

We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

185,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com



Modelling of Sediment Transport in Shallow Waters by Stochastic and Partial Differential Equations

Charles Wilson Mahera and Anton Mtega Narsis

Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/52237>

1. Introduction

Sediment transport in coastal waters take place in near-shore environments due to the motions of waves and currents resulting in the formation of characteristic coastal landforms such as beaches, barrier islands, and capes, etc. Though sediment transport modelling has been carried for over three decades[26], the study remains a challenging topic of research since a unified description of the process is still to be achieved[25]. A brief review of sediment transport modelling and the widely used classical as well as most recent methods and their limitations in applications are given by [22, 34]. It is observed that sediment transport studies (e.g. the underlying physics and methods used) and modes of sediment transport are yet to be fully studied. Both experimental approaches and mathematical modelling coupled with advanced numerical solutions are needed for better understanding of the how fundamental sediment transport processes is significant for environmental researchers to provide practical and scientifically sound solutions to hydraulic engineering problems. According to [34], the choice of a method for solving a specific problem depends on the nature and complexity of the problem itself, the capabilities of the chosen model to simulate the problem adequately, data availability for model calibration and verification and overall available time and budget for solving the problem. [34] also found that discrepancies between hydrodynamic/sediment transport model predictions and measurements can be attributed to different causes. They include over simplification of the problem by using an inappropriate model 1D versus 2D or 2D versus 3D, the use of inappropriate input data, lack of appropriate data for model calibration, unfamiliarity with the limitations of the hydrodynamic/sediment transport equations used in developing the model, and computational errors in source codes because of approximations in the numerical schemes used in solving the governing equations (boundary condition problems/truncation errors because of discretization).

Modelling sediment transport basically involves interaction of hydrological, hydrodynamic and sedimentology processes. The interactions cause variability of the process parameters which are partly deterministic (having a known structure in space/time, e.g. the yearly storm/calm weather period) and partly stochastic (e.g. sediment pick-up or turbulent viscosity, and also weather variations), both leading to slow spatial/temporal variations in the sand wave-dynamics[31]. Therefore, stochastic characteristics of the governing parameters (such as, suspended sediment inflow concentration) and parameters describing the beginning and rate of erosion and sedimentation respectively are considered as stochastic variables. The observations of the stochastic nature of bedload transport have given impetus to a probabilistic formulation of bedload transport equations.

In stochastic modelling spatio-temporal behavior of phenomena is modelled with random components. The random walk simulation model enables the observation of the phenomena in scales much smaller than the grid size, as well as the tracing of the movement of individual particles, thereby describing the natural processes more accurately. Concentrations of particles are easily calculated from the spatial positions of the particles and, more importantly, when and where required. Furthermore, errors due to numerical diffusion observed in methods such as Finite differences or Finite elements, are avoided and there is considerable reduction in computational time since the calculating load is restricted to the domain parts where the majority of the parcels are gathered.

In a random walk model the displacement of an arbitrary particle, at each time step consists of an advective, deterministic component and an independent, stochastic component. In a simplified one-dimensional transport model the Brownian motion of a particle can be described by the Langevin equation[32].

In order to investigate the fate of suspended sediment in coastal and estuarine waters as well as the evolution of sea or river beds, sediment dynamics need to be represented at a scale relevant to the numerical discretized solution, and significant effort is devoted to parameterize sediment processes. Sediment diffusivities, settling velocities, and cohesive processes such as flocculation all have an impact on suspended sediment throughout the water column. The approaches implemented in these coastal models may present distinct strengths and shortcomings with regard to some important issues for coastal zones, both numerical and physical. While these detailed limitations need to be considered as part of model assessment, more general issues also hinder present state of the art models. In particular, sediment transport is inherently highly empirical, which is further compounded by issues arising from turbulence closure schemes.

In this chapter we focus on deterministic (i.e., process based) coastal ocean models, which are being increasingly used to study coastal sediment dynamics and coastal morphological evolution[23, 27–29]. These models usually treat the short term (hours to days) to medium term (days to months) evolution. Historically, they were first based on depth averaged equations (two dimensional horizontal (2DH) models) and were applied both to riverine see [33] and coastal[24] environments.

We also study the methods based on probability concept appeared to be superior for predicting local transport of bedload. Although several deterministic methods show comparable performance for predicting total sectional transport rate, their performances are significantly reduced for predicting lateral variation of local transportation rates [22]. Development of a stochastic theory based model that can explicitly present the random term

of sediment concentrations could be achieved. With the development of better numerical techniques, the stochastic differential equations can be solved using Itô's integration technique without the need to rely on analytical solutions under simplified conditions.

It is known that bedform changes have a significant effect on the flow dynamics contributed by imbalance between sediments in and out from those areas. The imbalance can easily be disturbed by the external factors, such as extreme storm events, mean sea-level rise, changes in tidal regime, human interferences and so on. Therefore, a better prediction of these bedforms is required to be able to understand their sensitivity to external conditions. In this chapter we developed and describe a particle-based approach to simulate entrainment, transport, and settling of non-cohesive sediments in shallow waters. Sediment distributions are modelled as a set of particles that are tracked on an individual basis by solving Lagrangian transport equations that account for the drift part by the mean flow, settling, and random horizontal motions.

The rest of this chapter is organised as follows. The brief description of an Eulerian transport model is done in Section 3. The particle model for sediment transport is discussed in Section 4. The interpretation of the partial differential equation called Fokker-Planck equation into the well-known Eulerian transport model for sediment transport is described in Section 5. The description and discussion of the two dimensional channel for a test case of sediment transport is carried out in Section 7.

In our work we do carry out the estimation of the change in bedforms. Nevertheless, we do not yet recompute the flow velocities when a change in the shallow water depth occurs.

2. Shallow water flow equations

In order for particle models to describe transport problems in shallow waters, the inputs such as water flow velocities $[U(x, y, t), V(x, y, t)]^T$, water levels ξ , water depths $H(x, y, t)$ and so forth are required. In our application, the inputs are often computed by the hydrodynamic model, which can solve the depth-averaged shallow water equations or 3 dimensional shallow water [15]. The generated results in this case are written into a matlab format that can be loaded and read in the particle model for simulation of sediment transport. The inputs are assumed to satisfy the shallow water equations. The momentum equations are represented by the following equations:

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + g \frac{\partial \xi}{\partial x} - fV + g \frac{U(U^2 + V^2)^{\frac{1}{2}}}{(C_z)^2 H} = 0 \quad (1)$$

$$\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + g \frac{\partial \xi}{\partial y} + fU + g \frac{V(U^2 + V^2)^{\frac{1}{2}}}{(C_z)^2 H} = 0. \quad (2)$$

The velocity is uniform over the vertical, therefore, for that reason, the rise and fall of free surface is given by equations of conservation of mass called the continuity equation:

$$\frac{\partial H}{\partial t} + \frac{\partial(UH)}{\partial x} + \frac{\partial(VH)}{\partial y} = 0, \quad (3)$$

where

- $H = h + \zeta$ is the total depth;
- ζ is the water-level with respect to a reference;
- h depth of the water with respect to a reference;
- C_z bottom friction coefficient (Chezy coefficient);
- g acceleration of gravity;
- f Coriolis parameter.

The shallow water equations are entirely described by equations (1)-(3), provided the closed and open boundary conditions and initial fields are given [9].

3. Eulerian sediment transport model

In this section we briefly introduce the Eulerian model for sediment transport. We consider noncohesive type of sediment particle. The dynamics of the suspended particles can be described by well-known Eulerian transport model with the source and sink terms included. The following Eulerian sediment transport model is similar to that in [20], for example:

$$\begin{aligned} \frac{\partial(HC)}{\partial t} + \frac{\partial(HUC)}{\partial x} + \frac{\partial(HVC)}{\partial y} - \frac{\partial}{\partial x}\left(D\frac{\partial HC}{\partial x}\right) - \frac{\partial}{\partial y}\left(D\frac{\partial HC}{\partial y}\right) \\ = -\gamma HC + \mathcal{E}(U, V) \cdot \lambda_s. \end{aligned} \quad (4)$$

Where $C(x, y, t)$ is depth averaged concentration, γ is the deposition coefficient, $\mathcal{E}(U, V) = (U^2 + V^2)(m^2s^{-2})$ is a function of flow velocities and the term $\lambda_s \cdot \mathcal{E}(U, V)$ models erosion of sediment particles. The particle pick up function is parameterized as $\lambda_s \cdot \mathcal{E}(U, V)$, where, λ_s is the erosion coefficient, it can be related to sediment properties (grain size, grain shape). This parameterisation is motivated by the analysis of field observations reported in [14] and reference therein. Typically, $\lambda_s \approx 3 \times 10^{-2}(kgsm^{-4})$ for fine sand. In this article $\lambda_s = 0.0001(kgm^{-4}s)$ is within the range reported in literature (see e.g., [14, 16]). Note that the term γHC models the deposition of sediment and γ is the deposition coefficient, it is reported that $\gamma \approx 4 \times 10^{-3}s^{-1}$ [14] for fine sand.

3.1. Determination of bedlevel changes by Eulerian transport model

In addition to suspension and deposition processes, the following equation is used to determine the depth changes and therefore the change of bed-level in each grid cell i, j :

$$\frac{\partial h}{\partial t} = \frac{1}{(1 - po)\rho_s}(d_e - s_e). \quad (5)$$

Where $s_e = \gamma HC$ stands for deposition and $d_e = \lambda_s \cdot \mathcal{E}(U, V)$ stands for erosion (term responsible for suspending particles). Sea bed porosity is represented by po , ρ_s is the density of sediment particles.

In this section we have constructed a simplified transport model which is derived from the Eulerian transport model (4). This simple model is then made consistent with the simplified Lagrangian particle model. In this way it becomes easy to compare the bedlevel changes to see if they are similar. We simplified Equation (4) by assuming that the deposition and erosion processes balance:

$$\gamma CH = (U^2 + V^2) \cdot \lambda_s. \quad (6)$$

Quite often, transport in water is defined as the product of the concentration of sediment particles C and a velocity U or V as well the depth of water in that grid cell. Thus, transport along x and y directions is respectively given by;

$$q_x = UCH \text{ and } q_y = VCH,$$

where in vector form $\bar{q} = [q_x, q_y]^T$, using equation (6), it follows that

$$\bar{q} = [(U^2 + V^2)U \cdot f_d, (U^2 + V^2)V \cdot f_d]^T, \quad (7)$$

where $f_d = \frac{\lambda_s}{\gamma}$ stands for the drag force. This depends on the properties of a particle for example its size or its area. Thus, in order to determine how much mass exits or comes into a given location, it is important to consider the divergence. The divergence determines the average rate of how much mass comes into the cell (change of mass per second per area):

$$\frac{\partial m}{\partial t} = -\text{div}(\bar{q}), \quad (8)$$

where $\bar{q} = \frac{1}{T} \int_0^T q dt$, div stands for divergence. Consequently, Equation (8) represents the rate of how much mass stays behind or leaves the cell by assuming the absence of destruction or creation of a matter. Since we want to determine the effects of sediment transport on the sea bedforms, the equation for the bed level is represented by;

$$\frac{\partial h}{\partial t} = \frac{1}{(1 - po)\rho_s} \frac{\partial m}{\partial t}. \quad (9)$$

In the present application the determination of the bed level change using finite difference scheme is estimated by the following equations:

$$\frac{\partial h}{\partial t} = -\frac{1}{\rho_s(1 - po)} \cdot \text{div}(\bar{q}).$$

For cases where flow is in one direction for example when $v = 0$, transport is given by the following equation:

$$\frac{\partial h}{\partial t} = -\frac{f_d}{\rho_s(1 - p_o)} \cdot \text{div}(\overline{U^3}).$$

With the aid of Equation (8)–(9), accordingly the determination of bed level changes is now done by using the following equation:

$$\Delta h \approx \frac{-f_d T}{\rho_s(1 - p_o)} \cdot \left(\frac{\partial \mathbf{u}_m}{\partial x} + \frac{\partial \mathbf{v}_m}{\partial y} \right). \quad (10)$$

where $\mathbf{u}_m = \frac{1}{T} \int_0^T (U^2 + V^2) U dt$ and $\mathbf{v}_m = \frac{1}{T} \int_0^T (U^2 + V^2) V dt$. Next, let us now discuss the Lagrangian particle model in the following section.

4. A particle model for sediment transport in shallow waters

A particle model is a description of a transport process by means of random walk models. Random walk model is defined as the stochastic differential equation that describes the movement of a particle that subsequently undergoes a displacement, which consists of the drift part and a stochastic(diffusive) part [4, 13].

4.1. Integration of particle movement

In this section, the following 2-dimensional stochastic differential equations is developed:

$$\begin{aligned} dX(t) &\stackrel{\text{Itô}}{=} \left[U + \frac{D}{H} \left(\frac{\partial H}{\partial x} \right) + \frac{\partial D}{\partial x} \right] dt + \sqrt{2D} dW_1(t), \\ dY(t) &\stackrel{\text{Itô}}{=} \left[V + \frac{D}{H} \left(\frac{\partial H}{\partial y} \right) + \frac{\partial D}{\partial y} \right] dt + \sqrt{2D} dW_2(t), \end{aligned} \quad (11)$$

where the Brownian process $W_1(t)$ and $W_2(t)$ are Gaussian [3], and $D(x, y, t)$ is the horizontal dispersion coefficient for sediment transport. Typically, $D = \mathcal{O}(10 - 100) m^2/s$ [14]. Note that $U(x, y)$ and $V(x, y)$ are the flow velocities along the x and y direction respectively given in m/s , $H(x, y)$ is the averaged total depth plus relative water levels due to waves, $dW_1(t)$ and $dW_2(t)$ are independent increments of Brownian motions with mean $(0, 0)^T$ and covariance $E[dW_1(t)dW_2(t)^T] = Idt$ where I is an identity matrix ([3, 10]). The simulation of sediment transport is initiated with zero number of particles.

4.2. Deposition of sediment particles

We associate with each sediment particle a binary state which at any time t is given by

$$S_t = \begin{cases} 1 & \text{particle is in suspension} \\ 0 & \text{otherwise (particle is on the sea bed).} \end{cases}$$

Given a particle in suspension, we are interested in the transition of state 1 to state 0. In continuous form, this transition can be modelled by the following equation

$$\frac{dP(S_t = 1)}{dt} = -\gamma \cdot P(S_t = 1), \quad \text{where initially} \quad P(S_0 = 1) = 1 \quad (12)$$

where $\gamma(x, y, t)$ is the deposition coefficient, in this chapter $\gamma = \gamma(x, y, t)$ is constant, $P(S_t = 1)$ is the probability that the state of the particle at time t is 1. When the particle is in the flow, its evolution is described by the following transition probability equation in discrete form:

$$\begin{aligned} P(S_{t+\Delta t} = 1 \mid S_t = 1) &= P(S_0 = 1) \cdot [1 - \gamma \cdot \Delta t] \\ &= [1 - \gamma \Delta t] \end{aligned} \quad (13)$$

Assuming the system state (e.g. flow field and turbulence patterns) to be constant during the period of the time step, it follows that the probability that a particle will be sedimented is given by

$$P(S_{t+\Delta t} = 0 \mid S_t = 1) = 1 - [P(S_{t+\Delta t} = 1 \mid S_t = 1)]. \quad (14)$$

4.3. Suspension of sediment particles

Mass represents concentration of a group of particles at a certain location. A source term is included in our particle model such that the expected number of suspended particles (enp) in grid cell i, j at time t is given by

$$enp_{(i,j,t)} = \frac{\Delta x \cdot \Delta y \cdot \Delta t \cdot (U^2 + V^2) \cdot \lambda_s}{\mathcal{M}_p}, \quad (15)$$

where \mathcal{M}_p is the mass of each particle, Δx and Δy are the width of the grid cells along x and y directions respectively, Δt is the time step size, and λ_s is the erosion coefficient. For each grid cell i, j we use the expected number of particles from Equation (15) to determine the actual number of particles to be suspended. This is done by drawing a number from a Poisson distribution function.

5. The relationship between the Kolmogorov Forward Partial differential and The Advection Diffusion transport model

In this section the relationship between the Kolmogorov Forward Partial differential corresponding to the 2-dimensional Stochastic differential equations and the well known two dimensional advection diffusion equations for sediment transport is discussed in detail. Since we are interested in the particle being in suspension, we assume that the particle at position (x, y) at time t has expectation of their mass $\langle \cdot \rangle$ defined by;

$$\langle m(x, y, t) \rangle = f(x, y, t) \cdot P(S_t = 1). \quad (16)$$

This is known as mass density of particles per unit area of the grid box. The Kolmogorov Forward Partial differential is known as the Fokker-Planck equation(FPE)[3]. In this section we incorporate the processes such as suspension and sedimentation states of the particles in the model for transport of sediments in shallow water. We let D be the diffusion coefficient, $P(S_t = 1)$, is the probability that particle is in suspension, $(S_t = 1)$ denotes a state that a particle is in suspension and $(S_t = 0)$ denotes the state that the particle is deposited on the sea bed or bed of the shallow water. The stochastic process (X_t, Y_t) is a Markov process. The probability density function of the particle position $f(x, y, t)$ based on the two dimensional Stochastic Differential equations (SDEs) (11) evolves according to the following Fokker-Planck equation [9].

$$\begin{aligned} \frac{\partial f(x, y, t)}{\partial t} = & -\frac{\partial}{\partial x} \left(\left[U + \left(\frac{\partial H}{\partial x} D \right) / H + \frac{\partial D}{\partial x} \right] \cdot f(x, y, t) \right) \\ & - \frac{\partial}{\partial y} \left(\left[V + \left(\frac{\partial H}{\partial y} D \right) / H + \frac{\partial D}{\partial y} \right] \cdot f(x, y, t) \right) \\ & + \frac{1}{2} \frac{\partial^2}{\partial x^2} (f(x, y, t) \cdot 2D) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (f(x, y, t) \cdot 2D). \end{aligned} \quad (17)$$

The resulting sediment transport model (11) is an extension of two dimensional particle model for pollutant dispersion in the shallow waters developed by [9]. The extension in the present model includes the erosion and deposition terms. It is possible to derive the Fokker-Planck equation that describes the probabilistic transport of the sediments from one location to another. The derivation of the Fokker-Planck equation is done as follows, we first differentiate equation (16) with respect to time t , to obtain

$$\frac{\partial}{\partial t} \langle m(x, y, t) \rangle = P(S_t = 1) \frac{\partial}{\partial t} f(x, y, t) + f(x, y, t) \frac{\partial}{\partial t} P(S_t = 1),$$

next with the aid of Equation (12), it follows that,

$$\frac{\partial}{\partial t} \langle m(x, y, t) \rangle = P(S_t = 1) \frac{\partial}{\partial t} f(x, y, t) - \gamma f(x, y, t) \cdot P(S_t = 1). \quad (18)$$

Therefore, we also add the erosion term to Equation (18) and come up with

$$\frac{\partial}{\partial t} \langle m(x, y, t) \rangle = P(S_t = 1) \frac{\partial}{\partial t} f(x, y, t) - \gamma f(x, y, t) \cdot P(S_t = 1) + \lambda_s \cdot \mathcal{E}(U, V) \quad (19)$$

next we multiply on both sides of Equation (17) by $P(S_t = 1)$ to obtain

$$\begin{aligned} P(S_t = 1) \frac{\partial}{\partial t} f(x, y, t) = & -\frac{\partial}{\partial x} \left(\left[U + \frac{D}{H} \frac{\partial H}{\partial x} + \frac{\partial D}{\partial x} \right] f(x, y, t) \cdot P(S_t = 1) \right) \\ & - \frac{\partial}{\partial y} \left(\left[V + \frac{D}{H} \frac{\partial H}{\partial y} + \frac{\partial D}{\partial y} \right] f(x, y, t) \cdot P(S_t = 1) \right) \\ & + \frac{1}{2} \frac{\partial^2}{\partial x^2} (2D \cdot f(x, y, t) P(S_t = 1)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (2D \cdot f(x, y, t) \cdot P(S_t = 1)). \end{aligned} \quad (20)$$

The substitution of Equation (20) into equation (19) gives the following Fokker-Planck equation which now have extra terms that model the deposition and erosion of sediments.

$$\begin{aligned} \frac{\partial \langle m(x, y, t) \rangle}{\partial t} = & -\frac{\partial}{\partial x} \left(\left[U + \frac{D}{H} \frac{\partial H}{\partial x} + \frac{\partial D}{\partial x} \right] \cdot \langle m(x, y, t) \rangle \right) \\ & -\frac{\partial}{\partial y} \left(\left[V + \frac{D}{H} \frac{\partial H}{\partial y} + \frac{\partial D}{\partial y} \right] \cdot \langle m(x, y, t) \rangle \right) \\ & + \frac{1}{2} \frac{\partial^2}{\partial x^2} (2D \cdot \langle m(x, y, t) \rangle) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (2D \cdot \langle m(x, y, t) \rangle) \\ & - \gamma \cdot \langle m(x, y, t) \rangle + \lambda_s \cdot \mathcal{E}(U, V). \end{aligned} \quad (21)$$

The last two terms in the transport equation (21) represents the process of sediment deposition and erosion of sediments respectively. An average mass $\langle m(x, y, t) \rangle$ per unit area (kg/m^2) of a particle is related to a particle depth averaged concentration $C(x, y, t)$ in mass per unit volume (kg/m^3) ([9, 18]).

The concentration of materials is given in kg/m^3 , therefore, the expected mass of a particle at position (x, y) can be related by the concentration in that grid cell(location) as follows:

$$\langle m(x, y, t) \rangle = H(x, y, t) \cdot C(x, y, t). \quad (22)$$

Substitution of equation (22) into the Fokker-Planck equation (21) leads to the two dimensional Advection diffusion partial differential equation commonly known as Eulerian sediment transport model (4). Consequently, the transport equation (4) which was discussed in Section 3, is consistent with the particle model for sediment transport (11)-(26).

Therefore, after having constructed the particle model for sediment transport, it is now necessary to develop the equations that cater for the bed level changes using the particle model.

In the next section we shall briefly discuss the numerical approximation of our particle model.

6. Numerical approximation of the particle model

Euler scheme is used in the numerical implementation of the particle model. The scheme is convergent in the weak sense with accuracy of order $\mathcal{O}(\Delta t)$ and it is half order accurate in the strong sense. Higher order schemes for stochastic differential equations are described in [11]. The discretisations of the hydrodynamic flow models is widely discussed by [15], for example. The particle model (11)-(26) is discretised and uses the following Euler scheme to approximate the numerical solutions. We discretise the two dimensional stochastic differential equations for integrating the movement of the particle in similar way to that as in [9] with the modifications by the inclusion of the sedimentation and deposition parts:

$$\bar{X}(t_{k+1}) = \bar{X}(t_k) + \left[U + \left(\frac{\partial H}{\partial x} D \right) / H + \frac{\partial D}{\partial x} \right] \Delta t_k + \sqrt{2D} \Delta W_1(t_k) \quad (23)$$

$$\bar{Y}(t_{k+1}) = \bar{Y}(t_k) + \left[V + \left(\frac{\partial H}{\partial y} D \right) / H + \frac{\partial D}{\partial y} \right] \Delta t_k + \sqrt{2D} \Delta W_2(t_k) \quad (24)$$

$$P_{k+1}(S_t = 1) = (1 - \gamma(x, y, t) \Delta_k) P_k(S_t = 1). \quad (25)$$

Where $\bar{X}(t_{k+1})$ and $\bar{Y}(t_{k+1})$ are the numerical approximations of $X(t)$ and $Y(t)$ respectively, while $\bar{X}(t_0) = X(t_0) = x_0$ and $\bar{Y}(t_0) = Y(t_0) = y_0$ are initial locations of a particle. In addition to Eqns. (23)-(25), we also use Eqns. (15)-(26) to make the simulation of sediment transport complete. There are several schemes that can be used for simulation process for instance, Euler, Heun, Milstein scheme and Runge kutta methods. Much detailed work on numerical methods for stochastic models can be found in (e.g., [11]).

Numerical schemes such as the Euler scheme often show very poor convergence behaviour. This implies that, in order to get accurate results, small time steps are needed thus requiring much computation. Another problem with the Euler (or any other numerical scheme) is its undesirable behaviour in the vicinity of boundaries; a time step that is too large may result in particles unintentionally crossing boundaries. Therefore, the treatment of boundary condition for particle Models is often done in the following section as follows.

6.1. Boundaries

One problem with numerical integration of particle positions arises in the vicinity of boundaries. Given the current location, $(X(t), Y(t))$, we may find that the new location, $(X(t + \Delta t), Y(t + \Delta t))$, is on the other side of a boundary, i.e. the particle has crossed a boundary. Depending on the type of boundary this may be physically impossible. We consider two types of boundaries. The first type, closed boundaries, represents boundaries intrinsic to the domain such as banks, sea bed, and coastal lines. The second type of boundaries are open boundaries, which arise from the modeller's decision to artificially limit the domain because particles are not expected to reach any further or simply because no domain information is available at those locations. It is clear that it is undesirable to have particles cross the first type of boundary, whereas for the second type it is quite natural. Based on this classification, we apply the following rules to particles crossing borders during integration:

- In case an open boundary is crossed by a particle, the particle remains in the sea but is now outside the scope of the model and is therefore removed.
- In case a closed boundary is crossed by a particle during the drift step of integration, the step taken is cancelled and the time step halved until the boundary is no longer crossed. However, because of the halving, say n times, the integration time is reduced to $2^{-n} \Delta t$, leaving a remaining $(1 - 2^{-n}) \Delta t$ integration time, which, at a constant step size, requires at least another $2^n - 1$ steps in order to complete the full time-step Δt . Note that at each of these steps it may be needed to further reduce the step size. This further reduction applies only to the current time step, leaving the step size of following sub-steps unaffected. This method effectively models shear along the coastline.

- If a closed boundary is crossed during the diffusive part of integration, the step size halving procedure described above is maintained with the modification that in addition to the position, the white noise process is also restored to its state prior to the invalidated integration step. The process of halving the time step and continuing integration with the reduced step size is repeated until the full Δt time step has been integrated without crossing a boundary.

In addition, this chapter have also considered the pick up of sediment particles at the inflows this will be discussed in the next section.

6.2. Particle flux at open boundaries

We now consider particle flux at open boundaries. This flux is the difference between particles flowing into and out of the domain. The number of particles flowing out should not be controlled as it is a natural consequence of the movement of a particle. As soon as a particle crosses an open boundary it is considered gone and further integration is no longer possible as no data outside the domain is given. For this very reason, however, we do need to explicitly model the particles flowing in. We determine the expected number of particles entering the domain as follows:

$$enp_{(i,j,t)} = \begin{cases} \frac{\Delta y \cdot \Delta t \cdot V \cdot (U^2 + V^2) \cdot \lambda_s}{\gamma \mathcal{M}_p} & \text{inflow parallel to y-axis} \\ \frac{\Delta x \cdot \Delta t \cdot U \cdot (U^2 + V^2) \cdot \lambda_s}{\gamma \mathcal{M}_p} & \text{inflow parallel to x-axis,} \end{cases} \quad (26)$$

where, γ is the deposition coefficient. The actual number of particles added at the domain boundary at each iteration is obtained by drawing a value from the Poisson distribution parameterised by the above expectation value.

To determine the actual number of particles to be suspended in a grid cell i, j we draw a number from a Poisson distribution function with mean $enp_{(i,j,t)}$ determined by Equation (15). We assume that particles are infinitely many on the sea bed. However, the particle that is suspended is not the same as the one that is deposited.

Before implementing the particle model (11)-(26), we first required to show the consistence between the Fokker-Planck equation and its the Eulerian transport model. This will be described in the next Section.

7. Determination of bedlevel changes using particle models

Comparing with a simplified form of Equation (4), where in this case we assume for the local change in mass $\frac{\partial m}{\partial t} \approx d_e - s_e$. We also assume that the deposition and erosion processes balance (see Section 3.1). The approximation of the change in mass with respect to time, in each grid cell i, j in the this particle model is determined by the following equations:

$$\frac{\partial m}{\partial t} = \frac{\Delta \mathcal{N}_p}{\Delta t} \frac{1}{\Delta x \Delta y} \mathcal{M}_p. \quad (27)$$

Using equation (5) and the fact that $\frac{\partial m}{\partial t} \approx d_e - s_e$, the equation for the bed level change can be derived as follows,

$$\frac{\partial h}{\partial t} = \frac{\Delta \mathcal{N}_p}{\Delta t} \frac{1}{\rho_s(1 - p_0)\Delta x \Delta y} \mathcal{M}_p.$$

Where, \mathcal{M}_p is the mass of each particle, ρ_s and p_0 denote respectively the density of an individual grain particle and the bed porosity. From [14], we find $\rho_s = 2650 \text{ kg/m}^3$ and $p_0 \approx 0.5$. $\Delta \mathcal{N}_p$ is the difference between the number of deposited and suspended particles at each iteration in each grid cell i, j . Hence the cumulated (integrated) change of the level of the sea Δh in m for all time steps is determined by the following equation:

$$\begin{aligned} \Delta h &= \int_0^T \frac{\partial h}{\partial t} dt \\ \Delta h &\approx \sum \frac{\Delta \mathcal{N}_p}{\rho_s(1 - p_0)\Delta x \Delta y} \mathcal{M}_p. \end{aligned} \quad (28)$$

More information about the effect of parameters on the sea bed level changes can be found in [16], for example.

7.1. Primary input of the model

The initial field is defined as

$$depth = h_0 + h_1 \cdot \exp(-((x - 0.0)^2)/(2 \cdot (wd)^2)),$$

where the initial amplitude of the disturbance is $h_1 = 10.0$, width of disturbance (wd) = $2000m$, the tidal period $T = 720$ minutes. Constant for sediment transport rate $K = 0.16 \text{ kgs}^3/\text{m}^6$, porosity $p_0 = 0.5$, $\rho = 2600 [\text{kg}/\text{m}^3]$, density of sediment, horizontal domain length of x goes from $-10000m$ to $10000m$. The horizontal domain length of y goes from $2m$ to $4500m$.

Diffusion constant $D = 10 \text{ m}^2/\text{s}$, starting time $T_{start} = 0s$, number of seconds in a year $T_{year} = 365 \cdot 24 \cdot 60 \cdot 60s/\text{year}$. Final time s $T_{stop} = 100 \cdot T_{year}$, $M = 50$ is the number of grid points across the channel, $dt = 1.0 \cdot \text{year}$ is the time-step s . Tidal mean discharge per unit width m^2/s is given by $q_0 = h_0 \cdot u_0$, tidal amplitude discharge per unit width m^2/s is given by $q_1 = h_0 \cdot u_1$. We then determine the changes in the bedforms as described in Section 3.1. The rate of the changes in the level of the floor of the sea is described by the divergence of transport in each grid cell. For comparison purposes, the divergence of transport is computed by using the original data from the hydrodynamic model as described in this section.

Note that the two dimensional channel in a Matlab code is used whereby in the routine, we first compute the flow fields U, V as well as its depths in each grid cell. Then compute average load transport and finally compute the divergence using a finite difference methods in a Matlab code. Where the divergence of the U -vector field along x and the V -vector field

along y is evaluated using central differences wherever possible and forward or backward differences on the boundaries. Note that the average change of the bedforms in each grid cell are estimated by using Equation (10) in Section 3.1. The results obtained are eventually compared with those obtained when the estimations of the changes of bedforms are carried out using Equation (28) with the aid of the two dimensional SDEs or sometimes in this chapter we call it as the Lagrangian particle model. This stochastic model is discussed in Section 7. Note that both results due to Equation (10) and Equation (28) are similar in terms of shapes, (see Fig. 2).

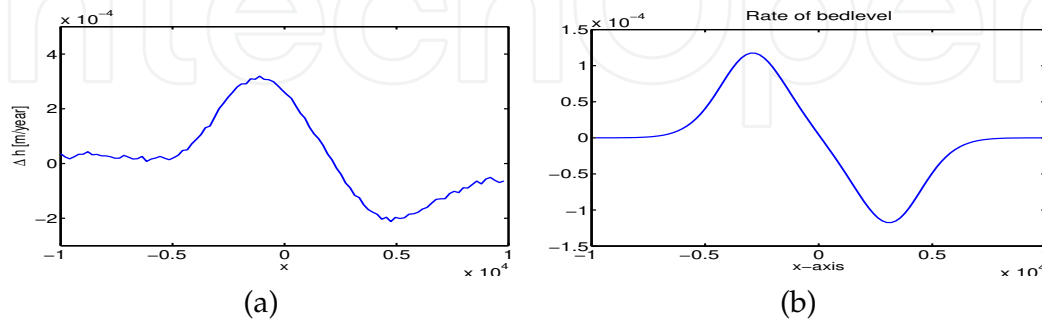


Figure 1. Change of bed level in $m/year$ for a two dimensional channel (a) is due to the particle model, while (b) the result is computed by using the Eulerian approach.

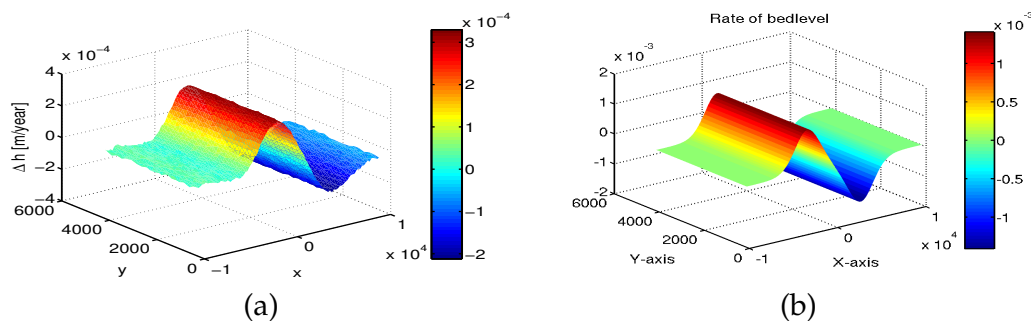


Figure 2. Change of bed level in $m/year$ for a two dimensional channel (a) is due to the particle model, while (b) the result is computed by using the Eulerian approach.

The Figures 1 and 2 represent the results obtained by solving the same problem by using two approaches. Fig. 1(a) and Fig. 2(a) are due to a simplified particle model with very small effect of the diffusion. On the other hand, in Figures 1(b) and Fig. 2(b) are due to a simplified Eulerian model, no diffusion is considered. We should expected to get deposition at the retardation of the flow and erosion at the acceleration of the flow. Resulting in the net migration of the channels in the direction of $(\overline{U^3})$ as in [19]. The positive sign on the colorbar in the figures imply that deposition is taking place while the negative sign implying the occurrence of erosion of sediments.

Some results, in Fig. 3(a,b), represent the local depth change in two selected cells. Part (a) shows a steady deposition in the grid cell at the location $(x, y) = (5km, 2km)$ whereas another grid cell in the location $(x, y) = (-2km, 0.8km)$, part (b), there is also a steady deposition. The diffusion coefficient in the test case for the particle model is $0.00001m^2/s$, $\gamma = 0.00013s^{-1}$.

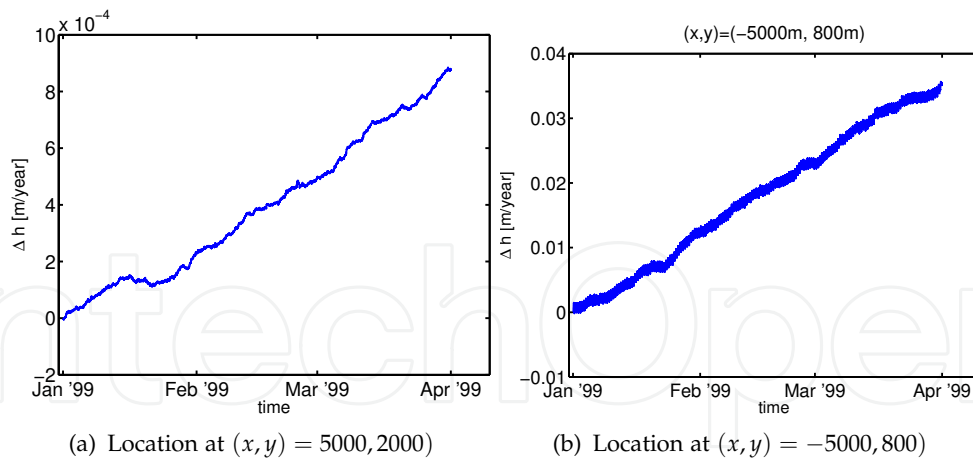


Figure 3. Cumulative local changes in depth by using data of 90 days for two selected grid cells in the ideal two dimensional channel domain light particles

The parameters for the test case experiments using ideal domain with the PM are found in (Table 1.)

Constant	Unit	Value	Constant	Unit	Value
porosity	-	0.50	γ	s^{-1}	0.00013
grid offset	m	$(-10000, 2)$	λ_s	$kg \cdot s \cdot m^{-4}$	0.001
grid size	-	22×101	D	$m^2 \cdot s^{-1}$	0.000010
cell size	m	200×200	δ	-	0
sand density	kgm^{-3}	2650	\mathcal{M}_p	kg	3000
initial location	m	$(1800, 400)$	$f = \frac{\lambda_s}{\gamma}$	$kg \cdot s^2 \cdot m^{-4}$	$\mathcal{O}(10^{-4})$

Table 1. Parameters used by particle model for the sediment transport in the test case

8. Transport of heavy particles in shallow water

As mentioned earlier, sediment transport is a complex process determined by various properties of the sediment materials. Let us in this section consider that we deal with heavy particles. Heavy particles unlike the lighter ones, tend to attain the equilibrium much faster than the tidal cycle. The description of the derivation of the equations for the heavy particles have followed similar lines using the same equations as those for the lighter particles.

$$\frac{\partial(CH)}{\partial t} + \frac{\partial(HUC)}{\partial x} + \frac{\partial(HVC)}{\partial y} = S_e - S_d \quad (29)$$

where in this case $S_e = \lambda_s U^k$ is erosion term, $-S_d = \gamma CH$, is the deposition term

where λ_s, γ, k are constants. In addition, an equation for the change of the bedlevel is presented:

$$\frac{\partial h}{\partial t} = \frac{1}{\rho_s(1 - po)} \cdot (S_e - S_d). \quad (30)$$

However, the difficulty with the heavy (coarser) material is that the parameter γ becomes larger (settling velocity) and this makes the equation (29) 'stiff'. The maximum allowed time step thus becomes very small and since we want to make long simulation this is not very convenient. Therefore, the solution is to make a first order approximation and assume that the two source terms S_e and S_d are much bigger than the right side terms of Equation (29). Therefore, we can write equation (29):

$$\delta \left(\frac{\partial(CH)}{\partial t} + \frac{\partial(HUC)}{\partial x} + \frac{\partial(HVC)}{\partial y} \right) = S_e - S_d, \quad (31)$$

now we substitute a Taylor series expansion of the depth averaged concentration CH

$$CH = Q = Q_0 + \delta Q_1 + \mathcal{O}(\delta^2),$$

into Equation (31) to get the following equations such that 0-order is given by :

$$S_e - S_d = 0, \quad (32)$$

while the 1st order is:

$$\frac{\partial(Q_0)}{\partial t} + \frac{\partial(UQ_0)}{\partial x} + \frac{\partial(VQ_0)}{\partial y} = -(\gamma Q_1). \quad (33)$$

In other words, in Equation (33), we are looking for a source term (γQ_1) that balances the advective transport of the 0 -order solution. Although this is easy in finite difference approach, however, in the particle model, Equation (33) can be approached as follows. The best we can do so far is to solve:

$$\frac{\partial Q}{\partial t} + \frac{\partial(UQ)}{\partial x} + \frac{\partial(VQ)}{\partial y} = 0, \quad (34)$$

with (32) as an initial condition for every time step separately and then set

$$\gamma Q_1 \approx \frac{\partial Q}{\partial t}. \quad (35)$$

In other words, Equation (33) should be more or less balanced. If we now omit the source term and measures the rate of changes, these should be approximately equal. Therefore for the heavy particles, between the beginning and end of the integration time loop should do the following

- (i) First we remove all deposited particles.
- ii) Followed by generating particles according to (see Equation 32)

- ii) Store net change in number of particles (concentration).
- v) Next we do one time step integration of the particle using Equation (34).
- v) Compute differences over the previous step using (iii).

For now we have assumed that

$$\left| \frac{\partial Q_0}{\partial t} \right| \ll |\gamma Q_1|.$$

Note that

$$\text{transport vector } q_x = UHC = \frac{\lambda_s}{\gamma} U(U^2 + V^2) \quad (36)$$

$$\text{transport vector } q_y = VHC = \frac{\lambda_s}{\gamma} V(U^2 + V^2), \quad (37)$$

since $\lambda_s U^2 - \gamma CH = 0$, the equation for the rate change of the bedlevel due to the transport coarse material is given by

$$\frac{\partial h}{\partial t} = - \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right) \cdot \frac{1}{\rho_s(1 - p_o)}, \quad (38)$$

where $f_d = \frac{\lambda_s}{\gamma}$ is the pick up function which depends into the characteristics of the sediment/sand materials. For example in case of a mixture of larger sand of volume= l^3 will have a mass= $\rho_s l^3$. While a sand of double size whose volume is $8l^3$ will have mass= $\rho_s 8l^3$. Therefore, the two particle require different value of the drag force particle f_d . But in this chapter we assume that all particles have the same mass, l =length.

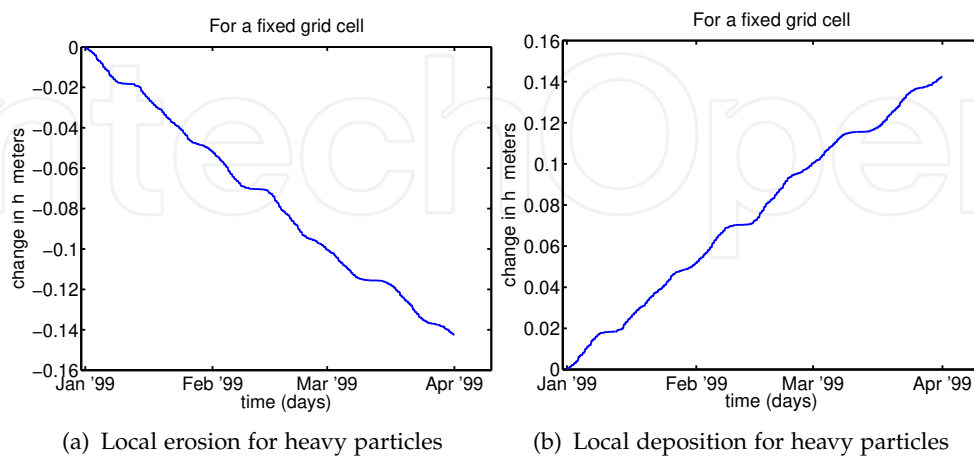


Figure 4. Cumulative local changes in depth by using data of 90 days for two selected grid cells in the ideal two dimensional channel domain for heavy particles

9. Discussion and conclusions

In this chapter we have developed a two dimensional stochastic differential equations. This type of Mathematical model is known as Particle model in some cases. In particular we have developed a two dimensional particle model for sediment transport in which we have added two more equations to model erosion and deposition processes. The description of Transport of materials in shallow waters or in atmosphere can be described by Eulerian approach. This is a deterministic approach in which Partial Differential Equations are used. Numerical methods are usually implemented to approximate the solutions of the PDEs. In this case, one can be faced with computation problems if the dimensional of the model is high. An in most cases if you modify the model to include in more processes it may not be possible to get a closed solution. An alternative approach called the Lagrangian approach can be also used to describe the transport of sediments[35]. This is a probabilistic approach that uses the transition probabilities to derive the state of the transported material. The transition probabilities or density functions is the solution of the Fokker -Planck equation which is also a type of PDEs.

The crucial part is to show that FPE is consistent with ADEs, once that is done then one can derive the underlying SDEs. The derived SDEs can be used as a particle model for the simulation of the transport of materials in the shallow or atmosphere.

In this chapter we have derived the Fokker-Planck equation and included the deposition and erosion terms based on the developed particle model. Moreover, we have also shown that by interpreting the Eulerian sediment transport as the Fokker Planck equation with the additional terms, it becomes possible to derived the underlying particle model that is consistent with the Eulerian sediment transport model. Furthermore, the results of sediment transport due to particle model has been compared to that obtained by computing the transport using Eulerian model in their simplest form. We have got some results for the changes in the bed level for the data of 90 days for an idealized two dimension domain. We have also used our model to test the prediction of bed-level changes by using the approach of parallel computations of bedforms using the real data of the Dutch North sea[5]. Therefore, at least for now we can say that we have solved the set of mathematical equations called particle model for sediment transport. These equations have given us reasonable results for the sea bedlevel changes. Nevertheless, the determination of the morphological changes is a complicated process that depends on several factors such as waves, the size of sand, mass and density[36]. The particle model in this work has been simplified, what we can say is that the results are reasonable. But more factors will have to be taken into account. For instance, in the particle model we have considered that all particles have equal mass while in reality each particle has different mass. For better predictions of the complex behaviour sediment transport that vary with time and space a feedback among **water motion, sediment transport and bottom changes**, we recommend the coupling of both the hydrodynamic and the transport models. Moreover, make sure the grid mesh are of the same form for the hydrodynamic and particle model. Sometimes you may also be required to change the number of particles into concentration. That can be done by using a function called point spread function(PSF).

Author details

Charles Wilson Mahera and Anton Mtega Narsis

1 University of Dar-es-salaam, College of Natural and Applied Sciences, Department of Mathematics, Dar-es-salaam, Tanzania

Department of General Studies, Dar es Salaam Institute of Technology, Dar es Salaam, Tanzania

References

- [1] P. S. Addison, Bo Qu, A. Nisbet, and G. Pender.(1997). A non-Fickian particle-tracking diffusion model based on fractional Brownian motion. *International Journal for numerical methods in fluids*, Vol, 25, No. 0, pp. 1373–1384.
- [2] B.Øksendal (2005). *Stochastic Differential Equations. An introduction with applications*, 6th Ed, Springer, New York,.
- [3] Arnold. L. (1974). *Stochastic differential equations: Theory and applications*, Wiley, London.
- [4] R. W. Barber & N. P. Volakos (2005). Modelling depth-integrated pollution dispersion in the gulf of thermaikos using a lagrangian particle technique, *Proceedings of Inter. Confer. on Water Resources Management*, pp. 173–184, ISBN, Portugal, April and 2005, WIT Trans. on Ecol. and the Env., Vol 80, Algarve
- [5] W. M. Charles, E.van den Berg, H. X. Lin, A. W. Heemink, and M. Verlaan (2008). Parallel and distributed simulation of sediment dynamics in shallow water using particle decomposition approach. *International Journal of Parallel and Distributed Computing*, Vol 68 717Ü728.
- [6] H. B. Fischer, E.J. List, R.C.Y. Koh, J. Imberger, & N.H. Brooks (1979). *Mixing in Inland and Coastal Waters*, 6th Ed, Academic Press, New York.
- [7] C. W. Gardiner, (2004). *Handbook of Stochastic Methods for physics, Chemistry and the Natural Sciences*, Springer-Verlag, Berlin.
- [8] J. Fredsøe and R. Deigaard, (1992). *Mechanics of Coastal Sediment transport*, World Scientific, Singapore.
- [9] A. W. Heemink (1990). Stochastic modeling of dispersion in shallow water *Stochastic Hydrology and hydraulics*, Vol. 4, :161-174
- [10] A. H. Jazwinski, (1970). *Stochastic Processes and Filtering Theory*, Academic Press, New York.
- [11] P. E. Kloeden and E. Platen, (1999). *Numerical solutions of Stochastic Differential equations. Application of Mathematics, Stochastic Modelling and applied probability*, Springer-Verlag, Berlin Heidelberg.
- [12] J. Fredsøe and R. Deigaard, (1992). *Mechanics of Coastal Sediment transport*, World Scientific, Singapore.

- [13] A. Matheja (2000). Sediment transport using the random walk method on top of cobra *Proc. of the 4th of Inter. Confer. on Hydroinformatics, Iowa City, USA*, 1–7
- [14] H. M. Schuttelaars and H. E. de Swart (1996). An idealized long-term morphodynamic model of a tidal embayment *Eur. J. Mech., B/Fluids* Vol. 15(1), :55–80
- [15] G. S. Stelling (1983). Communications on construction of computational methods for shallow water flow problems *PhD Dissertation*, Delft University of Technology, The Netherlands
- [16] K. de Jong (1998). Tidally Averaged Transport Models *PhD Dissertation*, Delft University of Technology, The Netherlands.
- [17] A. F. B. Tompson and L. W. Grelhar(1990). Numerical simulation of the solute transport in three-dimensional, randomly heterogeneous porous media *Water Resources Research*, Vol. 26(10) :2541–2562.
- [18] H. F. P. van den Boogaard, M. J. J . Hoogkamer, and A. W. Heemink (1993). Parameter identification in particle models *Stochastic Hydrology and hydraulics*, Vol. 7(2), :109-130.
- [19] J. van den Kreeke, S. E. Hoogewoning, and M. Verlaan, H. F. P. van den Boogaard, M. J. J . Hoogkamer, and A. W. Heemink (1993). Analytical model for the morphodynamics of a trench in the presence of tidal currents *Continental Shelf Research*, Vol. 22(11-12), :1811-1820.
- [20] L. C. van Rijn (1993). Principles of sediment transport in rivers, estuaries and
- [21] Amoudry, L. O., and A. J. Souza, A. J. (2011) .Deterministic coastal morphological and sediment transport modelling: A review and discussion", *Rev. Geophys.*, 49, RG2002, doi:10.1029/2010RG000341.
- [22] Bhuiyan, F. 2011 Stochastic and Deterministic Methods of Computing Graded Bedload Transport, *Sediment Transport - Flow and Morphological Processes*", Prof. Faruk Bhuiyan (Ed.), ISBN: 978-953-307-374-3.
- [23] Brown, J. M., & Davies, A. G. (2009) Methods for medium term prediction of the net sediment transport by waves and currents in complex coastal regions", *Cont. Shelf Res.*, 29, pp.1502-1514.
- [24] de Vriend, H. J. (1987) 2DH mathematical modelling of morphological evolutions in shallow water, *Coastal Eng.*, Vol 11, pp. 1-27.
- [25] Dewals, B, Rulot, F., Erpicum, S., Archambeau, P. & Pirotton, M. (2011) *Advanced Topics in Sediment Transport Modelling: Non-alluvial Beds and Hyperconcentrated Flows*, *Sediment Transport*, Dr. Silvia Susana Ginsberg (Ed.), ISBN: 978-953-307-189-3.
- [26] Frey, P. & Church, M. (2009) How River Beds Move, *Science*, Vol 325(5947), pp. 1509 - 1510.

- [27] Harris, C. K., Sherwood, C. R., Signell, R. P., Bever, A. J., & Warner, J. C. (2008) Sediment dispersal in the northwestern Adriatic Sea, *J. Geophys. Res.*, 113, C11S03, doi:10.1029/2006JC003868.
- [28] Hu, K., Ding, P., Wang, Z., and Yang, S. (2009) A 2D/3D hydrodynamic and sediment transport model for the Yangtze Estuary, China, *J. Mar. Syst.*, Vol 77, pp. 114-136.
- [29] Lumborg, U. (2005) Modelling the deposition, erosion, and flux of cohesive sediment through Øresund, *J. Mar. Syst.*, Vol 56, pp. 179-193.
- [30] Man, C. (2007) *Stochastic Modelling of Suspended Sediment Transport in Regular and Extreme Flow Environments*, PhD Dissertation, State University of New York at Buffalo, p. 182.
- [31] Németh, A. A. & Hulscher, S. J. M. H., (2003) Finite amplitude sand waves in shallow seas, *IAHR-symposium on River, Coastal and Estuarine Morphodynamics*, Barcelona, Spanje, editors Sánchez-Arcilla, S. en Bateman, A., pp. 435-444.
- [32] Rodean, H.C. (1996) Stochastic Lagrangian Models of Turbulence Diffusion, *American, Meteorological Society*, 84pp, Boston.
- [33] Struiksmā, N., Olesen, K. W., Flokstra, C., & de Vriend, H. J. (1985) Bed deformation in curved alluvial channels, *J. Hydraul. Res.*, Vol 23(1), pp. 57-79.
- [34] Papanicolaou, A. N., Elhakeem, M., Krallis, G., Prakash, S., & Edinger, J., (2008) Sediment Transport Modeling Review-Current and Future Developments, *Journal of Hydraulic Engineering, ASCE*, 14pgs
- [35] Hunter, J. R., Crais, P.D and Phillips H. E. (1993) On the use of random walk models with spatially variable diffusivity. *Computation Physics*, 106:366–376, 1993.
- [36] H. M. Schuttelaars and H. E. de Swart (2000) Multiple morphodynamic equilibria in tidal embayments. *J. Geophysical Research*, 105(c10):14,105–14,118.