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Frequency Transformation for Linear State-Space Systems and Its Application to High-Performance Analog/Digital Filters

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Additional information is available at the end of the chapter

1. Introduction

Frequency transformation is one of the well-known techniques for design of analog and digital filters [1, 2]. This technique is based on variable substitution in a transfer function and allows us to easily convert a given prototype low-pass filter into any kind of frequency selective filter such as low-pass filters of different cutoff frequencies, high-pass filters, band-pass filters, and band-stop filters. It is also well-known that the transformed filters retain some properties of the prototype filter such as the stability and the shape of the magnitude response. For example, if a prototype filter is stable and has the Butterworth magnitude response, any filter given by the frequency transformation is also stable and of the Butterworth characteristic. Due to this useful fact, the frequency transformation is suitable not only to the filter design but also to the real-time tuning of cutoff frequencies, which can be applied to design of variable filters [3] and to adaptive notch filtering [4, 5]. Hence the frequency transformation plays important roles in many modern applications of signal processing from both the theoretical and practical points of view.

The purpose of this chapter is to provide further insights into the theory of frequency transformation from the viewpoint of *internal* properties of filters. In many textbooks on digital signal processing, the frequency transformation is discussed in terms of only the input-output properties, i.e. properties on the transfer function. In other words, few results have been reported about the relationship between the frequency transformation and the internal properties. As is well-known, the internal properties of filters are closely related to the problem of how we should construct a filter structure of a given transfer function, and this problem must be carefully considered in order to obtain analog filters of high dynamic range and low sensitivity [6–12] or digital filters of high accuracy with respect to finite wordlength effects [13–25]. Hence it is worthwhile to investigate the frequency transformation from the viewpoint of the internal properties, and to extend the results to some practical applications.

In order to discuss the frequency transformation from the viewpoint of the internal properties of filters, we make use of the state-space representation. The state-space representation is one of the well-known internal descriptions of linear systems and, in addition, it provides a powerful tool for synthesis of analog/digital filter structures with the aforementioned high-performance. The results from our discussion are twofold. First, we reveal many useful properties of frequency transformation in terms of the state-space representation. The properties to be presented here are closely related to the following three elements of linear state-space systems: the controllability Gramian, the observability Gramian, and the second-order modes. These three elements are known to be very important in characterization of internal properties of analog/digital filters and synthesis of high-performance filter structures. Second, we apply this result to the technique of design and synthesis of analog and digital filters with high performance structures. To be more specific, we present simple and unified frameworks for design and synthesis of analog/digital filters that simultaneously realize the change of frequency characteristics and attain the aforementioned high-performance. Furthermore, we extend this result to variable filters with high-performance structures.

The chapter is organized as follows. Section 2 reviews the fundamentals of the state-space representation of linear systems, including analog filters and digital filters. Section 3 introduces the classical theory of frequency transformation. Sections 4 and 5 are the main theme of this chapter. In Section 4 we discuss the frequency transformation by using the state-space representation and reveal insightful relationships between the frequency transformation and the internal properties of filters. In Section 5 we extend this theory and present new useful methods for design and synthesis of high-performance analog/digital filters.

2. State-space representation, Gramians and second-order modes

In this section we introduce state-space representation of linear systems. In addition, we introduce the aforementioned three elements on the internal properties—controllability Gramian, observability Gramian, and second-order modes—and we address how these elements are applied to synthesis of high-performance filter structures. We will present these topics for digital filters and analog filters, respectively.

2.1. State-space representation of digital filters

Consider the following state-space equations for an N-th order stable single-input/single-output linear discrete-time system:

$$\begin{aligned}
\boldsymbol{x}(n+1) &= \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{b}u(n) \\
y(n) &= \boldsymbol{c}\boldsymbol{x}(n) + du(n)
\end{aligned} \tag{1}$$

where $u(n)$, $y(n)$ and $\boldsymbol{x}(n) \in \Re^{N \times 1}$ denote the scalar input, the scalar output and the state vector, respectively, and $\boldsymbol{A} \in \Re^{N \times N}$, $\boldsymbol{b} \in \Re^{N \times 1}$, $\boldsymbol{c} \in \Re^{1 \times N}$ and $d \in \Re^{1 \times 1}$ are constant coefficients. Throughout this chapter we assume that the system is stable, controllable and observable. If this state-space system represents a digital filter, each entry of $\boldsymbol{x}(n)$

corresponds to each output of delay elements of the filter. Taking the z-transform of (1), we have

$$zX(z) = AX(z) + bU(z)$$
$$Y(z) = cX(z) + dU(z) \tag{2}$$

from which the transfer function $H(z)$ is described in terms of (A, b, c, d) as

$$H(z) = d + c(zI_N - A)^{-1}b \tag{3}$$

where I_N denotes the $N \times N$ identity matrix.

It is well-known that the transfer function $H(z)$ is invariant under nonsingular transformation matrices $T \in \Re^{N \times N}$ of the state: if $x(n)$ is transformed into $\overline{x}(n) = T^{-1}x(n)$, then the state-space system (A, b, c, d) is also transformed into the following set $(\overline{A}, \overline{b}, \overline{c}, \overline{d})$:

$$(\overline{A}, \overline{b}, \overline{c}, \overline{d}) = (T^{-1}AT, T^{-1}b, cT, d). \tag{4}$$

It is easy to show that the transfer function of this new set is the same as that of (A, b, c, d). Therefore, many structures exist for a digital filter with a given transfer function $H(z)$. This nonsingular transformation is called similarity transformation.

We next introduce the controllability Gramian, the observability Gramian, and the second-order modes. For the system (A, b, c, d), the solutions K and W to the following Lyapunov equations are called the controllability Gramian and the observability Gramian, respectively:

$$K = AKA^T + bb^T$$
$$W = A^T WA + c^T c. \tag{5}$$

The Gramians K and W are symmetric and positive definite, i.e. $K = K^T > 0$ and $W = W^T > 0$, because the system (A, b, c, d) is assumed to be stable, controllable and observable. Then, the eigenvalues of the matrix product KW are all positive. We denote these eigenvalues as $\theta_1^2, \theta_2^2, \cdots, \theta_N^2$ and assume that $\theta_1^2 \geq \theta_2^2 \geq \cdots \geq \theta_N^2$. Their positive square roots $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_N$ are called the second-order modes of the system. In the literature on control system theory, the second-order modes are also called Hankel singular values because $\theta_1, \theta_2, \cdots, \theta_N$ are equal to the nonzero singular values of the Hankel operator of $H(z)$.

The two Gramians and the similarity transformation $\overline{x}(n) = T^{-1}x(n)$ are simply related as follows: the controllability/observability Gramians $(\overline{K}, \overline{W})$ of the system in (4) are given by

$$(\overline{K}, \overline{W}) = (T^{-1}KT^{-T}, T^T WT). \tag{6}$$

On the other hand, the second-order modes are invariant under similarity transformation because of the following relationship

$$\overline{KW} = T^{-1}(KW)T. \tag{7}$$

Hence it follows that the Gramians depend on realizations of the system, while the second-order modes depend only on the transfer function.

In the literature on synthesis of filter structures [13–25], it is shown that the two Gramians and the second-order modes play central roles in analysis and optimization of filter performance such as the roundoff noise and the coefficient sensitivity. In other words, given the transfer function of a digital filter, we can formulate some cost functions with respect to the aforementioned filter performance in terms of the two Gramians (K, W), and a filter structure of high performance can be obtained by constructing the two Gramians appropriately in such a manner that they optimize or sub-optimize the corresponding cost functions.

An example of high-performance digital filter structures is the balanced form [15, 16, 18, 23, 25]. This form consists of the two Gramians given by

$$K = W = \Theta \tag{8}$$

where Θ is the diagonal matrix consisting of the second-order modes, i.e.

$$\Theta = \mathrm{diag}(\theta_1, \theta_2, \cdots, \theta_N). \tag{9}$$

Another example is the minimum roundoff noise structure [13, 14, 16, 17], which consists of the two Gramians that satisfy the following relationships

$$W = \left(\frac{1}{N}\sum_{i=1}^{N}\theta_i\right)^2 K$$
$$K_{ii} = 1 \tag{10}$$

where K_{ii} denotes the i-th diagonal entry of K.

Finally, we address the significance of the second-order modes from two practical aspects. First, it is known in the literature that the second-order modes describe the optimal values of the aforementioned cost functions. Therefore, it follows that the optimal performance is determined by the second-order modes of a given transfer function. Another important feature of the second-order modes can be seen in the field of the balanced model reduction [26–28], where it is shown that the second-order modes provide the upper bound of the approximation error between the reduced-order system and the original system.

2.2. State-space representation of analog filters

An N-th order linear continuous-time system (including analog filter) can be described by the following state-space representation

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$$
$$y(t) = \boldsymbol{c}\boldsymbol{x}(t) + du(t) \tag{11}$$

where $u(t)$, $y(t)$ and $\boldsymbol{x}(t) \in \Re^{N \times 1}$ are the scalar input, the scalar output and the state vector of the system, respectively, and $\boldsymbol{A} \in \Re^{N \times N}, \boldsymbol{b} \in \Re^{N \times 1}, \boldsymbol{c} \in \Re^{1 \times N}$ and $d \in \Re^{1 \times 1}$ are constant coefficients. The system $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ is assumed to be stable, controllable and observable. If this system represents a continuous-time analog filter that comprises N integrators, the state vector corresponds to the output signals of these integrators.

Taking the Laplace transform of (11) leads to

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{b}U(s)$$
$$Y(s) = \boldsymbol{c}\boldsymbol{X}(s) + dU(s), \tag{12}$$

which results in the following transfer function

$$H(s) = d + \boldsymbol{c}(s\boldsymbol{I}_N - \boldsymbol{A})^{-1}\boldsymbol{b}. \tag{13}$$

As similar to the discrete-time case, the transfer function is invariant under similarity transformation: if $\boldsymbol{x}(t)$ is transformed by a nonsingular matrix $\boldsymbol{T} \in \Re^{N \times N}$ into $\boldsymbol{T}^{-1}\boldsymbol{x}(t)$, then the new state-space system $(\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{b}, \boldsymbol{c}\boldsymbol{T}, d)$ is an equivalent realization to $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ of the transfer function $H(s)$. Therefore, many circuit topologies exist for an analog filter with a given transfer function $H(s)$.

The controllability Gramian $\boldsymbol{K}$ and the observability Gramian $\boldsymbol{W}$ of a continuous-time state-space system are respectively obtained as the solutions to the following Lyapunov equations:

$$\boldsymbol{A}\boldsymbol{K} + \boldsymbol{K}\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T = \boldsymbol{0}_{N \times N}$$
$$\boldsymbol{A}^T\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A} + \boldsymbol{c}^T\boldsymbol{c} = \boldsymbol{0}_{N \times N} \tag{14}$$

where $\boldsymbol{0}_{N \times N}$ denotes the $N \times N$ zero matrix. By the assumption of the stability, controllability and observability of $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$, the Gramians $\boldsymbol{K}$ and $\boldsymbol{W}$ are shown to be symmetric and positive definite. Then, as in the discrete-time case, the second-order modes $\theta_1, \theta_2, \cdots, \theta_N$ are obtained as the positive square roots of the eigenvalues of $\boldsymbol{K}\boldsymbol{W}$.

The relationship of similarity transformations to the Gramians and the second-order modes in the continuous-time case is the same as that in the discrete-time case. The new Gramians $(\overline{\boldsymbol{K}}, \overline{\boldsymbol{W}})$ of the transformed continuous-time system given by a similarity transformation $\boldsymbol{T}$ are shown to be $(\boldsymbol{T}^{-1}\boldsymbol{K}\boldsymbol{T}^{-T}, \boldsymbol{T}^T\boldsymbol{W}\boldsymbol{T})$, and thus the Gramians depend on realizations of the system. On the other hand, the second-order modes are invariant because $\overline{\boldsymbol{K}\boldsymbol{W}} = \boldsymbol{T}^{-1}(\boldsymbol{K}\boldsymbol{W})\boldsymbol{T}$ holds.

As in the discrete-time case, the Gramians and the second-order modes of continuous-time systems play important roles in synthesis of filter structures of high performance [6–12]. A high-performance structure can be obtained by optimizing or sub-optimizing a prescribed cost function in terms of the controllability and observability Gramians. Such a cost function can be seen as a measure of the dynamic range and the sensitivity of an analog filter. In addition, the optimal values of such cost functions are determined by the second-order modes.

3. Frequency transformation

3.1. Frequency transformation of digital filters

Frequency transformation of digital filters can be seen in the work of Oppenheim [29] and Constantinides [2]. The work of Oppenheim is applied to finite impulse response (FIR) transfer functions, whereas the work of Constantinides is applied to infinite impulse response (IIR) transfer functions. In this chapter, the frequency transformation of digital filters is restricted to the work of Constantinides.

Now let $H(z)$ be the transfer function of a given N-th order digital low-pass filter. The frequency transformation in the discrete-time case is defined as

$$H(F(z)) = H(z)|_{z^{-1} \leftarrow 1/F(z)} \tag{15}$$

which results in a new composite transfer function $H(F(z))$. The function $1/F(z)$ for this transformation is defined as an M-th order stable all-pass function of the form

$$\frac{1}{F(z)} = \pm z^{-M} \frac{G(z^{-1})}{G(z)}$$

$$G(z) = 1 + \sum_{k=1}^{M} g_k z^{-k}. \tag{16}$$

The well-known typical frequency transformations make use of the following four types of all-pass functions

$$\frac{1}{F_{\mathrm{LP}}(z)} = \frac{z^{-1} - \xi}{1 - \xi z^{-1}}$$

$$\frac{1}{F_{\mathrm{HP}}(z)} = -\frac{z^{-1} + \xi}{1 + \xi z^{-1}}$$

$$\frac{1}{F_{\mathrm{BP}}(z)} = -\frac{z^{-2} - \frac{2\xi\eta}{\eta+1} z^{-1} + \frac{\eta-1}{\eta+1}}{1 - \frac{2\xi\eta}{\eta+1} z^{-1} + \frac{\eta-1}{\eta+1} z^{-2}}$$

$$\frac{1}{F_{\mathrm{BS}}(z)} = \frac{z^{-2} - \frac{2\xi\eta}{1+\eta} z^{-1} + \frac{1-\eta}{1+\eta}}{1 - \frac{2\xi\eta}{1+\eta} z^{-1} + \frac{1-\eta}{1+\eta} z^{-2}} \tag{17}$$

which respectively correspond to the low-pass-low-pass (LP-LP), low-pass-high-pass (LP-HP), low-pass-band-pass (LP-BP) and low-pass-band-stop (LP-BS) transformations. The parameters ξ and η determine the cutoff frequencies of the transformed filters. On the block diagram of a digital filter, the frequency transformation means that each delay element z^{-1} in $H(z)$ is replaced[1] with an all-pass filter $1/F(z)$.

3.2. Frequency transformation of analog filters

Let $H(s)$ be the transfer function of a given N-th order analog low-pass filter. The frequency transformation of analog filters is defined as the following variable substitution [1]

$$H(F(s)) = H(s)|_{s^{-1} \leftarrow 1/F(s)}. \tag{18}$$

Hence the frequency transformation yields a new composite transfer function $H(F(s))$ from the prototype transfer function $H(s)$. In general, the cutoff frequency of the prototype low-pass filter is set to be 1 rad/s. From a circuit point of view, the substitution $s^{-1} \leftarrow 1/F(s)$ means that each integrator $1/s$ in the prototype filter $H(s)$ is replaced with another system with the transfer function $1/F(s)$.

The transformation function $1/F(s)$ is defined as the following Foster reactance function [1]

$$\frac{1}{F(s)} = \frac{z(s)}{p(s)} = G \frac{(s^2 + \omega_{z1}^2)(s^2 + \omega_{z2}^2)(s^2 + \omega_{z3}^2) \cdots}{s(s^2 + \omega_{p1}^2)(s^2 + \omega_{p2}^2)(s^2 + \omega_{p3}^2) \cdots} \tag{19}$$

where $G > 0$ and $0 \leq \omega_{z1} < \omega_{p1} < \omega_{z2} < \omega_{p2} < \omega_{z3} < \omega_{p3} < \cdots$. The Foster reactance functions are determined in such a manner that the degree of difference of $p(s)$ and $z(s)$ is 1, i.e. $|\deg p(s) - \deg z(s)| = 1$. In the case of the well-known typical LP-LP, LP-HP, LP-BP and LP-BS transformations, the reactance functions are respectively given by

$$\frac{1}{F_{\text{LP}}(s)} = \frac{G}{s}$$

$$\frac{1}{F_{\text{HP}}(s)} = Gs$$

$$\frac{1}{F_{\text{BP}}(s)} = \frac{Gs}{s^2 + \omega_{p1}^2}$$

$$\frac{1}{F_{\text{BS}}(s)} = \frac{G(s^2 + \omega_{z1}^2)}{s}. \tag{20}$$

The parameters G, ω_{p} and ω_{s} determine the cutoff frequencies of the transformed filters.

---

[1] To be precise, replacing z^{-1} with another transfer function often yields a delay-free loop. In this case, some extra processing such as reformulation of the coefficients of the transformed filter is required after this replacement.

It is important to note that the Foster reactance functions are classified into two categories—strictly proper reactance functions and improper reactance functions[2]. In the typical frequency transformations of (20), $1/F_{\mathrm{LP}}(s)$ and $1/F_{\mathrm{BP}}(s)$ correspond to strictly proper reactance functions, whereas $1/F_{\mathrm{HP}}(s)$ and $1/F_{\mathrm{BS}}(s)$ are improper reactance functions.

4. State-space analysis of frequency transformation

In this section, we discuss the frequency transformation from the viewpoint of the internal properties. In other words, we show many interesting results of the frequency transformation in terms of the state-space representation.

This research has its roots in the work of Mullis and Roberts [30], where they presented a simple state-space formulation of frequency transformation for digital filters and they proved an important property of the second-order modes—they are invariant under frequency transformation. In addition, they provided practical impacts of these results on the design and synthesis of high-performance digital filters.

In this chapter we start with introducing this work, and then we further extend this result and present other theoretical results on the relationship between the frequency transformation and the state-space representation of discrete-time systems. In addition, we also present similar results for continuous-time systems.

4.1. State-space formulation of frequency transformation for digital filters and invariance of second-order modes

Mullis and Roberts [30] first presented an explicit state-space representation of frequency transformation as follows. Let (A, b, c, d) be a state-space representation of a given prototype filter $H(z)$. Then, the transfer function $H(F(z))$ that is given by the frequency transformation (15) with an M-th order all-pass function $1/F(z)$ can be explicitly described by

$$H(F(z)) = \mathcal{D} + \mathcal{C}(z I_{MN} - \mathcal{A})^{-1} \mathcal{B} \tag{21}$$

with the following coefficients

$$\begin{aligned}
\mathcal{A} &= I_N \otimes \alpha + [A(I_N - \delta A)^{-1}] \otimes (\beta \gamma) \\
\mathcal{B} &= [(I_N - \delta A)^{-1} b] \otimes \beta \\
\mathcal{C} &= [c(I_N - \delta A)^{-1}] \otimes \gamma \\
\mathcal{D} &= d + \delta c(I_N - \delta A)^{-1} b
\end{aligned} \tag{22}$$

where $(\alpha, \beta, \gamma, \delta)$ is an arbitrary state-space representation of $1/F(z)$, and $\otimes$ stands for the Kronecker product for matrices.

---

[2] A rational function $G(s) = N(s)/D(s)$ is called strictly proper if $\deg N(s) < \deg D(s)$. On the other hand, $G(s)$ is called improper if $\deg N(s) > \deg D(s)$. Since the Foster reactance functions given by (19) always satisfy $|\deg p(s) - \deg z(s)| = 1$, there does not exist any reactance function such that $\deg p(s) = \deg z(s)$.

The significance of the description given by (22) lies in the fact that, by using this description, we can easily carry out the frequency transformation on a state-space structure as well as a transfer function. Also, note that this description does not include any delay-free loop.

In addition to the above state-space formulation, Mullis and Roberts also described the Gramians and the second-order modes of the transformed system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. The two Gramians, which are respectively denoted by $\mathcal{K}$ and $\mathcal{W}$, are given as follows:

$$\begin{aligned} \mathcal{K} &= K \otimes Q \\ \mathcal{W} &= W \otimes Q^{-1} \end{aligned}$$ (23)

where Q is the controllability Gramian of the all-pass system $(\alpha, \beta, \gamma, \delta)$. From this relationship we easily see

$$\mathcal{K}\mathcal{W} = (KW) \otimes I_M$$ (24)

which means that the matrix product $\mathcal{K}\mathcal{W}$ have the same eigenvalues as KW with multiplicity M. This shows that the second-order modes of transformed filters are the same as those of a given prototype filter. Hence the second-order modes of digital filters are invariant under frequency transformation.

The practical benefit of this invariance property is discussed as follows. As stated in Section 2, the second-order modes determine the optimal values of cost functions with respect to finite wordlength effects. In [30], using the fact that the minimum roundoff noise is characterized by the second-order modes, it was proved that the minimum attainable value of the roundoff noise of digital filters is independent of the filter characteristics that are controlled by the frequency transformation. A similar conclusion can be drawn for the balanced model reduction: the upper bound of the approximation error due to the balanced model reduction is invariant under frequency transformation.

Furthermore, in the case of the LP-LP transformation, the work of [30] also presents the specific state-space-based frequency transformation that can preserve the optimal realizations. This specific transformation is given by

$$\begin{aligned} \mathcal{A} &= (\xi I_N + A)(I_N + \xi A)^{-1} \\ \mathcal{B} &= \sqrt{1 - \xi^2}(I_N + \xi A)^{-1}b \\ \mathcal{C} &= \sqrt{1 - \xi^2}c(I_N + \xi A)^{-1} \\ \mathcal{D} &= d - \xi c(I_N + \xi A)^{-1}b. \end{aligned}$$ (25)

By setting the prototype state-space filter (A, b, c, d) to be the optimal realization and applying (25), we can obtain arbitrary low-pass filters that have the same optimal realization as the prototype filter.

In the rest of this section, we will provide our results that are derived by further extending these results.
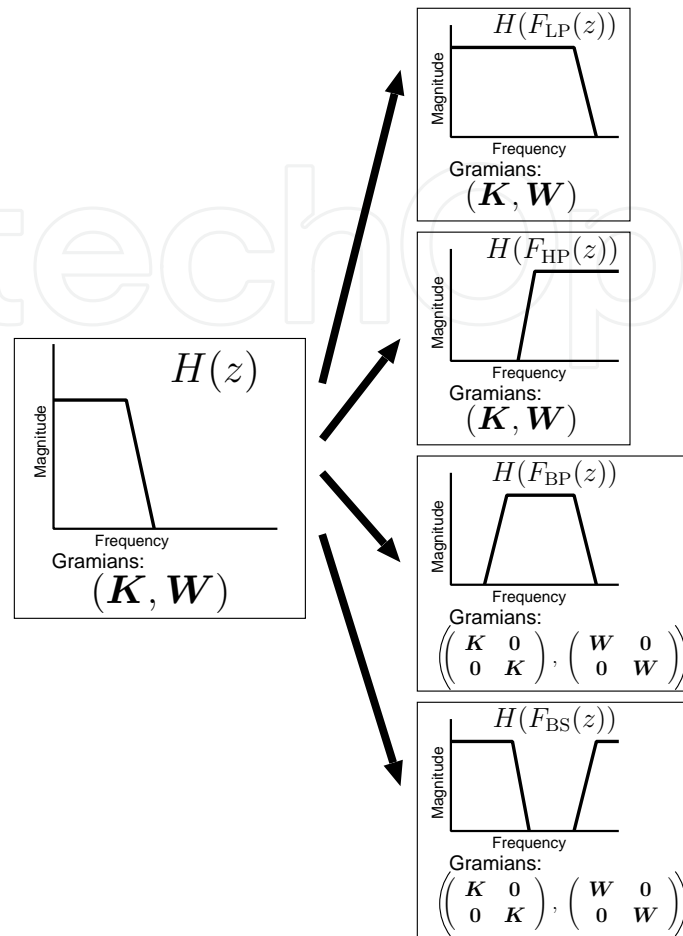
Figure 1. Gramian-preserving frequency transformation.

4.2. Gramian-preserving frequency transformation for digital filters

Here we pay special attention to the controllability and observability Gramians, and we provide a new state-space formulation of frequency transformation that can keep these Gramians invariant. This new state-space-based frequency transformation is called the Gramian-preserving frequency transformation [31] and includes the formulation of (25) as a special case.

Before showing the mathematical formulation of the Gramian-preserving frequency transformation, we first discuss how the Gramian-preserving frequency transformation is related to design and synthesis of digital filters. Simple examples for design/synthesis of low-pass, high-pass, band-pass and band-stop filters are given in Fig. 1. Here, suppose that we are given a prototype low-pass filter with the transfer function $H(z)$, as shown at the left of this figure. Also, let the controllability/observability Gramians of this prototype filter be $\boldsymbol{K}$ and $\boldsymbol{W}$, respectively. Then, by applying the Gramian-preserving frequency transformation to this prototype filter, we can convert this filter into other arbitrary low-pass, high-pass, band-pass and band-stop filters that consist of the *same*

controllability/observability Gramians as those of the prototype filter[3]. Now, recalling that high-performance structures can be obtained by appropriate choice of the Gramians, we notice that the Gramian-preserving frequency transformation is a very powerful technique for simultaneous design and synthesis of high-performance digital fitlers. That is, if we prepare the structure of a given prototype low-pass filter as a high-performance one such as the balanced form and the minimum roundoff noise form, the Gramian-preserving frequency transformation enables us to obtain other types of filters with the same high-performance structure. This fact is also true for analog filters, as will be shown later in the next subsection.

We now present the mathematical formulation of the Gramian-preserving frequency transformation. Given a prototype state-space digital filter (A, b, c, d) with the transfer function $H(z)$ and an M-th order all-pass function $1/F(z)$, the following description provides the Gramian-preserving frequency transformation to produce the composite transfer function $H(F(z))$:

$$
\begin{aligned}
\widetilde{\mathcal{A}} &= \widetilde{\alpha} \otimes I_N + (\widetilde{\beta}\widetilde{\gamma}) \otimes [A(I_N - \widetilde{\delta}A)^{-1}] \\
\widetilde{\mathcal{B}} &= \widetilde{\beta} \otimes [(I_N - \widetilde{\delta}A)^{-1}b] \\
\widetilde{\mathcal{C}} &= \widetilde{\gamma} \otimes [c(I_N - \widetilde{\delta}A)^{-1}] \\
\widetilde{\mathcal{D}} &= d + \widetilde{\delta}c(I_N - \widetilde{\delta}A)^{-1}b
\end{aligned}
\tag{26}
$$

where the set $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta})$ is a state-space representation of $1/F(z)$ with the controllability/observability Gramians equal to the identity matrix, i.e.

$$
\widetilde{\alpha}\widetilde{\alpha}^T + \widetilde{\beta}\widetilde{\beta}^T = \widetilde{\alpha}^T\widetilde{\alpha} + \widetilde{\gamma}^T\widetilde{\gamma} = I_M.
\tag{27}
$$

This relationship means that the set $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta})$ is a balanced form. It should be noted that such a set always exists if $1/F(z)$ is stable.

Now we turn our attention to the mathematical formulation of the Gramians of $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$, which are respectively denoted by $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{W}}$. They are given in terms of the Gramians of the prototype filter as follows:

$$
\begin{aligned}
\widetilde{\mathcal{K}} &= I_M \otimes K \\
\widetilde{\mathcal{W}} &= I_M \otimes W
\end{aligned}
\tag{28}
$$

which means that $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{W}}$ become block diagonal matrices with M diagonal blocks all equal to K and W. Therefore, as stated earlier, $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{W}}$ respectively become the same as K and W with multiplicity M. Hence (26) preserves the Gramians under frequency transformation.

---

[3] In the case of LP-BP and LP-BS transformations, the transformed filters have the same Gramians with multiplicity 2 as those of the prototype filter. This is because the all-pass functions $1/F_{BP}(z)$ and $1/F_{BS}(z)$ are second-order functions and the order of $H(F_{BP}(z))$ and $H(F_{BS}(z)$ become twice as high as that of $H(z)$.

We next discuss the Gramian-preserving frequency transformation from a realization point of view. From (27), we first see that realization of the Gramian-preserving frequency transformation requires us to construct the structure of the all-pass filter $1/F(z)$ appropriately such that its state-space representation becomes a balanced form. Although formulation of the balanced form is known to be non-unique for a given transfer function, we presented a useful technique [31]: given an all-pass transfer function $1/F(z)$, its normalized lattice structure becomes a balanced form, which enables us to realize the Gramian-preserving frequency transformation. This is derived from the fact that $1/F(z)$ is all-pass. Now, recall that the frequency transformation of digital filters means that each delay element in a prototype filter is replaced with an all-pass filter (and delay-free loops, if any, are eliminated after this replacement)[4]. In view of this, we can conclude that the Gramian-preserving frequency transformation is interpreted as the replacement of each delay element in the prototype filter with the all-pass filter that has the normalized lattice structure. Figure 2 illustrates this scheme. Given a state-space prototype filter as in Fig. 2(a), we carry out the aforementioned replacement and we obtain the transformed state-space filter as in Fig. 2(b). The all-pass filter that is included in this structure consists of M lattice sections $\Phi_1, \cdots, \Phi_M$, and each section Φ_i is given as in Fig. 2(c). The variable ξ_i for $1 \leq i \leq M$ denotes the i-th lattice coefficient for $1/F(z)$, and $\widehat{\xi}_i = \sqrt{1 - \xi^2}$.

Finally, we provide the mathematical formulation of the Gramian-preserving frequency transformation based on the normalized lattice structure. The normalized lattice structure of $1/F(z)$ can be given by the following state-space representation:

$$
\widetilde{\alpha} = \begin{pmatrix}
-\xi_1 & -\widehat{\xi}_1\xi_2 & -\widehat{\xi}_1\widehat{\xi}_2\xi_3 & \cdots & -\widehat{\xi}_1\widehat{\xi}_2\widehat{\xi}_3\cdots\widehat{\xi}_{M-3}\xi_{M-2} & -\widehat{\xi}_1\widehat{\xi}_2\widehat{\xi}_3\cdots\widehat{\xi}_{M-2}\xi_{M-1} & -\widehat{\xi}_1\widehat{\xi}_2\widehat{\xi}_3\cdots\widehat{\xi}_{M-1}\xi_M \\
\widehat{\xi}_1 & -\xi_1\xi_2 & -\xi_1\widehat{\xi}_2\xi_3 & \cdots & -\xi_1\widehat{\xi}_2\widehat{\xi}_3\cdots\widehat{\xi}_{M-3}\xi_{M-2} & -\xi_1\widehat{\xi}_2\widehat{\xi}_3\cdots\widehat{\xi}_{M-2}\xi_{M-1} & -\xi_1\widehat{\xi}_2\widehat{\xi}_3\cdots\widehat{\xi}_{M-1}\xi_M \\
0 & \widehat{\xi}_2 & -\xi_2\xi_3 & \cdots & -\xi_2\widehat{\xi}_3\widehat{\xi}_4\cdots\widehat{\xi}_{M-3}\xi_{M-2} & -\xi_2\widehat{\xi}_3\widehat{\xi}_4\cdots\widehat{\xi}_{M-2}\xi_{M-1} & -\xi_2\widehat{\xi}_3\widehat{\xi}_4\cdots\widehat{\xi}_{M-1}\xi_M \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \widehat{\xi}_{M-1} & -\xi_{M-1}\xi_M
\end{pmatrix}
$$

$$
\widetilde{\beta} = \begin{pmatrix}
\widehat{\xi}_1\widehat{\xi}_2\widehat{\xi}_3\cdots\widehat{\xi}_{M-1}\widehat{\xi}_M \\
\xi_1\widehat{\xi}_2\widehat{\xi}_3\cdots\widehat{\xi}_{M-1}\widehat{\xi}_M \\
\xi_2\widehat{\xi}_3\cdots\widehat{\xi}_{M-1}\widehat{\xi}_M \\
\vdots \\
\xi_{M-2}\widehat{\xi}_{M-1}\widehat{\xi}_M \\
\xi_{M-1}\widehat{\xi}_M
\end{pmatrix}
$$

$$
\widetilde{\gamma} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \pm\widehat{\xi}_M \end{pmatrix}
$$

$$
\widetilde{\delta} = \pm\xi_M. \tag{29}
$$

Therefore, substitution of (29) into (26) carries out the Gramian-preserving frequency transformation. Note that the state-space representation $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ given in this way becomes sparse due to many zero entries in $\widetilde{\alpha}$ and $\widetilde{\gamma}$. To be precise, the set $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$

---

[4] Note that the mathematical formulation of the Gramian-preserving frequency transformation (26) is derived after elimination of delay-free loops. Therefore, (26) does not have the problem of delay-free loops. See [30] for the details.
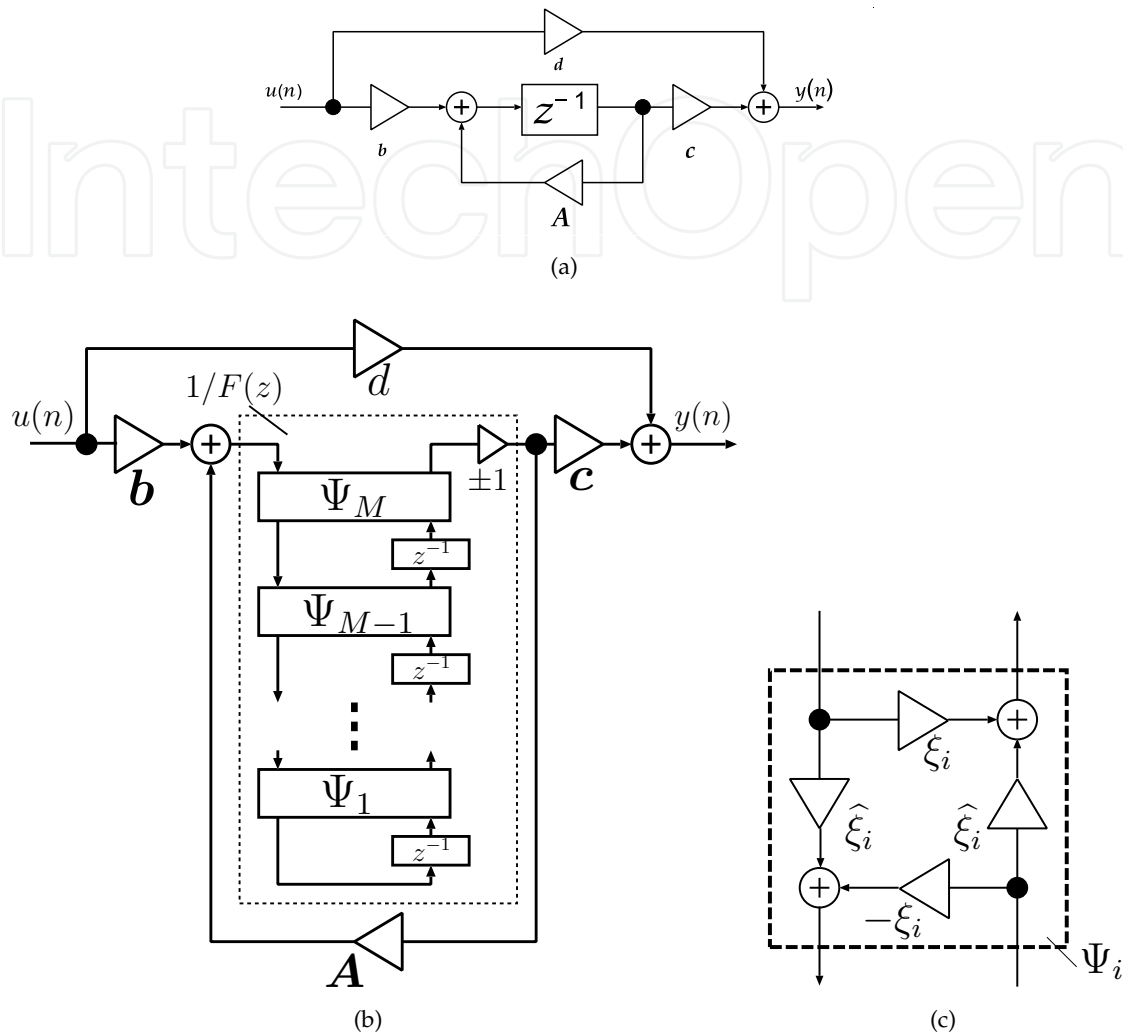
Figure 2. Gramian-preserving frequency transformation: (a) prototype state-space filter, (b) transformed state-space filter, and (c) a normalized lattice section Ψ_i.

has in total $(M-1)N(MN-M/2)$ zero entries. Hence this state-space filter is very suitable to implementation.

4.3. Results for analog filters

In the case of analog filters, little had been reported about the state-space analysis of frequency transformation. On the other hand, our work [32–34] has derived many results that are similar to the discrete-time case. Here we will introduce these results.

We first present a state-space formulation of frequency transformation for analog filters. One thing to be noted here is that, as stated in Section 3.2, the frequency transformation functions (i.e. Foster reactance functions) are classified into strictly proper functions and improper

functions. In this chapter we focus on the case of strictly proper reactance functions, which include the LP-LP and the LP-BP transformations.

Now consider a state-space representation (A, b, c, d) of a given prototype low-pass filter with the transfer function $H(s)$. Also, let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be a state-space representation of $H(F(s))$, where $1/F(s)$ denotes a strictly proper Foster reactance function. Then, $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ can be given in terms of (A, b, c, d) as follows:

$$\begin{aligned}
\mathcal{A} &= I_N \otimes \alpha + A \otimes (\beta\gamma) \\
\mathcal{B} &= b \otimes \beta \\
\mathcal{C} &= c \otimes \gamma \\
\mathcal{D} &= d
\end{aligned} \tag{30}$$

where the set (α, β, γ) shown here is an arbitrary state-space representation of $1/F(s)$, i.e.

$$1/F(s) = \gamma(sI_M - \alpha)^{-1}\beta \tag{31}$$

and M is the order of $1/F(s)$, i.e. $M = \deg p(s)$ in (19). Note that the d-term in a state-space representation of $1/F(s)$ becomes zero because the reactance function is strictly proper. Therefore, the state-space-based frequency transformation given here is simpler than the discrete-time case (22).

Next we discuss the second-order modes of analog filters under frequency transformation. Let (K, W) and $(\mathcal{K}, \mathcal{W})$ be the controllability/observability Gramians of (A, b, c, d) and $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, respectively. Using (30), we can prove the following property:

$$\begin{aligned}
\mathcal{K} &= K \otimes P^{-1} \\
\mathcal{W} &= W \otimes P
\end{aligned} \tag{32}$$

where P is the positive definite matrix that satisfies the following relationship called the lossless positive-real lemma:

$$\begin{aligned}
\alpha^T P + P\alpha &= 0_{M \times M} \\
P\beta &= \gamma^T.
\end{aligned} \tag{33}$$

From (32) we easily see

$$\mathcal{K}\mathcal{W} = (KW) \otimes I_M \tag{34}$$

which proves that the second-order modes of analog filters are invariant under frequency transformation.

We now present the Gramian-preserving frequency transformation for analog filters. Let $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ be the state-space filter that is given by this transformation. Then, $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ is formulated as

$$\widetilde{\mathcal{A}} = \widetilde{\alpha} \otimes I_N + (\widetilde{\beta}\widetilde{\gamma}) \otimes A$$
$$\widetilde{\mathcal{B}} = \widetilde{\beta} \otimes b$$
$$\widetilde{\mathcal{C}} = \widetilde{\gamma} \otimes c$$
$$\widetilde{\mathcal{D}} = d \tag{35}$$

where $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$ is a state-space representation of $1/F(s)$ that satisfies $P = I_M$ in (33), i.e.

$$\widetilde{\alpha}^T + \widetilde{\alpha} = \mathbf{0}_{M \times M}$$
$$\widetilde{\beta} = \widetilde{\gamma}^T. \tag{36}$$

For $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ described as above, the controllability/observability Gramians $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{W}})$ are found to be

$$\widetilde{\mathcal{K}} = I_M \otimes K$$
$$\widetilde{\mathcal{W}} = I_M \otimes W. \tag{37}$$

Needless to say, this relationship is the same as in the discrete-time case (28). Hence the Gramians of a prototype state-space filter are preserved under this transformation.

As in the discrete-time case, formulation of $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$ is known to be non-unique. In [34], we presented a closed-form representation of $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$ that will be very suitable to circuit implementation. In order to derive this representation, we first rewrite the Foster reactance function (19) as the following partial fraction

$$\frac{1}{F(s)} = \sum_{i=1}^{L} \frac{G_i s}{s^2 + \omega_{\mathrm{p}i}^2} + \frac{G_0}{s} \tag{38}$$

where $G_1, \cdots G_L$ and G_0 are all real and nonnegative, and $L = \lfloor M/2 \rfloor$, i.e. L is the largest integer less than or equal to $M/2$. Note that $G_0 = 0$ holds if M is even. Also, note that the first term on the right-hand side of (38) vanishes if $M = 1$. Now we can formulate the desired state-space representation of $1/F(s)$ by using the parameters of (38). The formulation depends on the value of M, i.e. the order of $1/F(s)$. For even M, we give the desired state-space representation, which is denoted by $(\widetilde{\alpha}_{\mathrm{even}}, \widetilde{\beta}_{\mathrm{even}}, \widetilde{\gamma}_{\mathrm{even}})$, as follows:

$$\widetilde{\alpha}_{\mathrm{even}} = \mathrm{block\ diag}\left(\Omega_{\mathrm{p}1}, \Omega_{\mathrm{p}2}, \cdots \Omega_{\mathrm{p}L}\right)$$
$$\widetilde{\beta}_{\mathrm{even}} = \left(\widetilde{\psi}_1^T\ \widetilde{\psi}_2^T\ \cdots\ \widetilde{\psi}_L^T\right)^T$$
$$\widetilde{\gamma}_{\mathrm{even}} = \widetilde{\beta}_{\mathrm{even}}^T \tag{39}$$

where $\Omega_{\mathrm{p}i} \in \Re^{2 \times 2}$ and $\widetilde{\psi}_i \in \Re^{2 \times 1}$ for $M = 1, 2, \cdots, L$ are respectively given by

$$\Omega_{\mathrm{p}i} = \begin{pmatrix} 0 & \omega_{\mathrm{p}i} \\ -\omega_{\mathrm{p}i} & 0 \end{pmatrix}$$

$$\widetilde{\psi}_i = \begin{pmatrix} \sqrt{G_i} \\ 0 \end{pmatrix}. \tag{40}$$

If M is odd, we give the desired state-space representation $(\widetilde{\alpha}_{\mathrm{odd}}, \widetilde{\beta}_{\mathrm{odd}}, \widetilde{\gamma}_{\mathrm{odd}})$ as

$$\widetilde{\alpha}_{\mathrm{odd}} = \begin{pmatrix} \widetilde{\alpha}_{\mathrm{even}} & \mathbf{0}_{2L \times 1} \\ \mathbf{0}_{1 \times 2L} & \mathbf{0}_{1 \times 1} \end{pmatrix}$$

$$\widetilde{\beta}_{\mathrm{odd}} = \left(\widetilde{\beta}_{\mathrm{even}}^T \quad \sqrt{G_0} \right)^T$$

$$\widetilde{\gamma}_{\mathrm{odd}} = \widetilde{\beta}_{\mathrm{odd}}^T. \tag{41}$$

Note that the above expression reduces to $(\widetilde{\alpha}_{\mathrm{odd}}, \widetilde{\beta}_{\mathrm{odd}}, \widetilde{\gamma}_{\mathrm{odd}}) = (0, \sqrt{G_0}, \sqrt{G_0})$ if $M = 1$. By direct calculation it is easy to prove that the state-space representations (39) and (41) satisfy the transfer function $1/F(s)$ given by (38) for even M and odd M, respectively, and that they also satisfy $P = I_M$ in the lossless positive-real lemma, i.e.

$$\widetilde{\alpha}_{\mathrm{even}}^T + \widetilde{\alpha}_{\mathrm{even}} = \mathbf{0}_{M \times M}$$

$$\widetilde{\beta}_{\mathrm{even}} = \widetilde{\gamma}_{\mathrm{even}}^T$$

$$\widetilde{\alpha}_{\mathrm{odd}}^T + \widetilde{\alpha}_{\mathrm{odd}} = \mathbf{0}_{M \times M}$$

$$\widetilde{\beta}_{\mathrm{odd}} = \widetilde{\gamma}_{\mathrm{odd}}^T. \tag{42}$$

This result shows that (39) and (41) offer the closed-form expression for the Gramian-preserving frequency transformation.

Finally, we discuss the physical interpretation of the Gramian-preserving frequency transformation, which will bring further insight into the circuit theory. As in the discrete-time case, we first discuss the Gramian-preserving frequency transformation in terms of the block diagram. As illustrated in Fig. 3, the Gramian-preserving frequency transformation for analog filters is derived from the model of Fig. 3(b), which is given by replacing the integrators in the prototype filter of Fig. 3(a) with an appropriate state-space representation $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$ of the Foster reactance function $1/F(s)$. Here, we have to consider how the circuit topology of the set $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$ is constructed. In order to answer this, consider again the partial fraction of strictly proper Foster reactance functions $1/F(s)$ as in (38). This expression is well-known as the *LC* driving-point impedance functions corresponding to the first Foster canonical form [1], which is realized by the series connection of a capacitor of capacitance $1/G_0$ and L parallel combinations of an inductor of inductance $G_i/\omega_{\mathrm{p}i}^2$ and a capacitor of capacitance $1/G_i$.

Figure 4(a) shows the circuit representation of $1/F(s)$, where $1/F(s)$ is related to V and I as $1/F(s) = V(s)/I(s)$. This circuit is easily expressed in state-space form as
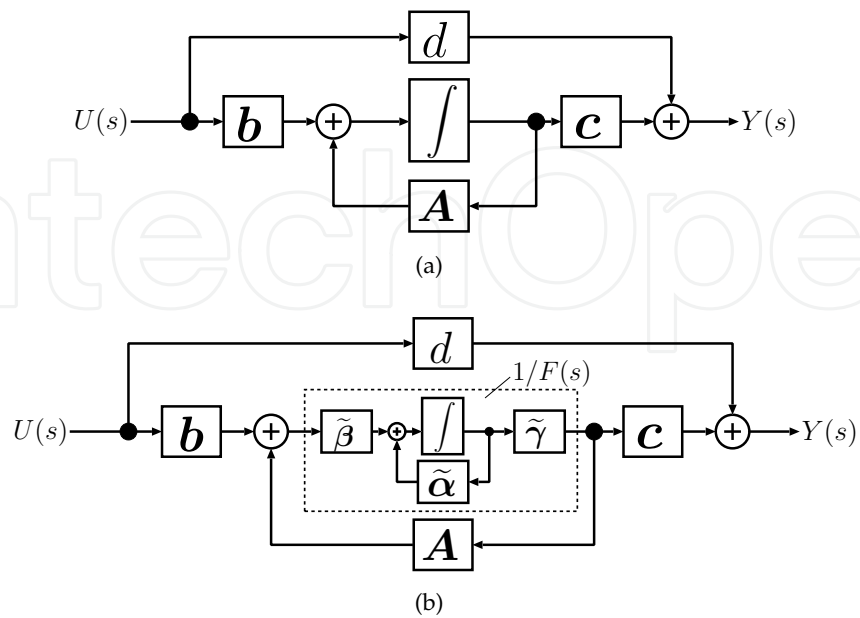
Figure 3. Gramian-preserving frequency transformation for analog filters: (a) prototype state-space filter, and (b) transformed state-space filter.

$$\frac{1}{F(s)} = \sum_{i=1}^{L} \gamma_i (sI_2 - \alpha_i)^{-1} \beta_i$$
$$+ \gamma_0 (sI_1 - \alpha_0)^{-1} \beta_0 \tag{43}$$

where the subsystems $(\alpha_i, \beta_i, \gamma_i)$ for $1 \leq i \leq L$ and $(\alpha_0, \beta_0, \gamma_0)$ are found to be

$$\alpha_i = \begin{pmatrix} 0 & G_i \\ -\frac{\omega_{pi}^2}{G_i} & 0 \end{pmatrix}$$
$$\beta_i = \begin{pmatrix} G_i \\ 0 \end{pmatrix}$$
$$\gamma_i = \begin{pmatrix} 1 & 0 \end{pmatrix}$$
$$\alpha_0 = 0$$
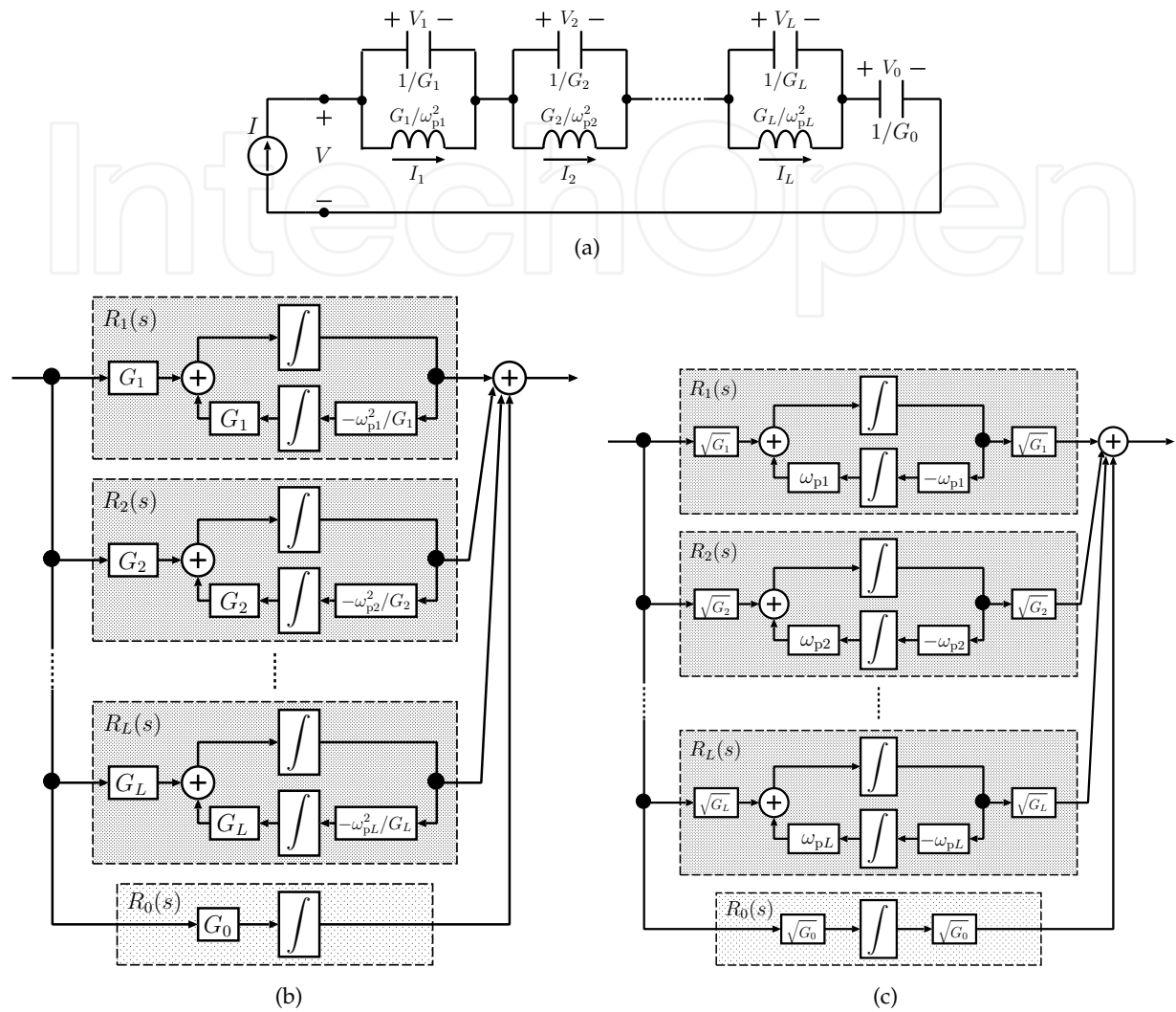$$\beta_0 = G_0$$
$$\gamma_0 = 1 \tag{44}$$

Figure 4. Construction of desired state-space model of $1/F(s)$ for Gramian-preserving frequency transformation: (a) LC circuit representation of $1/F(s)$, (b) state-space model of the LC circuit, and (c) desired state-space model.

with their state vectors $\boldsymbol{X}_i(s)$ and $\boldsymbol{X}_0(s)$ defined as

$$\boldsymbol{X}_i(s) = \left(V_i(s) \ -I_i(s) \right)^T$$
$$\boldsymbol{X}_0(s) = V_0. \tag{45}$$

Figure 4(b) shows the state-space model of $1/F(s)$ described as above. Substituting (44) into (33), we obtain the solutions $\boldsymbol{P}_i$ and $\boldsymbol{P}_0$ to the lossless positive-lemma for $(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \boldsymbol{\gamma}_i)$ and

$(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, \boldsymbol{\gamma}_0)$ as follows:

$$
\begin{aligned}
\boldsymbol{P}_i &= \mathrm{diag}(1/G_i, \ G_i/\omega_{\mathrm{p}i}^2) \\
\boldsymbol{P}_0 &= 1/G_0.
\end{aligned}
\tag{46}
$$

From (46) we see that the state-space model of Fig. 4(b) does not satisfy $\boldsymbol{P} = \boldsymbol{I}_M$ in (33). Hence it is necessary to modify the structure of this model such that $\boldsymbol{P}_i = \boldsymbol{I}_2$ and $\boldsymbol{P}_0 = \boldsymbol{I}_1$ hold. To this end, we consider the following nonsingular matrices

$$
\begin{aligned}
\boldsymbol{T}_i &= \mathrm{diag}(\sqrt{G_i}, \ \omega_{\mathrm{p}i}/\sqrt{G_i}), \quad 1 \le i \le L \\
\boldsymbol{T}_0 &= \sqrt{G_0}.
\end{aligned}
\tag{47}
$$

Note that these matrices satisfy $\boldsymbol{T}_i \boldsymbol{T}_i^T = \boldsymbol{P}_i^{-1}$ and $\boldsymbol{T}_0 \boldsymbol{T}_0^T = \boldsymbol{P}_0^{-1}$. Using these matrices, we apply the similarity transformation to (44), which results in the new structure $(\boldsymbol{\alpha}_i', \boldsymbol{\beta}_i', \boldsymbol{\gamma}_i')$ and $(\boldsymbol{\alpha}_0', \boldsymbol{\beta}_0', \boldsymbol{\gamma}_0')$ as

$$
\begin{aligned}
\boldsymbol{\alpha}_i' &= \boldsymbol{T}_i^{-1} \boldsymbol{\alpha}_i \boldsymbol{T}_i = \begin{pmatrix} 0 & \omega_{\mathrm{p}i} \\ -\omega_{\mathrm{p}i} & 0 \end{pmatrix} \\
\boldsymbol{\beta}_i' &= \boldsymbol{T}_i^{-1} \boldsymbol{\beta}_i = \begin{pmatrix} \sqrt{G_i} \\ 0 \end{pmatrix} \\
\boldsymbol{\gamma}_i' &= \boldsymbol{\gamma}_i \boldsymbol{T}_i = \begin{pmatrix} \sqrt{G_i} & 0 \end{pmatrix} \\
\boldsymbol{\alpha}_0' &= \boldsymbol{T}_0^{-1} \boldsymbol{\alpha}_0 \boldsymbol{T}_0 = 0 \\
\boldsymbol{\beta}_0' &= \boldsymbol{T}_0^{-1} \boldsymbol{\beta}_0 = \sqrt{G_0} \\
\boldsymbol{\gamma}_0' &= \boldsymbol{\gamma}_0 \boldsymbol{T}_0 = \sqrt{G_0}
\end{aligned}
\tag{48}
$$

and its corresponding model is given by Fig. 4(c). Then, it immediately follows that this modified structure satisfies $\boldsymbol{P} = \boldsymbol{I}_M$ in (33) and coincides with the desired state-space representations (39) and (41).

The above discussion shows that the desired structure of Fig. 4(c) is obtained by applying the similarity transformation based on (47) to the first Foster canonical form for LC impedance networks. Here, it turns out that the nonsingular matrices $\boldsymbol{T}_i$'s and $\boldsymbol{T}_0$ serve as the scaling matrices that convert the matrices $\boldsymbol{P}_i$'s and $\boldsymbol{P}_0$ into the identity matrices. Therefore, we conclude that our proposed Gramian-preserving frequency transformation is derived from a state-space system of which integrators are replaced with $1/F(s)$, where the structure of $1/F(s)$ is constructed as the scaled version of the first Foster canonical form for LC impedance networks. It is interesting to note that this construction of Fig. 4(c) is similar to the realization of orthonormal ladder filters [7]: the orthonormal ladder filters are obtained by applying the

L_2 scaling to the structure of singly-terminated *LC* ladder networks, whereas the structures of Fig. 4(c) is obtained by applying another type of scaling, which makes use of the solutions to the lossless positive-real lemma, to the Foster canonical form for *LC* networks.

Before concluding this section, it should be noted again that the above results apply to the case of strictly proper reactance functions that include the LP-LP and the LP-BP transformations. For details of the improper reactance functions such as the LP-HP and the LP-BS transformations, see [32–34].

5. Application to design and synthesis of high-performance filters

This section applies the results of the previous section to design and synthesis of high-performance analog and digital filters. Emphasis is on the tunable filters, and we present a simple method to obtain state-space-based tunable filters with high-performance structures.

5.1. High-performance digital filters

Here we apply the Gramian-preserving frequency transformation to design and synthesis of a variable band-pass filter of high-performance structure [35]. The variable band-pass filter to be presented here is assumed to have the fixed bandwidth and the tunable center-frequency. Such a band-pass filter requires the simplified LP-BP transformation with the following all-pass function:

$$\frac{1}{F_{\mathrm{BP}}(z)} = -z^{-1}\frac{z^{-1} - \xi_{\mathrm{BP}}}{1 - \xi_{\mathrm{BP}}z^{-1}} \tag{49}$$

where $\xi_{\mathrm{BP}} = \cos\omega_{\mathrm{BP}}$ and ω_{BP} is the desired center-frequency of the passband in the variable band-pass filter. The desired state-space representation of (49) in order to carry out the Gramian-preserving frequency transformation (i.e. the state-space representation of (49) with the normalized lattice structure) is found to be

$$\widetilde{\boldsymbol{\alpha}} = \begin{pmatrix} \xi_{\mathrm{BP}} & 0 \\ \sqrt{1 - \xi_{\mathrm{BP}}^2} & 0 \end{pmatrix}$$

$$\widetilde{\boldsymbol{\beta}} = \begin{pmatrix} \sqrt{1 - \xi_{\mathrm{BP}}^2} \\ -\xi_{\mathrm{BP}} \end{pmatrix}$$

$$\widetilde{\boldsymbol{\gamma}} = \begin{pmatrix} 0 & -1 \end{pmatrix}$$

$$\widetilde{\delta} = 0. \tag{50}$$

Substituting (50) into (26), we obtain the state-space representation of the variable band-pass filter as

$$\widetilde{\mathcal{A}} = \begin{pmatrix} \xi_{\mathrm{BP}} \boldsymbol{I}_N & -\sqrt{1 - \xi_{\mathrm{BP}}^2} \boldsymbol{A} \\ \sqrt{1 - \xi_{\mathrm{BP}}^2} \boldsymbol{I}_N & \xi_{\mathrm{BP}} \boldsymbol{A} \end{pmatrix}$$

$$\widetilde{\mathcal{B}} = \begin{pmatrix} \sqrt{1 - \xi_{\mathrm{BP}}^2} \boldsymbol{b} \\ -\xi_{\mathrm{BP}} \boldsymbol{b} \end{pmatrix}$$

$$\widetilde{\mathcal{C}} = \begin{pmatrix} \boldsymbol{0}_{1 \times N} & -\boldsymbol{c} \end{pmatrix}$$

$$\mathcal{D} = d \tag{51}$$

and we can easily control the center-frequency of this filter by changing the value of ξ_{BP} in (51).

Now we present a design/synthesis example. The prototype filter used here is the fourth-order elliptic low-pass filter with the following transfer function:

$$H(z) = \frac{0.0101 - 0.0362z^{-1} + 0.0524z^{-2} - 0.0362z^{-3} + 0.0101z^{-4}}{1 - 3.7895z^{-1} + 5.4142z^{-2} - 3.4553z^{-3} + 0.8310z^{-4}}. \tag{52}$$

The peak-to-peak ripple, the minimum stopband attenuation and the passband-edge frequency of this filter are 0.5 dB, 40 dB and 0.05π rad, respectively. We choose the state-space representation (A, b, c, d) of this prototype filter as follows:

$$A = \begin{pmatrix} 0.9838 & -0.1007 & -0.0165 & -0.0171 \\ 0.1007 & 0.9582 & -0.1029 & -0.0273 \\ -0.0165 & 0.1029 & 0.9336 & -0.1015 \\ 0.0171 & -0.0273 & 0.1015 & 0.9139 \end{pmatrix}$$

$$b = \begin{pmatrix} 0.1490 & -0.1953 & 0.1669 & -0.0995 \end{pmatrix}^T$$

$$c = \begin{pmatrix} 0.1490 & 0.1953 & 0.1669 & 0.0995 \end{pmatrix}$$

$$d = 0.0101. \tag{53}$$

The controllability/observability Gramians of this realization are calculated as

$$K = W = \mathrm{diag}(0.8850, 0.6124, 0.2761, 0.0817), \tag{54}$$

which shows that this realization is the balanced realization.

Applying (51) to (53) yields the eighth-order variable band-pass filter. It can be easily checked that, for any ξ_{BP}, the Gramians of this band-pass filter become the same as (54) with multiplicity 2, i.e.

$$K = W = \mathrm{diag}(0.8850, 0.6124, 0.2761, 0.0817, 0.8850, 0.6124, 0.2761, 0.0817). \tag{55}$$

Therefore, the variable band-pass filter keeps the balanced form regardless of the location of the center-frequency.

Figures 5(a), (b), (c) and (d) show the magnitude responses of our proposed variable filter for $\zeta_{\mathrm{BP}} = -0.8$, -0.4, 0.5 and 0.9, respectively. For comparison purpose, the magnitude responses in the case of the cascaded direct form are also shown here, and all the coefficients of these two variable filters are quantized to 10 fractional bits. From Figs. 5(a), (b), (c) and (d) we know that our proposed variable filter shows very good agreement with the ideal magnitude responses for all ζ_{BP}. This result confirms that, our proposed variable filter exhibits high accuracy for all tunable characteristics by constructing the state-space representation of the prototype filter appropriately with respect to the Gramians. On the other hand, the magnitude responses of the cascaded direct form are degraded in all cases and the degradation is extremely large for $\zeta_{\mathrm{BP}} = 0.9$. As is well-known, direct form digital filters are very sensitive to quantization effects. In addition, since variable digital filters with direct form do not take into account the controllability/observability Gramians, the performance of the direct form with respect to quantization effects highly depends on the frequency characteristics. These facts show the utility of our proposed method.
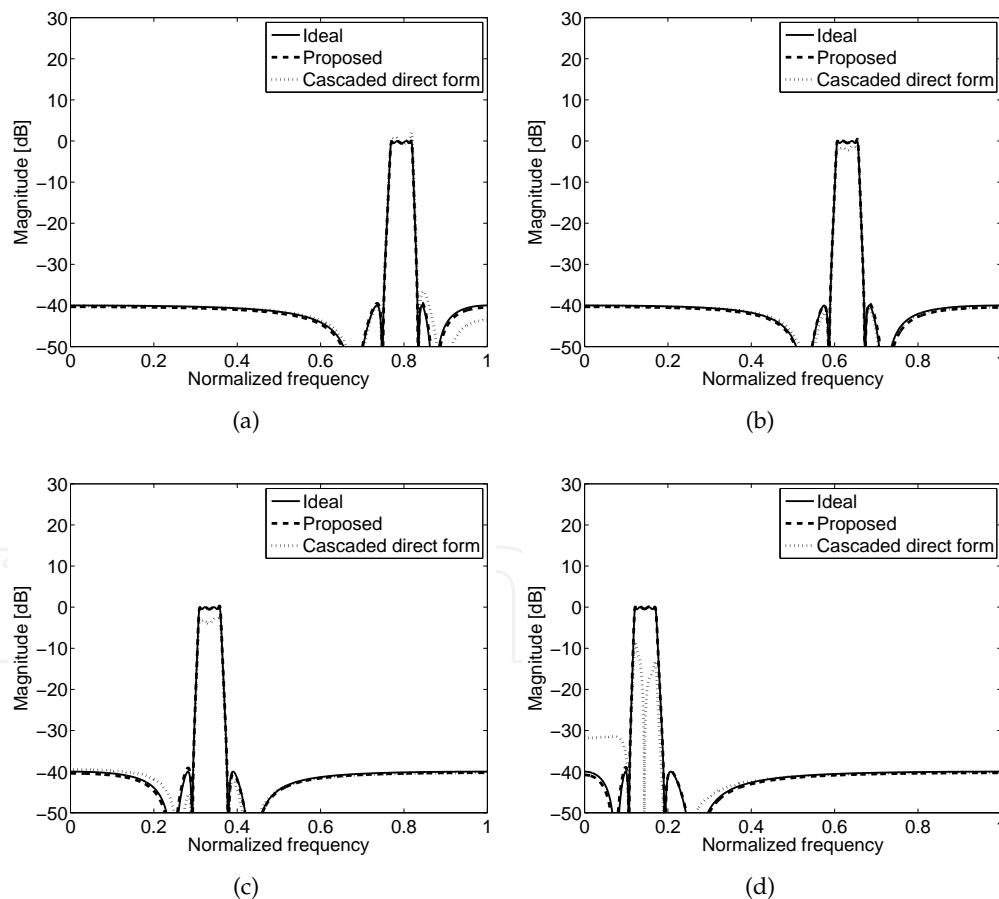

Figure 5. Magnitude responses of the eighth-order variable band-pass digital filters: (a) Responses for $\zeta_{\mathrm{BP}} = -0.8$. (b) Responses for $\zeta_{\mathrm{BP}} = -0.4$. (c) Responses for $\zeta_{\mathrm{BP}} = 0.5$. (d) Responses for $\zeta_{\mathrm{BP}} = 0.9$.

5.2. High-performance analog filters

Here we will design and synthesize a variable analog band-pass filter by using the Gramian-preserving frequency transformation. In the LP-BP transformation, we use the second-order Foster reactance function $1/F_{\mathrm{BP}}(s)$ as in (20). Therefore we apply (39) to (35), which results in the following state-space formulation of the desired variable analog band-pass filter:

$$
\begin{aligned}
\widetilde{\mathcal{A}} &= \begin{pmatrix} G A & \omega_{\mathrm{p1}} I_N \\ -\omega_{\mathrm{p1}} I_N & \mathbf{0}_{N \times N} \end{pmatrix} \\
\widetilde{\mathcal{B}} &= \begin{pmatrix} \sqrt{G} b \\ \mathbf{0}_{N \times 1} \end{pmatrix} \\
\widetilde{\mathcal{C}} &= \begin{pmatrix} \sqrt{G} c & \mathbf{0}_{1 \times N} \end{pmatrix} \\
\widetilde{\mathcal{D}} &= d.
\end{aligned}
\tag{56}
$$

As a design/synthesis example, here we use the following prototype low-pass filter

$$
H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}.
\tag{57}
$$

This transfer function is the third-order Butterworth low-pass filter with a cutoff frequency of 1 rad/s. We give the state-space representation of this prototype filter as the following orthonormal ladder structure [7]:

$$
\begin{aligned}
A &= \begin{pmatrix} 0 & a_1 & 0 \\ -a_1 & 0 & a_2 \\ 0 & -a_2 & -a_3 \end{pmatrix} \\
b &= \begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix} \\
c &= \begin{pmatrix} c_1 & 0 & 0 \end{pmatrix} \\
d &= 0
\end{aligned}
\tag{58}
$$

with

$$
(a_1, a_2, a_3, b_3, c_1) = (0.7071, 1.2247, 2.0000, 0.7979, 1.4472).
\tag{59}
$$

From (58) and (59), the controllability/observability Gramians of this filter are found to be

$$
\begin{aligned}
K &= I_3 \\
W &= \begin{pmatrix} 16.4493 & 9.3052 & 3.7988 \\ 9.3052 & 9.8696 & 5.3723 \\ 3.7988 & 5.3723 & 3.2899 \end{pmatrix}.
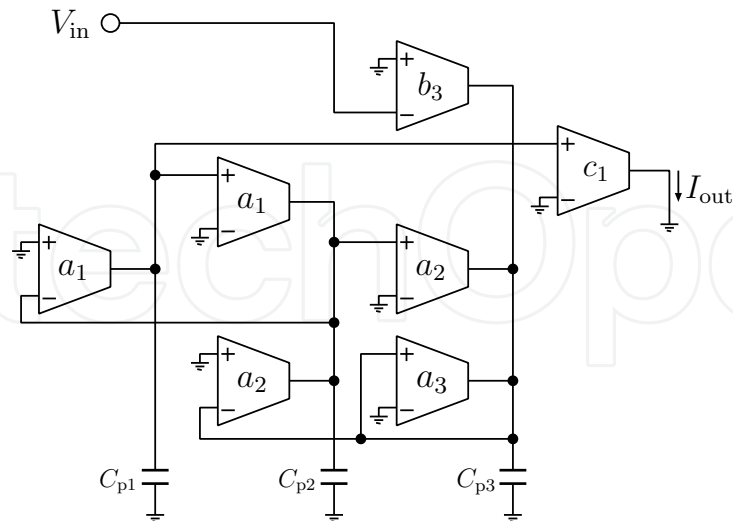\end{aligned}
\tag{60}
$$

Figure 6. Prototype filter based on transconductance-capacitor integrators.

As seen above, the controllability Gramian of the orthonormal ladder structure becomes the identity matrix. This property brings the high-performance with respect to the dynamic range and the sensitivity. Figure 6 illustrates the block diagram of this filter structure based on transconductance-capacitor integrators, where the normalized capacitance distribution is given by

$$(C_{\text{p1}}, C_{\text{p2}}, C_{\text{p3}}) = C_{\text{p}}(0.3091, 0.3957, 0.2952) \tag{61}$$

and C_{p} is the unit-less value of the total capacitance when expressed in F. The specification of (61) is determined according to the following rule [10]:

$$C_{\text{p}i} = \frac{\sqrt{\eta_i w_{ii} k_{ii}}}{\sum_j \sqrt{\eta_j w_{jj} k_{jj}}}$$

$$\eta_i = \sum_j |a_{ij}|. \tag{62}$$

As is seen from (58) and Fig. 6, the structure of this prototype filter is very sparse and suitable for circuit implementation. This is another benefit of the orthonormal ladder structure.

Applying (56) to this prototype filter, we finally obtain the state-space representation of the variable band-pass filter, and its corresponding circuit realization is given by Fig. 7. It can be easily shown that the controllability/observability Gramians $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{W}})$ of this band-pass filter become

$$\widetilde{\mathcal{K}} = \text{block diag}(K, K) = I_6$$
$$\widetilde{\mathcal{W}} = \text{block diag}(W, W) \tag{63}$$
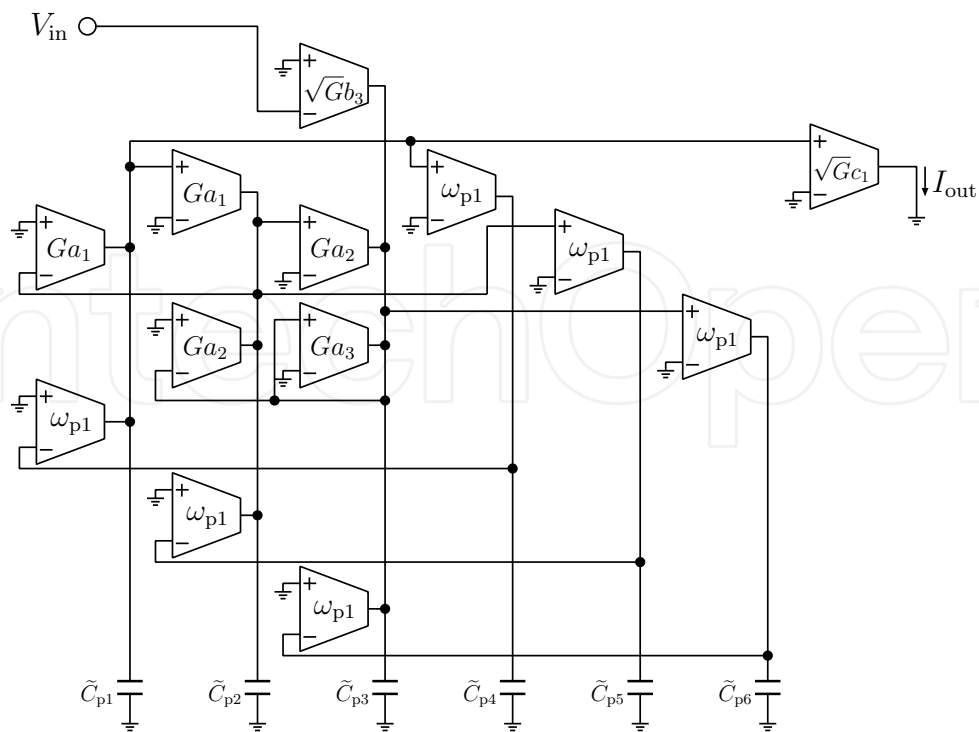
Figure 7. Band-pass filter given by Gramian-preserving frequency transformation.

for arbitrary values of G and ω_{p1}. It follows from this result that the Gramian-preserving frequency transformation easily produces the band-pass filter with the orthonormal ladder structure for arbitrary center frequency and bandwidth. Therefore, by controlling the parameters of G and ω_{p1}, we can realize tunable band-pass filters with the orthonormal ladder structure.

The high-performance of this band-pass filter can be demonstrated by not only calculation of the Gramians, but also numerical evaluation of the dynamic range. For details, see [34] and the references therein.

6. Conclusion

In this chapter we have introduced insightful and useful results on the classical frequency transformation of analog filters and digital filters. While most of the known results on the frequency transformation are described in terms of the transfer functions, the results given in this chapter are based on the state-space representation, which have revealed many useful properties with respect to the performance of filters that is dominated by the internal properties as well as the input-output relationship. In particular, the Gramian-preserving frequency transformation is very attractive to design and synthesis of high-performance filters. Using this new frequency transformation, we have presented variable analog/digital filters that retain high-performance regardless of the change of the frequency characteristics.

In addition to the aforementioned work, some other results on the frequency transformation have been reported in the literature. One of them is the state-space formulation of 2-D frequency transformation [36], which presents an explicit state-space-based frequency transformation for 2-D digital filters. Also, Yan et al. [37, 38] extended this work to

formulations of more general 2-D frequency transformation. Moreover, in [39] we have revealed the invariance property of the second-order modes of 2-D separable denominator digital filters under frequency transformation. Proof of this invariance property in the case of 2-D non-separable denominator digital filters is still an open problem. Derivation of the Gramian-preserving frequency transformation in the 2-D case is also an open problem.

Another interesting topic is the transformations based on "lossy" functions. In both the cases of analog frequency transformation and digital frequency transformation, the required transformation functions have the lossless property. On the other hand, it is theoretically possible to use lossy functions for transformation. Motivated by this, in [33, 40] we presented the state-space analysis of lossy transformations and revealed that the second-order modes are decreased under such transformations. Development of a practical application of this property is a future work.

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