

We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

185,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com



Geometry and Dynamics of a Quantum Search Algorithm for an Ordered Tuple of Multi-Qubits

Yoshio Uwano

Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/53187>

1. Introduction

1.1. Geometry and dynamics viewpoints to quantum search algorithms

Quantum computation has been one of the hottest interdisciplinary research areas over some decades, where informatics, physics and mathematics are crossing with (see [1] including an excellent historical overview and [2–4] as later publications for general references). In the middle of 1990's, two great discoveries are made by Shor [5] in 1994 and by Grover [6] in 1996 that roused bubbling enthusiasm to quantum computation. As one of those, Grover found in 1996 the quantum search algorithm for the linear search through unsorted lists [6, 7], whose efficiency exceeds the theoretical bound of the linear search in classical computing: For an unsorted list of N data, the Grover search algorithm needs only $O(\sqrt{N})$ trials to find the target with high probability, while the linear search in classical computing needs $O(N)$ trials. Throughout this paper, the term *classical computing* means the computation theory based on the conventional binary-code operations. The adjective *classical* here is used as an antonym of *quantum*; like quantum mechanics vs classical mechanics.

Though the classical linear search is not of high complexity, the speedup by Grover's algorithm is exciting due to its wide applicability to other search-based problems; G-BBHT algorithm, the quantum counting problem, the minimum value search, the collision problem and the SAT problem, for example [8, 9]. A number of variations and extensions of the Grover algorithm have been made (see [10–13], for example): As far as the author made a search for academic articles in 2012 with keywords 'Grover', 'quantum' and 'search' by Google scholar (accessed 5 September 2012), more than five hundred 'hits' are available. Many of those can be traced from the preprint archive [14].

Among numbers of studies concerning Grover's quantum search algorithm, a pioneering geometric study on the algorithm is made by Miyake and Wadati in 2001 [15]: The sequence of quantum states generated by the Grover algorithm in 2^n data is shown to be on a

geodesic in $(2^{n+1} - 1)$ -dimensional sphere. Further, the reduced search sequence is given rise to the complex projective space \mathbb{CP}^{2^n-1} through a geometric reduction, which is also shown to be on a geodesic in \mathbb{CP}^{2^n-1} . Roughly speaking, the reduction in [15] is made through the phase-factor elimination from quantum states, so that \mathbb{CP}^{2^n-1} is thought of as the space of rays. Note that the geodesics above are associated with the standard metric on $(2^{n+1} - 1)$ -dimensional sphere and with the Fubini-Study metric on \mathbb{CP}^{2^n-1} , respectively. The Fubini-Study metric on \mathbb{CP}^{2^n-1} is utilized also in [15] to measure the minimum distance from each state involved in the search sequence to the submanifold consisting of non-entangled states, which characterizes the entanglement of the states along the search.

As expected benefits of geometric and dynamical views on quantum algorithmic studies like [15], the following would be worth listed;

- 1 By revealing underlying geometry of quantum algorithms (not necessarily universal), numbers of results in geometry are expected to be applied to make advances in quantum computation and information.
- 2 On looked upon the iterations made in algorithms as (discrete) time-evolutions of states, numbers of results in dynamical systems are expected to be applied to make advances in quantum computation and information.
- 3 In view of a close connection between geometry and dynamical systems, geometric and dynamical-systems studies on quantum algorithms may provide interesting examples of dynamical systems.

It would be worth noting here that there exists another approach to quantum searches using adiabatic evolution [16–18]. That approach, however, is outside the scope of this chapter since the search dealt with in this chapter is organized on the so-called amplitude magnification technique [8] which differs from the adiabatic evolution.

1.2. Quantum search for an ordered tuple of multi-qubits – a brief history –

Motivated by the work [15], the author studied in [19] a Grover-type search algorithm for an ordered tuple of multi-qubits together with a geometric reduction other than the reduction made in [15]. While the search algorithm is organized as a natural extension of Grover's original algorithm, the reduction of the search space made in [19] provides a nontrivial result: On denoting the degree of multi-qubits by n and the number of multi-qubits enclosed in each ordered tuple by ℓ , the space of $2^n \times \ell$ complex matrices with unit norm denoted by $M_1(2^n, \ell)$ is taken as the extended space of ordered tuples of multi-qubits (ESOT), where the collection of all the ordered tuples denoted by $M_1^{OT}(2^n, \ell)$ is included. The reduction is applied to the regular part, $\dot{M}_1(2^n, \ell)$, of the ESOT, $M_1(2^n, \ell)$, to give rise to the space denoted by \dot{P}_ℓ of regular density matrices of degree ℓ which plays a key role in quantum information theory. Roughly speaking, the reduction applied in [19] is made by the elimination of 'complex rotations' leaving the relative configuration of multi-qubit states placed in each ordered tuples, so that the reduction is understood to be a very natural geometric projection of $\dot{M}_1(2^n, \ell)$ to the space, \dot{P}_ℓ .

A significant result arising from the reduction is that the Riemannian metric on \dot{P}_ℓ is shown to be derived 'consistently' from the standard metric on $\dot{M}_1(2^n, \ell)$, which coincides with the SLD-Fisher metric on \dot{P}_ℓ up to a constant multiple [19]. Namely, as a Riemannian manifold,

$M_1(2^n, \ell)$ is reduced to the space of regular density matrices of degree ℓ endowed with the SLD-Fisher metric, so that \dot{P}_ℓ is referred to as the quantum information space (QIS). Put another way, the reduction made in [19] reveals a direct nontrivial connection between the ESOT and the QIS. The former is a stage of quantum computation and the latter the stage of quantum information theory.

Due to the account given below, however, geometric studies were not made in [19] either on the search sequence in the ESOT generated by the Grover-type search algorithm or on the reduced search sequence in the QIS: Instead of geometric studies on the search sequences, it is the gradient dynamical system associated with the negative von Neumann entropy as the potential that is discussed in [19] on inspired by a series works of Nakamura [20–22] on complete integrability of algorithms arising in applied mathematics. The result on the gradient system in [19] has drawn the author's interest to publish [23, 24] on the gradient systems on the QIS realizing the Karmarkar flow for linear programming and a Hebb-type learning equation for multivariate analysis, while geometric studies on the search sequences were left undone.

1.3. Chapter purpose, summary and organization

The purpose of this chapter is therefore to study the Grover-type search sequence for an ordered tuple of multi-qubits from geometric and dynamical viewpoints, which has been left since [19]. In particular, the reduced search sequence in the QIS is intensively studied from the viewpoint of quantum information geometry.

As an extension of [15] on the original search sequence, the Grover-type search sequence in the ESOT, $M_1(2^n, \ell)$, is shown to be on a geodesic. As a nontrivial result on the reduced search sequence in the QIS, the sequence is characterized in terms of an important geometric object in quantum information geometry:

Main Theorem *Through the reduction of the regular part, $\dot{M}_1(2^n, \ell)$, of the extended space of ordered tuples of multi-qubits (ESOT) to the quantum information space (QIS), \dot{P}_ℓ , the reduced search sequence is on a geodesic in the QIS with respect to the m -parallel transport.*

Note that the m -parallel transport is the abbreviation of the *mixture* parallel transport [25–27], which is characteristic of the QIS.

To those who are not familiar to differential geometry, an important remark should be made on the term *geodesic* before the outline of chapter organization: One might hear that the geodesic between a pair of points is understood to be the shortest path connecting those points. This is true if geodesics are discussed on a Riemannian manifold endowed with the Levi-Civita (or Riemannian) parallel transport. As a reference accessible to potential readers, the book [28] is worth cited. Geodesics in the ESOT, $M_1(2^n, \ell)$, discussed in this chapter are the case. In general, however, geodesics are *not* characterized by shortest-path property *but* by autoparallel curves which have the shortest paths only in the case of the Levi-Civita parallel transport. What is needed to define geodesics is a parallel transport, while Riemannian metrics are not always necessary. The m -parallel transport of the QIS is the very example of parallel transport whose geodesics do not have the shortest-path property. As another crucial parallel transport in the QIS, the *exponential* parallel transport (the e -parallel transport) is well-known [25–27], whose geodesics do not have the shortest-path property either, though it is not dealt with in this chapter.

The organization of this chapter is outlined in what follows. Section 2 is for the quantum search for an ordered tuple of multi-qubits. The section starts with a brief review of the

classical linear search in unsorted lists. The second subsection is for preliminaries to the quantum search: Mathematics for multi-qubits and ordered tuples of them is introduced. In the third subsection, the Grover-type quantum search algorithm is organized for an ordered tuple of multi-qubits along with the idea of Grover [6]. Dynamical behavior of the search sequence thus obtained is studied in the fourth subsection from the geometric viewpoint: The search sequence in the ESOT, $M_1(2^n, \ell)$, is shown to be on a geodesic in the ESOT. Section 3 is devoted to a study on the reduced search sequence in the QIS from geometric and dynamical points of view. The first subsection is a brief introduction of the QIS. The geometric reduction of the ESOT to the QIS is made in the second subsection: To be precise, our interest is focused on the reduction of the regular part, $\dot{M}_1(2^n, \ell)$, of the ESOT to simplify our geometric analysis. The third subsection starts with the standard parallel transport in the Euclidean space as a very familiar and intuitive example of the parallel transport. After the Euclidean case, the m -parallel transport in the QIS is introduced. It is shown that the reduced search sequence in the QIS is on a geodesic in the QIS with respect to the m -parallel transport. Section 4 is for concluding remarks, in which a significance of the main theorem (or Theorem 3.3) and some questions for future studies are included. A mathematical detail of Sec. 3 is consigned to Appendices following Sec. 4. Many symbols are introduced for geometric setting-up and analysis, which are listed in Appendix 1.

2. Quantum search for an ordered tuple of multi-qubits

2.1. Classical search: Review

The classical linear search in unsorted lists is outlined very briefly in what follows.

Let N be the number of unsorted data in a list, so that the data are labeled as d_1, d_2, \dots, d_N . The N is assumed to be large enough. We start with a very figurative description of the search by taking the counter-consultation of a thick telephone book as an example; namely, the identification of the subscriber of a given telephone number. In the telephone book, the superscripts, $j = 1, 2, \dots, N$, of the data, $\{d_j\}_{j=1,2,\dots,N}$, correspond to the names of subscribers sorted alphabetically and each d_j shows the telephone number of the j -th subscriber. Among the data, there assumed to be one telephone number, say d_M , that we wish to know its subscriber. The d_M is referred to as the target or the marked datum. A very naive way of finding d_M is to check the telephone number from d_1 in ascending order whether or not it is the same as the target datum until we find d_M . The label M turns out to be the subscriber who we wish to identify. In average, this way requires $\frac{N}{2}$ trials of checking to find d_M .

The linear search is described in a smarter form than above in terms of the oracle function. In the same setting above, the oracle function, denoted by f , is defined to be the function of $\{1, 2, \dots, N\}$ to $\{0, 1\}$ subject to

$$f(j) = \begin{cases} 1 & (j = M) \\ 0 & (j \neq M) \end{cases} \quad (1)$$

for $j = 1, 2, \dots, N$. Namely, $f(j) = 1$ means the ‘hit’ while $f(j) = 0$ a ‘miss’. In theory, the evaluation of $f(j)$ is assumed to be done instantaneously, so that the evaluations does not affect the complexity of the problem. The search is therefore made by evaluating $f(j)$ from $j = 1$ in ascending order until we have $f(j) = 1$. The expected number of evaluations is $\frac{N}{2}$, linear in N , so that we say the classical search needs $O(N)$ evaluations. It is well known that the estimate $O(N)$ is the theoretical lowest bound of the classical linear search.

2.2. Quantum search: Preliminaries

2.2.1. Single-qubit

As known well, information necessary to classical computing is encoded into sequences of '0' and '1'. The minimum unit carrying '0' or '1' is said to be a *bit*. A quantum analogue of a *bit* is called a *qubit*, that takes 2-dimensional-vector form with complex-valued components. In particular, the basis vectors, $(1,0)^T$ and $(0,1)^T$, are taken to play the role of symbols '0' and '1' for classical computing, so that they are referred to as the *computational basis vectors*. We note here that the superscript T indicates the transpose operation to vectors and matrices henceforth. A significant difference between *qubit* and *bit* is that superposition of the computational basis vectors is allowed in qubit while it is not so of '0' and '1' in bit. Namely, superposition, $\alpha(1,0)^T + \beta(0,1)^T$ ($\alpha, \beta \in \mathbb{C}$), is allowed in qubit, so that we refer to the space of 2-dimensional column complex vectors, denoted by \mathbb{C}^2 , as the single-qubit space. The \mathbb{C}^2 is endowed with the natural Hermitian inner product, say $\phi^\dagger \psi$ for $\phi, \psi \in \mathbb{C}^2$, where the superscript † indicates the Hermitian conjugate operation to vectors and matrices.

2.2.2. Multi-qubits

In order to express classical n -bit information in quantum computing, it is clearly necessary to prepare 2^n computational basis vectors, which span 2^n -dimensional complex vector space \mathbb{C}^{2^n} : For any integer x subject to $1 \leq x \leq 2^n$, let us denote by $e(x)$ the canonical basis vector in \mathbb{C}^{2^n} , whose x -th component equals 1 ($x = 1, 2, \dots, 2^n$), while the others are naught. Then every $e(x)$ corresponds to the binary sequence $x_1 x_2 \dots x_n$ ($x_j = 0, 1, j = 1, 2, \dots, n$) with $x - 1 = \sum_{j=1}^n x_j 2^{n-j}$, so that the basis $\{e(x)\}_{x=1,2,\dots,2^n}$ turns out to be the computational basis. To be precise mathematically, \mathbb{C}^{2^n} should be understood as n -tensor product,

$$\mathbb{C}^{2^n} \cong (\mathbb{C}^2)^{\otimes n} = \overbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}^n, \quad (2)$$

of the single-qubit spaces (\mathbb{C}^2 s). The $\mathbb{C}^{2^n} (\cong (\mathbb{C}^2)^{\otimes n})$ is called the n -qubit space (more generally, the multi-qubit space), which usually thought of as a Hilbert space for a combined quantum system consisting of n single-qubit systems. In the n -qubit space, any vectors with unit length are called *state vectors*;

$$\phi = \sum_{x=1}^{2^n} \alpha_x e(x) \quad (\alpha_x \in \mathbb{C}, x = 1, 2, \dots, 2^n) \quad \text{with} \quad \sum_{x=1}^{2^n} |\alpha_x|^2 = 1. \quad (3)$$

It is worth noting here that, in a context of quantum computing or of quantum information, the n -qubit space, $(\mathbb{C}^2)^{\otimes n}$, is often assumed to be a 2^n -dimensional subspace of a complex Hilbert space (usually of infinite dimension) where a quantum dynamical system is described.

2.2.3. Ordered tuples of multi-qubits

We move on to introduce ordered tuples of multi-qubits: The degree of multi-qubits is set to be n and the number of multi-qubit data enclosed in any tuple to be ℓ henceforth. Let $M(2^n, \ell)$ be the set of $2^n \times \ell$ matrices, which is made into a complex Hilbert space of dimension $2^n \times \ell$ endowed with the Hermitian inner product

$$\langle \Phi, \Phi' \rangle = \frac{1}{\ell} \text{trace } \Phi^\dagger \Phi' \quad (\Phi, \Phi' \in M(2^n, \ell)). \quad (4)$$

The $M(2^n, \ell)$ with \langle, \rangle is the Hilbert space for our quantum search. The subset,

$$M_1(2^n, \ell) = \{\Phi \in M(2^n, \ell) \mid \langle \Phi, \Phi \rangle = 1\}, \quad (5)$$

of $M(2^n, \ell)$ is what we are going to deal with henceforth. An ordered tuple of multi-qubits is a matrix in $M_1(2^n, \ell)$ of the form

$$\Phi = (\phi_1, \phi_2, \dots, \phi_\ell) \quad \text{with} \quad \phi_j^\dagger \phi_j = 1 \quad (\phi_j \in \mathbb{C}^{2^n}, j = 1, 2, \dots, \ell). \quad (6)$$

Namely, every column vector of an ordered tuple of multi-qubits stands for a n -qubit state vector. Then the subset of $M_1(2^n, \ell)$ defined by

$$M_1^{OT}(2^n, \ell) = \{\Phi = (\phi_1, \phi_2, \dots, \phi_\ell) \in M_1(2^n, \ell) \mid \phi_j^\dagger \phi_j = 1 \ (j = 1, 2, \dots, \ell)\} \quad (7)$$

is the space of ordered tuples of multi-qubits, so that we refer to $M_1(2^n, \ell)$ including $M_1^{OT}(2^n, \ell)$ as the *extended* space of ordered tuples of multi-qubits (ESOT).

On closing this subsection, a remark on a vector-space structure of the ESOT, $M_1(2^n, \ell)$, is made: As a vector space, the ESOT allows the following isomorphisms,

$$M_1(2^n, \ell) \cong \mathbb{C}^{2^n} \otimes \mathbb{C}^\ell \cong \overbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}^n \otimes \mathbb{C}^\ell, \quad (8)$$

which is usually looked upon as a Hilbert space of the combined system consisting of n two-level particle systems (single-qubit systems) and an ℓ -level particle system. The structure (8) will be a clue to think about a physical realization of the present algorithm.

2.3. Quantum search for an ordered tuple

We are now in a position to present a Grover-type algorithm for an ordered tuple of multi-qubits. Our recipe traces, in principle, Grover's original scenario for the single-target state search [6]. We start with the initial state denoted by A and the target state W , which are defined to be

$$A = \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad W = (e(\sigma_1), e(\sigma_2), \dots, e(\sigma_\ell)), \quad (9)$$

where σ is an injection of $\{1, 2, \dots, \ell\}$ into $\{1, 2, \dots, 2^n\}$. On recalling that the state vector $e(x)$ corresponds to the binary sequence $x_1 x_2 \cdots x_n$, the target W corresponds to the ordered tuple of the binary sequences, $\sigma_{j,1} \sigma_{j,2} \cdots \sigma_{j,\ell}$, associated with $e(\sigma_j)$ ($j = 1, 2, \dots, \ell$). Note that σ is not necessarily injective in general, but, for simplicity in the succeeding section, it is required to be an injection. Through this chapter, we further assume that n is sufficiently larger than ℓ , so that, in W , the number ℓ of binary sequences is relatively quite smaller than the length n of each binary sequence.

Like in many literatures on quantum computation, we apply the description without the *oracle qubit* below. A treatment and a role of the oracle qubit can be seen, for example, in [1].

The quantum search is proceeded by applying iteratively the unitary transformation

$$I_G = (-I_A) \circ I_W \quad (10)$$

of $M(2^n, \ell)$ looked upon as a Hilbert space, where I_A and I_W are the unitary transformations defined to be

$$I_A : \Phi \in M(2^n, \ell) \mapsto \Phi - 2\langle A, \Phi \rangle A \in M(2^n, \ell), \quad (11)$$

$$I_W : \Phi \in M(2^n, \ell) \mapsto \Phi - 2\langle W, \Phi \rangle W \in M(2^n, \ell). \quad (12)$$

A very crucial remark is that, on implementation, I_W will be of course *not* realized with the target W (see [1] for example).

To express the action of I_G to the initial state A , it is convenient to introduce the $2^n \times \ell$ matrix,

$$R = \sqrt{\frac{2^n}{2^n - 1}} A - \sqrt{\frac{1}{2^n - 1}} W \in M_1(2^n, \ell). \quad (13)$$

The pair $\{W, R\}$ forms an orthonormal basis of the subspace, denoted by $\text{span}\{W, R\}$, of $M(2^n, \ell)$ consisting of all the superpositions of the initial state A and the target W . The action of the operator I_G leaves the subspace, $\text{span}\{W, R\}$, invariant; $I_G(\text{span}\{W, R\}) = \text{span}\{W, R\}$. The action of I_G can be therefore restricted on $\text{span}\{W, R\}$ to be

$$I_G : (W, R) \mapsto (W, R) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (14)$$

where θ is defined by

$$\sin \frac{\theta}{2} = \sqrt{\frac{1}{2^n}}, \quad \cos \frac{\theta}{2} = \sqrt{\frac{2^n - 1}{2^n}}, \quad 0 < \theta < \pi. \quad (15)$$

On putting (10), (13) and (14) together, the k -times iteration I_G^k of I_G applied to A results in

$$I_G^k(A) = \left(\sin\left(k + \frac{1}{2}\right)\theta \right) W + \left(\cos\left(k + \frac{1}{2}\right)\theta \right) R \quad (k = 1, 2, 3, \dots). \quad (16)$$

Hence $I_G^k(A)$ gets closed to the target W if $(k + \frac{1}{2})\theta$ does to $\frac{\pi}{2}$. Indeed, under the assumption $n \gg 1$, Eq. (15) yields $\theta \simeq 2\sqrt{\frac{1}{2^n}}$, so that the probability of observing the state W from the state $I_G^k(A)$ gets the highest (closed to one) at the iteration number nearest to $\frac{\pi}{4}\sqrt{2^n} - \frac{1}{2}$. Namely, like Grover's original search algorithm, complexity of the quantum search presented above for an ordered tuple of multi-qubits is of the order of square root of 2^n , the length of binary sequences allowed to be expressed in multi-qubits. In the case of $\ell = 1$, our search of course becomes Grover's original ones, so that our search is thought of as a natural generalization of Grover's original one [6] based on the amplitude magnification technique (see [8], for example).

On closing this subsection, a remark should be made in what follows, which would be of importance to think of a physical implementation in future: We have organized the Grover-type search algorithm I_G as a unitary transformation of the ESOT, $M_1(2^n, \ell)$. Since physically acceptable tuples, however, are in the subset, $M_1^{OT}(2^n, \ell)$, of the ESOT, it is worth checking whether or not I_G leaves $M_1^{OT}(2^n, \ell)$ invariant. By a straightforward calculation with (9), (13) and (14), I_G indeed leaves $M_1^{OT}(2^n, \ell)$ invariant. Though this fact is very basic and simple, this supports, to an extent, a physical feasibility of the present algorithm.

2.4. Geodesic property of the search sequence

We show that the search sequence $\{I_G^k(A)\}$ generated by (16) is on the geodesic starting from the initial state A to the target state W , like in Wadati and Miyake [15] on Grover's original search.

2.4.1. Geometric setting-up

As is briefly mentioned of in Sec. 1, the term 'geodesics' can deal with a wider class of curves in differential geometry than that in usual sense. In usual sense especially among non-geometers, for example, one might have an experience of hearing a phrase like 'the shortest path between a pair of points is a geodesic'. In contrast with phrases like this, geodesics are defined to be autoparallel curves in differential geometry. Put in another way, we have to fix a parallel transport to discuss geodesics in the geometric framework. We have a variety of parallel transports, among which the Levi-Civita (or Riemannian) parallel transport can provide the shortest-path property. Note here that the Levi-Civita parallel transport is defined as the parallel transport that leaves the Riemannian metric endowed

with the space. The geodesics to be mentioned of in this subsection can be understood as the familiar *shortest paths*.

Our discussion is made on the ESOT, $M_1(2^n, \ell)$ defined by (5), with which the standard Riemannian metric is endowed in the following way. To those who are not familiar to geometry, it is recommended to think of the 2-dimensional unit-radius sphere, S^2 , in place of $M_1(2^n, \ell)$, since $M_1(2^n, \ell)$ is a $(2^{n+1}\ell - 1)$ -dimensional analogue of S^2 . A Riemannian metric of $M_1(2^n, \ell)$ has a role of an inner product in every tangent space,

$$T_\Phi M_1(2^n, \ell) = \{X \in M(2^n, \ell) \mid \Re(\text{trace } \Phi^\dagger X) = 0\} \quad (\Phi \in M_1(2^n, \ell)), \quad (17)$$

of $M_1(2^n, \ell)$ at Φ as follows, where \Re indicates the operation of taking the real part of complex numbers: On recalling the intuitive case of S^2 , the tangent space at a point $p \in S^2 \subset \mathbf{R}^3$ is thought of as the collection of all the vectors normal to the radial vector p , which can be understood as all the velocity vectors from the dynamical viewpoint. The Riemannian metric, denoted by $((\cdot, \cdot))^{ESOT}$, of $M_1(2^n, \ell)$ is defined to give the inner product

$$((X, X'))_\Phi^{ESOT} = \frac{1}{\ell} \Re(\text{trace } X^\dagger X') \quad (X, X' \in T_\Phi M_1(2^n, \ell), \Phi \in M_1(2^n, \ell)) \quad (18)$$

in each tangent space $T_\Phi M_1(2^n, \ell)$.

2.4.2. Geodesics

We are to give an explicit form of geodesics in a very intuitive manner as follows. Let us recall the 2-dimensional case, in which a geodesic with the initial position $p \in S^2 \subset \mathbf{R}^3$ is known well to be realized as a big circle passing through p . By the initial velocity, say $v \in \mathbf{R}^3$, always normal to p , the geodesic is uniquely determined as the intersection of S^2 and the plane spanned by the vectors p and v . The same story is valid for geodesics in $M_1(2^n, \ell)$, so that we get an explicit form,

$$\Phi(s) = (\cos \sqrt{\ell} s) \Phi_0 + (\sin \sqrt{\ell} s) X_0 \quad (s \in \mathbf{R}), \quad (19)$$

of the geodesic with the initial position $\Phi_0 \in M_1(2^n, \ell)$ and the initial vector $X_0 \in T_{\Phi_0} M_1(2^n, \ell)$ of unit length tangent to the geodesic. In (19), s is taken to be the length parameter measured from the initial point Φ_0 . To be precise from differential geometric viewpoint, the geodesics given by (19) are said to be associated with the Levi-Civita (or Riemannian) parallel transport in $M_1(2^n, \ell)$.

We are to determine a geodesic which the search sequence $\{I_G^k(A)\}$ is placed on. From (13), (16) and (19), we can construct the geodesic from the big circle passing both W and R , so that we obtain

$$\begin{aligned} \Psi(s) &= (\cos \sqrt{\ell} s) \left(\sqrt{\frac{1}{2^n}} W + \sqrt{\frac{2^n - 1}{2^n}} R \right) + (\sin \sqrt{\ell} s) \left(\sqrt{\frac{2^n - 1}{2^n}} W - \sqrt{\frac{1}{2^n}} R \right) \\ &= \left(\cos(\sqrt{\ell} s + \frac{\theta}{2}) \right) W + \left(\sin(\sqrt{\ell} s + \frac{\theta}{2}) \right) R \quad (s \in \mathbf{R}) \end{aligned} \quad (20)$$

as the desired geodesic, where s is the length parameter and θ is defined by (15). Setting the parameter sequence $\{s_k\}_{k=0,1,2,\dots}$ to be $s_k = \frac{k}{\sqrt{\ell}}\theta$, Eq. (20) with $s = s_k$ indeed provides the search sequence $\{I_G^k(A)\}_{k=0,1,2,\dots}$; $\Psi(s_k) = I_G^k(A)$ (see (16)). To summarize, we have the following.

Theorem 2.1. *The Grover-type search sequence $\{I_G^k(A)\}$ given by (16) for an ordered tuple of multi-qubits is on the geodesic curve $\Psi(s)$ given by (20) in the ESOT, $M_1(2^n, \ell)$.*

As the closing remark of this section, it should be pointed out that in the case of $\ell = 1$, Theorem 2.1 reproduces the result of Miyake and Wadati on Grover's original search sequence on $S^{2^{n+1}-1}$ in [15].

3. Geometry and dynamics of the projected search sequence in the QIS

In this section, the reduced search sequence in the QIS is shown to be on a geodesic with respect to the m -parallel transport, one of the two significant parallel transports of the QIS. The reduced search sequence is derived from the Grover-type sequence $\{I_G^k(A)\}$ along with the reduction of the regular part of the ESOT to the QIS. The reduction method applied here is entirely different from that in Miyake and Wadati [15].

3.1. The QIS

This subsection is devoted to a brief introduction of the quantum information space (QIS), the space of regular density matrices endowed with the quantum SLD-Fisher metric (see also [19] for another brief introduction and [25, 26] for a detailed one).

Let us consider the space of $\ell \times \ell$ density matrices

$$P_\ell = \{\rho \in M(\ell, \ell) \mid \rho^\dagger = \rho, \text{trace } \rho = 1, \rho : \text{positive semidefinite}\}, \quad (21)$$

and its regular part

$$\dot{P}_\ell = \{\rho \in M(\ell, \ell) \mid \rho^\dagger = \rho, \text{trace } \rho = 1, \rho : \text{positive definite}\}, \quad (22)$$

where $M(\ell, \ell)$ denotes the set of $\ell \times \ell$ complex matrices. The tangent space of \dot{P}_ℓ at ρ can be described by

$$T_\rho \dot{P}_\ell = \{\Xi \in M(\ell, \ell) \mid \Xi^\dagger = \Xi, \text{trace } \Xi = 0\}. \quad (23)$$

In this chapter, the regular part \dot{P}_ℓ of P_ℓ plays a central role, while P_ℓ is usually dealt with as the quantum information space. A plausible account for taking \dot{P}_ℓ is that we can be free from dealing with the boundary of P_ℓ which requires us an extra effort especially in differential calculus.

To any tangent vector $\Xi \in T_{\rho}\dot{P}_{\ell}$, the symmetric logarithmic derivate (SLD) is defined to provide the Hermitian matrix $\mathcal{L}_{\rho}(\Xi) \in M(\ell, \ell)$ subject to

$$\frac{1}{2} \{ \rho \mathcal{L}_{\rho}(\Xi) + \mathcal{L}_{\rho}(\Xi) \rho \} = \Xi \quad (\Xi \in T_{\rho}\dot{P}_{\ell}). \quad (24)$$

The quantum SLD-Fisher metric, denoted by $((\cdot, \cdot))^{QF}$, is then defined to be

$$((\Xi, \Xi'))_{\rho}^{QF} = \frac{1}{2} \text{trace} [\rho (L_{\rho}(\Xi) L_{\rho}(\Xi') + L_{\rho}(\Xi') L_{\rho}(\Xi))] \quad (\Xi, \Xi' \in T_{\rho}\dot{P}_{\ell}) \quad (25)$$

(see [25, 26]), which plays a central role in quantum information theory.

A more explicit expression of $((\cdot, \cdot))^{QF}$ is given in what follows. Let $\rho \in \dot{P}_{\ell}$ be expressed as

$$\begin{aligned} \rho &= h \Theta h^{\dagger}, \quad h \in U(l) \\ \Theta &= \text{diag}(\theta_1, \dots, \theta_{\ell}) \quad \text{with} \quad \text{trace } \Theta = 1, \quad \theta_k > 0 \quad (k = 1, 2, \dots, \ell), \end{aligned} \quad (26)$$

where $U(l)$ denotes the group of $\ell \times \ell$ unitary matrices,

$$U(l) = \{h \in M(\ell, \ell) \mid h^{\dagger} h = I_{\ell}\}, \quad (27)$$

and I_{ℓ} the identity matrix of degree- ℓ . On expressing $\Xi \in T_{\rho}\dot{P}_{\ell}$ as

$$\Xi = h \chi h^{\dagger} \quad (28)$$

with $h \in U(l)$ in (26), the SLD $\mathcal{L}_{\rho}(\Xi)$ to $\Xi \in T_{\rho}\dot{P}_{\ell}$ takes an explicit expression [19]

$$(h^{\dagger} \mathcal{L}_{\rho}(\Xi) h)_{jk} = \frac{2}{\theta_j + \theta_k} \chi_{jk} \quad (j, k = 1, 2, \dots, \ell). \quad (29)$$

Putting (26)-(29) into (25), we have

$$((\Xi, \Xi'))_{\rho}^{QF} = 2 \sum_{j,k=1}^{\ell} \frac{\bar{\chi}_{jk} \chi'_{jk}}{\theta_j + \theta_k} \quad (30)$$

[19], where $\Xi' \in T_{\rho}\dot{P}_{\ell}$ is expressed as

$$\Xi' = h \chi' h^{\dagger}. \quad (31)$$

The space of $\ell \times \ell$ regular density matrices, \dot{P}_{ℓ} , endowed with the quantum SLD-Fisher metric $((\cdot, \cdot))^{QF}$ defined above is what we are referring to as the quantum information space (QIS) in the present chapter, which will be denoted also as the pair $(\dot{P}_{\ell}, ((\cdot, \cdot))^{QF})$ henceforth.

3.2. Geometric reduction of the regular part of the ESOT to the QIS

We move to show how the regular part, denoted by $\dot{M}_1(2^n, \ell)$, of the ESOT is reduced to the QIS through the geometric way, where $M_1(2^n, \ell)$ is defined to be

$$\dot{M}_1(2^n, \ell) = \{\Phi \in M_1(2^n, \ell) \mid \text{rank } \Phi = \ell\}. \quad (32)$$

A key to the reduction is the $U(2^n)$ action on $M_1(2^n, \ell)$,

$$\alpha_g : \Phi \in M_1(2^n, \ell) \mapsto g\Phi \in M_1(2^n, \ell) \quad (g \in U(2^n)), \quad (33)$$

where $U(2^n)$ stands for the group of $2^n \times 2^n$ unitary matrices,

$$U(2^n) = \{g \in M(2^n, 2^n) \mid g^\dagger g = I_{2^n}\}, \quad (34)$$

with $M(2^n, 2^n)$ denoting the set of $2^n \times 2^n$ complex matrices and I_{2^n} the identity matrix of degree- 2^n . The $U(2^n)$ action (33) is well-defined also on $\dot{M}_1(2^n, \ell)$ since it leaves $\dot{M}_1(2^n, \ell)$ invariant; $\alpha_g(\dot{M}_1(2^n, \ell)) = \dot{M}_1(2^n, \ell)$.

The $U(2^n)$ action given above provides us with the equivalence relation \sim both on $M_1(2^n, \ell)$ and on $\dot{M}_1(2^n, \ell)$;

$$\begin{aligned} \Phi \sim \Phi' & \quad \text{if and only if} \quad \exists g \in U(2^n) \text{ s.t. } \alpha_g \Phi = \Phi' \\ & \quad (\Phi, \Phi' \in M, M = M_1(2^n, \ell), \dot{M}_1(2^n, \ell)). \end{aligned} \quad (35)$$

The subset of M defined by

$$[\Phi] = \{\Phi' \in M \mid \Phi \sim \Phi'\} \quad (M = M_1(2^n, \ell), \dot{M}_1(2^n, \ell)) \quad (36)$$

is called the equivalence class whose representative is $\Phi \in M$ ($M = M_1(2^n, \ell), \dot{M}_1(2^n, \ell)$). Note that $[\Phi] = [\Phi']$ holds true if and only if $\Phi \sim \Phi'$. The collection of the equivalence classes is called the quotient space, denoted by M/\sim , of M by \sim ($M = M_1(2^n, \ell), \dot{M}_1(2^n, \ell)$).

To describe a geometric structure of the quotient spaces, M/\sim ($M = M_1(2^n, \ell), \dot{M}_1(2^n, \ell)$), let us introduce the group of $(2^n - \ell) \times (2^n - \ell)$ unitary matrices,

$$U(2^n - \ell) = \{\kappa \in M(2^n - \ell, 2^n - \ell) \mid \kappa^\dagger \kappa = I_{2^n - \ell}\}, \quad (37)$$

with $M(2^n - \ell, 2^n - \ell)$ denoting the set of $(2^n - \ell) \times (2^n - \ell)$ complex matrices and $I_{2^n - \ell}$ the identity matrix of degree- $(2^n - \ell)$. We have the following lemma [19].

Lemma 3.1. *The quotient space $M_1(2^n, \ell) / \sim$ is realized as P_ℓ defined by (21), where the projection of $M_1(2^n, \ell)$ to P_ℓ is given by*

$$\pi^{(n, \ell)} : \Phi \in M_1(2^n, \ell) \mapsto \frac{1}{\ell} \Phi^\dagger \Phi \in P_\ell. \quad (38)$$

Similarly, the quotient space $\dot{M}_1(2^n, \ell) / \sim$ is realized as \dot{P}_ℓ defined by (22). The projection is given by $\pi^{(n, \ell)}$ restricted to $\dot{M}_1(2^n, \ell)$. The $\dot{M}_1(2^n, \ell)$ admits the fibered manifold structure with the fiber $U(2^n) / U(2^n - \ell)$. Namely, the inverse image $(\pi^{(n, \ell)})^{-1}(\rho) = \{\Phi \in \dot{M}_1(2^n, \ell) \mid \pi^{(n, \ell)}(\Phi) = \rho\}$ of any $\rho \in \dot{P}_\ell$ is diffeomorphic to $U(2^n) / U(2^n - \ell)$.

Note that the fibered manifold structure of $\dot{M}_1(2^n, \ell)$ allows us to proceed differential calculus on $\dot{P}_\ell \cong \dot{M}_1(2^n, \ell) / \sim$ freely, while not on P_ℓ due to a collapse of the fibered structure on the boundary.

What is an intuitive interpretation of the quotient spaces, $M_1(2^n, \ell) / \sim$ and $\dot{M}_1(2^n, \ell) / \sim$? Let us consider any pair of points Φ and $\Phi' = \exists g \Phi$ in M ($M = M_1(2^n, \ell), \dot{M}_1(2^n, \ell)$). Then since $g \in U(2^n)$, the inner products between column vectors in Φ (see (6)) are kept invariant under α_g :

$$\langle \phi'_j, \phi'_k \rangle = \langle g\phi_j, g\phi_k \rangle = \langle \phi_j, \phi_k \rangle \quad (j, k = 1, 2, \dots, \ell). \quad (39)$$

This implies that the relative configuration of column vectors (namely multi-qubits) is kept invariant under the $U(2^n)$ action. Hence each of the quotient spaces of $M_1(2^n, \ell)$ and of $\dot{M}_1(2^n, \ell)$, is understood to be a space of relative configurations of multi-qubits [19]. We wish to explain the relative configurations in more detail in a very simple setting-up with $n = 6$ and $\ell = 6$. Let us consider the set, $S = \{A, B, \dots, Z, a, \dots, z, 0, 1, \dots, 9, ", ", ", "\}$, consisting of the capital Roman letters, the small ones, the arabic digits, a comma and a period. The correspondence of the 2^6 -computational basis vectors, $e(x)$ ($x = 1, \dots, 2^6 = 64$) (see subsubsec. 2.2.2), to the elements of S starts from $e(1) \mapsto A$ in ascending order. Then, under the equivalence relation \sim defined by (35), the word 'Search' is identified with 'Vhduk' since the latter can be obtained from the former through α_g with the three-step shift matrix g . On choosing $g \in U(2^n)$ to exchange the capital letters for the small ones, 'Search' is identified with 'sEARCH'.

We are now in a position to show that the QIS $(\dot{P}_\ell, ((\cdot, \cdot))^{QF})$ is a very natural outcome of the reduction of $(\dot{M}_1(2^n, \ell), ((\cdot, \cdot))^{ESOT})$. Note here that the Riemannian metric $((\cdot, \cdot))^{ESOT}$ of $M_1(2^n, \ell)$ naturally turns out to be a metric of $\dot{M}_1(2^n, \ell)$ under the restriction $\Phi \in \dot{M}_1(2^n, \ell)$, so that we apply the same symbol, $((\cdot, \cdot))^{ESOT}$, to the metric of $\dot{M}_1(2^n, \ell)$. A crucial key is the direct-sum decomposition of the tangent space,

$$T_\Phi \dot{M}_1(2^n, \ell) = \{X \in M(2^n, \ell) \mid \Re(\text{trace } \Phi^\dagger X) = 0\} \quad (\Phi \in \dot{M}_1(2^n, \ell)), \quad (40)$$

of $\dot{M}_1(2^n, \ell)$ at Φ , which is associated with the fibered-manifold structure of $\dot{M}_1(2^n, \ell)$ mentioned of in Lemma 3.1. Note that $T_\Phi \dot{M}_1(2^n, \ell)$ is identical with $T_\Phi M_1(2^n, \ell)$ if $\Phi \in \dot{M}_1(2^n, \ell)$.

Let us consider the pair of subspaces, $\text{Ver}(\Phi)$ and $\text{Hor}(\Phi)$, of $T_\Phi \dot{M}_1(2^n, \ell)$, which are defined by

$$\text{Ver}(\Phi) = \{X \in T_\Phi \dot{M}_1(2^n, \ell) \mid X = \zeta \Phi, \zeta \in u(2^n)\} \quad (41)$$

and

$$\text{Hor}(\Phi) = \{X \in T_\Phi \dot{M}_1(2^n, \ell) \mid ((X', X))_\Phi^{ESOT} = 0, X' \in \text{Ver}(\Phi)\}. \quad (42)$$

The $u(2^n)$ is the Lie algebra of $U(2^n)$ consisting of all the $2^n \times 2^n$ anti-Hermitian matrices,

$$u(2^n) = \{\zeta \in M(2^n, 2^n) \mid \zeta^\dagger = -\zeta\}. \quad (43)$$

The $\text{Ver}(\Phi)$ and $\text{Hor}(\Phi)$ are often called the vertical subspace and the horizontal subspace of $T_\Phi \dot{M}_1(2^n, \ell)$, respectively. The $\text{Ver}(\Phi)$ is understood to be the tangent space at Φ of the fiber space,

$$U(2^n) \cdot \Phi = \{\Phi' \in \dot{M}_1(2^n, \ell) \mid \Phi' = \alpha_g(\Phi), g \in U(2^n)\}, \quad (44)$$

passing Φ , and $\text{Hor}(\Phi)$ to be the subspace of $T_\Phi \dot{M}_1(2^n, \ell)$ normal to $\text{Ver}(\Phi)$ with respect to $((\cdot, \cdot))_\Phi^{ESOT}$. Thus the orthogonal direct-sum decomposition

$$T_\Phi \dot{M}_1(2^n, \ell) = \text{Ver}(\Phi) \oplus \text{Hor}(\Phi) \quad (\Phi \in \dot{M}_1(2^n, \ell)) \quad (45)$$

with respect to the inner product $((\cdot, \cdot))_\Phi^{ESOT}$ is allowed to the tangent space $T_\Phi \dot{M}_1(2^n, \ell)$.

On using (45), the horizontal lift of any tangent vector of the QIS is given as follows: Let us fix $\rho \in \dot{P}_\ell$ arbitrarily and any $\Phi \in \dot{M}_1(2^n, \ell)$ subject to $\pi^{(n,l)}(\Phi) = \rho$. For any tangent vector $\Xi \in T_\rho \dot{P}_\ell$ (see (23)), the horizontal lift of Ξ at Φ is the unique tangent vector, denoted by Ξ^* , in $T_\Phi \dot{M}_1(2^n, \ell)$ that satisfies

$$(\pi^{(n,l)})_{*\Phi}(\Xi^*) = \Xi \quad \text{and} \quad \Xi^* \in \text{Hor}(\Phi), \quad (46)$$

where $(\pi^{(n,l)})_{*\Phi} : T_\Phi \dot{M}_1(2^n, \ell) \rightarrow T_{\pi^{(n,l)}(\Phi)} \dot{P}_\ell = T_\rho \dot{P}_\ell$ is the differential map of $\pi^{(n,l)}$ at Φ . For a detail of differential maps, see Appendix 2. Recalling, further, the orthogonal direct-sum decomposition (45), we can understand that the horizontal lift Ξ^* of $\Xi \in T_\rho \dot{P}_\ell$ is of minimum length among vectors, say X s, in $T_\Phi \dot{M}_1(2^n, \ell)$ subject to $(\pi^{(n,l)})_{*\Phi}(X) = \Xi$.

Accordingly, the horizontal lift (46) and the Riemannian metric $((\cdot, \cdot))_\Phi^{ESOT}$ are put together to give rise the Riemannian metric, denoted by $((\cdot, \cdot))_\rho^{RS}$, of \dot{P}_ℓ , which is defined to satisfy

$$((\Xi, \Xi'))_\rho^{RS} = ((\Xi^*, \Xi'^*))_\Phi^{ESOT} \quad \text{with} \quad \pi^{(n,l)}(\Phi) = \rho \quad (\Xi, \Xi' \in T_\rho \dot{P}_\ell, \rho \in \dot{P}_\ell). \quad (47)$$

The Ξ^* and Ξ'^* are the horizontal lift at Φ of Ξ and Ξ' , respectively, and the superscript RS implies that $((\cdot, \cdot))^{RS}$ is the Riemannian metric of \dot{P}_ℓ looked upon as the reduced space. Note here that the rhs of (47) is well-defined owing to the invariance,

$$((\alpha_g)_*\Phi(X), (\alpha_g)_*\Phi(X'))_{\alpha_g(\Phi)}^{ESOT} = ((X, X'))_\Phi^{ESOT} \quad (X, X' \in T_\Phi \dot{M}_1(2^n, \ell), \Phi \in \dot{M}_1(2^n, \ell)), \quad (48)$$

of $((\cdot, \cdot))^{ESOT}$ and the equivariance,

$$\text{Hor}(\alpha_g(\Phi)) = (\alpha_g)_*\Phi(\text{Hor}(\Phi)) \quad (\Phi \in \dot{M}_1(2^n, \ell)), \quad (49)$$

of $\text{Hor}(\Phi)$ under the $U(2^n)$ action, where $(\alpha_g)_*\Phi$ is the differential map of α_g at Φ (see (33) and Appendix 2). In view of (47), we say that the projection $\pi^{(n,l)} : \dot{M}_1(2^n, \ell) \rightarrow \dot{P}_\ell$ is a Riemannian submersion [29].

We have the following on the coincidence of $((\cdot, \cdot))^{RS}$ and $((\cdot, \cdot))^{QF}$ [19].

Theorem 3.2. *The Riemannian metric $((\cdot, \cdot))^{RS}$ defined by (47) to make $\pi^{(n,l)}$ the Riemannian submersion coincides with the SLD-Fisher metric defined by (25) up to the constant multiple 4; $4((\cdot, \cdot))^{RS} = ((\cdot, \cdot))^{QF}$.*

On closing this subsection, a comparison between the reduction here and the one by Miyake and Wadati is made. The reduction methods are essentially different since our reduction is made under 'left' $U(2^n)$ action while 'right' $U(1)$ action is dealt with in [15]. The resultant spaces, namely the reduced spaces, are of course different mutually.

3.3. Geodesic property of the reduced search sequence

We are now in a position to show that the reduced search sequence $\{\pi^{(n,l)}(I_G^k(A))\}$ in the QIS is on an m -geodesic, a geodesic with respect to the m -parallel transport, of the QIS.

3.3.1. Intuitive example of parallel transports: The Euclidean case

Let us start with thinking of parallel transport in the 3-dimensional Euclidean space \mathbf{R}^3 , the conventional model space not only for basic mathematics and physics but for our daily life. In \mathbf{R}^3 , the notion of *parallel* seems to be a trivial one, which is usually not presented in a differential geometric framework to those who are not familiar to differential geometry. As minimum geometric knowledge necessary in this subsection, we introduce below a coordinate expression of tangent vectors. As known well, \mathbf{R}^3 is endowed with the Cartesian coordinates $y = (y_1, y_2, y_3)^T$ valid globally in \mathbf{R}^3 . The tangent vectors at any point $p \in \mathbf{R}^3$ can be understood to be the infinitesimal limit of displacements from p . The tangent vector understood to be the displacement-limit $\lim_{\epsilon \rightarrow 0} (p + \epsilon e^{(j)})$ is then written as $\left(\frac{\partial}{\partial y_j}\right)_p$, where $e^{(j)}$ s are the orthonormal vectors along the j -th axis ($j = 1, 2, 3$). The account for the expression $\left(\frac{\partial}{\partial y_j}\right)_p$ is that we have $\lim_{\epsilon \rightarrow 0} F(p + \epsilon e^{(j)}) = \left(\frac{\partial F}{\partial y_j}\right)(p)$ for any differentiable functions F . The parallel transport is defined to be a rule to transfer the tangent vectors at $p \in \mathbf{R}^3$ to those at another $p' \in \mathbf{R}^3$, which is of course have to be subject to several

mathematical claims not gotten in detail here: The well-known parallel transport in the conventional Euclidean space, \mathbf{R}^3 , is clearly expressed as

$$\sum_{j=1}^3 v_j \left(\frac{\partial}{\partial y_j} \right)_p \in T_p \mathbf{R}^3 \mapsto \sum_{j=1}^3 v_j \left(\frac{\partial}{\partial y_j} \right)_{p'} \in T_{p'} \mathbf{R}^3 \quad (v_j \in \mathbf{R}). \quad (50)$$

An important note is that parallel transports in general differentiable manifolds (including the familiar sphere S^2) are defined in terms of curves specifying the way of point-translations (see Appendix 3 for the case of S^2). The Euclidean case (50) is hence understood to be a curve-free case.

Once the parallel transport (50) is given to \mathbf{R}^3 , geodesics in \mathbf{R}^3 are defined to be autoparallel curves: Let $\gamma(t)$ ($t \in \exists[t_0, t_1] \in \mathbf{R}$) be a curve in \mathbf{R}^3 , whose tangent vector at $t = \tau$ of the curve is given by $\frac{d\gamma}{dt}(\tau)$. The curve $\gamma(t)$ is autoparallel if the tangent vector at each point is equal to the parallel transport of the initial tangent vector $\frac{d\gamma}{dt}(t_0)$. Accordingly, every autoparallel curve turns out to be a straight line or its segment as widely known. Geodesics in \mathbf{R}^3 discussed here have the shortest-path property with respect to the Euclidean metric since the parallel transport (50) leaves the metric invariant. Note that parallel transports other than (50) can exist whose geodesics of course lose the shortest-path property with respect to the Euclidean metric.

3.3.2. The m -parallel transport in the QIS

We move on to the m -parallel transport in the QIS. Fortunately, the m -parallel transport can be described in a similar setting-up to that for the transport (50). Let us start with the space of $\ell \times \ell$ complex matrices, $M(\ell, \ell)$, that includes the QIS, \dot{P}_ℓ , as a subset. The $M(\ell, \ell)$ admits the matrix-entries as global (complex) coordinates like the Cartesian coordinates of \mathbf{R}^3 . The tangent space $T_\rho \dot{P}_\ell$ at $\rho \in \dot{P}_\ell$ can be identified with the set of $\ell \times \ell$ traceless Hermitian matrices (see (23)), which can be dealt with in a similar way to the Euclidean parallel transport setting-up. Indeed, in view of the definition (22), \dot{P}_ℓ is understood to be a fragment of an affine subspace of $M(\ell, \ell)$. Hence the tangent space $T_\rho \dot{P}_\ell$ at every $\rho \in \dot{P}_\ell$ admits the structure (23), which is looked upon as a linear subspace of $M(\ell, \ell)$.

According to quantum information theory [25, 26], the m -parallel transport is written in a simple form

$$\Xi \in T_\rho \dot{P}_\ell \mapsto \Xi \in T_{\rho'} \dot{P}_\ell. \quad (51)$$

The geodesic from ρ_0 to ρ_1 with respect to the m -parallel transport is therefore characterized as an autoparallel curve,

$$\rho^{mg}(t) = (1 - t)\rho_0 + t\rho_1 \quad (0 \leq t \leq 1), \quad (52)$$

which takes a very similar form to the Euclidean case. The parameter t in (52) can be chosen arbitrarily up to affine transformations; $t \rightarrow at + b$ ($a, b \in \mathbf{R}$).

A very important remark should be made here. From a very naive viewpoint, the geodesic $\rho^{mg}(t)$ in (52) looks like ‘straight’. This is *not true*, however, since the QIS is not Euclidean due to the SLD-Fisher metric $((\cdot, \cdot))^{QF}$ endowed in the QIS. Precisely, $\rho^{mg}(t)$ has to be understood to be ‘curved’ in the QIS.

3.3.3. The reduced search sequence is on a geodesic

We are at the final stage to show that the search sequence $\{I_G^k(A)\}$ is reduced through $\pi^{(n,l)}$ on an m -geodesic, a geodesic with respect to the m -transport, of the QIS. We start with calculating the reduced sequence $\{\pi^{(n,l)}(I_G^k(A))\}$ explicitly. Though the initial states A for the search sequence $\{I_G^k(A)\}$ is out of the range $M_1(2^n, \ell)$, we apply $\pi^{(n,l)}$ to $\{I_G^k(A)\}$ in the manner (38). Since we have

$$(W^\dagger W)_{jh} = \delta_{jh} \quad (j, h = 1, 2, \dots, \ell), \quad (53)$$

$$(R^\dagger R)_{jh} = \frac{2^n - 2 + \delta_{jh}}{2^n - 1} \quad (j, h = 1, 2, \dots, \ell), \quad (54)$$

$$(W^\dagger R)_{jh} = (R^\dagger W)_{jh} = \frac{1 - \delta_{jh}}{\sqrt{2^n - 1}} \quad (j, h = 1, 2, \dots, \ell), \quad (55)$$

$$(A^\dagger A)_{jh} = 1 \quad (j, h = 1, 2, \dots, \ell), \quad (56)$$

the reduced search sequence $\{\pi^{(n,l)}(I_G^k(A))\}$ takes the form

$$\begin{aligned} & \pi^{(n,l)}(I_G^k(A)) \\ &= \frac{1}{\ell} \left\{ \left(\sin\left(k + \frac{1}{2}\right)\theta \right) W + \left(\cos\left(k + \frac{1}{2}\right)\theta \right) R \right\}^\dagger \left\{ \left(\sin\left(k + \frac{1}{2}\right)\theta \right) W + \left(\cos\left(k + \frac{1}{2}\right)\theta \right) R \right\} \\ &= \frac{1}{\ell} \left(\sin^2\left(k + \frac{1}{2}\right)\theta \right) W^\dagger W + \frac{1}{\ell} \left(\cos^2\left(k + \frac{1}{2}\right)\theta \right) R^\dagger R \\ & \quad + \frac{1}{\ell} \left(\cos\left(k + \frac{1}{2}\right)\theta \right) \left(\sin\left(k + \frac{1}{2}\right)\theta \right) (R^\dagger W + W^\dagger R) \\ &= (1 - \tau_k) \left(\frac{1}{\ell} A^\dagger A \right) + \tau_k \left(\frac{1}{\ell} I_\ell \right) \quad (k = 1, 2, \dots) \end{aligned} \quad (57)$$

with

$$\begin{aligned} \tau_k &= 1 - \frac{2^n - 2}{2^n - 1} \cos^2\left(\left(k + \frac{1}{2}\right)\theta\right) \\ & \quad - \frac{2}{\sqrt{2^n - 1}} \left(\cos\left(k + \frac{1}{2}\right)\theta \right) \left(\sin\left(k + \frac{1}{2}\right)\theta \right) \quad (k = 1, 2, \dots), \end{aligned} \quad (58)$$

where I_ℓ stands for the $\ell \times \ell$ identity matrix. The δ_{jh} in (53)-(55) indicates Kronecker’s delta and θ is defined already to satisfy (15).

The expression (52) and (57) are put together to inspire us to consider the m -geodesic in the QIS of the form

$$\rho^G(t) = (1-t) \left(\frac{1}{\ell} A^\dagger A \right) + t \left(\frac{1}{\ell} I_\ell \right) \quad (\varepsilon \leq t \leq 1) \quad (59)$$

where ε is a sufficiently small positive number subject to $0 < \varepsilon < \tau_1$ (see (58) with $k = 1$ for τ_1). Note here that the reduction $\frac{1}{\ell} A^\dagger A \in \mathcal{P}_\ell$ of the initial states $A \in \mathcal{M}_1(2^n, \ell)$ turns out to be placed as the limit point of the geodesic $\rho^G(t)$ in the sense that

$$\lim_{\varepsilon \rightarrow +0} \rho^G(\varepsilon) = \frac{1}{\ell} A^\dagger A. \quad (60)$$

Combining (59) with (57) and (58), we have

$$\pi^{(n,l)}(I_G^k(A)) = \rho^G(\tau_k) \quad (k = 1, 2, \dots, K_n) \quad (61)$$

where K_n is the integer nearest to $\frac{\pi}{4}\sqrt{2^n} - \frac{1}{2}$. The reduction of the initial state is placed at the limit point of the m -geodesic $\rho^G(t)$ in the sense (60). Thus we have the following outlined as Main Theorem in Sec. 1:

Theorem 3.3. *Through the reduction of the regular part, $\mathcal{M}_1(2^n, \ell)$, of the extended space of ordered tuples of multi-qubits (ESOT) to the quantum information space (QIS), \mathcal{P}_ℓ , the reduced search sequence $\{\pi^{(n,l)}(I_G^k(A))\}_{k=1,2,\dots,K_n}$ is on the m -geodesic $\rho^G(t)$ of the QIS given by (59).*

4. Concluding remarks

We have studied the Grover-type search sequence for an ordered tuple of multi-qubits. The search sequence itself is shown to be on a geodesic with respect to the Levi-Civita parallel transport in the ESOT. Further, the reduced search sequence in the QIS is shown to be on a geodesic with respect to the m -parallel transport in the QIS. The m -geodesics do not have the shortest-path property but they are very important geodesics in the QIS together with those with respect to e -parallel transport. The geometric reduction method applied this chapter is entirely different from the method in Miyake and Wadati [15].

A significance of this chapter is the discovery of a novel geometric pathway that connects directly the search sequence in the ESOT with an m -geodesic in the QIS. According to a crucial role of the m -geodesics and the e -geodesics together with their mutual duality, the pathway will be a key to further studies on the search in the ESOT from the quantum information geometry viewpoint. Further, since the QIS is well-known to be the stage for describing dynamics of quantum-state ensembles of quantum systems [2, 3], the pathway shown in this chapter will be of good use to connect the search in the ESOT with dynamics of a certain quantum system.

An direct application of the search in the ESOT is not yet found: However, if a problem with a strong relation to relative ordering of data (see around Eq. (39)) exists, our search will be worth applying to the problem.

On closing this section, three questions are posed below, which would be of interest from the viewpoint of the expected benefits listed in Sec. 1.

- 1 In view of the results in this chapter, we are able to clarify that the ‘Grover search orbit’ given by a continuous-time version of (16) is an m -geodesic. A question thereby arises as to ‘Is it possible to characterize the m -geodesics by orbits of a certain dynamical system on $M_1(2^n, \ell)$?’ To this direction, a variation of the free particle system on $M_1(2^n, \ell)$ would be a candidate (Benefits 1 and 2).
- 2 Accordingly, another question would be worth posed: ‘Is it possible to characterize the e -geodesics by orbits of a certain dynamical system on $M_1(2^n, \ell)$?’ (Benefits 1 and 2).
- 3 The celebrated fact on the duality between the m -transport and the e -transport (see [25] and [26]) may provide us with a further question: ‘If there exists a pair of dynamical systems on $M_1(2^n, \ell)$ whose reduced orbits characterize the m -geodesics and e -geodesics respectively, which kind of relation does it exist between those systems?’ (Benefit 3).

Acknowledgements

The author would like to thank the editor for his valuable comments to improve the earlier manuscript of this chapter. This work is partly supported by Special Fund for Strategic Research No. 4 (2011) in Future University Hakodate.

Appendices

Appendix 1. Glossary of symbols and notation

Acronyms

- ESOT: The abbreviation of the extended space of ordered tuples of multi-qubits, which is denoted by $M_1(2^n, \ell)$ (see Eq. (5)).
- QIS: The abbreviation of the quantum information space, which is realized as \dot{P}_ℓ , the set of $\ell \times \ell$ positive definite Hermitian matrices with unit trace, endowed with the quantum SLD-Fisher metric $((\cdot, \cdot))^{QF}$ (see Eq. (22) for \dot{P}_ℓ , and (24)-(31) for $((\cdot, \cdot))^{QF}$).
- SLD: The abbreviation of the symmetric logarithmic derivative (see Eq. (24)).

Sets and spaces

- $\text{Hor}(\Phi)$: The horizontal subspace of $T_\Phi \dot{M}_1(2^n, \ell)$ (see Eq. (42) with (41)).
- $M(\ell, \ell)$: The set of $\ell \times \ell$ complex matrices.
- $M(2^n, \ell)$: The set of $2^n \times \ell$ complex matrices.
- $M(2^n, 2^n)$: The set of $2^n \times 2^n$ complex matrices.
- $M(2^n - \ell, 2^n - \ell)$: The set of $(2^n - \ell) \times (2^n - \ell)$ complex matrices.

- $M_1(2^n, \ell)$: The subset of $M(2^n, \ell)$ consisting of $2^n \times \ell$ complex matrices with unit norm referred to as the extended space of ordered tuples of multi-qubits (see Eq. (5)), which is abbreviated to the ESOT.
- $\dot{M}_1(2^n, \ell)$: The subset of $M_1(2^n, \ell)$ consisting of the elements of $M_1(2^n, \ell)$ with the maximum rank equal to ℓ (see Eq. (32)).
- $M_1^{OT}(2^n, \ell)$: The subset of $M_1(2^n, \ell)$ consisting of $2^n \times \ell$ complex matrices whose columns are of unit length (see Eq. (7)).
- P_ℓ : The set of $\ell \times \ell$ positive semidefinite Hermitian matrices with unit trace; the space of $\ell \times \ell$ density matrices (see Eq. (21)).
- \dot{P}_ℓ : The set of $\ell \times \ell$ positive definite Hermitian matrices with unit trace; the space of $\ell \times \ell$ regular density matrices (see Eq. (22)).
- $T_\Phi M_1(2^n, \ell)$: The tangent space of $M_1(2^n, \ell)$ at $\Phi \in M_1(2^n, \ell)$ (see Eq. (17)).
- $T_\Phi \dot{M}_1(2^n, \ell)$: The tangent space of $\dot{M}_1(2^n, \ell)$ at $\Phi \in \dot{M}_1(2^n, \ell)$ (see Eq. (40)), which is identical with $T_\Phi M_1(2^n, \ell)$ if $\Phi \in \dot{M}_1(2^n, \ell)$.
- $T_\rho \dot{P}_\ell$: The tangent space of \dot{P}_ℓ at a point $\rho \in \dot{P}_\ell$ (see Eq. (23)).
- $U(1)$: The group of $\ell \times \ell$ unitary matrices (see Eq. (27)).
- $U(2^n)$: The group of $2^n \times 2^n$ unitary matrices (see Eq. (34)).
- $u(2^n)$: The Lie algebra of the group $U(2^n)$ (see Eq. (43)).
- $U(2^n - \ell)$: The group of $(2^n - \ell) \times (2^n - \ell)$ unitary matrices (see Eq. (37)).
- $\text{Ver}(\Phi)$: The vertical subspace of $T_\Phi \dot{M}_1(2^n, \ell)$ (see Eq. (41)).

Maps, operators and transformations

- α_g : The unitary transformation of $M_1(2^n, \ell)$ associated with $g \in U(2^n)$ (see Eq. (33)).
- $(\alpha_g)_*\Phi$: The differential map of α_g at $\Phi \in \dot{M}_1(2^n, \ell)$. See also Appendix 2 for the definition.
- I_A : The unitary transformation of $M_1(2^n, \ell)$ defined by (11).
- I_G : The unitary transformation composed of $-I_A$ and I_W (see Eq. (10)).
- I_W : The unitary transformation of $M_1(2^n, \ell)$ defined by (12).
- \mathcal{L}_ρ : The symmetric logarithmic derivative (SLD) (see Eqs. (24) and (29)).
- $\pi^{(n, l)}$: The projection of $\dot{M}_1(2^n, \ell)$ to \dot{P}_ℓ (the QIS) (see Eq. (38)).
- $(\pi^{(n, l)})_*\Phi$: The differential map of $\pi^{(n, l)}$ at $\Phi \in \dot{M}_1(2^n, \ell)$. See also Appendix 2 for the definition.
- T : The transpose operation to vectors and matrices.
- † : The Hermitian conjugate operation to vectors and matrices.

Metrics

- $((\cdot, \cdot))^{ESOT}$: The Riemannian metric of the ESOT and of its regular part (see Eq. (18)).
- $((\cdot, \cdot))^{QF}$: The quantum SLD-Fisher metric of the QIS (see Eqs. (25) and (30)).
- $((\cdot, \cdot))^{RS}$: The Riemannian metric of the QIS other than $((\cdot, \cdot))^{QF}$, that makes the projection $\pi^{(n, l)}$ a Riemannian submersion (see Eq. (47) with (46)).

Others

- A : The matrix expressing the initial state for the Grover-type search in the ESOT (see Eq. (9)).
- R : The matrix with which forms an orthonormal basis of the subspace consisting of all the superpositions of A and W (see Eq. (13)).
- W : The matrix expressing the target state (namely the marked state) for the Grover-type search in the ESOT (see Eq. (9)).
- \sim : The equivalence relation both on $M_1(2^n, \ell)$ and on $\dot{M}_1(2^n, \ell)$ (see Eq. (35)).

Appendix 2. Differential maps

We here give a detailed explanation of differential maps, $(\pi^{(n,l)})_{*\Phi}$ and $(\alpha_g)_{*\Phi}$. For any $X \in T_\Phi M_1(2^n, \ell)$, we can always find a curve $\gamma(t)$ ($-\tau < t < \tau$, $\tau > 0$) on $M_1(2^n, \ell)$ subject to $\gamma(0) = \Phi$ and $\frac{d\gamma}{dt}(0) = X$. The differential map $(\pi^{(n,l)})_{*\Phi}$ is defined to be

$$(\pi^{(n,l)})_{*\Phi}(X) = \left. \frac{d}{dt} \right|_{t=0} \pi^{(n,l)}(\gamma(t)), \quad (62)$$

which turns out to take the explicit form

$$(\pi^{(n,l)})_{*\Phi}(X) = \frac{1}{\ell}(X^\dagger \Phi + \Phi^\dagger X). \quad (63)$$

The differential map $(\alpha_g)_{*\Phi}$ of α_g at Φ is defined in the same way: On the same setting-up to the curve $\gamma(t)$ with $X \in T_\Phi M_1(2^n, \ell)$, the $(\alpha_g)_{*\Phi}$ is defined by

$$(\alpha_g)_{*\Phi}(X) = \left. \frac{d}{dt} \right|_{t=0} \alpha_g(\gamma(t)), \quad (64)$$

which yields

$$(\alpha_g)_{*\Phi}(X) = gX. \quad (65)$$

Appendix 3. The standard parallel transport in S^2

In this appendix, the standard parallel transport is concisely reviewed. In particular, we present the fact that the transport depends on the choice of the paths connecting a pair of points in S^2 . Below, S^2 is realized as the set,

$$S^2 = \{y \in \mathbf{R}^3 \mid y^T y = 1\}, \quad (66)$$

in the 3-dimensional Euclidean space \mathbf{R}^3 .

Let us fix a pair of distinct points, y_0 and y_1 , in S^2 arbitrarily, which connect by a smooth curve $\gamma(s)$, where s is the length parameter. Namely, the $\gamma(s)$ satisfies

$$\gamma(0) = y_0, \quad \gamma(L) = y_1 \quad (L : \text{the full curve length}). \quad (67)$$

Again, we remark that $\gamma(s)$ takes 3-dimensional vector form. To express tangent vectors of S^2 at $\gamma(s)$, we prepare the orthonormal basis $\{v_1(s), v_2(s)\}$ of $T_{\gamma(s)}S^2$ subject to

$$v_1(s) = \dot{\gamma}(s), \quad v_2(s) = \gamma(s) \times \dot{\gamma}(s) \quad (68)$$

where the overdot $\dot{}$ stands for the derivation by s and \times the vector product operation. We note here that the vector, $\dot{\gamma}(s)$, tangent to $\gamma(s)$ is always of unit length since s is the length parameter. This ensures that the basis $\{v_1(s), v_2(s)\}$ is orthonormal. In terms of the basis $\{v_1(s), v_2(s)\}$, any tangent vector at $\gamma(s)$ can be expressed in the form of linear combination, $c_1 v_1(s) + c_2 v_2(s)$ ($c_1, c_2 \in \mathbf{R}$). Accordingly, the parallel transport along the curve $\gamma(s)$ is understood to be the way of connecting $\{v_1(L), v_2(L)\}$ at $y_1 = \gamma(L)$ to $\{v_1(0), v_2(0)\}$ at $y_0 = \gamma(0)$.

To express the parallel transport concisely, it is of good use to introduce the one-parameter rotation matrix,

$$\Gamma(s) = (v_0(s), v_1(s), v_2(s)), \quad v_0(s) = \gamma(s) \quad (0 \leq s \leq L). \quad (69)$$

On denoting by $P_\gamma(v_1(0), v_2(0))$ the parallel transport of $(v_1(0), v_2(0))$ along the curve $\gamma(s)$, we have

$$(v_1(L), v_2(L)) = P_\gamma(v_1(0), v_2(0)) \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix} \quad (70)$$

with

$$a = - \int_0^L \frac{1}{2} \text{trace} (N \dot{\Gamma}(s) \Gamma(s)^T) ds, \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (71)$$

If we choose $\gamma(s)$ to be a big circle or its segment, the $\gamma(s)$ is autoparallel since a in (70) and (71) vanishes, so that big circles and their segments turn out to be geodesics.

Author details

Yoshio Uwano

Department of Complex and Intelligent Systems, Faculty of Systems Information Science, Future University Hakodate, Kameda Nakano-cho, Hakodate, Japan

References

- [1] Nielsen MA., Chuang IL. Quantum Computation and Quantum Information. Cambridge: Cambridge U.P.; 2000.
- [2] Benenti G., Casati G., Strini G. Principles of Quantum Computation and Information Volume I. Singapore: World Scientific; 2004.
- [3] Benenti G., Casati G., Strini G. Principles of Quantum Computation and Information Volume II. Singapore: World Scientific; 2007.
- [4] Mermin ND. Quantum Computer Science An Introduction. Cambridge: Cambridge U.P.; 2007.
- [5] Shor PW. Algorithms for quantum computation: discrete logarithms and factoring. In: Proceedings of the 35th Annual Symposium on Foundations of Computer Science. Los Alamitos: IEEE Press; 1994.
- [6] Grover L. A fast quantum mechanical algorithm for database search. In: Proceedings of the 28th Annual ACM Symposium on the Theory of Computing. New York: ACM Press; 1996.
- [7] Grover L. Quantum mechanics helps in searching for a needle in a haystack. Phys. Rev. Lett. 1997; 79 p325-328.
- [8] Gruska J. Quantum Computing. London: McGraw-Hill International; 1999.
- [9] Ambainis A. Guest column: Quantum search algorithms. ACM SIGACT News 2004; 35 p22-33.
- [10] Galindo A., Martin-Delgado MA. Family of Grover's search algorithms. Phys. Rev. A 2000; 62 062303 p1-6.
- [11] Grover LK. A different kind of quantum search. arXiv:quant-ph/0503205v1 2005.
- [12] Grover LK. Fixed-point quantum search. Phys. Rev. Lett. 2005; 95 150501 p1-4.
- [13] Tulsi T., Grover LK., Patel A. A new algorithm for fixed point quantum search. Quantum Info. and Comput. 2006; 6 p483-494.
- [14] arXiv: <http://arXiv.org> (accessed 5 September 2012).
- [15] Miyake A., Wadati M. Geometric strategy for the optimal quantum search. Phys. Rev. A 2001; 64 042317 p1-9.
- [16] Farhi E., Goldstone J., Gutmann S., Sipser M. Quantum computation by adiabatic evolution. arXiv:quant-ph/0001106v1 2000.
- [17] Roland J., Cerf NJ. Quantum search by local adiabatic evolution. Phys. Rev. A 2002; 65 042308 p1-6.

- [18] Rezakhani AT., Abasto DF., Lidar DA., Zanardi P. Intrinsic geometry of quantum adiabatic evolution and quantum phase transitions. *Phys. Rev. A* 2010; 82 012321 p1-16.
- [19] Uwano Y., Hino H., Ishiwatari Y. Certain integrable system on a space associated with a quantum search algorithm. *Physics of Atomic Nuclei* 2007; 70 p784-791.
- [20] Nakamura Y. Completely integrable dynamical systems on the manifolds of Gaussian and multinomial distributions. *Japan J. Indust. Appl. Math.* 1993; 10 p179-189.
- [21] Nakamura Y. Lax pair and fixed point analysis of Karmarkar's projective scaling trajectory for linear programming. *Japan J. Indust. Appl. Math.* 1994; 11 p1-9.
- [22] Nakamura Y. Neurodynamics and nonlinear integrable systems of Lax type. *Japan J. Indust. Appl. Math.* 1994; 11 p11-20.
- [23] Uwano Y., Yuya H. A gradient system on the quantum information space realizing the Karmarkar flow for linear programming – A clue to effective algorithms –. In: Sakaji A., Licata I., Singh J., Felloni S. (eds.) *New Trends in Quantum Information (Special Issue of Electronic Journal of Theoretical Physics)*. Rome: Aracne Editorice; 2010 p257-276.
- [24] Uwano Y., Yuya H. A Hebb-type learning equation on the quantum information space – A clue to a fast principal component analyzer. *Far East Journal of Applied Mathematics* 2010; 47 p149-167.
- [25] Hayashi M. *Quantum Information*. Berlin: Springer-Verlag; 2006.
- [26] Amari S., Nagaoka H. *Methods of Information Geometry*. Providence: American Mathematical Society; 2000.
- [27] Bengtsson I., Życzkowski K. *Geometry of Quantum States*. Cambridge: Cambridge U.P.; 2006.
- [28] Nakahara M. *Geometry, Topology and Physics*. Bristol: IOP Publishing; 1990.
- [29] Montgomery R.A. *Tour of Subriemannian Geometries, Their Geodesics and Applications*. Providence: American Mathematical Society; 2002.