

We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

186,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com



Coherent Upper Conditional Previsions Defined by Hausdorff Outer Measures to Forecast in Chaotic Dynamical Systems

Serena Doria

Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/50326>

1. Introduction

Coherent conditional previsions and probabilities are tools to model and quantify uncertainties; they have been investigated in de Finetti [3], [4], Dubins [10] Regazzini [13], [14] and Williams [20]. Separately coherent upper and lower conditional previsions have been introduced in Walley [18], [19] and models of upper and lower conditional previsions have been analysed in Vicig et al. [17] and Miranda and Zaffalon [12].

In the subjective probabilistic approach coherent probability is defined on an arbitrary class of sets and any coherent probability can be extended to a larger domain. So in this framework no measurability condition is required for random variables. In the sequel, bounded random variables are bounded real-valued functions (these functions are called *gambles* in Walley [19] or *random quantities* in de Finetti [3]). When a measurability condition for a random variable is required, for example to define the Choquet integral, it is explicitly mentioned through the paper.

Separately coherent upper conditional previsions are functionals on a linear space of bounded random variables satisfying the axioms of separate coherence. They cannot always be defined as an extension of conditional expectation of measurable random variables defined by the Radon-Nikodym derivative, according to the axiomatic definition. It occurs because one of the defining properties of the Radon-Nikodym derivative, that is to be measurable with respect to the σ -field of the conditioning events, contradicts a necessary condition for coherence (see Doria [9, Theorem 1], Seidenfeld [16]).

So the necessity to find a new mathematical tool in order to define coherent upper conditional previsions arises.

In Doria [8], [9] a new model of coherent upper conditional prevision is proposed in a metric space. It is defined by the Choquet integral with respect to the s -dimensional Hausdorff

outer measure if the conditioning event has positive and finite Hausdorff outer measure in its dimension s . Otherwise if the conditioning event has Hausdorff outer measure in its dimension equal to zero or infinity it is defined by a 0-1 valued finitely, but not countably, additive probability. Coherent upper and lower conditional probabilities are obtained ([6]) when only 0-1 valued random variables are considered.

If the conditioning event B has positive and finite Hausdorff outer measure in its Hausdorff dimension then the given upper conditional prevision defined on a linear lattice of bounded random variables is proven to be a functional, which is monotone, submodular, comonotonically additive and continuous from below. Moreover all these properties are proven to be a sufficient condition under which the upper conditional probability defined by Hausdorff outer measure is the unique monotone set function, which represent a coherent upper conditional prevision as Choquet integral. The given model of coherent upper conditional prevision can be applied to make prevision in chaotic systems.

Many complex systems are strongly dependent on the initial conditions, that is small differences on the initial conditions lead the system to entirely different states. These systems are called *chaotic systems*. Thus uncertainty in the initial conditions produces uncertainty in the final state of the system. Often the final state of the system, called *strange attractor* is represented by a fractal set, i.e., a set with non-integer Hausdorff dimension. The model of coherent upper prevision, introduced in this chapter, can be proposed to forecast in a chaotic system when the conditional prevision of a random variable is conditioned to the attractor of the chaotic system.

The outline of the chapter is the following.

In Section 2 The notion of separately coherent conditional previsions and their properties are recalled.

In Section 3 separately coherent upper conditional previsions are defined in a metric space by the Choquet integral with respect to Hausdorff outer measure if the conditioning event has positive and finite Hausdorff outer measure in its dimension. Otherwise they are defined by a 0-1 valued finitely, but not countably, additive probability.

In Section 4 results are given such that a coherent upper conditional prevision, defined on a linear lattice of bounded random variables containing all constants, is uniquely represented as the Choquet integral with respect to its associated Hausdorff outer measure if and only if it is monotone, submodular and continuous from below.

2. Separately coherent upper conditional previsions

Given a metric space (Ω, d) the Borel σ -field is the σ -field generated by the open sets of Ω . Let $\mathbf{B}$ be a Borel-measurable partition of Ω , i.e. all sets of the partition are Borel sets.

For every $B \in \mathbf{B}$ let us denote by $X|B$ the restriction to B of a random variable defined on Ω and by $\sup(X|B)$ the supremum value that X assumes on B .

Separately coherent upper conditional previsions $\bar{P}(\cdot|B)$ are functionals, defined on a linear space of bounded random variables, i.e. bounded real-valued functions, satisfying the axioms of separate coherence [19].

Definiton 1. Let (Ω, d) be a metric space and let $\mathbf{B}$ be a Borel-measurable partition of Ω . For every $B \in \mathbf{B}$ let $\mathbf{K}(B)$ be a linear space of bounded random variables on B . Separately coherent upper conditional previsions are functionals $\bar{P}(\cdot|B)$ defined on $\mathbf{K}(B)$, such that the following conditions hold for every X and Y in $\mathbf{K}(B)$ and every strictly positive constant λ :

- 1) $\bar{P}(X|B) \leq \sup(X|B)$;
- 2) $\bar{P}(\lambda X|B) = \lambda \bar{P}(X|B)$ (positive homogeneity);
- 3) $\bar{P}(X + Y|B) \leq \bar{P}(X|B) + \bar{P}(Y|B)$ (subadditivity);
- 4) $\bar{P}(B|B) = 1$.

Coherent upper conditional previsions can always be extended to coherent upper previsions on the class $\mathbf{L}(B)$ of all bounded random variables defined on B . If coherent upper conditional previsions are defined on the class $\mathbf{L}(B)$ no measurability condition is required for the sets B of the partition $\mathbf{B}$.

Suppose that $\bar{P}(X|B)$ is a coherent upper conditional prevision on a linear space $\mathbf{K}(B)$ then its conjugate coherent lower conditional prevision is defined by $\underline{P}(X|B) = -\bar{P}(-X|B)$. If for every X belonging to $\mathbf{K}(B)$ we have $P(X|B) = \underline{P}(X|B) = \bar{P}(X|B)$ then $P(X|B)$ is called a coherent *linear* conditional prevision (de Finetti [?]) and it is a linear positive functional on $\mathbf{K}(B)$.

Definition 2. Let (Ω, d) be a metric space and let $\mathbf{B}$ be a Borel-measurable partition of Ω . For every $B \in \mathbf{B}$ let $\mathbf{K}(B)$ be a linear space of bounded random variables on B . Then linear coherent conditional previsions are functionals $P(\cdot|B)$ defined on $\mathbf{K}(B)$, such that the following conditions hold for every X and Y in $\mathbf{K}(B)$ and every strictly positive constant λ :

- 1') if $X \geq 0$ then $P(X|B) \geq 0$ (positivity);
- 2') $P(\lambda X|B) = \lambda P(X|B)$ (positive homogeneity);
- 3') $P(X + Y|B) = P(X|B) + P(Y|B)$ (linearity);
- 4') $P(B|B) = 1$.

A class of bounded random variables is called a *lattice* if it is closed under point-wise maximum $\vee$ and point-wise minimum $\wedge$.

Two random variables X and Y defined on B are *comonotonic* if, $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0 \forall \omega_1, \omega_2 \in B$.

Definition 3. Let (Ω, d) be a metric space and let $\mathbf{B}$ be a Borel-measurable partition of Ω . For every $B \in \mathbf{B}$ let $\mathbf{K}(B)$ be a linear lattice of bounded random variables defined on B and let $\bar{P}(\cdot|B)$ be a coherent upper conditional prevision defined on $\mathbf{K}(B)$ then for every X, Y, X_n in $\mathbf{K}(B)$ $\bar{P}(\cdot|B)$ is

- i) *monotone* iff $X \leq Y$ implies $\bar{P}(X|B) \leq \bar{P}(Y|B)$;
- ii) *comonotonically additive* iff $\bar{P}(X + Y|B) = \bar{P}(X|B) + \bar{P}(Y|B)$ if X and Y are comonotonic;
- iii) *submodular* iff $\bar{P}(X \vee Y|B) + \bar{P}(X \wedge Y|B) \leq \bar{P}(X|B) + \bar{P}(Y|B)$;
- iv) *continuous from below* iff $\lim_{n \rightarrow \infty} \bar{P}(X_n|B) = \bar{P}(X|B)$ if X_n is an increasing sequence of random variables converging to X .

A bounded random variable is called *B-measurable* or measurable with respect to the partition $\mathbf{B}$ [19, p.291] if it is constant on the atoms B of the partition. Let $G(\mathbf{B})$ be the class of all $\mathbf{B}$ -measurable random variables.

Denote by $\bar{P}(X|\mathbf{B})$ the random variable equal to $\bar{P}(X|B)$ if $\omega \in B$.

Separately coherent upper conditional previsions $\bar{P}(X|B)$ can be extended to a common domain $\mathbf{H}$ so that the function $\bar{P}(\cdot|\mathbf{B})$ can be defined from $\mathbf{H}$ to $G(\mathbf{B})$ to summarize the collection of $\bar{P}(X|B)$ with $B \in \mathbf{B}$.

$\bar{P}(\cdot|\mathbf{B})$ is assumed to be separately coherent if all the $\bar{P}(\cdot|B)$ are separately coherent. In Theorem 1 [9] the function $\bar{P}(X|\mathbf{B})$ is compared with the Radon-Nikodym derivative.

It is proven that, every time that the σ -field of the conditioning events is properly contained in the σ -field of the probability space and it contains all singletons, the Radon-Nikodym derivative cannot be used as a tool to define coherent conditional previsions. This is due to the fact that one of the defining properties of the Radon-Nikodym derivative, that is to be measurable with respect to the σ -field of the conditioning events, contradicts a necessary condition for the coherence.

Analysis done points out the necessity to introduce a different tool to define coherent conditional previsions.

3. Separately coherent upper conditional previsions defined by Hausdorff outer measures

In this section coherent upper conditional previsions are defined by the Choquet integral with respect to Hausdorff outer measures if the conditioning event B has positive and finite Hausdorff outer measure in its dimension. Otherwise if the conditioning event B has Hausdorff outer measure in its dimension equal to zero or infinity they are defined by a 0-1 valued finitely, but not countably, additive probability.

3.1. Hausdorff outer measures

Given a non-empty set Ω , let $\wp(\Omega)$ be the class of all subsets of Ω . An *outer measure* is a function $\mu^* : \wp(\Omega) \rightarrow [0, +\infty]$ such that $\mu^*(\emptyset) = 0$, $\mu^*(A) \leq \mu^*(A')$ if $A \subseteq A'$ and $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Examples of outer set functions or outer measures are the Hausdorff outer measures [11], [15].

Let (Ω, d) be a metric space. A topology, called the *metric topology*, can be introduced into any metric space by defining the open sets of the space as the sets G with the property:

if x is a point of G , then for some $r > 0$ all points y with $d(x, y) < r$ also belong to G .

It is easy to verify that the open sets defined in this way satisfy the standard axioms of the system of open sets belonging to a topology [15, p.26].

The Borel σ -field is the σ -field generated by all open sets. The Borel sets include the closed sets (as complement of the open sets), the F_σ -sets (countable unions of closed sets) and the G_σ -sets (countable intersections of open sets), etc.

The diameter of a non empty set U of Ω is defined as $|U| = \sup \{d(x, y) : x, y \in U\}$ and if a subset A of Ω is such that $A \subset \bigcup_i U_i$ and $0 < |U_i| < \delta$ for each i , the class $\{U_i\}$ is called a δ -cover of A .

Let s be a non-negative number. For $\delta > 0$ we define $h_{s,\delta}(A) = \inf \sum_{i=1}^{+\infty} |U_i|^s$, where the infimum is over all δ -covers $\{U_i\}$.

The *Hausdorff s -dimensional outer measure* of A , denoted by $h^s(A)$, is defined as

$$h^s(A) = \lim_{\delta \rightarrow 0} h_{s,\delta}(A).$$

This limit exists, but may be infinite, since $h_{s,\delta}(A)$ increases as δ decreases because less δ -covers are available. The *Hausdorff dimension* of a set A , $\dim_H(A)$, is defined as the unique value, such that

$$\begin{aligned} h^s(A) &= +\infty & \text{if } 0 \leq s < \dim_H(A), \\ h^s(A) &= 0 & \text{if } \dim_H(A) < s < +\infty. \end{aligned}$$

We can observe that if $0 < h^s(A) < +\infty$ then $\dim_H(A) = s$, but the converse is not true.

Hausdorff outer measures are *metric* outer measures:

$$h^s(E \cup F) = h^s(E) + h^s(F) \text{ whenever } d(E, F) = \inf \{d(x, y) : x \in E, y \in F\} > 0.$$

A subset A of Ω is called *measurable* with respect to the outer measure h^s if it decomposes every subset of Ω additively, that is if

$$h^s(E) = h^s(A \cap E) + h^s(E - A) \text{ for all sets } E \subseteq \Omega.$$

All Borel subsets of Ω are measurable with respect to any metric outer measure [11, Theorem 1.5]. So every Borel subset of Ω is measurable with respect to every Hausdorff outer measure h^s since Hausdorff outer measures are metric.

The restriction of h^s to the σ -field of h^s -measurable sets, containing the σ -field of the Borel sets, is called Hausdorff s -dimensional measure. In particular the Hausdorff 0-dimensional measure is the counting measure and the Hausdorff 1-dimensional measure is the Lebesgue measure.

The Hausdorff s -dimensional measures are *modular* on the Borel σ -field, that is

$$h^s(A \cup B) + h^s(A \cap B) = h^s(A) + h^s(B)$$

for every pair of Borelian sets A and B ; so that [5, Proposition 2.4] the Hausdorff outer measures are *submodular*

$$h^s(A \cup B) + h^s(A \cap B) \leq h^s(A) + h^s(B).$$

In [15, p.50] and [11, Theorem 1.6 (a)] it has been proven that if A is any subset of Ω there is a G_σ -set G containing A with $h^s(A) = h^s(G)$. In particular h^s is an *outer regular* measure.

Moreover Hausdorff outer measures are *continuous from below* [11, Lemma 1.3], that is for any increasing sequence of sets $\{A_i\}$ we have

$$\lim_{i \rightarrow \infty} h^s(A_i) = h^s(\lim_{i \rightarrow \infty} A_i).$$

h^s -Measurable sets with finite Hausdorff s -dimensional outer measure can be approximated from below by closed subsets [15, p.50] [11, Theorem 1.6 (b)] or equally the restriction of every Hausdorff outer measure h^s to the class of all h^s -measurable sets with finite Hausdorff outer measure is *inner regular* on the class of all closed subsets of Ω .

In particular any h^s -measurable set with finite Hausdorff s -dimensional outer measure contains an F_σ -set of equal measure, and so contains a closed set differing from it by arbitrary small measure.

Since every metric space is a Hausdorff space then every compact subset of Ω is closed; denote by $\mathbf{O}$ the class of all open sets of Ω and by $\mathbf{C}$ the class of all compact sets of Ω , the restriction of each Hausdorff s -dimensional outer measure to the class $\mathbf{H}$ of all h^s -measurable sets with finite Hausdorff outer measure is *strongly regular* [5, p.43] that is it is regular:

a) $h^s(A) = \inf \{h^s(U) | A \subset U, U \in \mathbf{O}\}$ for all $A \in \mathbf{H}$ (outer regular);

b) $h^s(A) = \sup \{h^s(C) | C \subset A, C \in \mathbf{C}\}$ for all $A \in \mathbf{H}$ (inner regular)

with the additional property:

c) $\inf \{h^s(U - A) | A \subset U, U \in \mathbf{O}\} = 0$ for all $A \in \mathbf{H}$

Any Hausdorff s -dimensional outer measure is translation invariant, that is, $h^s(x + E) = h^s(E)$, where $x + E = \{x + y : y \in E\}$ [11, p.18].

3.2. The Choquet integral

We recall the definition of the Choquet integral [5] with the aim to define upper conditional previsions by Choquet integral with respect to Hausdorff outer measures and to prove their properties. The Choquet integral is an integral with respect to a monotone set function. Given a non-empty set Ω and denoted by S a set system, containing the empty set and properly contained in $\wp(\Omega)$, the family of all subsets of Ω , a monotone set function $\mu: S \rightarrow \mathbb{R}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is such that $\mu(\emptyset) = 0$ and if $A, B \in S$ with $A \subseteq B$ then $\mu(A) \leq \mu(B)$. Given a monotone set function μ on S , its *outer set function* is the set function μ^* defined on the whole power set $\wp(\Omega)$ by

$$\mu^*(A) = \inf \{\mu(B) : B \supset A; B \in S\}, A \in \wp(\Omega)$$

The inner set function of μ is the set function μ_* defined on the whole power set $\wp(\Omega)$ by

$$\mu_*(A) = \sup \{\mu(B) | B \subset A; B \in S\}, A \in \wp(\Omega)$$

Let μ be a monotone set function defined on S properly contained in $\wp(\Omega)$ and $X: \Omega \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ an arbitrary function on Ω . Then the set function

$$G_{\mu, X}(x) = \mu \{\omega \in \Omega : X(\omega) > x\}$$

is decreasing and it is called *decreasing distribution function* of X with respect to μ . If μ is continuous from below then $G_{\mu,X}(x)$ is right continuous. In particular the decreasing distribution function of X with respect to the Hausdorff outer measures is right continuous since these outer measures are continuous from below. A function $X : \Omega \rightarrow \overline{\mathbb{R}}$ is called upper μ -measurable if $G_{\mu^*,X}(x) = G_{\mu,X}(x)$. Given an upper μ -measurable function $X : \Omega \rightarrow \overline{\mathbb{R}}$ with decreasing distribution function $G_{\mu,X}(x)$, if $\mu(\Omega) < +\infty$, the *asymmetric Choquet integral* of X with respect to μ is defined by

$$\int X d\mu = \int_{-\infty}^0 (G_{\mu,X}(x) - \mu(\Omega)) dx + \int_0^{\infty} G_{\mu,X}(x) dx$$

The integral is in $\mathbb{R}$, can assume the values $-\infty$, $+\infty$ or is undefined when the right-hand side is $+\infty - \infty$.

If $X \geq 0$ or $X \leq 0$ the integral always exists. In particular for $X \geq 0$ we obtain

$$\int X d\mu = \int_0^{+\infty} G_{\mu,X}(x) dx$$

If X is bounded and $\mu(\Omega) = 1$ we have that

$$\int X d\mu = \int_{\inf X}^0 (G_{\mu,X}(x) - 1) dx + \int_0^{\sup X} G_{\mu,X}(x) dx = \int_{\inf X}^{\sup X} G_{\mu,X}(x) dx + \inf X.$$

3.3. The model

A new model of coherent upper conditional prevision is defined in [9, Theorem 2].

Theorem 1. Let (Ω, d) be a metric space and let $\mathbf{B}$ be a Borel-measurable partition of Ω . For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s -dimensional outer measure. Let $\mathbf{K}(B)$ be a linear space of bounded random variables on B . Moreover, let m be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B . Then for each $B \in \mathbf{B}$ the functional $\overline{P}(X|B)$ defined on $\mathbf{K}(B)$ by

$$\overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s \text{ if } 0 < h^s(B) < +\infty$$

and by

$$\overline{P}(X|B) = m(XB) \text{ if } h^s(B) = 0, +\infty$$

is a coherent upper conditional prevision.

The lower conditional previsions $\underline{P}(A|B)$ can be define as in the previous theorem if h_s denotes the Hausdorff s -dimensional inner measure.

Given an upper conditional prevision $\overline{P}(X|B)$ defined on a linear space the lower conditional prevision $\underline{P}(X|B)$ is obtained as its conjugate, that is $\underline{P}(X|B) = -\overline{P}(-X|B)$. If B has positive and finite Hausdorff outer measure in its Hausdorff dimension s and we denote by h_s the Hausdorff s -dimensional inner measure we have

$$\underline{P}(X|B) = -\overline{P}(-X|B) = -\frac{1}{h^s(B)} \int_B (-X) dh^s =$$

$$\frac{1}{h^s(B)} \int_B X dh_s = \frac{1}{h_s(B)} \int_B X dh_s.$$

The last equality holds since each B is h^s -measurable, that is $h^s(B) = h_s(B)$.

The unconditional upper prevision is obtained as a particular case when the conditioning event is Ω , that is $\bar{P}(A) = \bar{P}(A|\Omega)$ and $\underline{P}(A) = \underline{P}(A|\Omega)$.

Coherent upper conditional probabilities are obtained when only 0-1 valued random variables are considered; they have been defined in [6] :

Theorem 2. Let (Ω, d) be a metric space and let $\mathbf{B}$ be a Borel-measurable partition of Ω . For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s -dimensional outer measure. Let m be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B . Thus, for each $B \in \mathbf{B}$, the function defined on $\wp(B)$ by

$$\bar{P}(A|B) = \frac{h^s(AB)}{h^s(B)} \quad \text{if } 0 < h^s(B) < +\infty$$

and by

$$\bar{P}(A|B) = m(AB) \quad \text{if } h^s(B) = 0, +\infty$$

is a coherent upper conditional probability.

Let B be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s . Denote by h^s the s -dimensional Hausdorff outer measure and for every $A \in \wp(B)$ by $\mu_B^*(A) = \bar{P}(A|B) = \frac{h^s(AB)}{h^s(B)}$ the upper conditional probability defined on $\wp(B)$. From Theorem 1 we have that the upper conditional prevision $\bar{P}(\cdot|B)$ is a functional defined on $L(B)$ with values in $\mathfrak{R}$ and the upper conditional probability μ_B^* integral represents $\bar{P}(X|B)$ since $\bar{P}(X|B) = \int X d\mu_B^* = \frac{1}{h^s(B)} \int X dh^s$. The number $\frac{1}{h^s(B)}$ is a normalizing constant.

4. Examples

Example 1 Let $B = [0, 1]$. The Hausdorff dimension of B is 1 and the Hausdorff 1-dimensional measure h^1 is the Lebesgue measure. Moreover $h^1[0, 1] = 1$ then the coherent upper conditional prevision is defined for every $X \in \mathbf{K}(B)$ by

$$\bar{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s = \int_B X dh^1$$

We recall the definition of the Cantor set. Let $E_0 = [0, 1]$, $E_1 = [0, 1/3] \cup [2/3, 1]$, $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, etc., where E_n is obtained by removing the open middle third of each interval in E_{n-1} , so E_n is the union of 2^n intervals, each of length $\frac{1}{3^n}$. The Cantor's set is the perfect set $E = \bigcap_{n=0}^{\infty} E_n$.

Example 2 Let B be the Cantor set. The Hausdorff dimension of the Cantor set is $s = \frac{\ln 2}{\ln 3}$ and $h^s(B) = 1$. Then the coherent upper conditional prevision is defined for every $X \in \mathbf{K}(B)$ by

$$\bar{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s = \int_B X dh^{\frac{\ln 2}{\ln 3}}$$

Example 3 Let $B = \{\omega_1, \omega_2, \dots, \omega_n\}$. The Hausdorff dimension of B is 0 and the Hausdorff 0-dimensional measure h^0 is the counting measure. Moreover $h^0(B) = n$ then the coherent upper conditional prevision is defined for every $X \in \mathbf{K}(B)$ by

$$\bar{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s = \frac{1}{n} \sum_1^n X(\omega_i)$$

5. Upper envelope

A necessary and sufficient condition for an upper prevision $\bar{P}$ to be coherent is to be the *upper envelope* of linear previsions, i.e. there is a class M of linear previsions such that $\bar{P} = \sup\{P(X) : P \in M\}$ [19, 3.3.3].

Given a coherent upper prevision $\bar{P}$ defined on a domain $\mathbf{K}$ the maximal coherent extension of $\bar{P}$ to the class of all bounded random variables is called [19, 3.1.1] *natural extension* of $\bar{P}$.

The linear extension theorem [19, 3.4.2] assures that the class of all linear extensions to the class of all bounded random variables of a linear prevision P defined on a linear space $\mathbf{K}$ is the class $M(P)$ of all linear previsions that are dominated by P on $\mathbf{K}$. Moreover the upper and lower envelopes of $M(P)$ are the natural extensions of P [19, Corollary 3.4.3].

Let $P(\cdot|B)$ be the coherent upper conditional prevision on the the class of all bounded Borel-measurable random variables defined on B defined in Theorem 1.

In Doria [9, Theorem 5] it is proven that, for every conditioning event B , the given upper conditional prevision is the upper envelope of all linear extensions of $P(\cdot|B)$ to the class of all bounded random variables on B .

Theorem 3. Let (Ω, d) be a metric space and let $\mathbf{B}$ be a Borel-measurable partition of Ω . For every conditioning event $B \in \mathbf{B}$ let $\mathbf{L}(B)$ be the class of all bounded random variables defined on B and let $P(\cdot|B)$ be the coherent upper conditional prevision on the class of all bounded Borel-measurable random variables defined in Theorem 1. Then the coherent upper conditional prevision defined on $\mathbf{L}(B)$ as in Theorem 1 is the upper envelope of all linear extensions of $P(\cdot|B)$ to the class $\mathbf{L}(B)$.

In the same way it can be proven that the conjugate of the coherent upper conditional prevision $\bar{P}(\cdot|B)$ is the *lower envelope* of $M(P)$, the class of all linear extension of $P(\cdot|B)$ dominating $P(\cdot|B)$.

6. Main results

For each B in $\mathbf{B}$, denote by s the Hausdorff dimension of B then the Hausdorff s -dimensional outer measure is called the Hausdorff outer measure *associated* with the coherent upper prevision $\bar{P}(\cdot|B)$. Let $B \in \mathbf{B}$ be measurable with respect to the Hausdorff outer measure associated with $\bar{P}(\cdot|B)$.

The Choquet integral representation of a coherent upper conditional prevision with respect to its associated Hausdorff outer measure has been investigated in [7]. In [9] necessary and sufficient conditions are given such that a coherent upper conditional prevision is uniquely represented as the Choquet integral with respect to its associated Hausdorff outer measure.

In [9, Theorem 4] it is proven that, if the conditioning event has positive and finite Hausdorff outer measure in its dimension s and $\mathbf{K}(B)$ is a linear lattice of bounded random variables defined on B , necessary conditions for the functional $\bar{P}(X|B)$ to be represented as Choquet integral with respect to the upper conditional probability μ_B^* , i.e. $\bar{P}(X|B) = \frac{1}{h^s(B)} \int X dh^s$, are that $\bar{P}(X|B)$ is monotone, comonotonically additive, submodular and continuous from below.

Theorem 4. Let (Ω, d) be a metric space and let $\mathbf{B}$ be a Borel-measurable partition of Ω . For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s -dimensional outer measure. Let $\mathbf{K}(B)$ be a linear lattice of bounded random variables defined on B . If the conditioning event B has positive and finite Hausdorff s -dimensional outer measure then the coherent upper conditional prevision $\bar{P}(\cdot|B)$ defined on $\mathbf{K}(B)$ as in Theorem 2 is:

- i) monotone;
- ii) comonotonically additive;
- iii) submodular;
- iv) continuous from below.

Moreover if the conditioning event B has positive and finite Hausdorff s -dimensional outer measure, from the properties of the Choquet integral ([5, Proposition 5.1]) the coherent upper conditional prevision $\bar{P}(\cdot|B)$ is

- v) translation invariant;
- vi) positively homogeneous;

So the functional $\bar{P}(\cdot|B)$ can be used to defined a *coherent risk measure* [1]. since it is monotone, subadditive, translation invariant and positively homogeneous.

In [9, Theorem 6] sufficient conditions are given for a coherent upper conditional prevision to be uniquely represented as Choquet intergral with respect to its associated Hausdorff outer measure.

Theorem 5. Let (Ω, d) be a metric space and let $\mathbf{B}$ be a Borel-measurable partition of Ω . For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s -dimensional outer measure. Let $\mathbf{K}(B)$ be a linear lattice of bounded random variables on B containing all constants. If B has positive and finite Hausdorff outer measure in its dimension and the coherent upper conditional prevision $\bar{P}(\cdot|B)$ on $\mathbf{K}(B)$ is monotone, comonotonically additive, submodular and continuous from below then $\bar{P}(\cdot|B)$ is representable as Choquet integral with respect to a monotone, submodular set function which is continuous from below. Furthermore all monotone set functions on $\wp(B)$ with these properties agree on the set system of weak upper level sets $M = \{\{X \geq x\} | X \in \mathbf{K}(B), x \in \mathbb{R}\}$ with the upper conditional probability $\mu_B^*(A) = \frac{h^s(AB)}{h^s(B)}$ for $A \in \wp(B)$. Let β be a monotone set function on $\wp(B)$, which is submodular, continuous from below and such that represents $\bar{P}(\cdot|B)$ as Choquet integral. Then the following equalities hold

$$\bar{P}(X|B) = \int_B X d\beta = \int_B X d\mu_B^* = \frac{1}{h^s(B)} \int_B X dh^s.$$

An example is given in the particular case where $\mathbf{K}(B)$ is the linear space of all bounded Borel-measurable random variables on B and the restriction of the Hausdorff s -dimensional outer measure to the Borel σ -field of subsets of B is considered.

Example 2. Let (Ω, d) be a metric space and let $\mathbf{B}$ be a Borel-measurable partition of Ω . For every $B \in \mathbf{B}$ let $\mathbf{K}(B)$ be the linear space of all bounded Borel-measurable random variables on B and let S be the Borel σ -field of subsets of B . Denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s -dimensional outer measure. If $0 < h^s(B) < +\infty$ define $\mu_B(A) = \frac{h^s(AB)}{h^s(B)}$, for every $A \in S$; $\mu_B(A)$ is modular and continuous from below on S since each Hausdorff s -dimensional (outer) measure is σ -additive on the Borel σ -field. Moreover let $P(\cdot|B)$ be a coherent linear conditional prevision, which is continuous from below. Then $P(\cdot|B)$ can be uniquely represented as the Choquet integral with respect to the coherent upper conditional probability μ_B , that is

$$P(X|B) = \int X d\mu_B = \frac{1}{h^s(B)} \int X dh^s.$$

The previous example can be obtained as a consequence of the Daniell-Stone Representation Theorem [5, p. 18].

7. Conclusions

In this chapter a model of coherent upper conditional precision is introduced. It is defined by the Choquet integral with respect to the s -dimensional Hausdorff outer measure if the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension s . Otherwise if the conditioning event has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity it is defined by a 0-1 valued finitely, but not countably, additive probability. If the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension the given upper conditional prevision, defined on a linear lattice of bounded random variables which contains all constants, is uniquely represented as the Choquet integral with respect Hausdorff outer measure if and only if it is a functional which is monotone, submodular, comonotonically additive and continuous from below.

Coherent upper conditional prevision based on the Hausdorff s -dimensional measure permits to analyze complex systems where information represented by sets with Hausdorff dimension less than s , have no influence on the situation; information represented by sets with the same Hausdorff dimension of the conditioning event can influence the system.

Coherent upper previsions defined by Hausdorff outer measures can also be applied in decision theory, to asses preferences between random variables defined on fractal sets and to defined coherent risk measures.

Author details

Serena Doria

Department of Engineering and Geology, University G.d'Annunzio, Chieti-Pescara, Italy

8. References

- [1] P. Artzner, F. Delbaen, J. Elber, D. Heath.(1999) Coherent measures of risk. *Mathematical Finance*, 9, 203-228.

- [2] P. Billingsley. (1986). *Probability and measure*, Wiley, USA.
- [3] B. de Finetti. (1970). *Teoria della Probabilita'*, Einaudi Editore, Torino.
- [4] B. de Finetti. (1972). *Probability, Induction and Statistics*, Wiley, New York.
- [5] D. Denneberg. (1994). *Non-additive measure and integral*. Kluwer Academic Publishers.
- [6] S. Doria. (2007). Probabilistic independence with respect to upper and lower conditional probabilities assigned by Hausdorff outer and inner measures. *International Journal of Approximate Reasoning*, 46, 617-635.
- [7] S. Doria. (2010). Coherent upper conditional previsions and their integral representation with respect to Hausdorff outer measures, In *Combining Soft Computing and Statistical Methods in Data Analysis* (C. Borgelt et al. editors), *Advances in Intelligent and Soft Computing* 77, 209-216, Springer.
- [8] S. Doria. (2011). Coherent upper and lower conditional previsions defined by Hausdorff outer and inner measures, In *Modeling, Design and Simulation of Systems with Uncertainties* (A. Rauh and E. Auer editors), *Mathematical Engineering*, Volume 3, 175-195, Springer.
- [9] S. Doria. (2012). Characterization of a coherent upper conditional prevision as the Choquet integral with respect to its associated Hausdorff outer measure. *Annals of Operations Research*, Vol 195, 33-48.
- [10] L.E. Dubins. (1975). Finitely additive conditional probabilities, conglomerability and disintegrations. *The Annals of Probability*, Vol. 3, 89-99.
- [11] K.J. Falconer. (1986). *The geometry of fractals sets*. Cambridge University Press.
- [12] E. Miranda, M.Zaffalon. (2009). Conditional models: coherence and inference through sequences of joint mass functions, *Journal of Statistical Planning and Inference* 140(7), 1805-1833.
- [13] E. Regazzini. (1985). Finitely additive conditional probabilities, *Rend. Sem. Mat. Fis., Milano*, 55, 69-89.
- [14] E. Regazzini. (1987). De Finetti's coherence and statistical inference, *The Annals of Statistics*, Vol. 15, No. 2, 845-864.
- [15] C.A. Rogers. (1998). *Hausdorff measures*. Cambridge University Press.
- [16] T. Seidenfeld, M.J. Schervish, J.B. Kadane. (2001). Improper regular conditional distributions, *The Annals of Probability*, Vol. 29, No. 4, 1612-1624.
- [17] P. Vicig, M. Zaffalon, F.G. Cozman. (2007). Notes on "Notes on conditional previsions", *International Journal of Approximate Reasoning*, Vol 44, 358-365.
- [18] P. Walley. (1981). Coherent lower (and upper) probabilities, *Statistics Research Report*, University of Warwick.
- [19] P. Walley. (1991). *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London.
- [20] P.M. Williams. (2007). Notes on conditional previsions, *International Journal of Approximate Reasoning*, Vol 44, 3, 366-383.