjugate to ψ^a. Using the standard properties of Poisson brackets we can write the resulting canonical equations in the bracket form:

$$\frac{d\psi^a}{dt} = \{\psi^a, \mathcal{H}\}_M, \qquad \frac{d\overline{\psi}^{\overline{a}}}{dt} = \{\overline{\psi}^{\overline{a}}, \mathcal{H}\}_M. \tag{44}$$

This implies, of course, the following well-known equation:

$$i\hbar\frac{d\psi^a}{dt} = H^a{}_b\psi^b + \frac{1}{2}\Gamma^{a\overline{b}}\frac{\partial\mathcal{V}}{\partial\overline{\psi}^{\overline{b}}} \tag{45}$$

and, equivalently, its complex conjugate.

Let us stress that due to the $\mathcal{V}$-term, this is a nonlinear Schrödinger equation. The nonlinearity and its possible consequences for the decoherence and measurement problems depend on our invention in constructing the $\mathcal{V}$-model. Of course, the procedure is more promising for large, in particular infinite, systems, and the above finite-level framework is rather a toy model. This concerns both the first- and second-order Schrödinger equations. Nevertheless, in the above models nonlinearity was more or less introduced "by hand", as an additional perturbation term. Our main idea, we are going to describe it now, consists in introducing of nonlinearity in analogy to the passing from special to general relativity.

3. Non-direct nonlinearity and the dynamical scalar product

Let us remind some other, well-established nonlinearities of intrinsically geometric origin, appearing in physics. One of them is Einstein-Hilbert general relativity. It is well known that majority of well-established field theories is originally linear, and the nonlinearity appears in a consequence of their mutual interactions and symmetry principles. But there is one exceptional nonlinearity, namely that of gravitation theory. In special-relativistic physics the space-time arena is given by the flat Minkowski space. Its geometry is an absolute factor which restricts the symmetry to the Poincare group. But it is a strange and originally surprizing fact that physics does not like absolute objects. In general relativity the metric tensor becomes a dynamical quantity with the dynamics ruled by the Hilbert Lagrangian. It is so-to-speak an essentially nonlinear centre of physical reality. Its dynamics is essentially, non-perturbatively nonlinear and invariant under the infinite-dimensional group of the space-time diffeomorphisms (general covariance group). And automatically it becomes the group of symmetry of the whole physics. The dynamics is quasilinear, nevertheless by necessity nonlinear. The relationship between essential nonlinearity and large symmetry groups seems to be a general rule. Let us mention now two another, simpler examples from different branches of physics.

The first example belongs to mechanics of continua, first of all to plasticity theory, although elastic applications are also possible [11]. Let us consider a real linear space V and the set $\mathrm{Sym}\,(V^* \otimes V^*)$ of symmetric metric tensors on V. It is obviously non-connected and consists of components characterized by the signature. Let us consider the manifold $\mathrm{Sym}^+\,(V^* \otimes V^*)$ of positively definite metrics. And now, assuming that the metrics elements of $\mathrm{Sym}^+\,(V^* \otimes V^*)$ describe some physical reality, let us ask for the metric structures, i.e., kinetic energy forms on $\mathrm{Sym}^+\,(V^* \otimes V^*)$. Of course, the simplest possibility is

$$ds^2 = G^{ijkl} dg_{ij} dg_{kl}, \qquad \text{i.e.,} \qquad G = G^{ijkl} dg_{ij} \otimes dg_{kl}, \tag{46}$$

where G^{ijkl} is constant and satisfies the natural nonsingularity and symmetry conditions:

$$G^{ijkl} = G^{klij}, \qquad G^{ijkl} = G^{jikl} = G^{ijlk}. \tag{47}$$

This metric on the manifold of metrics is flat. But this is rather strange and non-aesthetic. The natural question appears why not to use the following intrinsic metric:

$$ds^2 = \lambda g^{jk} g^{li} dg_{ij} dg_{kl} + \mu g^{ji} g^{lk} dg_{ij} dg_{kl}, \tag{48}$$

or, in more sophisticated terms:

$$G = \lambda g^{jk} g^{li} dg_{ij} \otimes dg_{kl} + \mu g^{ji} g^{lk} dg_{ij} \otimes dg_{kl}, \tag{49}$$

where λ, μ are constants and $\left[g^{ij}\right]$ is the contravariant inverse of $\left[g_{ij}\right]$, i.e., $g^{ik} g_{kj} = \delta^i{}_j$. This metric structure is evidently non-Euclidean, Riemannian in $\mathrm{Sym}^+ \left(V^* \otimes V^*\right)$, but it does not contain anything a priori fixed, but λ, μ. To be more precise, it is only λ that is essential up to normalization, because the μ-term, being degenerate, is only an auxiliary correction. The corresponding kinetic energy of the g-process will be

$$T = \frac{\lambda}{2} g^{jk} g^{li} \frac{dg_{ij}}{dt} \frac{dg_{kl}}{dt} + \frac{\mu}{2} g^{ji} g^{lk} \frac{dg_{ij}}{dt} \frac{dg_{kl}}{dt}. \tag{50}$$

Expressions of this type are used, e.g., in incremental approaches to plasticity. They are also interesting in certain elastic problems and in defect theory.

Let us also mention about some other application. Consider the motion of material point with the mass m and internal g-degrees of freedom. The corresponding kinetic energy will be given by

$$T = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{\lambda}{2} g^{jk} g^{li} \frac{dg_{ij}}{dt} \frac{dg_{kl}}{dt} + \frac{\mu}{2} g^{ji} g^{lk} \frac{dg_{ij}}{dt} \frac{dg_{kl}}{dt}, \tag{51}$$

obviously x^i are here coordinates of the centre of mass. One can also introduce some potential term built of x^a, g_{ij}. In a sense, the structure of (50), (51) resembles that of generally-relativistic Lagrangians, obviously with the proviso that only the time derivatives occur, as we are dealing here with a system which does not possess any other continuous independent variables. Indeed, the main term of Hilbert Lagrangian begins from the expression proportional to

$$g^{\nu\varkappa} g^{\mu\lambda} g^{\alpha\beta} g_{\mu\nu,\alpha} g_{\varkappa\lambda,\beta}, \tag{52}$$

where $g_{\mu\nu}$ is the space-time metric and $g^{\alpha\beta}$ is its contravariant inverse. Differentiation with respect to the space-time coordinates is meant here. The structural similarity to the prescription (50), (51) is obvious.

It is important that the both last expressions for T are invariant under the total group $\mathrm{GL}(V)$, or rather under the semi-direct product $\mathrm{GL}(V) \times_s V$. Again the "large" symmetry group is responsible for the essential nonlinearity even of the geodetic models described by the expressions for T.

It is interesting to ask what changes appear when we assume V to be a complex linear space and g a sesquilinear Hermitian form. Obviously, instead of (50), (51) we will have then

$$T = \frac{\lambda}{2} g^{j\bar{k}} g^{l\bar{i}} \frac{dg_{\bar{i}j}}{dt} \frac{dg_{\bar{k}l}}{dt} + \frac{\mu}{2} g^{j\bar{i}} g^{l\bar{k}} \frac{dg_{\bar{i}j}}{dt} \frac{dg_{\bar{k}l}}{dt}, \tag{53}$$

$$T = \frac{m}{2} g_{\bar{i}j} \frac{dx^{\bar{i}}}{dt} \frac{dx^j}{dt} + \frac{\lambda}{2} g^{j\bar{k}} g^{l\bar{i}} \frac{dg_{\bar{i}j}}{dt} \frac{dg_{\bar{k}l}}{dt} + \frac{\mu}{2} g^{j\bar{i}} g^{l\bar{k}} \frac{dg_{\bar{i}j}}{dt} \frac{dg_{\bar{k}l}}{dt}. \tag{54}$$

As usual, the matrix $\left[g^{i\bar{j}}\right]$ is reciprocal to $\left[g_{\bar{k}l}\right]$.

Let us observe that the models (50), (51), (53), (54) are structurally similar to our earlier affinely-invariant models of the affinely-rigid body [2, 3], i.e., roughly speaking to affinely-invariant geodetic models on the affine group. The idea there was that the material point was endowed with additional internal or collective degrees of freedom described by the attached linear basis $(\ldots, e_A, \ldots)$, or equivalently its dual $\left(\ldots, e^A, \ldots\right)$. The affinely-invariant kinetic energy was given by

$$T = \frac{m}{2} C_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{A}{2} \Omega^i{}_j \Omega^j{}_i + \frac{B}{2} \Omega^i{}_i \Omega^j{}_j, \tag{55}$$

where $C_{ij} = \eta_{AB} e^A{}_i e^B{}_j$ is the Cauchy deformation tensor, η_{AB} is the fixed reference (material) metric, and $\Omega^i{}_j$ is so-called affine velocity (affine generalization of angular velocity),

$$\Omega^i{}_j = \frac{de^i{}_A}{dt} e^A{}_j. \tag{56}$$

This expression for T is affinely invariant and in spite of its apparently strange structure it is dynamically applicable, due to its strongly non-quadratic prescription (nonlinearity of equations of motion).

Let us now go back to our quantum problem. First of all, let us notice that even in our finite-level system with the nonlinearity directly introduced to the Schrödinger equation, the procedure is in general non-trivial. There are two reasons for that: the possible time-dependence of the Hamiltonian $H^a{}_b$, and the non-quadratic term $\mathcal{V}\left(\psi, \overline{\psi}\right)$. Nevertheless, it is still a provisional solution.

Much more geometric is the following reasoning. To give up the fixed scalar product and to introduce instead the dynamical one, in analogy to general relativity and continuum mechanics. And then to define the kinetic energy for Γ in analogy to (53):

$$T = L[\Gamma] = \frac{A}{2} \Gamma^{b\bar{c}} \Gamma^{d\bar{a}} \dot{\Gamma}_{\bar{a}b} \dot{\Gamma}_{\bar{c}d} + \frac{B}{2} \Gamma^{b\bar{a}} \Gamma^{d\bar{c}} \dot{\Gamma}_{\bar{a}b} \dot{\Gamma}_{\bar{c}d}. \tag{57}$$

Therefore, the configuration space of our system consists of pairs $(\psi^a, \Gamma_{\bar{a}b})$. The Lagrangian may be given by

$$L[\psi, \Gamma] = L(1,2)[\psi, \Gamma] + \mathcal{V}[\psi, \Gamma] + L[\Gamma], \tag{58}$$

where $L(1,2)\,[\psi,\Gamma]$, $\mathcal{V}\,[\psi,\Gamma]$ are just the previously introduced models (16), (21), however with the dynamical, non-fixed Γ subject to the variational procedure. Obviously, this complicates the Euler-Lagrange equations even in their parts following only from $L(1,2)+\mathcal{V}$.

The resulting theory is essentially nonlinear and invariant under the group $\mathrm{GL}(W)$, instead of the unitary group $\mathrm{U}(W,\Gamma)$ which preserves the traditional quantum mechanics. In a sense, the gap between quantum and classical mechanics diffuses. Not everything is quantum, not everything is classical. And the effective nonlinearity creates some hope for explaining the quantum paradoxes. Let us observe that the variation of $I[\Gamma]=\int L[\Gamma]dt$ leads to the following equations:

$$-A\Gamma^{b\overline{n}}\left(\ddot{\Gamma}_{\overline{n}k}-\dot{\Gamma}_{\overline{n}l}\Gamma^{l\overline{c}}\dot{\Gamma}_{\overline{c}k}\right)\Gamma^{k\overline{a}}-B\Gamma^{l\overline{n}}\left(\ddot{\Gamma}_{\overline{n}l}-\dot{\Gamma}_{\overline{n}k}\Gamma^{k\overline{c}}\dot{\Gamma}_{\overline{c}l}\right)\Gamma^{b\overline{a}}=0. \tag{59}$$

As mentioned, the second term is merely a correction; the first term is essential. Let us notice that we did not perform variation in any other term of Lagrangian (58). It is so as if the ψ^{a}-degrees of freedom were non-excited. Of course, this is more than academic assumption, nevertheless convenient as a toy model. It is clear that the above equation (59) possesses solutions of the form:

$$\Gamma_{\overline{r}s}=G_{\overline{r}z}\exp(Et)^{z}{}_{s}=\exp(Ft)_{\overline{r}}{}^{\overline{z}}G_{\overline{z}s}, \tag{60}$$

where the initial condition for the scalar product $G=\Gamma(0)$ is a Hermitian sesquilinear form, $G\in\mathrm{Herm}\left(\overline{W}^{*}\otimes W^{*}\right)$. But this Hermitian property is to be preserved during the whole evolution. This will be the case when the linear mappings $E\in L(W)$, $F\in L\left(\overline{W}^{*}\right)$ will be G-Hermitian, i.e., when the sesquilinear forms analytically given by

$${}_{G}E_{\overline{r}s}=G_{\overline{r}z}E^{z}{}_{s},\qquad (F_{G})_{\overline{r}s}=F_{\overline{r}}{}^{\overline{z}}G_{\overline{z}s} \tag{61}$$

are Hermitian. Such solutions are analogous to our geodetic solutions in affinely-invariant models of the homogeneously deformable body [2, 3]. Obviously, there is a deep geometric difference, because in mechanics of homogeneously deformable bodies one deals with real mixed tensors describing configurations, while here we are doing with complex sesquilinear forms. Nevertheless, the general philosophy is the same. Let us observe some interesting facts. Namely, the above evolution of Γ may show all possible modes: it may be exponentially increasing, exponentially decaying, and even oscillatory. The point is how the initial data for $\Gamma(0)=G$, E, F are fixed. Obviously, the academic model of the evolution of Γ when ψ is fixed is rather non-physical, nevertheless, there is a hope that the mentioned ways of behaviour may have something to do with decoherence and measurement paradoxes. Obviously, this hypothesis may be confirmed only a posteriori, by solving, at least approximately, the total system of equations derived from (58) and (63) below. In any case, it is almost sure that the supposed dynamics of Γ should be based on (57) in (58). This follows from our demand of $\mathrm{GL}(W)$-invariance and from the analogy with general relativity and affine body dynamics. For example, we could try to use some fixed background metric G and assume:

$$L[G,\Gamma]=\frac{I}{2}G^{b\overline{c}}G^{d\overline{a}}\dot{\Gamma}_{\overline{a}b}\dot{\Gamma}_{\overline{c}d}+\frac{K}{2}G^{b\overline{a}}G^{d\overline{c}}\dot{\Gamma}_{\overline{a}b}\dot{\Gamma}_{\overline{c}d}. \tag{62}$$

When having G at disposal, we can also define potential-like terms Tr $\left({}^{G}\Gamma^{p}\right)$, where ${}^{G}\Gamma^{r}{}_{s} := G^{r\overline{z}}\Gamma_{\overline{z}s}$. But of course, it seems aesthetically superfluous to fix some scalar products G taken from nowhere, when the dynamical one is used. And it is only $L[\Gamma]$ (57) that seems to have a chance for solving the decoherence problem due to its strong, geometrically motivated nonlinearity.

In any case, the simplest $\mathrm{GL}(W)$-invariant Lagrangian seems to have the form:

$$\begin{aligned} L = {} & i\alpha_1\Gamma\left(\overline{\psi}^{\overline{a}}\dot{\psi}^{b} - \dot{\overline{\psi}}^{\overline{a}}\psi^{b}\right) + \alpha_2\Gamma_{\overline{a}b}\dot{\overline{\psi}}^{\overline{a}}\dot{\psi}^{b} + \left(\alpha_3\Gamma_{\overline{a}b} + \alpha_4 H_{\overline{a}b}\right)\overline{\psi}^{\overline{a}}\psi^{b} \\ & + \alpha_5\Gamma^{d\overline{a}}\Gamma^{b\overline{c}}\dot{\Gamma}_{\overline{a}b}\dot{\Gamma}_{\overline{c}d} + \alpha_6\Gamma^{b\overline{a}}\Gamma^{d\overline{c}}\dot{\Gamma}_{\overline{a}b}\dot{\Gamma}_{\overline{c}d} - \mathcal{V}(\psi,\Gamma), \end{aligned} \tag{63}$$

where, e.g.,

$$\mathcal{V}(\psi,\Gamma) = \frac{\varkappa}{2}\left(\Gamma_{\overline{a}b}\overline{\psi}^{\overline{a}}\psi^{b} - b\right)^{2}. \tag{64}$$

This expression (63) contains all the structural terms mentioned above and is $\mathrm{GL}(W)$-invariant. All quantities in it (except real constants) are dynamical variables and are subject to the variational procedure. We do not investigate in detail the resulting equations of motion. They are very complicated and describe the mutual interaction between ψ^{a}, $\Gamma\overline{ab}$. Nevertheless, their structure is interesting and instructive. Let us quote them for the above Lagrangian (63):

$$\begin{aligned} \frac{\partial L}{\partial\overline{\psi}^{\overline{a}}} = {} & \left(2i\alpha_1\Gamma_{\overline{a}b} - \alpha_2\dot{\Gamma}_{\overline{a}b}\right)\dot{\psi}^{b} - \alpha_2\Gamma_{\overline{a}b}\ddot{\psi}^{b} \\ & + \left(i\alpha_1\dot{\Gamma}_{\overline{a}b} + \alpha_3\Gamma_{\overline{a}b} + \alpha_{4\Gamma}H_{\overline{a}b} - \mathcal{V}'\Gamma_{\overline{a}b}\right)\psi^{b} = 0, \end{aligned} \tag{65}$$

$$\begin{aligned} \frac{\partial L}{\partial\Gamma_{\overline{a}b}} = {} & -A\Gamma^{b\overline{n}}\left(\ddot{\Gamma}_{\overline{n}k} - \dot{\Gamma}_{\overline{n}l}\Gamma^{l\overline{c}}\dot{\Gamma}_{\overline{c}k}\right)\Gamma^{k\overline{a}} - B\Gamma^{l\overline{n}}\left(\ddot{\Gamma}_{\overline{n}l} - \dot{\Gamma}_{\overline{n}k}\Gamma^{k\overline{c}}\dot{\Gamma}_{\overline{c}l}\right)\Gamma^{b\overline{a}} \\ & + i\alpha_1\left(\overline{\psi}^{\overline{a}}\dot{\psi}^{b} - \dot{\overline{\psi}}^{\overline{a}}\psi^{b}\right) + \alpha_2\dot{\overline{\psi}}^{\overline{a}}\dot{\psi}^{b} + \left(\alpha_3 - \mathcal{V}'\right)\overline{\psi}^{\overline{a}}\psi^{b} = 0. \end{aligned} \tag{66}$$

In spite of their relatively complicated structure, these nonlinear equations are readable. For any case we have retained the direct nonlinearity term derived from $\mathcal{V}$. But the main idea of nonlinearity and large $\mathrm{GL}(W) \simeq \mathrm{GL}(n,\mathbb{C})$-symmetry is just the interaction between ψ and Γ. And it is just the interaction of the generally-relativistic type. As seen, at the same time it is structurally similar to affinely-invariant geodetic models of elastic vibrations of the homogeneously deformable body [2, 3]. Even independently on our quantum programme, this model is interesting in itself as an example of highly-symmetric dynamical system on a homogeneous space. Nonlinearity of the system is rational because the inverse matrix $\left[\Gamma^{a\overline{b}}\right]$ is a rational function of $[\Gamma_{\overline{c}d}]$. Therefore, there are some hopes for a solvability, perhaps at least qualitative or approximate, of the system (65), (66).

Let us notice that the α_3-controlled term may be included into the α_4-expression. We simply decided to write it separately to stress the special role of the Hamiltonian terms proportional to the identity operator. Let us stress that the Lagrangian (63) is not the only expression with the above enumerated properties. Rather, it is the simplest one. For example, one might replace $\Gamma^{a\overline{b}}$ by $\Gamma^{a\overline{b}} + \alpha_7 \psi^a \overline{\psi}^{\overline{b}}$, etc. Perhaps this will modify somehow the resulting equations, but in a rather non-essential way. All important features are already predicted by equations following from (63). Let us remind that in (63) we have according to quantum mechanics

$$\alpha_1 = \frac{\hbar}{2}, \qquad \alpha_4 = -1. \tag{67}$$

The nonlinearity contained in (63) resembles in a sense the Thomas-Fermi approximation in quantum mechanics [12, 13]. It may be considered as an alternative way of describing open systems.

When concentrating on the dynamical system model of Schrödinger equation, one is faced with the interesting question as to what might be described by admitting non-Hamiltonian forces. Perhaps nothing physical. There is nevertheless some possibility that such forces might be useful for describing quantum dissipative phenomena. However, at this stage we have no idea concerning this problem.

The $\mathrm{GL}(W) \simeq \mathrm{GL}(n, \mathbb{C})$-invariance of the Hamiltonian system implies the existence of n^2 complex constants of motion. We do not quote here their explicit form to avoid writing unnecessary and complicated formulas. Nevertheless, their existence is a remarkable property of the theory.

Let us stress that the strong and geometrically implied nonlinearity of equations following from (63) gives a chance for the macroscopic reinforcement and enhancement of quantum events.

It is very important to remember that in the model (63), (65), (66), based on the mutual interaction between ψ and Γ, the "scalar product" Γ is not a constant of motion. And there is an exchange of energy between ψ- and Γ-degrees of freedom, especially in situations when the Γ-motion is remarkably excited. Therefore, if $\alpha_5 \neq 0$, $\alpha_6 \neq 0$, the quantity Γ, although fundamental for physical interpretation, in a sense loses its physical meaning of the scalar product. If $\alpha_2 = 0$, $\alpha_5 = 0$, $\alpha_6 = 0$, then the invariance of $\mathcal{L}$ under the $\mathrm{U}(1)$-group

$$\psi \mapsto \exp\left(-\frac{i}{\hbar} e \chi\right) \psi \tag{68}$$

implies, via Noether theorem, that

$$e\frac{2\alpha_1}{\hbar} \langle \psi | \psi \rangle = e\frac{2\alpha_1}{\hbar} \Gamma_{\overline{a}b} \overline{\psi}^{\overline{a}} \psi^b \tag{69}$$

is a constant of motion. And then the usual polarization formula for quadratic forms implies that $\langle \psi | \varphi \rangle = \Gamma_{\overline{a}b} \overline{\psi}^{\overline{a}} \varphi^b$ is preserved. If $\alpha_2 \neq 0$, this procedure leads to the scalar product

$$\Gamma_{\overline{a}b} \overline{\psi}^{\overline{a}} \varphi^b + \frac{i\alpha_2}{2\alpha_1} \Gamma_{\overline{a}b} \left(\overline{\psi}^{\overline{a}} \dot{\varphi}^b - \dot{\overline{\psi}}^{\overline{a}} \varphi^b \right). \tag{70}$$

It fails to be positively definite, nevertheless, it seems to be physically interesting [1]. And finally, when $\alpha_5 \neq 0$, $\alpha_6 \neq 0$, Noether theorem tells us that the U(1)-constant of motion differs form the above ones by the term proportional to $\Gamma^{a\bar{b}}\dot{\Gamma}_{\bar{b}a}$. But now Γ is a dynamical variable and the last expression is not quadratic in $\left(\Gamma_{\bar{a}b}, \dot{\Gamma}_{\bar{a}b}\right)$. Because of this there is no polarization procedure and no well-defined Noether-based scalar product at all.

4. Towards general systems

The main ideas of our nonlinearity model in quantum mechanics were formulated and presented in the simplest case of finite-level systems. Let us mention now about a more general situation. We concentrate mainly on the non-relativistic Schrödinger mechanics. It is true that on the very fundamental level of theoretical physics one is interested mainly in relativistic theory. Nevertheless, the non-relativistic counterparts are also interesting at least from the methodological point of view; they just enable us to understand deeper the peculiarity of relativistic theory. And of course they may be also directly useful physically in situations where in condensed matter theory, relativistic effects are not relevant, e.g., in superconductivity and superfluidity.

Galilei group and Galilei space-time are structurally much more complicated than Poincare (inhomogeneous Lorentz) group and Minkowski space-time. Because of this, the construction of Lagrangian for the Schrödinger theory need some comments and a few steps of reasoning.

One is rather used to start with wave-mechanical ideas of Schrödinger equation, i.e.,

$$i\hbar \frac{\partial \psi}{\partial t} = \mathbb{H}\psi \tag{71}$$

rather than with some yet unspecified precisely field theory on the Galilei space-time. The standard scalar product of Schrödinger amplitudes on the three-dimensional Euclidean space will be denoted as usually by

$$\langle \psi_1 | \psi_2 \rangle = \int \overline{\psi_1(x)} \psi_2(x) d_3 x, \tag{72}$$

where, obviously, orthogonal Cartesian coordinates are meant. More generally, in curvilinear coordinates we would have

$$\langle \psi_1 | \psi_2 \rangle = \int \overline{\psi_1(q)} \psi_2(q) \sqrt{|g(3)|} d_3 q, \tag{73}$$

where $|g(3)|$ is an abbreviation for the determinant of the matrix of Euclidean metric tensor $g(3)$, and q^i are generalized coordinates. The same formula is valid in the Riemann space, where all coordinates are "curvilinear". However, here we do not get into such details.

Hermitian conjugation of operators, $\mathbb{A} \mapsto \mathbb{A}^+$ is meant in the usual sense of $L^2\left(\mathbb{R}^3\right)$ with the above scalar product,

$$\langle \psi_1 | \mathbb{A}\psi_2 \rangle = \langle \mathbb{A}^+ \psi_1 | \psi_2 \rangle, \tag{74}$$

obviously with some additional remarks concerning domains. The Hamilton operator is self-adjoint, $\mathbb{H}^{+} = \mathbb{H}$, also with some care concerning the domains.

The Schrödinger equation is again variational with the Lagrangian

$$\mathcal{L} = \mathcal{L}_T + \mathcal{L}_H, \tag{75}$$

where, $\mathcal{L}_T$, $\mathcal{L}_H$ denote respectively the terms depending linearly on first-order time derivatives and independent on them,

$$\mathcal{L}_T = \frac{i\hbar}{2}\left(\overline{\psi}\frac{\partial\psi}{\partial t} - \frac{\partial\overline{\psi}}{\partial t}\psi\right) = -\mathrm{Im}\left(\hbar\overline{\psi}\frac{\partial\psi}{\partial t}\right) = \mathrm{Re}\left(i\hbar\overline{\psi}\frac{\partial\psi}{\partial t}\right), \tag{76}$$

$$\mathcal{L}_H = -\overline{\psi}\left(\mathbb{H}\psi\right). \tag{77}$$

The total action is

$$I = I_T + I_H = \int \mathcal{L}_T dt d_3 x + \int \mathcal{L}_H dt d_3 x, \tag{78}$$

and the resulting variational derivative of I equals

$$\frac{\delta I}{\delta\overline{\psi}(t,x)} = -\mathbb{H}\psi + i\hbar\frac{\partial\psi}{\partial t}, \tag{79}$$

therefore, the stationary points are given indeed by the solutions of (71). In the very important special case of a material point moving in the potential field V,

$$\mathbb{H} = -\frac{\hbar^2}{2m}g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + V = -\frac{\hbar^2}{2m}\Delta + V, \tag{80}$$

it is convenient and instructive to rewrite the Lagrangian term $\mathcal{L}_H$ in a variationally equivalent form structurally similar to the Klein-Gordon Lagrangian,

$$\mathcal{L}'_H = -\frac{\hbar^2}{2m}g^{ij}\frac{\partial\overline{\psi}}{\partial x^i}\frac{\partial\psi}{\partial x^j} + V\overline{\psi}\psi, \tag{81}$$

which differs from the original one by a total divergence term. In particular, there are no artificial second derivatives in $\mathcal{L}'_H$. This resembles the situation one is faced with in General Relativity. For a free particle the total Lagrangian equals

$$\mathcal{L}' = \frac{i\hbar}{2}\left(\overline{\psi}\frac{\partial\psi}{\partial t} - \frac{\partial\overline{\psi}}{\partial t}\psi\right) - \frac{\hbar^2}{2m}g^{ij}\frac{\partial\overline{\psi}}{\partial x^i}\frac{\partial\psi}{\partial x^j}. \tag{82}$$

In this form, structurally as similar as possible to the Lagrangian for the Klein-Gordon field, the essential geometrical similarities and differences between Galilei and Poincare quantum symmetries are visible.

The well-known expressions for the probability density and probability current are obtained in non-relativistic quantum mechanics on the basis of some rather rough and intuitive statistical concepts. However, when the Schrödinger equation is interpreted within the framework of field theory on the Galilean space-time, those concepts appear as direct consequences of Noether theorem just as said above in the finite number of levels part. The Schrödinger Lagrangian for a free particle is invariant under the group U(1) of global gauge transformations,

$$\psi \mapsto \exp\left(-\frac{i}{\hbar}e\chi\right)\psi, \tag{83}$$

where e denotes the coupling constant (elementary charge) and χ is the gauge parameter. Within the Kaluza-Klein formulation of electrodynamics of point charges, the variable χ and electric charge e are canonically conjugate quantities. The invariance under this gauge group implies the conservation law for the Galilean current $\left(j^t, \bar{j}\right)$, where

$$j^t = \varrho = e\overline{\psi}\psi, \qquad j^a = \frac{e\hbar}{2im}\left(\overline{\psi}\frac{\partial\psi}{\partial x^a} - \frac{\partial\overline{\psi}}{\partial x^a}\psi\right). \tag{84}$$

The resulting continuity equation has the usual form:

$$\frac{\partial\varrho}{\partial t} + \frac{\partial j^a}{\partial x^a} = 0, \qquad \text{i.e.,} \qquad \frac{\partial\varrho}{\partial t} + \mathrm{div}\bar{j} = 0. \tag{85}$$

Let us stress that all those formulas hold in rectilinear orthonormal coordinates. In general coordinates we would have to multiply the above expressions by $\sqrt{|g(3)|}$ so as to turn them respectively into scalar and contravariant vector densities. And so one does in a general Riemann space. Continuity equation is satisfied if ψ is a solution of the Schrödinger equation.

The functional

$$\psi \mapsto Q_t[\psi] = \int j^t d_3x = \int \varrho(t,x)d_3x \tag{86}$$

describes the total charge at the time instant t. On the basis of field equations it is independent on time,

$$\frac{d}{dt}Q_t[\psi] = 0. \tag{87}$$

This is the global law of the charge conservation. Let us observe that Q is a functional quadratic form of ψ. Its polarization reproduces the usual scalar product as a sesquilinear Hermitian form (up to a constant factor):

$$4e\langle\psi|\varphi\rangle = Q[\psi+\varphi] - Q[\psi-\varphi] - iQ[\psi+i\varphi] + iQ[\psi-i\varphi]. \tag{88}$$

In a consequence of charge conservation law $\langle\psi|\varphi\rangle$ is time-independent if ψ, φ are solutions of the same Schrödinger equation.

One can do the same for multiplets of scalar Schrödinger fields ψ^a, i.e., for the continuum of systems described in the previous section. Then we have

$$\mathcal{L}_T = \frac{i\hbar}{2}\Gamma_{\overline{a}b}\left(\overline{\psi}^{\overline{a}}\frac{\partial\psi^b}{\partial t} - \frac{\partial\overline{\psi}^{\overline{a}}}{\partial t}\psi^b\right), \tag{89}$$

$$\langle\psi|\varphi\rangle = \int \Gamma_{\overline{a}b}\overline{\psi}^{\overline{a}}\varphi^b\sqrt{|g(3)|}d_3q, \tag{90}$$

$$L_T = \mathrm{Re}\left\langle\psi|i\hbar\frac{\partial\psi}{\partial t}\right\rangle = -\mathrm{Im}\left\langle\psi|\hbar\frac{\partial\psi}{\partial t}\right\rangle, \tag{91}$$

$$\mathcal{L}_H = -\Gamma_{\overline{a}b}\overline{\psi}^{\overline{a}}\left(\mathbb{H}\psi\right)^b, \qquad L_H = -\langle\psi|\mathbb{H}\psi\rangle. \tag{92}$$

And again, for systems with scalar potentials, when

$$(\mathbb{H}\psi)^a = -\frac{\hbar^2}{2m}\Delta\psi^a + V^a{}_b\psi^b, \tag{93}$$

we can remove second derivatives as a divergence term and obtain the modified Lagrangian

$$\mathcal{L}'_H = -\frac{\hbar^2}{2m}\Gamma_{\overline{a}b}\frac{\partial\overline{\psi}^{\overline{a}}}{\partial x^i}\frac{\partial\psi^b}{\partial x^j}g(3)^{ij} - V_{\overline{a}b}\overline{\psi}^{\overline{a}}\psi^b, \tag{94}$$

where

$$V_{\overline{a}b} = \Gamma_{\overline{a}c}V^c{}_b. \tag{95}$$

We have used here Cartesian coordinates for simplicity. Using the operator of linear momentum,

$$\mathbb{P}_a = \frac{\hbar}{i}\frac{\partial}{\partial x^a}, \tag{96}$$

we can write

$$L_H = -\frac{1}{2m}\langle\mathbb{P}_a\psi|\mathbb{P}_b\psi\rangle\, g(3)^{ab} - \langle\psi|V\psi\rangle. \tag{97}$$

In analogy to (16), (17), (21) we can write the direct-nonlinearity Lagrangian as follows:

$$\begin{aligned}\mathcal{L} = {} & i\alpha_1\Gamma_{\overline{a}b}\left(\overline{\psi}^{\overline{a}}\dot{\psi}^b - \dot{\overline{\psi}}^{\overline{a}}\psi^b\right) + \alpha_2\Gamma_{\overline{a}b}\dot{\overline{\psi}}^{\overline{a}}\dot{\psi}^b \\ & + \alpha_3\Gamma_{\overline{a}b}\frac{\partial\overline{\psi}^{\overline{a}}}{\partial x^i}\frac{\partial\psi^b}{\partial x^j}g(3)^{ij} + \alpha_4 V_{\overline{a}b}\overline{\psi}^{\overline{a}}\psi^b - \mathcal{V}(\psi,\Gamma),\end{aligned} \tag{98}$$

where, e.g., the non-quadratic term $\mathcal{V}$ responsible for nonlinearity may be chosen in a quartic form (22). The combination of the α_2-, α_3-terms may be purely academic, as explained above, but it may also describe something like the Klein-Gordon phenomena, when α_2 and α_3 are appropriately suited. Indeed, at an appropriate ratio $\alpha_2 : \alpha_3$ the above expression for $\mathcal{L}$ becomes:

$$\mathcal{L} = i\alpha_1 \Gamma_{\overline{a}b} \left(\overline{\psi}^{\overline{a}} \dot{\psi}^b - \dot{\overline{\psi}}^{\overline{a}} \psi^b \right) + \alpha_{231} \Gamma_{\overline{a}b} \frac{\partial \overline{\psi}^{\overline{a}}}{\partial x^\mu} \frac{\partial \psi^b}{\partial x^\nu} g^{\mu\nu} - \alpha_{232} \Gamma_{\overline{a}b} \overline{\psi}^{\overline{a}} \psi^b + \alpha_4 V_{\overline{a}b} \overline{\psi}^{\overline{a}} \psi^b - \mathcal{V}(\psi, \Gamma), \tag{99}$$

where g is the Minkowskian metric. For the notational simplicity we used Cartesian coordinates in this formula, to avoid the multiplying of $\mathcal{L}$ by $\sqrt{|g|}$. Of course, within relativistic framework it is rather artificial to superpose the g-Minkowskian terms with the non-relativistic α_1-term. The consequent relativistic theory should rather combine the Klein-Gordon and Dirac terms. Therefore, $\mathcal{L}$ should be then postulated as:

$$\mathcal{L} = i\alpha_{11} \Gamma^\mu{}_{\overline{a}b} \left(\overline{\psi}^{\overline{a}} \frac{\partial \psi^b}{\partial x^\mu} - \frac{\partial \overline{\psi}^{\overline{a}}}{\partial x^\mu} \psi^b \right) + \alpha_{231} \Gamma_{\overline{a}b} \frac{\partial \overline{\psi}^{\overline{a}}}{\partial x^\mu} \frac{\partial \psi^b}{\partial x^\nu} g^{\mu\nu} - \alpha_{232} \Gamma_{\overline{a}b} \overline{\psi}^{\overline{a}} \psi^b + \alpha_4 V_{\overline{a}b} \overline{\psi}^{\overline{a}} \psi^b - \mathcal{V}(\psi, \Gamma). \tag{100}$$

Here we do not fix entities and constants. As usual in the Dirac framework, Γ in the α_{231}-, α_{232}-terms is the sesquilinear Hermitian form of the neutral signature $(++--)$. And Γ^μ are sesquilinear Hermitian Dirac forms, i.e., raising their first index with the help of the reciprocal contravariant $\Gamma^{a\overline{b}}$, $\Gamma^{a\overline{c}} \Gamma_{\overline{c}b} = \delta^a{}_b$, one obtains the Dirac matrices $\gamma^{\mu a}{}_b = \Gamma^{a\overline{c}} \Gamma^\mu{}_{\overline{c}b}$ satisfying the anticommutation rules:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I_4. \tag{101}$$

Strictly speaking, the quantities $\Gamma_{\overline{a}b}$ at α_{231}-, α_{232}-terms need not be the same. Nevertheless, in the fundamental quantum studies it is convenient to identify them. Let us mention, e.g., some ideas connected with the Dirac-Klein-Gordon equation appearing in certain problems of mathematical physics [2, 3]. In particular, it turns out that the $\mathrm{SU}(2,2)$-ruled (conformally ruled) theory of spinorial geometrodynamics has a specially-relativistic limit based on Lagrangian similar to (100). In any case, the specially-relativistic theory based on the superposition of the Klein-Gordon and Dirac Lagrangians is interesting from the point of view of some peculiar kinship between pairs of fundamental particles. We mean pairs of fermions and pairs of quarks which occur in the standard model [9, 14, 15].

Let us go back to our general model (98), not necessarily relativistic one. Its nonlinearity is contained only in the non-quadratic potential term $\mathcal{V}(\psi, \Gamma)$. Lagrangian $\mathcal{L}$ and the action are local in the x-space when the Hamilton operator is a sum of the position- and momentum-type operators. However, in general the corresponding contribution to action is given by

$$I_\chi[\psi] = -\gamma \int \overline{\psi}^{\overline{a}}(t,x) \chi_{\overline{a}b}(x,y) \psi^b(t,y) dt dx dy, \tag{102}$$

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therefore,

$$\frac{\delta I_{\chi}}{\delta \overline{\psi}^{\overline{a}}(t,x)} = -\gamma \int \chi_{\overline{a}b}(x,y)\psi^{b}(t,y)dy. \tag{103}$$

This integral expression implies that our equations of motion will be integro-differential ones. If we seriously assume that also the scalar product is given by the double spatial integral, so that, e.g.,

$$I_1 = i\alpha \int \Gamma_{\overline{a}b}(x,y) \left(\overline{\psi}^{\overline{a}}(t,x)\dot{\psi}^{b}(t,y) - \dot{\overline{\psi}}^{\overline{a}}(t,x)\psi^{b}(t,y) \right) dtdxdy, \tag{104}$$

and in general

$$\Gamma_{\overline{a}b}(x,y) \neq \Gamma_{\overline{a}b}\delta(x-y), \tag{105}$$

then the variational derivative of I_1 also leads to the integro-differential equation, because

$$\frac{\delta I_1}{\delta \overline{\psi}^{\overline{a}}(t,x)} = 2i\alpha \int \Gamma_{\overline{a}b}(x,y)\dot{\psi}^{b}(t,y)dy. \tag{106}$$

The necessity of using integro-differential equations is embarrassing. However, the main difficulty appears when we wish to follow the finite-level systems dynamics in the general case. Namely, it was relatively easy to write formally something like the elements $\left[\Gamma^{a\overline{b}}\right]$ of the matrix reciprocal to $[\Gamma_{\overline{c}d}]$ for the finite-dimensional system. And one can try to follow this procedure for the dynamical scalar product in the general infinite-dimensional case. However, such a hybrid is not convincing. It would be then only "internal degrees of freedom" subject to the procedure of the dynamical scalar product. No doubt that this does not seem satisfactory. Let us write the dynamical scalar product of the Schrödinger-like quantum mechanics in the form:

$$\langle \psi | \varphi \rangle = \int \Gamma_{\overline{a}b}(x,y)\overline{\psi}^{\overline{a}}(x)\varphi^{b}(y)dxdy, \tag{107}$$

where $\overline{\Gamma_{\overline{a}b}(x,y)} = \Gamma_{\overline{b}a}(y,x)$. Assuming in addition the translational invariance, we have $\Gamma(x,y) = \Gamma(x-y)$, i.e.,

$$\langle \psi | \varphi \rangle = \int \Gamma_{\overline{a}b}(x-y)\overline{\psi}^{\overline{a}}(x)\varphi^{b}(y)dxdy. \tag{108}$$

A simplifying assumption would be factorization

$$\Gamma_{\overline{a}b}(x,y) = \Gamma_{\overline{a}b}\mathcal{K}(x,y) = \Gamma_{\overline{a}b}\mathcal{K}(x-y). \tag{109}$$

Let us remind that in the usual quantum mechanics without the dynamical scalar product, we have simply

$$\Gamma_{\overline{a}b}(x,y) = \Delta_{\overline{a}b}\delta(x-y), \tag{110}$$

where Δ is a positively definite algebraic scalar product for internal modes. In appropriately chosen basis we have simply $\Delta_{\overline{a}b} = \delta_{\overline{a}b}$. In the mentioned half-a-way approach, (110) is still valid but with the dynamical, time-dependent $\Delta_{\overline{a}b}$.

Much more reasonable, although incomparatively more difficult, would be a consequent approach based on the dynamical scalar product (107). Here we only mention some ideas. The main point is the construction of the full inverse of Γ in (107), with the time-dependent Γ, i.e., analytically $\Gamma^{a\overline{b}}(x,y;t)$ such that

$$\int \Gamma^{a\overline{c}}(x,u;t)\Gamma_{\overline{c}b}(u,y;t)du = \delta^{a}{}_{b}\delta(x-y). \tag{111}$$

Obviously, in general it is a rather very difficult problem to find explicitly the formula for that inverse. One can try to discretize it by choosing some appropriate finite (or perhaps countable) family Ω of vectors a_p such that

$$\Gamma^{a\overline{b}}(x,y;t) = \sum_{p\in\Omega} \Gamma^{a\overline{b}}(p;t)\delta(x-y+a_p). \tag{112}$$

This may be used as a basis for some discretization procedure like the finite-element method for finding $\Gamma^{a\overline{b}}(x,y;t)$.

In any case, everything said in the former section about the dynamical scalar product and about the failure of the absolute scalar product (72), (73) remains true in wave mechanics on the differential configuration manifold. Although it is true that in this case everything becomes much more complicated.

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