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Microscopic Formulation of Fractional Theory of Viscoelasticity

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1. Introduction

Viscoelasiticity refers to the phenomenon in which a material body, when deformed exhibits both elastic (akin to solids)and viscous (akin to liquids) behavior. The body stores mechanical energy (elastic behavior) and dissipates it simultaneously(viscous behavior). Linear theory of viscoelasticity treats the body as a linear system which when subjected to an excitation responds with a response function. If the excitation is a stress, the response is a strain and if the excitation is a strain, the response is a stress. Mechanical models involving a spring-mass connected to a dashpot have been used to explain the elastic and viscous behavior.The mathematical structure of the theory and the spring-dash-pot type of mechanical models used and the so called Standard Linear Solid have all been only too well known [1-4]. In recent years methods of fractional calculus have been applied to develop viscoelastic models especially by Caputo and Mainardi [5,6] , Glockle and Nonenmacher [7], and Gorenflo and Mainardi [8]. A recent monograph by Mainardi[9] gives extensive list of references to the literature connecting fractional calculus, linear viscoelasticity and wave motion.All these works treat the phenomenon of viscoelasticity as a macroscopic phenomenon exhibited by matter treated as an elastic continuum albeit including a viscous aspect as well. It should be recognized that matter has an atomic structure and is fundamentally discrete in nature. A microscopic approach would recognize this aspect and a theoretical model would yield the results for the continuum as a limit.

It is well established that lattice dynamics provides a microscopic basis for understanding a host of phenomena in condensed matter physics, including mechanical, thermal, dielectric and optical phenomena, which are macroscopic, generally described from the continuum point of view. Since the pioneering work of Born and Von Karman [10], lattice dynamics has developed into a veritable branch of condensed matter physics. Lucid treatment of various

aspects of lattice dynamical theory of condensed matter can be found in renowned treatises [11-13]. Of particular interest to the present paper is the work of Askar [14] on the lattice dynamical foundations of continuum theories of elasticity, piezoelectricity, viscoelasticity and plasticity.

The objective of the chapter is to develop a lattice dynamical model based on the methods of fractional calculus so as to provide a microscopic basis for the theory of viscoelasticity. The plan of the chapter is as follows. First a brief review of the mechanical models of viscoelasticity is given. This is followed by an account of conventional lattice dynamical theory of viscoelaticity based on a model of linear chain of coupled oscillators with dissipative elements is given [14]. In the next section, lattice dynamical methods are extended to the model of linear chain of coupled fractional oscillators (requiring no additional dissipative elements) developed by the authors [15]. The response to sinusoidal forcing of the linear chain of fractional oscillators starting from a quiescent state is studied in the continuum limit and expressions for phase velocity, absorption and dispersion and specific dissipation function are derived. The results are discussed and numerical applications are presented in the final section

2. Mechanical models of viscoelasticity

According to the linear theory of viscoelasticity, the body may be considered to be a system responding linearly to an excitation. A fundamental aspect of the theory both from a mathematical and a physical point of view is the response of the system to an excitation usually applied as a Heaviside step function. If a unit step of stress $\sigma(t)$, is applied the material responds by undergoing a strain $\varepsilon(t)$. The test is called creep test and the material function characterizing the response is called the creep compliance and is denoted by $J(t)$. If on the other hand, if a unit strain is applied and the system responds with a stress $\sigma(t)$, the test is called a relaxation test and the material function characterizing the system response, $G(t)$, is called the *relaxation modulus*. The creep compliance and the relaxation modulus are defined through linear hereditary integrals of the Stieltjes type:

$$\varepsilon(t) = \int_{0^-}^{t} J(t-\tau)d\sigma(\tau)$$

$$\sigma(t) = \int_{0^-}^{t} G(t-\tau)d\varepsilon(\tau)$$

(1)

Simple mechanical models consisting of springs and dashpots incorporating elastic and dissipative functions have been advanced to explain the behaviour of viscoelastic solids and a good summary of the models of viscoelasticity can be found in the works of Caputo and Mainardi [5,6], Gorenflo and Mainardi[8], and Mainardi[9]. A simple model constituted by a spring in parallel with a dashpot, known as the Kelvin-Voigt model is characterized by the constitutive relation

$$\sigma(t) = m\varepsilon(t) + b\frac{d\varepsilon}{dt} \qquad (2)$$

and by the material functions

$$J(t) = \frac{1}{m}\left[1 - e^{-t/\tau_\varepsilon}\right] \qquad \text{(a)}$$

$$G(t) = m + b\delta(t) \qquad \text{(b)} \qquad (3)$$

where $\tau_\varepsilon = b/m$ is referred to as the retardation time.

A model consisting of a spring in series with a dashpot known as the Maxwell model is characterized by the constitutive relation

$$\sigma(t) + a\frac{d\sigma}{dt} = b\frac{d\varepsilon}{dt} \qquad (4)$$

and by the material functions

$$J(t) = \frac{a}{b} + \frac{t}{b} \qquad \text{(a)}$$

$$G(t) = \frac{b}{a}e^{-t/\tau_\sigma} \qquad \text{(b)} \qquad (5)$$

where $\tau_\sigma = a$ is referred to as the relaxation time.

Another model introduced by Zener [4], known as the Standard Linear Solid model is a combination of the above two models and is characterized by the constitutive relation

$$\left[1 + a\frac{d}{dt}\right]\sigma(t) = \left[1 + \frac{d}{dt}\right]\varepsilon(t) \qquad (6)$$

and by the material functions

$$J(t) = J_g + \chi_+\left[1 - e^{-t/\tau_\varepsilon}\right] \qquad \text{(a)}$$

$$G(t) = G_e + \chi_-e^{-t/\tau_\sigma} \qquad \text{(b)} \qquad (7)$$

Here J_g and G_e are known as the glass compliance and the equilibrium modulus respectively, and χ_+ and χ_- are the limiting values of creep for $t \to \infty$ and of relaxation for $t = 0$ respectively.

More recently methods of fractional calculus have been used [5-8] to generalize these models leading to the so-called Scott-Blair model characterized by the operator equation

$$\left[1 + a\frac{d^\alpha}{dt^\alpha}\right]\sigma(t) = \left[1 + \frac{d^\alpha}{dt^\alpha}\right]\varepsilon(t), 0 < \alpha \leq 1 \qquad (8)$$

The details of the characteristic material functions and their behavior with respect to time are discussed in detail in reference [8].

3. Lattice dynamical theory of viscoelasticity

3.1. The elastic continuum

Consider a one dimensional string of mass density ϱ per unit length and E the Young's modulus. If the string is homogeneously stressed and released, elastic forces set up vibrations in the string corresponding to elastic waves propagating in the string. The equation governing this propagation is given by 9a and these waves propagate with a velocity given by 9b

$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} \qquad (a)$$

$$c^2 = \frac{E}{\rho} \qquad (b) \tag{9}$$

The conventional model of lattice dynamics consists of a chain of N identical masses m connected by springs of spring constant k, shown in Figure 1. The equation of motion for the nth mass is given in the standard notation by

$$m \frac{d^2 u_n}{dt^2} = -k(2u_n - u_{n+1} - u_{n-1})$$

$$\text{or } \frac{d^2 u_n}{dt^2} = -\omega_0^2 (2u_n - u_{n+1} - u_{n-1}) \tag{10}$$

where u_n is the displacement from the equilibrium position of the nth mass. The right hand side represents the elastic restoring force on the nth atom. The atoms n=1 and n= N define the boundaries.

The frequency of oscillation is given by $\omega_0^2 = k/m$.
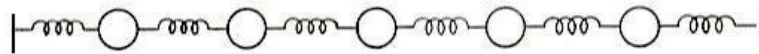
Figure 1. Linear chain of coupled oscillators

The elastic behavior of a continuous chain in equation (9a) can be obtained from the equation of motion of the linear chain of masses connected to each other by springs considered above in equation (9). In the limit when the number of masses N→∞ and the separation between the masses a→0, such that the product Na→ L, a finite length, the linear chain of coupled oscillators reduces to a continuous loaded string and is referred to as the long wavelength limit.

This process of taking the continuum limit is illustrated by considering the term on the right hand side of equation (10):

the factor

$$\omega_0^2 \left((2u_n(t) - u_{n+1}(t) - u_n(t)) \right)$$

can be written as

$$a^2 \omega_0^2 \left(\frac{1}{a} \left(\frac{(u_{n+1}(t) - u_n(t))}{a} - \frac{(u_n(t) - u_{n-1}(t))}{a} \right) \right)$$

In the limit of $N \to \infty$ and $a \to 0$, the chain of atoms can be treated as a continuum and $u_n(t) \to u(x,t)$, a continuous function. The term in the brackets can be written as $\frac{\partial^2 u}{\partial x^2}$ and $a^2 \omega_0^2$ can be expressed as $= a^2 \frac{k}{m} = \frac{ka}{m/a}$. In the limit $ka \to E$ and $m/a \to \rho$ The   term $a^2 \omega_0^2$ reduces to the square of the wave velocity. Thus the equation of motion in the limit can be written as $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ yielding the equation of motion of a continuum, equation (9a)

3.2. Viscoelasticity: Lattice dynamical approach

In order to develop a theory of viscoelasticity, the model of linear chain of atoms has been generalized to include dissipative effects [14] by incorporating dashpots either in parallel to the springs as shown in figure 2 to yield the lattice dynamical version of the Kelvin-Voigt model, or the dashpots in series with the springs as shown in figure 3 to yield the lattice dynamical version of the Maxwell model.
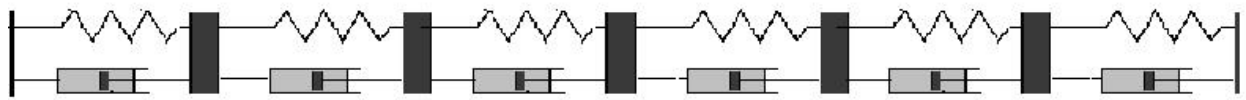
Figure 2. Kelvin-Voigt model for viscoelasticity

Figure 3. Maxwell model for viscoelasticity

It has been shown [14] that the models in the continuum limit lead to the stress-strain relations

$$\sigma = E\varepsilon + \zeta \frac{d\varepsilon}{dt} \qquad (a)$$

$$\frac{d\varepsilon}{dt} = \frac{\sigma}{E} + \frac{1}{E}\frac{d\sigma}{dt} \qquad (b)$$

$$(11)$$

respectively for the two models. Harmonic waves are found to propagate with decaying amplitude indicating dissipation. The details can be found in reference [14].

4. Fractional calculus approach

In this paper, the lattice dynamical approach is generalized by using the methods of fractional calculus, leading to the so-called linear chain of coupled fractional oscillators. However, this generalization does not require the dissipative elements, namely the dashpots to be explicitly considered, for the dissipation is intrinsic to the fractional oscillators [16-19]. As the method of approach is quite different, further reference will be made not to the work of Askar [14], but to the work of the authors on a linear chain of coupled fractional oscillators [19] and the approach is outlined below.

4.1. Linear chain of fractional oscillators

Instead of starting with equation (10), we consider the integral equation of motion for the nth mass interacting with only its nearest neighbors, which can be written without loss of generality as:

$$u_n(t) = u_n(0) + \dot{u}_n(0)t - \omega_0^2 \int_0^t \left(2u_n(\tau) - u_{n+1}(\tau) - u_{n-1}(\tau)\right)(t-\tau)d\tau \quad 2 \le n \le N-1 \qquad (12)$$

Here $u_n(0)$ and $\dot{u}_n(0)$ refer to the displacement from equilibrium and velocity of the nth mass at t=0. For the masses at the ends of the chain, the equations of motion are given by

$$u_1(t) = u_1(0) + \dot{u}_1(0)t - \omega_0^2 \int_0^t \left(2u_1(\tau) - u_2\right)(t-\tau)d\tau + \int_0^t f(\tau)(t-\tau)d\tau \qquad (13)$$

and

$$u_N(t) = u_N(0) + \dot{u}_N(0)t - \omega_0^2 \int_0^t \left(2u_N(\tau) - u_{N-1}(\tau)\right)(t-\tau)d\tau, \qquad (14)$$

respectively. Of course, it is $\omega_0^2 = k/m$, in the usual notation. The last term in equation (13) indicates a periodic forcing on the end atom.

The integrals on the right hand side of equations (11) -(13) are generalized to fractional integrals of order α (defined by equation (19) below) to yield the equations of motion of a chain of coupled fractional oscillators. However, as has been well known that a

fractional oscillator behaves like a damped harmonic oscillator [15-18], the oscillations of the linear chain of fractional oscillators can be sustained only if it is driven by an external force. Hence a periodic force, taken to be sinusoidal without loss of generality, acting only on the first member of the chain is included. Thus the equations of motion are given by

$$u_n = u_n(0) + \dot{u}_n(0)t - \frac{\omega_0^\alpha}{\Gamma(\alpha)} \int_0^t \left(2u_n(\tau) - u_{n+1}(\tau) - u_{n-1}(\tau)\right)(t-\tau)^{\alpha-1} d\tau, \, 2 \le n \le N-1 \quad (15)$$

$$u_1 = u_1(0) + \dot{u}_1(0)t - \frac{\omega_0^\alpha}{\Gamma(\alpha)} \int_0^t \left(2u_1(\tau) - u_2(\tau)\right)(t-\tau)^{\alpha-1} d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau \quad (16)$$

$$u_N = u_N(0) + \dot{u}_N(0)t - \frac{\omega_0^\alpha}{\Gamma(\alpha)} \int_0^t \left(2u_n(\tau) - u_{N-1}(\tau)\right)(t-\tau)^{\alpha-1} d\tau \quad (17)$$

The last term on the right hand side of equation (16) represents the effect of the periodic forcing as already noted.

4.2. Formal solution

The set of equations (15) through (17) can be solved formally by using the Laplace transform technique [20-24]. In the standard notation, the Laplace transform is defined by

$$L\{u_n(t)\} = \tilde{u}_n(s) = \int_0^\infty e^{-st} u_n(t) dt. \quad (18)$$

The Laplace transform of a fractional integral of order α, $(\text{Re}\,\alpha > 0)$ defined by [20] and can be evaluated by considering the fractional integral as a convolution of two functions (19b),

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau \quad \text{(a)}$$

$$\phi(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \text{ and } f(t) \quad \text{(b)} \quad (19)$$

i.e.,

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau = \phi(t) * f(t), , \quad (20)$$

and using the formula for the Laplace transform of the convolution. Thus

$$L\{I_{0+}^\alpha f(t)\} = L\{\phi(t) * f(t)\} = \tilde{\phi}(s)\tilde{f}(s); \quad (21)$$

But as discussed in [23],

$$\tilde{\phi}(s) = L\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\} = s^{-\alpha}, \; \mathrm{Re}\,\alpha > 0 \tag{22}$$

Taking Laplace transforms on both sides of the set of equations (15) through (17) yields

$$\tilde{u}_n(s) = u_n(0)s^{-1} + \dot{u}_n(0)s^{-2} - \omega_0^\alpha s^{-\alpha}\big(2\tilde{u}_n(s) - \tilde{u}_{n+1}(s) - \tilde{u}_{n-1}(s)\big), \; 2 \le n \le N-1 \tag{23}$$

$$\tilde{u}_1(s) = u_1(0)s^{-1} + \dot{u}_1(0)s^{-2} - \omega_0^\alpha s^{-\alpha}\big(2\tilde{u}_1(s) - \tilde{u}_2(s)\big) + \tilde{f}(s)s^{-\alpha} \tag{24}$$

$$\tilde{u}_N(s) = u_N(0)s^{-1} + \dot{u}_N(0)s^{-2} - \omega_0^\alpha s^{-\alpha}\big(2\tilde{u}_N(s) - \tilde{u}_{N-1}(s)\big) \tag{25}$$

The set of equations can be rewritten in the following matrix form:

$$\tilde{A}(s)\tilde{U}(s) = U(0)s^{-1} + \dot{U}(0)s^{-2} + \tilde{F}(s) \tag{26}$$

where the N x N matrix $\tilde{A}(s)$ is given by

$$\tilde{A}(s) = \begin{pmatrix}
1+2\omega_0^\alpha s^{-\alpha} & -\omega_0^\alpha s^{-\alpha} & 0 & 0 & . & . & 0 \\
-\omega_0^\alpha s^{-\alpha} & 1+2\omega_0^\alpha s^{-\alpha} & -\omega_0^\alpha s^{-\alpha} & 0 & 0 & . & . \\
. & -\omega_0^\alpha s^{-\alpha} & 1+2\omega_0^\alpha s^{-\alpha} & -\omega_0^\alpha s^{-\alpha} & 0 & 0 & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & 0 \\
. & . & . & . & -\omega_0^\alpha s^{-\alpha} & 1+2\omega_0^\alpha s^{-\alpha} & -\omega_0^\alpha s^{-\alpha} \\
0 & 0 & . & . & 0 & -\omega_0^\alpha s^{-\alpha} & 1+2\omega_0^\alpha s^{-\alpha}
\end{pmatrix} \tag{27}$$

and the N x 1 column vectors are given by

$$\tilde{U}(s) = \begin{pmatrix} \tilde{u}_1(s) \\ \tilde{u}_2(s) \\ \tilde{u}_3(s) \\ . \\ . \\ \tilde{u}_{N-1}(s) \\ \tilde{u}_N(s) \end{pmatrix} \quad U(0) = \begin{pmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ . \\ . \\ u_{N-1}(0) \\ u_N(0) \end{pmatrix} \quad \dot{U}(0) = \begin{pmatrix} \dot{u}_1(0) \\ \dot{u}_2(0) \\ \dot{u}_3(0) \\ . \\ . \\ \dot{u}_{N-1}(0) \\ \dot{u}_N(0) \end{pmatrix} \quad \text{and } \tilde{F}(s) = \begin{pmatrix} \tilde{f}(s)s^{-\alpha} \\ 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \end{pmatrix} \tag{28}$$

respectively. It may be noted that among the four column vectors, only the first and the last refer to Laplace transformed quantities, but the middle two refer to constants describing the initial conditions. The solution to Eq. (25) can be formally written as

$$\tilde{U}(s) = \tilde{A}^{-1}(s)\big(U(0)s^{-1} + \dot{U}(0)s^{-2} + \tilde{F}(s)\big) \tag{29}$$

Then the set of linear equations can be explicitly written down and the inverse Laplace transform applied to these equations on both sides term by term to obtain the displacements as functions of time. This formal procedure appears to be straight enough, however, it is not possible to carry out further simplifications in closed form of the expressions in equation. (28). Nevertheless, this set of equations can be numerically solved for specific values of N, the number of oscillators in the chain. Such calculations have been carried out and the details of numerical applications may be found in [19]. In the next section, the continuum limit of these equations is studied.

5. The continuum limit

It would appear to be a straight forward procedure to extend the same considerations to the equations of motion of a chain of coupled fractional oscillators given in equations (15)-(17).

However, it is not so straight forward and due caution has to be exercised in extending such a procedure. The reason is the occurrence of time derivatives of fractional order. Douglas has shown [25] that in the context of fractals, inhomogeneity in space results in the appearance of fractional order time derivatives and inhomogeneity in time implies fractional order derivatives of space. Since fractional order time dependence is being investigated in the present work, the question arises whether inhomogenneity in space can be ignored and whether space derivatives can be taken. The question is one of scale. The long wavelength limit implies that the lengths involved are very much larger than the separation distance between masses. On this scale the inhomogeneity in space can be ignored and space derivative can be interpreted as an average of some sort and a hand waving justification can be made. The limiting form of the equations (12) through (14) can be considered.

Thus the following factor in equation (15)

$$-\omega_0^\alpha \left((2u_n(\tau) - u_{n+1}(\tau) - u_{n-1}(\tau) \right)$$

can be written as

$$a^2 \omega_0^\alpha \left[\frac{1}{a} \left(\frac{(u_{n+1}(\tau) - u_n(\tau))}{a} - \frac{(u_n(\tau) - u_{n-1}(\tau))}{a} \right) \right] \tag{30}$$

and

$$a^2 \omega_0^\alpha = a^2 \frac{k}{m} = \frac{ka}{m/a} \tag{31}$$

In the limit a→ 0, $u_n(\tau)$ becomes a continuous variable $u(x,\tau)$ and the expression in parenthesis in (30) reduces to $\frac{\partial^2 u(x,\tau)}{\partial^2 x}$ and $ka \to \kappa$, $m/a \to \rho$ represent the tension and

the mass density respectively. The expression in (30) can be written in terms of a quantity c_0^α, where c_0 has the dimensions of 'velocity'. Assuming that at t=0 the displacement at the free end is subject to sinusoidal forcing, $u(0,t) = f(t) = A\sin(\omega t)$, the equation of motion reduces in the continuum limit to

$$u(x,t) = \frac{c_0^\alpha}{\Gamma(\alpha)} \int_0^t \frac{\partial^2 u(x,\tau)}{\partial^2 x}(t-\tau)^{\alpha-1}d\tau \tag{32}$$

with the initial conditions

$$u(x,0) = 0 \text{ and } \dot{u}(x,0) = 0 \text{ for } x > 0 \text{ and } u(0,t) = f(t).$$

Taking the Laplace transform on both sides of equation. (32) yields

$$\tilde{u}(x,s) = \frac{c_0^\alpha}{s^\alpha}\frac{d^2\tilde{u}(x,s)}{dx^2} \tag{33}$$

The equation. (33) can be rewritten as

$$\frac{d^2\tilde{u}(x,s)}{dx^2} - \frac{s^\alpha}{c_0^\alpha}\tilde{u}(x,s) = 0 \text{ for } x \neq 0 \text{ and } \tilde{u}(0,s) = \tilde{f}(s) \tag{34}$$

This is an ordinary differential equation and can be solved to yield

$$\tilde{u}(x,s) = \tilde{f}(s)e^{-\sqrt{\frac{s^\alpha}{c_0^\alpha}}x} \tag{35}$$

Substituting for $\tilde{f}(s) = \frac{A\omega}{s^2 + \omega^2}$ corresponding to a sinusoidal forcing $f(t) = A\sin(\omega t)$,

The eq. (35) can be inverted as a Bromwich integral

$$u(x,t) = \frac{A\omega}{2\pi i}\int_{Br}\frac{e^{st}e^{-\sqrt{\frac{s^\alpha}{c_0^\alpha}}x}}{(s^2 + \omega^2)}ds \tag{36}$$

The Bromwich integral in eq.(36) can be evaluated by appealing to the theory of complex variables [22-24]. By considering a Hankel-Bromwich path, it can be evaluated as the sum of two contributions:

$$u(x,t) = u_{tr}(x,t) + u_{st}(s,t) \tag{37}$$

representing a transient part and a steady state part respectively.

The transient part, $u_{tr}(x,t)$, arises from the Hankel loop consisting of the small circle and two lines parallel to the negative x-axis (as shown in figure 1in [17]) and is given by

$$u_{tr}(x,t) = \int_0^\infty e^{-rt} K_\alpha(r,\omega,c_0^\alpha,x)dr \qquad (38)$$

with

$$K_\alpha(r,\omega,c_0^\alpha,x) = \frac{A\omega}{\pi(r^2+\omega^2)} e^{-(\frac{r}{c_0})^{\alpha/2} x\cos(\pi\alpha/2)} \sin\left[(\frac{r}{c_0})^{\alpha/2} x\sin(\pi\alpha/2)\right] \qquad (39)$$

The steady part, $u_{st}(x,t)$, arises from the residues of the poles of the integrand in eq. (36) at $s = \pm i\omega$ and is given by

$$u_{st}(x,t) = Ae^{-\left(\frac{\omega}{c_0}\right)^{\alpha/2} x\cos(\pi\alpha/4)} \sin\omega(t - \frac{x}{\omega}\left(\frac{\omega}{c_0}\right)^{\alpha/2} \sin(\pi\alpha/4)) \qquad (40)$$

This completes the formal analysis of the response to sinusoidal forcing of a chain of fractional oscillators in the continuum limit. Numerical results will be presented in the next section.

6. Results and discussion

The results of the last section in equations (36)-(40) are new. Applications of fractional calculus to oscillation and dissipation problems and solutions to integral and fractional order differential equations can be found in Gorenflo and Mainardi [8] Impulse response functions are also discussed in their work. The same methodology has been extended to sinusoidal forcing of a linear chain in the present work. The response of a chain of fractional oscillators in the continuum limit, when subject to a sinusoidal forcing starting from a quiescent state consists of a transient part and a steady part. The transient part decays in time and approaches zero as $t \to \infty$. In the limit $\alpha \to 2$ the transient part vanishes entirely. Furthermore, it exhibits attenuation as a function of distance from the end as indicated by the spatial dependence of the kernel in eq. (39). No simple closed form expressions can be obtained, the only recourse is through numerical integration. Closed form solutions are not available even for simple mechanical models such as the standard linear solid, which are not based on fractional calculus, for the reason that the solutions are deemed mathematically cumbersome as has been discussed by Caputo and Mainardi [5]. These authors have applied methods of fractional calculus to the linear theory of viscoelasticity based on mechanical models and have discussed the propagation of viscoelastic waves[6]. Their work is macroscopic in its approach and is not based on the continuum limit of lattice dynamics. However, some concepts developed by these authors are quite insightful and the results of

the present study can be related to these concepts as follows. Equation (38) can be rewritten by changing the variable $r \to 1/\tau$ as

$$u_{tr} = \frac{A}{c_0^\alpha} \int_0^\infty R(\tau) e^{-t/\tau} d\tau \tag{41}$$

to bring out clearly the decay in time. It is obvious that the decay in time is characterized by a distribution of relaxation times. The relaxation spectrum is given by

$$R(\tau) = \frac{\omega}{\pi(1+\omega^2\tau^2)} e^{-\frac{x}{(c_0\tau)^{\alpha/2}}\cos(\pi\alpha/2)} \times \sin(\frac{x}{(c_0\tau)^{\alpha/2}}\sin(\pi\alpha/2)) \tag{42}$$

It may be noted that the relaxation spectrum depends also on the distance x, which appears in a similarity combination $x/(c_0\tau)^{\alpha/2}$ and that the factor in the exponent, $\cos(\pi\alpha/2)$, may become negative. Hence caution should be exercised in a strict interpretation in terms of relaxation times. The relaxation spectrum is displayed in figure 4 for several values of the parameter α. The values of the other parameters have been chosen to be $x = 1.0$, $c_0 = 1.0$ and $\omega = 1.88$ for this display.

The steady state solution given by equation (40) can be written as

$$u_{st}(x,t) = \frac{A}{c_0^\alpha} e^{-\delta x} \sin\omega(t - x/v) \tag{43}$$

explicitly showing that it as an attenuated wave, propagating with a velocity (phase velocity)

$$v = \frac{\omega c_0^{\alpha/2}}{\omega^{\alpha/2}\sin(\pi\alpha/4)} \tag{44}$$

and characterized by an absorption parameter

$$\delta = \left(\frac{\omega}{c_0}\right)^{\alpha/2}\cos(\pi\alpha/4) \tag{45}$$

In the limit $\alpha \to 2$, $v \to c_0$ and $\delta \to 0$ as can be seen from equations (44) and (45).

Phase velocity as a function of frequency for several values of α is shown in Fig. 5. The variation of the absorption parameter as a function of the driving frequency is shown in Fig.6.
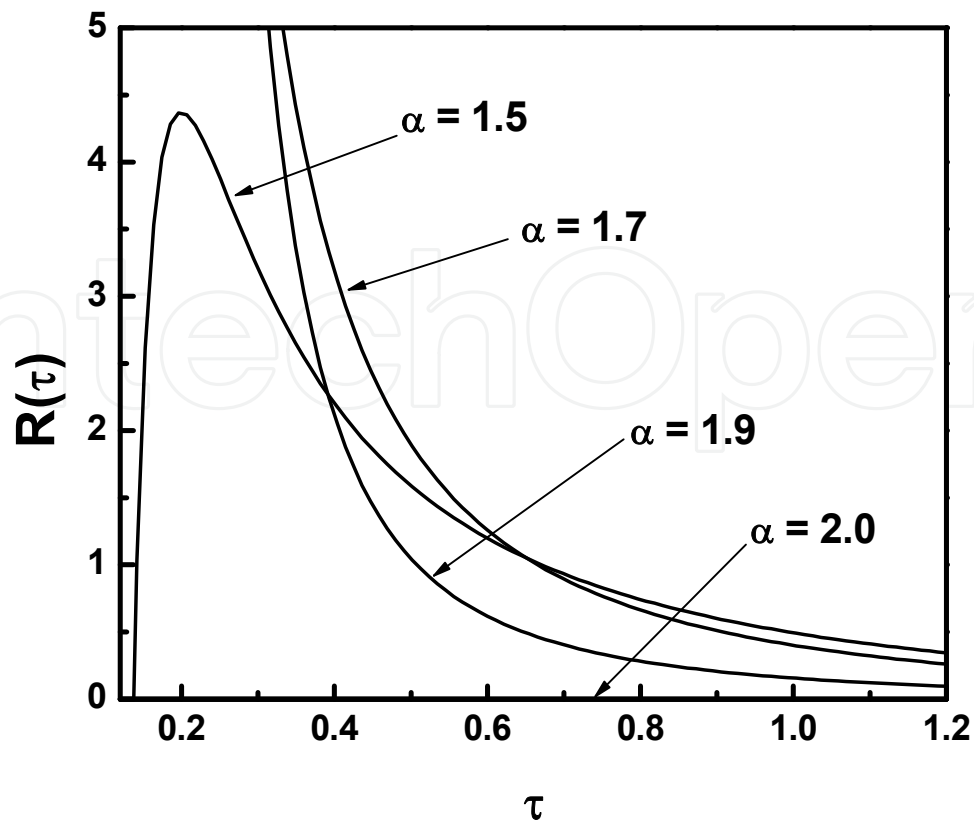
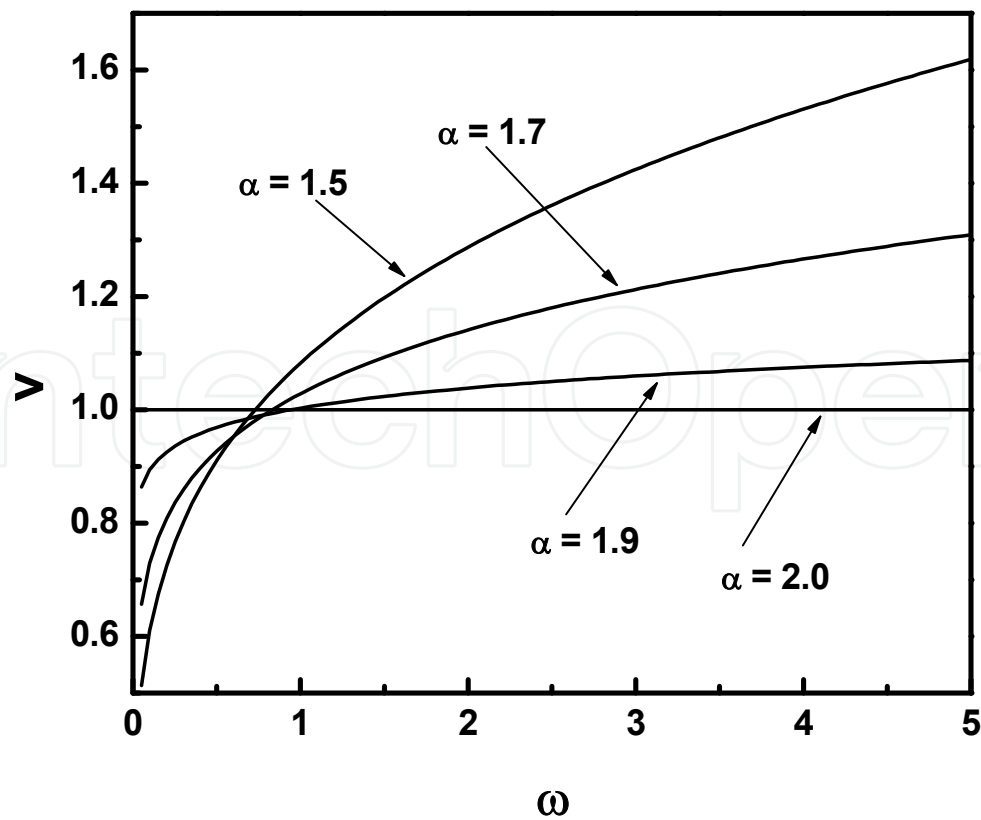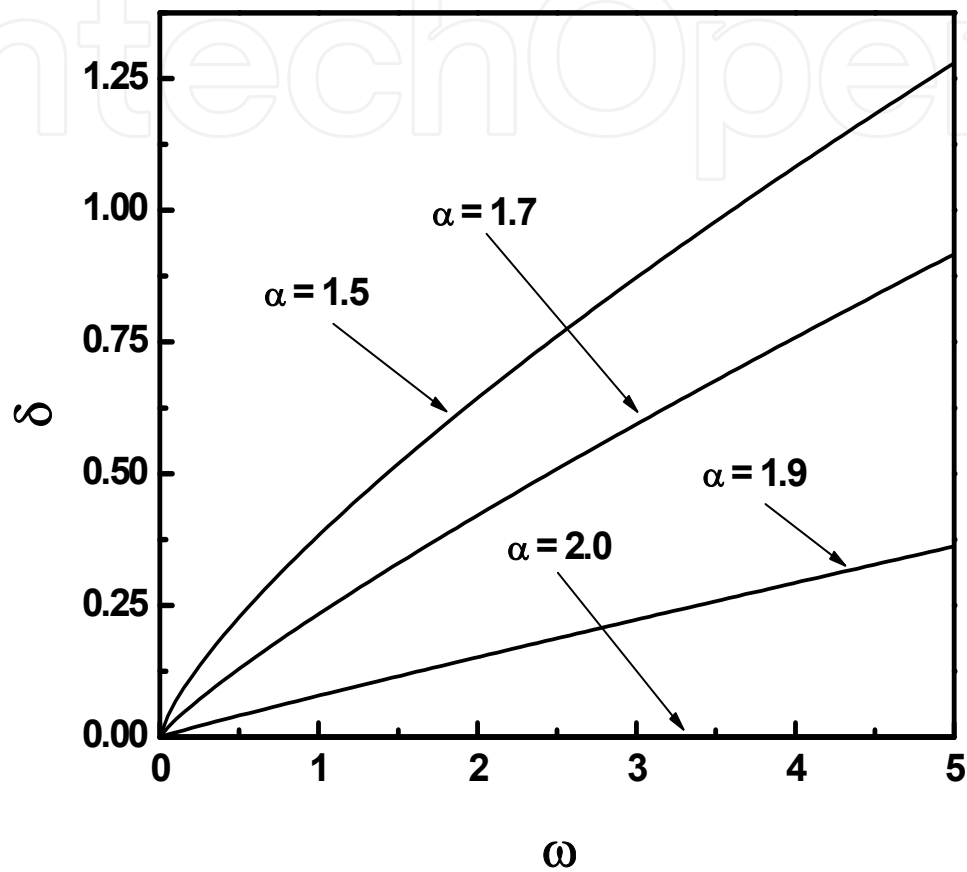**Figure 4.** Relaxation Spectrum for different values of α

Figure 5. Phase velocity as a function of driving frequency (in arbitrary units).

Figure 6. Absorption parameter δ as a function of driving Frequency (in arbitrary units).

According to equation. (40), the amplitude of the response wave decays with distance as $\sim Ae^{-\left(\frac{\omega}{c_0}\right)^{\alpha/2} x \cos(\pi\alpha/4)}$ and hence the energy as $A^2 e^{-2\left(\frac{\omega}{c_0}\right)^{\alpha/2} x \cos(\pi\alpha/4)}$. The specific dissipation function can be defined in terms of the distance over which the energy decays by a factor e^{-1}, which is given by

$$x = \frac{c_0^{\alpha/2}}{2\omega^{\alpha/2}\cos(\pi\alpha/4)} \tag{46}$$

The specific dissipation function is given by Q^{-1} where Q is defined (in analogy with the quality factor 'Q' of harmonic oscillators) as the phase change in radians experienced as the wave travels the distance x in equation. (45) and is given by

$$Q = x \left(\frac{\omega}{c_0} \right)^{\alpha/2} \sin(\pi\alpha / 4) = \frac{1}{2} \tan(\pi\alpha / 4) \tag{47}$$

An alternate definition of 'Q^{-1}' has been given [6] by Caputo and Mainardi, who introduced in analogy with the optical case, a complex 'refractive index, $n = n_r - in_i$. The real and imaginary parts of the refractive index for the wave in eq. (40) are given by

$$n_r = \frac{c_0}{v} = \frac{\omega^{\alpha/2} \sin(\pi\alpha / 4)}{\omega c_0^{\alpha/2-1}} \tag{48}$$

and

$$n_i = \frac{c_{0\delta}}{\omega} = \frac{c_0}{\omega} \left(\frac{\omega}{c_0} \right)^{\alpha/2} \cos(\pi\alpha / 4) \tag{49}$$

According to Caputo and Mainardi [6], the specific dissipation function is given by

$$Q^{-1} = \frac{2n_i}{n_r} = 2\cot(\pi\alpha / 4) \tag{50}$$

which is equivalent to the expression in equation (47).

Kolsky's definition [26, equation 5.22] of Q^{-1}, $(= 2v\delta / \omega$, in our notation), also yields the same expression as in equation (50).

6.1. Further discussion

As explained earlier, the classical theory of viscoelasticity is based on the so called Standard Linear Solid Model characterized by the constitutive relation given in equation (6). This model gives a good qualitative description of the deformation behavior of viscoelastic solids, but has been found to be inadequate to give quantitative account as has been discussed in detail by Gorenflo and Mainardi [8]. More complex mechanical models lead to difficulties in formulation and solution of the resulting complicated differential equations and no satisfactory resolution has been possible [8]. Quantitative agreement is secured only when recourse is made to the so called fractional viscoelastic models. The stress-strain relations are then expressed in terms of fractional order integrals in both creep- and relaxation- representations mentioned earlier and further details can be obtained in reference [9]. In the space-time domain, the approach based on

fractional calculus leads to the propagation of viscoelastic waves and the details can be found in Mainardi[9]. The approach developed in the present work is based on lattice dynamics and hence is a dynamical model. It is microscopic in approach and hence macroscopic properties are described in the so called long wave limit. Further deformation properties which are 'static' in nature are described in the zero frequency limit. As has been shown above the microscopic approach leads to the macroscopic properties in the appropriate limit. It has been noted [26] that the presence of spatial in-homogeneity in polymeric materials gives rise to fractional differential operators in time in the relevant evolution equations, while temporal in-homogeneity leads to fractional differential operators in space. Since only fractional order operation in time has been considered in the present work, there is an implicit in-homogeneity in space which results in a tendency of the particles to cluster and move in a collective fashion. Such collective motion can be considered as elastic waves of very large wavelength, much larger than the scale of the in-homogeneity. This provides a sort of justification for the limiting long wavelength approximation employed.

7. Summary

A microscopic approach based in lattice dynamics, but from the fractional calculus point of view has been developed for the theory of viscoelasticity. The system considered is a linear chain of fractional oscillators subject to a sinusoidal forcing at one end. For the system starting from a quiescent state, the response to sinusoidal forcing consists of a transient and a steady state part. The decay of the transient is characterized by a distribution of relaxation times and the expression for the relaxation spectrum has been obtained. The steady state is essentially a attenuated sinusoidal wave, whose phase velocity and attenuation have been studied and an expression for the specific dissipation function has been obtained. The results obtained are consistent with the results obtained by Mainardi and others from a macroscopic point of view.

Author details

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