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Stochastic Analysis of Heat Conduction and Thermal Stresses in Solids: A Review

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1. Introduction

In general, accurately predicting the thermal or mechanical loads acting on structural components is very difficult. There is no question that their material properties are random variables and thus are usually stated in terms of average values with attached uncertainties. Taking these factors into consideration, the *factor of safety* was introduced into structural design. However, current design methods based on the factor of safety cannot quantitatively estimate the safety of structures. In order to circumvent such an issue, the probability theory and mathematical statistics have been applied to many engineering problems. This allows us to determine the safety both quantitatively and objectively on the basis of the concept of reliability.

Currently, the application of probabilistic methods to engineering problems, which stems from the random vibration theory, has been broadened to the field of heat transfer. As structures subjected to extreme thermal load currently hold a prominent position in industries, the stochastic analysis of heat conduction and related thermal stresses in solids has drawn attention. In addition, the stochastic analyses of heat conduction in not only homogeneous but also nonhomogeneous bodies are being carried out more frequently because of the fabrication of advanced heat-resistant materials characterized by nonhomogeneity in recent years owing to advances in material manufacturing technology.

This article reviews research achievements for the stochastic analysis of heat conduction and related thermal stresses in solids. The objective of this review is to provide researchers and engineers, mainly in the field of heat transfer and thermoelasticity, with basic information useful for assessing the reliability of high-temperature apparatus. It is beyond the scope of this article to provide basic knowledge about the theory of probability and random

processes, which is necessary to describe randomness mathematically. Readers not familiar with this discipline are recommended to refer to textbooks related to stochastic modeling.

2. Overview

Table 1 summarizes existing studies that used probabilistic methods for heat conduction and thermal stress analysis. The existing studies are organized according to the type of parameters considered as stochastic quantities; the classification also distinguishes (i) homogeneity/nonhomogeneity of object materials to be analyzed and (ii) presence/absence of the analysis of thermal stress fields (including displacement fields due to thermal deformation). Note that the analysis of thermal stress fields includes the analysis of heat conduction as a prerequisite. Table 1 indicates that among the studies for homogeneous bodies, many treated heat conduction problems only, but studies that also investigated the effects of random parameters on thermal stresses (or thermal deformation) are limited. Moreover, studies that focused on nonhomogeneous bodies are far fewer than those that targeted homogeneous bodies, although the former have gradually increased since the early 1990s, coupled with the emergence of functionally graded materials (FGMs) [1].

Samuels [2] was the first to conduct a seminal study on heat conduction analysis using probabilistic methods. He analyzed a plate and sphere with randomly fluctuating surface temperature and spatiotemporally random internal heat generation. Parkus [3] was the first to study random thermal stresses; he successfully analyzed the thermoelastic problem of a semi-infinite body using probabilistic methods.

With regard to parameters considered as stochastic quantities, many papers have presented the analysis of problems where the surface temperature of an object or the temperatures of its surrounding media are regarded as stochastic quantities, i.e., random heating problems. This is probably because random heating problems are strongly related to the design of thermal insulating systems for equipment sensitive to temperature changes, for example. Moreover, quite a few studies considered the material properties of analysis objects as stochastic quantities. This is because the fact that any materials show variability in their properties to a greater or lesser extent has become public knowledge. An FGM, which is a typical nonhomogeneous material, includes more factors to produce large variability in the material properties, as compared to other materials. From early on, Poterasu *et al.* [4] focused on the large variability in the material properties of FGMs and attempted a stochastic analysis of thermal stresses in consideration of their randomness. However, their study unfortunately remained confined to a formulation based on the stochastic finite element method. It was ten years before full-scale studies on thermal stresses in nonhomogeneous bodies whose material properties were assumed to be stochastic quantities began to be conducted.

In the rest of this article, existing papers related to this topic are classified into six groups according to the type of random or uncertain parameters considered in the analysis, and an extensive literature review is presented for each group. The review gives special emphasis to analytical methods used in the respective papers.

Random parameter	Homogeneous bodies		Nonhomogeneous bodies	
	Heat conduction	Thermal stresses	Heat conduction	Thermal stresses
Surface temperature (or ambient temperature)	[2, 5-23]	[3, 24-37]		[38-41]
Heat flux	[23]			
Initial temperature	[11, 15, 21, 22, 42-47]	[48]		[49]
Material properties	[13, 21, 23, 47, 50-68]	[37, 69-74, 127]	[75-77]	[4, 78-84]
Heat transfer coefficients	[11, 12, 14, 21, 47, 56, 65, 85, 86]	[33, 35, 36, 87-89]	[90]	[91]
Emissivity	[23, 61, 65]			
Heat generation rate	[2, 12, 15, 43, 46, 57, 66, 92-94]	[95]		
Geometry	[86, 96]	[70, 73, 97, 98]		[78]

* There are also some review articles [99-102].

Table 1. Summary of stochastic heat conduction/thermal stress studies*

3. Case of random surface temperature or ambient temperature

Recently, as reliability and safety gain increasing importance in the design phase of high-temperature apparatuses or heat-resistant structures, conventional deterministic thermal stress analysis alone is not sufficient; analysis that considers uncertainties included in the analysis objects themselves and/or thermal environments (e.g., temperature of the surrounding media) is required. In general, accurately predicting the thermal or mechanical loads acting on structural components is very difficult [103]. Representative examples of such situations are random high-cycle temperature fluctuations observed at the upper core structure of fast-breeder reactors [34] and random variations in heat transfer coefficients (HTCs) around the stator vanes of gas turbines [87]. When uncertain factors are involved in thermal environments, the temperature and thermal stresses in objects should be evaluated stochastically.

We focused on existing studies that examined cases of random thermal boundary conditions. Samuels [2] analyzed the temperature field of a plate and sphere whose surface temperature fluctuated randomly; this was a pioneering work on heat conduction analysis using probabilistic methods. He applied the theory of random processes to determine the mean square temperature of bodies under the random heat conditions. Hung [6] analyzed the heat conduction of straight and circular fins whose root temperatures fluctuate randomly, and Yoshimura *et al*. [10] analyzed the temperature field for a rectangular fin whose ambient temperature fluctuates; the former acknowledged the approach adopted by Samuels [6].

Heller [7] derived the frequency response function for the temperature of an infinite plate subjected to random heating, from which the standard deviation of the temperature was estimated. Using a similar method, Heller also addressed the stochastic analysis for two-dimensional non-axisymmetric heat conduction of an infinite multilayered cylinder subjected to random heating at the outer surface [9]. Novichkov *et al.* [8], with the aid of the Monte Carlo method, evaluated the correlation function and spectral density function of the temperature for a double-layered infinite plate subjected to temporally random-varying heating. Gaikovich [16, 17] analytically obtained the covariance functions of temperature and *brightness* temperature in a homogeneous semi-infinite body for which the boundary temperature is given by a stationary random process. Subsequently, he studied the same problem in terms of spectral densities and presented significantly simpler results for the statistics [18, 19]. Nicola *et al.* [20] developed a variance propagation algorithm to investigate the effect of ambient temperature modeled as a random process on the variability of the steady-state temperature in a complex-shaped body cooled by convection. Chantasiriwan [22] solved the stochastic heat conduction problem under random boundary and initial conditions using a meshless method—the multiquadric collocation method. He demonstrated that the value of "the shape parameter" strongly influences the solution accuracy.

In contrast, studies on thermal stresses for randomly heated bodies originated from Parkus' work [3] on semi-infinite bodies. Heller [27] analyzed thermal stresses for a steel pipe with a concrete cylinder as a core material for which the surface temperature is expressed as a narrow-band random process. Lenyuk *et al.* [28] investigated the non-Fourier heat conduction and related thermal stresses in a semi-infinite body in contact with a medium whose temperature is a random process. The same analytical method was applied to the stochastic thermal stress problem of infinite cylinders [29]. Miller [31] derived the power spectrum of stress intensity factors from the temperature spectrum through the response function of temperature while supposing that the temperature variation in the analysis object can be modeled as a stationary Gaussian random process; he calculated the mean growth rate of a crack due to thermal fatigue using a statistical method. Singh *et al.* [30] evaluated the characteristics of random thermal stresses in a long hollow cylinder whose temperature at the outer surface is a random process, on the basis of the concept of a complex frequency response function.

Amada [32] stochastically analyzed the temperature and thermal stresses in an infinite plate, a solid sphere, and a solid cylinder where the surface temperature was assumed to be a stationary process. Consider an infinite plate of thickness h, where the temperature of one side of the surfaces T_∞ fluctuates randomly. The initial temperature of the plate is assumed to be zero. The heat conduction equation is expressed in a dimensionless form by Eq. (1).

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} \tag{1}$$

where $\theta = T/T_0$, $\tau = \kappa t / h^2$, $\xi = x/h$, T_0 denotes a reference temperature, κ is the thermal diffusivity, t is the time, and x is the through-thickness coordinate. The initial and boundary conditions are given by Eq. (2a) and Eqs. (2b), (2c), respectively.

$$\theta = 0 \qquad \text{at } \tau = 0 \qquad (a)$$
$$\theta = \theta_\infty(\tau) \text{ at } \zeta = 0 \qquad (b) \qquad (2)$$
$$\theta = 0 \qquad \text{at } \zeta = 1 \qquad (c)$$

where $\theta_\infty = T_\infty / T_0$. An analytical solution to Eqs. (1) and (2) is obtained in the form of Eq. (3).

$$\theta(\xi,\tau) = 2\pi \sum_{n=1}^{\infty} n \sin(n\pi\xi) \int_0^\tau \theta_\infty(\tau-\eta)\exp[-(n\pi)^2\eta]\mathrm{d}\eta \qquad (3)$$

If θ_∞ is a stationary process, the autocorrelation function of the temperature, R_θ, is given by Eq. (4).

$$R_\theta(\xi,\lambda) = \left\langle \theta(\xi,\tau)\cdot\theta(\xi,\tau+\lambda) \right\rangle =$$
$$4\pi^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mn\sin(m\pi\xi)\sin(n\pi\xi)\int_0^\infty\int_0^\infty R_{\theta_\infty}(\lambda+\mu-\eta)\exp[-(m\pi)^2\mu-(n\pi)^2\eta]\mathrm{d}\eta\mathrm{d}\mu \qquad (4)$$

where < > denotes the expectation operator and R_{θ_∞} and λ represent the autocorrelation function of θ_∞ and an arbitrary nondimensionalized time interval, respectively. The spectral density of the temperature, S_θ, is expressed as Eq. (5).

$$S_\theta(\xi,\omega) = 4\pi^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mn\sin(m\pi\xi)\sin(n\pi\xi)\int_0^\infty\int_0^\infty S_{\theta_\infty}(\omega)\exp\{-[(m\pi)^2-i\omega]\mu-[(n\pi)^2+i\omega]\eta\}\mathrm{d}\eta\mathrm{d}\mu \quad (5)$$

where S_{θ_∞} denotes the spectral density of θ_∞; ω, an angular frequency; and i, the imaginary number. For a solid sphere of radius a whose surface temperature is a stationary process, T_∞, the autocorrelation function of the temperature is derived as Eq. (6).

$$R_\theta(\rho,\lambda) =$$
$$\frac{4\pi^2}{\rho^2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mn(-1)^{m+n}\sin(m\pi\rho)\sin(n\pi\rho)\int_0^\infty\int_0^\infty R_{\theta_\infty}(\lambda+\mu-\eta)\exp[-(m\pi)^2\mu-(n\pi)^2\eta]\mathrm{d}\eta\mathrm{d}\mu \qquad (6)$$

where $\rho = r/a$ and r denotes the radial coordinate. For a solid cylinder of radius a whose surface temperature is a stationary process, T_∞, R_θ is obtained in the form of Eq. (7).

$$R_\theta(\rho,\lambda) = 4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_m\gamma_n\frac{J_0(\gamma_m\rho)}{J_1(\gamma_m)}\frac{J_0(\gamma_n\rho)}{J_1(\gamma_n)}\int_0^\infty\int_0^\infty R_{\theta_\infty}(\lambda+\mu-\eta)\exp[-\gamma_m^2\mu-\gamma_n^2\eta]\mathrm{d}\eta\mathrm{d}\mu \qquad (7)$$

where $J_0(\ )$ and $J_1(\ )$ are Bessel functions of the first kind of order 0 and 1, respectively. γ_n is the n-th positive root of the transcendental Eq. (8).

$$J_0(\gamma_n) = 0, \quad n = 1, 2, 3,\ldots,\infty \qquad (8)$$

The mean square temperature is obtained by substituting $\lambda = 0$ into Eqs. (4), (6), and (7).

Tanaka *et al*. [34] proposed an analytical method for obtaining not only the probability distribution of the residual life of an infinite plate with cracks but also the statistical properties of the crack length from the power spectral density of random surface temperature variations. The temperature variation is modeled as a narrow band stationary Gaussian process.

With regard to nonhomogeneous bodies, Sugano et al. analyzed the stochastic thermal stress problems of a nonhomogeneous plate [38] and disk [39] with randomly fluctuating surface temperature. They derived analytical solutions to the statistics of temperature and thermal stresses, assuming that the material properties of the objects vary in a certain way along one direction. Consider a nonhomogeneous annular disk of inner radius r_0 and outer radius r_1 with zero initial temperature, as shown in Figure 1. At the inner radius, the disk is subjected to non-axisymmetric heating due to the boundary temperature $T_\infty \cdot f(\phi)$, which varies randomly with respect to time and is symmetric about the x-axis. At the outer radius, heat dissipates to the surrounding medium of zero temperature via an HTC h_1. Given that the specific heat c and density d of the disk are assumed to be constant, the heat conduction equation, initial condition, and boundary conditions for this nonhomogeneous disk are expressed by Eq. (9a), Eq. (9b), and Eqs. (9c), (9d), respectively.

$$
\begin{align}
\frac{1}{r}\frac{\partial}{\partial r}\left[rK(r)\frac{\partial T}{\partial r}\right] + \frac{K(r)}{r^2}\frac{\partial^2 T}{\partial \phi^2} = dc\frac{\partial T}{\partial t} \quad &\text{(a)} \\
T = 0 \text{ at } t = 0 \quad &\text{(b)} \\
T = T_\infty \cdot f(\phi) \text{ at } r = r_0 \quad &\text{(c)} \\
K(r)\frac{\partial T}{\partial r} + h_1 T = 0 \text{ at } r = r_1 \quad &\text{(d)}
\end{align}
\tag{9}
$$

where the thermal conductivity K is given by a power function of r as Eq. (10).

$$
K(r) = K_0\left(\frac{r}{r_0}\right)^\beta
\tag{10}
$$

If T_∞ is a stationary process, the autocorrelation function of the temperature, R_T, is given by Eq. (11).

$$
\begin{aligned}
&R_T(r,\phi,\lambda) = \\
&\left(\frac{\kappa_0^*}{\pi}\right)^2\left(\frac{r}{r_0}\right)^{-\beta}\sum_{m,n=0}^{\infty}\sum_{k,l=1}^{\infty}\varepsilon_m\varepsilon_n\frac{L_{mk}(r)L_{nl}(r)}{H_{mk}H_{nl}}\Theta_m\Theta_n\cos m\phi\cos n\phi\int_0^{\infty}\int_0^{\infty}R_{T_\infty}(\lambda+p-q)\exp\left[-\frac{\kappa_0^*}{4}(w_{mk}^2 q + w_{nl}^2 p)\right]dpdq
\end{aligned}
\tag{11}
$$

where $\kappa_0^* = (K_0/dc)r_0^{-\beta}$, $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = \cdots\cdots = 2$, $\Theta_n = \int_0^\pi f(\phi)\cos n\phi\, d\phi$, and $L_{nl}(r)$ and H_{nl} are an explicit function of r and a constant (without going into detail) determined for each

set of n and l, respectively. Furthermore, w_{nl} denotes the l-th positive root satisfying a transcendental equation determined for each value of n. For more details, see [39]. The mean square temperature is obtained by substituting $\lambda = 0$ into Eq. (11).
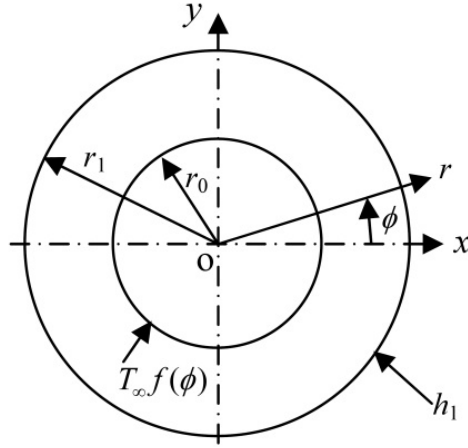

Figure 1. Schematic of a nonhomogeneous annular disk subjected to temporally random heating [39]

As a numerical example, Sugano *et al.* considered the case where the autocorrelation function of T_∞ and function f are given by Eqs. (12a) and (12b), respectively.

$$R_{T_\infty}(\lambda) = \exp(-V \mid \lambda \mid) \qquad \text{(a)}$$

$$f(\phi) = \begin{cases} 1 - \phi^2 / \alpha^2 & \text{for } 0 \le \phi \le \alpha \\ 0 & \text{for } \alpha \le \phi \le \pi - \alpha \qquad \text{(b)} \\ 1 - (\phi - \pi)^2 / \alpha^2 & \text{for } \pi - \alpha \le \phi \le \pi \end{cases} \qquad (12)$$

Figure 2 shows the spatial distribution of the mean square temperature for $V = 1$, $\alpha = \pi/6$, $r_1/r_0 = 2$, and $h_1 r_0/K_0 = 0.1$.
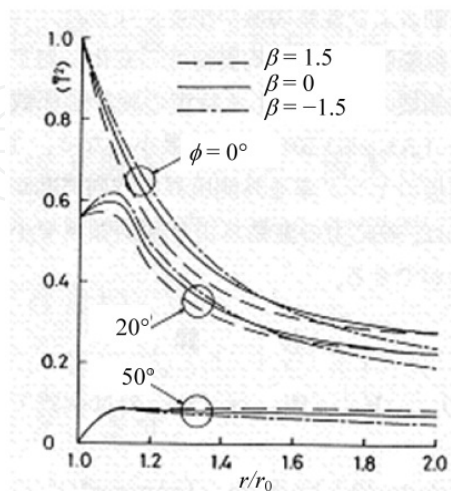
Figure 2. Effects of location-dependent thermal conductivity on mean square temperature [39]

Chiba *et al.* [41] addressed the stochastic thermal stress problem of functionally graded plates convectively heated from the surrounding medium whose temperature fluctuates randomly. The material properties of the plates are allowed to vary arbitrarily along the thickness direction. Under the condition that the surrounding medium temperature is expressed as a stationary process, they derived analytical solutions of statistics for this problem.
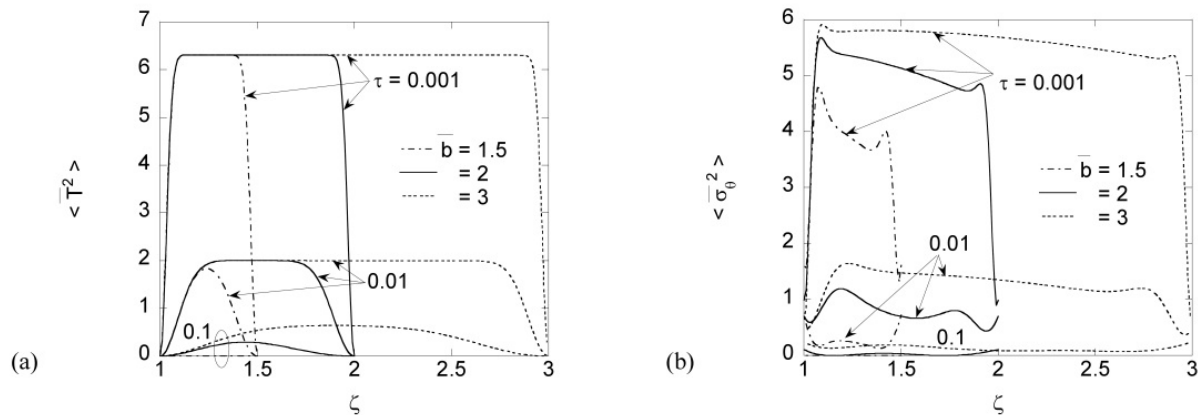
4. Case of random initial temperature

The initial temperature of structures is often uncertain (or random) in a real environment. For example, when space planes or space shuttles reenter the atmosphere, the initial temperature distribution of the fuselages is always uncertain [104]. Moreover, the temperature distribution in high-temperature apparatus, such as gas turbines, at the time of the resumption of operation is an uncertain factor in the design phase because of the time elapsed from shutdown and heat transfer from/to the surrounding media, such as a working fluid [103]. In order to investigate the effects of such randomness included in the initial temperature on the temperature and thermal stresses, stochastic analysis is absolutely imperative.

Thus far, stochastic studies on the heat conduction and thermal stress problems of solids with a random initial temperature have been limited. Ahmadi [42] studied the temperature field of an infinite plate and a semi-infinite body for which the initial temperature is a random field and showed that the randomness in the temperature diminishes over time. Subsequently, Grigorkiv *et al.* [44] conducted a similar study from the viewpoint of non-Fourier heat conduction. Tasaka [45] studied the convergence of statistical finite element solutions to one-dimensional heat conduction under a random initial condition. He also presented three different approaches for obtaining the statistics of temperature for the one-dimensional heat conduction problem in which the initial temperature is a random field and the internal heat generation is spatiotemporally random (i.e., a random wave) [46]: an analytical solution, a semi-analytical solution, and a numerical solution based on the finite difference method (FDM). He compared the numerical results of these different approaches. Nicolai *et al.* [47] investigated the transient behavior of temperature variance in bodies with random initial temperature using the stochastic finite element method (FEM). Scheerlinck *et al.* [21] analyzed the coupled heat and mass transfer problem in which the material properties, initial condition, and boundary conditions are random fields. In [21], a first-order perturbation algorithm based on the Galerkin finite-element discretization of Luikov's heat and mass transfer equations for capillary porous bodies was developed.

However, very few existing studies have dealt with the thermal stress problems for a random initial temperature. Chiba *et al.* [48] extended the work of Ahmadi [42] to thermal stress fields; they analytically obtained the autocorrelation functions of temperature and thermal stresses in seven simple-shaped bodies with the initial temtperature modeled as a homogeneous random field. These include an infinite plate, an infinite strip, a hollow sphere, an infinite body with a spherical hole, an infinite hollow cylinder, an annular disk,

and an infinite body with a circular cylindrical hole. As a numerical example, the mean square temperature and the mean square thermal stresses were also calculated for when the initial temperature is given by white noise. The transient behavior of these statistics was graphically represented, as shown in Figure 3. Note that τ denotes the Fourier number; ζ, the dimensionless radial coordinate ($= r/r_0$); and $\bar{b}$, the ratio of the inner and outer radii ($= r_1/r_0$). The temperatures at the inner and outer radii are both considered to be deterministic; therefore, the mean square temperatures at the surfaces are zero.

Figure 3. Transient behavior of (a) mean square temperature and (b) mean square tangential stress in a hollow sphere with random initial temperature modeled as white noise [48]

Sugano *et al*. [49] used probability theory to analyze the heat conduction and thermal stress problems of functionally graded plates with the initial temperature assumed to be a homogeneous random field. The material properties of the plates were allowed to vary arbitrarily along the thickness direction. The autocorrelation functions of temperature and thermal stresses were derived in an explicit form. Numerical calculations were performed for when the initial temperature is white noise or a Markov random field. The relationships between the through-thickness variation in the material composition and the statistics of the temperature and the thermal stresses were discussed.

## 5. Case of random material properties

As can be found in Table 1, there are a relatively large number of stochastic studies on the heat conduction and thermal stress problems of objects with random material properties. Some examples of studies that employed analytical (mathematical) methods are as follows: Chen *et al*. [50] analyzed the temperature field of a semi-infinite body with random thermal diffusivity by a perturbation method; Keller [54], Ahmadi [51, 52], Fox *et al*. [53], and Tzou [58] analyzed the heat conduction of approximately homogeneous bodies with random thermal conductivity; and Barrett [78] and Tzou [75] analyzed nonhomogeneous bodies with random thermal conductivity. Srivastava *et al*. [63] analyzed the one-dimensional steady heat conduction for thermal conductivity given by a random field and obtained analytical solutions to the mean and variance of the temperature in Earth's crust. Subsequently, Srivastava [67] presented analytical solutions to the mean and variance of heat flux for the

same problem; numerical results demonstrated that a decrease in the correlation length scale of the thermal conductivity increases the variability in the heat flux. Kotulski [69] investigated the thermoelastic wave propagation in a solid with a random coefficient of thermal expansion.

Examples that employed numerical methods are given below. Nakamura *et al*. [13] analyzed the heat conduction when the thermal properties of analysis objects and ambient temperature are random, and Emery [65] treated the case where the thermal conductivity, HTC, and emissivity are random. The stochastic FEM [105] was used in both works. Manolis *et al*. [77] investigated the stochastic steady-state heat conduction in a nonhomogeneous solid whose thermal conductivity varies linearly along one direction at the macroscopic level but is spatially random at the microscopic level (see Figure 4). They mathematically described the random thermal conductivity with Eq. (13).

$$k(x, \gamma) = k_0 + [k_1 + \varepsilon k_2(\gamma)]x \tag{13}$$

where k_0 and k_1 are constants, ε is a small parameter ($\varepsilon \ll 1$), k_2 is a zero-mean random field, and γ is a random variable. Note that the slope of the thermal conductivity consists of a constant plus a zero-mean random part. In their study, stochastic analysis of the heat conduction was carried out using a boundary integral equation approach.
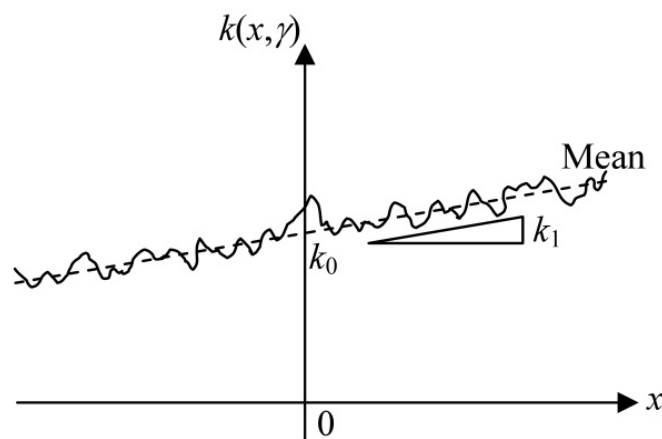
Figure 4. Linearly varying thermal conductivity [77]

Hien *et al*. [62] analyzed, using the stochastic FEM, the nonlinear heat conduction in solids whose thermal conductivity or specific heat is temperature-dependent and is considered to be a random field. Emery *et al*. [61] also analyzed the nonlinear heat conduction, which is attributed to not only random material properties but also random boundary conditions. Kaminski *et al*. [64] studied the heat conduction in composite materials whose thermal conductivity and specific heat are given as a random field by using the stochastic FEM. In [64], numerical results offered an interesting insight that whereas the randomness of the thermal conductivity monotonically increases the variability of temperature with time, the randomness of the specific heat increases the variability of temperature in the beginning but decreases it after a certain period of time. Liu [66] presented an analytical solution to the

relative variability of temperature in bodies for when the thermal properties and internal heat generation are random. Although this is not an exact statistic, it is useful to roughly understand the extent to which the temperature distribution is influenced by random input parameters. Nakamura *et al.* [23] performed stochastic heat conduction analysis of atmospheric reentry vehicles. They considered the aerodynamic heat flux, heat shield emissivity, and insulator thermal conductivity to be random variables. They analyzed the transient behavior of the statistics of temperature at several locations by using the FEM and Monte Carlo simulation.

Aida *et al.* [72] estimated the probability of crack initiation due to thermal stresses by using an FEM-based Monte Carlo simulation method where the elastic modulus and the tensile and compressive strengths have uncertainties. Nakamura *et al.* [37] conducted a similar study for the case of uncertain Young's modulus, tensile and compressive strengths, and ambient temperature by using the first-order approximation theory. In addition, an interesting study on the stochastic finite element analysis of a thermal deformation problem was reported for a carbon fiber reinforced plastic (CFRP)-laminated plate whose fiber orientation angle is random [71]. Using the stochastic FEM, Sluzalec [73] analyzed the thermoelastic deflection of a rectangular plate subjected to a thermal and mechanical load concentrated at the center, where the plate has material properties and a thickness given by a two-dimensional random field.

With regard to the stochastic analysis of FGMs, which considers the uncertainties of material properties, Poterasu *et al.* [4] formulated the stochastic FEM for the thermoelastic problem of FGMs whose thermomechanical properties are homogeneous random fields. Ferrante *et al.* [80] considered the volume fraction and porosity to be random fields with spatial correlation in an FGM plate having linearly varying material composition distribution and analyzed the steady-state thermal stresses using Monte Carlo simulation. The analysis of the results showed that deviations in the ceramic/metal volume fraction produce significant randomness in the thermal stress and safety factor distribution of the plate. Chiba *et al.* [79, 82] stochastically analyzed the transient heat conduction and thermal stress problems of infinite FGM plates with an uncertain thermal conductivity and coefficient of thermal expansion. The FGM plates were assumed to have arbitrary thermal and mechanical nonhomogeneities along the thickness direction. Two methods were used for the analysis: the direct Monte Carlo simulation method [79] and a perturbation method [82]. Sugano *et al.* [81] analyzed the thermoelastic problem of nonhomogeneous plates with a random thermal conductivity and coefficient of thermal expansion.

Hosseini *et al.* [74] analyzed thermoelastic waves in a thick hollow cylinder for which some material properties are independently random. The statistics of temperature, stresses, and displacement were obtained by combining a hybrid numerical method, which consists of the Galerkin FEM and the Newmark FDM, and Monte Carlo simulation. Their numerical results demonstrated that the peak positions of the variances of the temperature and displacement progress with time in response to the progression of the heat wave front. Unfortunately, the sample size for the Monte Carlo simulation was unspecified (150 samples according to

histograms shown therein); therefore, the degree of reliability of the presented numerical results is unclear. Using the same method, Hosseini *et al*. also conducted the thermoelastic wave analysis of an FGM thick hollow cylinder with random material properties [83]. Fairly recently, the same research group carried out this stochastic analysis using the meshless local Petrov-Galerkin method accompanied with Monte Carlo simulation [84]. This approach does not require the functionally graded cylinder to be assumed to be a multilayered cylinder with different material properties in each layer.

6. Case of random heat transfer coefficient

In real thermal environments, the HTCs of object surfaces are known to vary both temporally and spatially. For example, in the spinning process of light fiber wires, unsteady gas flow in furnaces has been reported to vary the spatial distribution of the HTCs of the fiber wire surface, which results in the variability of the wire diameter [106]. However, accurately predicting this spatial distribution of the HTCs is very difficult. In addition, the HTCs of turbine disk surfaces are influenced by many factors: the disk rotational speed, the presence or absence of shroud and neighboring disks, the distance from them, the velocity of cooling air, and its flow pattern. To make matters worse, these influences are nonlinear and change rapidly. Thus, accurate prediction of the HTCs is quite difficult. Moreover, there seems to be a measurement uncertainty of over 50% for the overall heat transfer coefficients of heat transfer surfaces of heat exchangers [107]. As long as the predicted values of HTCs include the uncertainties described above, a quantitative evaluation of the statistics of temperature and thermal stresses is needed to maintain an appropriate level of product quality or structure reliability. Hence, the temperature and thermal stresses in objects should be analyzed on the basis of the probability theory.

Stochastic studies on the heat conduction and thermal stress problems that consider spatial or temporal randomness in HTCs are scarce. Using a stochastic boundary element method, Drewniak [56] analyzed the steady-state heat conduction in solids for which the thermal conductivity or HTC is modeled as a random field. Madera [14] and Emery [65] analyzed the stochastic heat conduction problems of a rectangular fin for which they expressed the HTCs of the heat transfer surfaces as Gaussian white noise and a random field, respectively. The former derived partial differential equations for the expected value and covariance of temperature to present analytical solutions to these statistics under steady conditions. The latter used a higher-order perturbation method to analyze the problem and concluded that the use of first-order estimation for the standard deviation of temperature and second-order estimation for the mean response is preferable. Furthermore, the scale of correlation has been shown to have a strong effect on the statistics of the response. Kuznetsov [85] analyzed the stochastic heat conduction problem of an infinite strip for which the HTC is not a random field but is spatially random. Chiba [90] analytically obtained the second-order statistics, i.e., the mean and standard deviation, of the temperature for axisymmetrically heated FGM annular disks for which the surface HTCs are random fields. However, the abovementioned studies did not consider thermal stresses induced by the temperature changes.

In contrast, Mori *et al.* [87] numerically analyzed the statistics of steady thermal stresses produced in the stator vane of a gas turbine heated from the surroundings via random HTCs by using the stochastic FEM. Klevtsov *et al.* [33] investigated random thermal stress fluctuations in steam generation pipes, which are induced by randomly fluctuating HTCs/ambient temperature. Subsequently, the same authors [35, 36] estimated the variability of temperature and thermal stresses in a pipe for which the HTC of the inner surface and the temperature of the medium flowing in the pipe are random processes. They proposed a high-accuracy technique for predicting the material fatigue life that reflects the estimated variability. Ishikawa [88] used the FDM to analyze the coupled thermoelastic problem of a beam having HTCs described by a random field.

Chiba [89] analytically derived the second-order statistics of temperature and thermal stresses by a perturbation method in homogeneous annular disks for which the HTCs of the major surfaces are random fields. He assumed that the disks are subjected to a deterministic axisymmetric heat load. Numerical calculations were performed for the case where the surface HTCs are band-limited white noise random fields. The mean E[θ] and standard deviation S[θ] of the dimensionless temperature in the disks supposed as annular fins are shown in Figure 5. The ratio of the inner and outer radii of the disks is 0.2, and the coefficient of variation of the random HTCs is 0.1. In Figure 5, τ denotes a dimensionless time (Fourier number), and m is a nondimensionalized HTC (Biot number). The results are shown for the two cases in which the HTC mean is uniform throughout the disk surfaces and varies linearly along the radial direction.

Chiba [91] then analyzed the second-order statistics of the temperature and thermal stresses in FGM annular disks of variable thickness via Monte Carlo simulation; the HTCs of the disk surfaces were considered to be random fields and the disks were subjected to axisymmetric heat loads at the inner and outer radii.

7. Case of random internal heat generation/sink

Samuels [2] obtained analytical solutions of the mean and mean square value of the temperature field in a plate and sphere with randomly fluctuating surface temperature and spatiotemporally random internal heat generation. Becus [43] presented a solution to the heat conduction problem with a random heat source and random initial and boundary conditions. He also conducted a series of analytical studies on random heat conduction [108-111], which contributed greatly to the subsequent growth of this field. Vasseur *et al.* [92] analytically obtained relationships between the autocorrelation functions of heat generation and heat flow in the three-dimensional steady heat conduction of a homogeneous body, where the internal heat generation is expressed as a two-dimensional homogeneous random field. Nielsen [57] extended this work to address the case where thermal conductivity and heat generation are given by a random field and a cross-correlation exists between them. Val'kovskaya *et al.* [15] analyzed the stochastic temperature field in a two-layer solid disk subjected to heat sources for which power is a random function of time and radial coordinates. They also considered the randomness included in the initial temperature and ambient temperature. Ishikawa [93] analyzed the one-dimensional non-Fourier heat

conduction in a solid with internal heat generation of white noise. Srivastava *et al*. [94] analyzed the one-dimensional steady-state heat conduction in solids with internal heat generation described by a random field. They derived exact solutions for the mean and variance of the temperature.
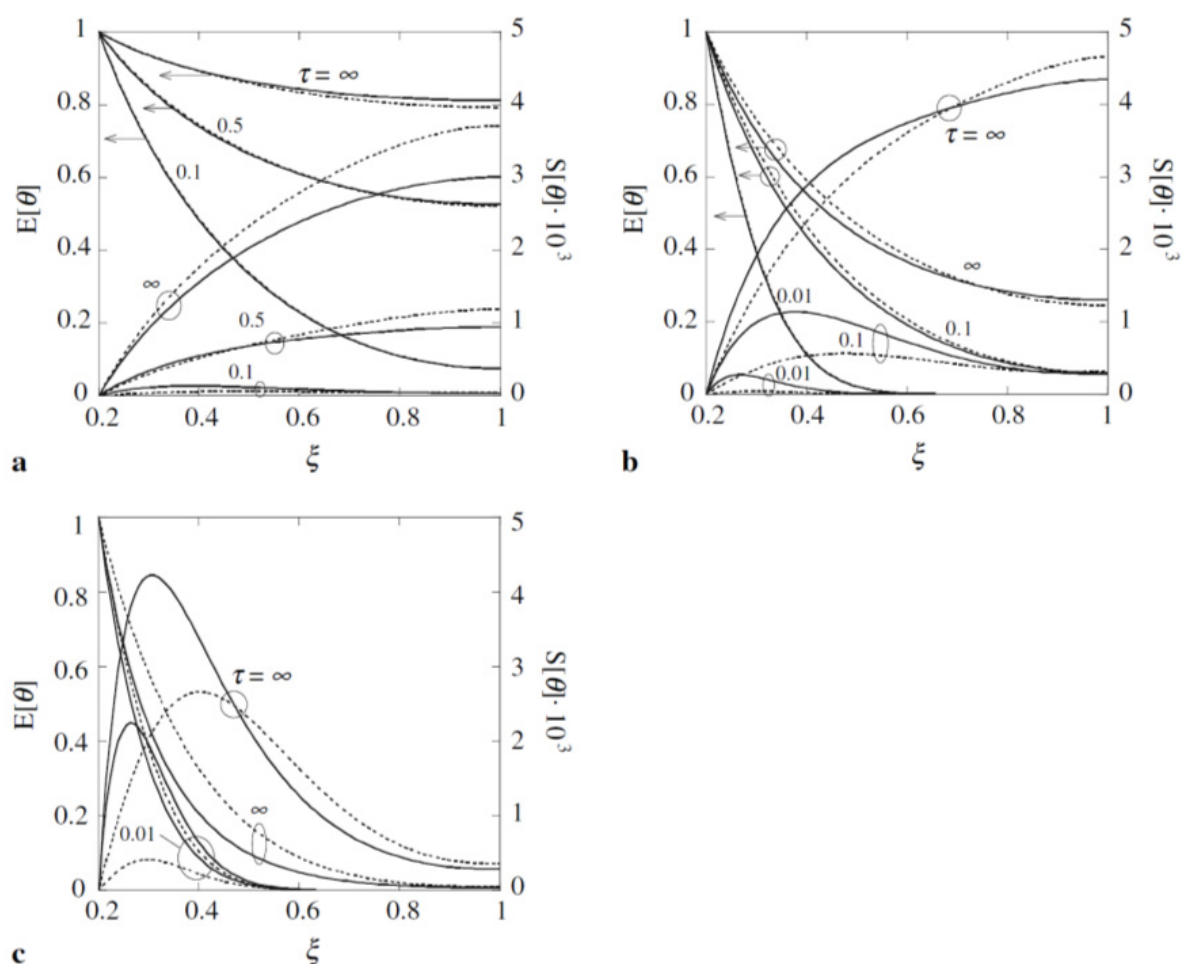
Figure 5. Transient distribution of the mean and standard deviation of the dimensionless temperature for uniform HTCs (solid curves) and linearly varying HTCs (broken curves) where (**a**) $m = 0.01$, (**b**) $m = 0.1$, and (**c**) $m = 1$ [89]

8. Case of random geometry

Shvets *et al*. [97] analytically evaluated the temperature field and thermal stresses in cylinders for which the radius fluctuates randomly along the circumferential direction. A perturbation method and Laplace transform were used for the analysis. Smith [70] discussed the structural reliability of hollow cylinders or hollow spheres; their inner and outer radii and material properties were random, and they were heated by the surroundings. Clarke [96] analytically solved the heat conduction problem of a layered plate for which the stacking sequence of layers and the layer thickness are random. This problem was later re-examined by Willis *et al*. [112]. Mori *et al*. [98] discussed the effects of the shape irregularity and thickness variability at a welded joint all around the body on stresses produced under

internal pressure in a large-sized, thin-walled pressure vessel. The stochastic FEM was used to solve this random geometry problem.

Emery *et al*. [86] estimated the standard deviations of the heat fluxes in an object for which the heated surface roughness or HTC at the heated surface was assumed to be a random field. They adopted the FEM and direct Monte Carlo simulations. In order to treat this class of stochastic heat conduction problems, the problem of a stochastic region was converted to one in which the conductivity is stochastic through a coordinate transformation. Figure 6 shows the standard deviations for the x and y components of the heat flux in a slab where heat is mainly transferred along the width direction (x-direction) and the heated surface roughness is a random field with a certain spatial correlation length.
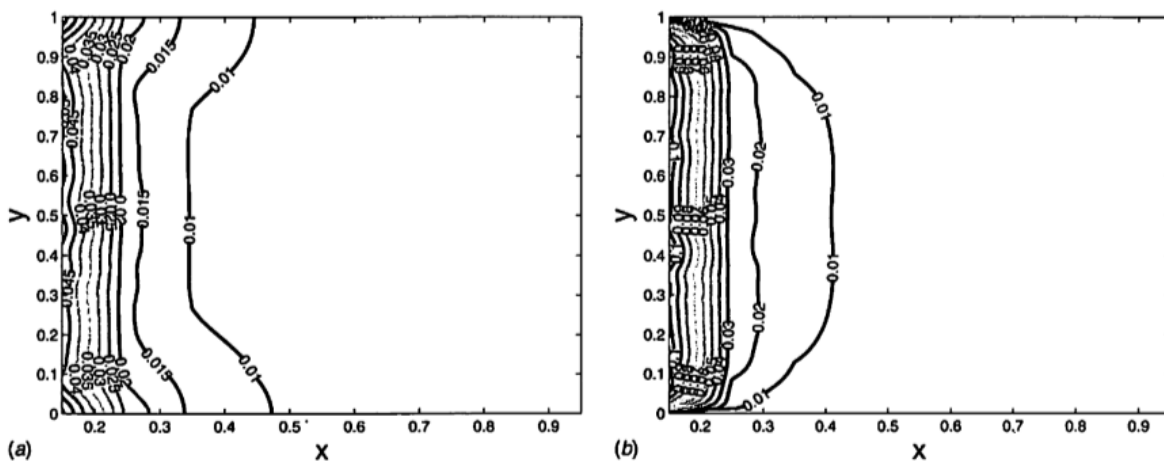
Figure 6. Contours of the standard deviations of the heat fluxes for an edge roughness of 2% and a correlation length of 5% of the slab mean width: (a) $\sigma(q_x)$ and (b) $\sigma(q_y)$ [86]

9. Numerical methods for stochastic heat conduction problems

In this section, we present numerical methods proposed thus far for the stochastic analysis of heat conduction. Case *et al*. [113] used the stochastic FEM to obtain the statistics of displacement due to thermal deformation in a solid for which the temperature distribution is expressed by simple functions of random variables. The Young's modulus and the coefficient of thermal expansion were assumed to depend on temperature. Nicolai *et al*. [11] proposed a numerical method for the computation of the statistics of the transient temperature field in heated foods with random initial and boundary condition parameters. This method is based on space discretization by the FEM and the stochastic systems theory. The same authors also developed a method for computing the statistics of the temperature field in objects with random thermophysical parameters; this method is based on incorporating perturbations of the thermal parameters in the FEM [60].

Madera [12, 114] obtained partial differential equations for the expectation and correlation of the stochastic temperature field for a solid subjected to random heating expressed by Gaussian white noise. These equations were solved by analytical methods (e.g., Green's function method) or numerical methods (e.g., control volume method). Madera [76] also

developed a numerical method for determining three-dimensional transient temperature fields described by stochastic heat conduction equations with random coefficients and by stochastic initial and boundary conditions. This method is based on stochastic mathematical model discretization by the FEM or the FDM and on the solution of the Volterra stochastic integral equations.

Nicolai *et al*. [115] extended the perturbation algorithm of Sluzalec [59] and Fadale *et al*. [116] for linear heat conduction problems with random field parameters to the algorithm used for the analysis of nonlinear heat conduction under random conditions. This extended algorithm can consider the effects of (i) random field initial conditions, (ii) correlated thermophysical parameters, and (iii) correlated boundary condition parameters. In addition, Nicolai *et al*. [117] presented a variance propagation algorithm for heat conduction problems with parameters involving random fluctuations in time. This algorithm is based on the FEM and involves the numerical solution of Sylvester and Lyapunov differential systems. This algorithm was also applied to calculate the mean and variance of the temperature at arbitrary positions in heated cylinders with random wave parameters [118]. Nicolai *et al*. [119] described how to use variance propagation algorithms for calculating the statistical characteristics of the stochastic temperature field of heated solids in detail. Moreover, they developed an extended variance propagation algorithm for stochastic coupled heat and mass transfer problems under random process boundary conditions [120]. The numerical results of their study demonstrated that the random fluctuations of the process conditions may cause considerable variability in the temperature and moisture content in solids undergoing a drying process.

Liu *et al*. [121] extended the stochastic FEM to analyze the heat conduction problems that simultaneously consider randomness in the material properties and heat load conditions. Le Maitre *et al*. [122] developed a computationally efficient numerical technique for solving two-dimensional stochastic diffusion equations, in which a multigrid technique is applied to the system of equations arising from the polynomial chaos and the FDM.

Xiu *et al*. [123] solved the two-dimensional transient heat conduction subject to random inputs by generalized polynomial chaos expansion. This study is a natural extension of their earlier work on stochastic steady-state diffusion problems [124]. Xiu *et al*. [125] also treated diffusion problems in domains with rough boundaries considered as random fields. They proposed a novel computational framework based on the use of stochastic mappings to transform the original problem in a random domain into a stochastic problem in a deterministic domain and solved the transformed stochastic problem using a stochastic Galerkin method and Monte Carlo simulations.

Saleh *et al*. [68] analyzed stochastic one-dimensional heat conduction with random heat capacity or random thermal conductivity, which they modeled as a random field. They employed the stochastic FEM based on the Karhunen-Loeve decomposition and the projection of the solution on chaos polynomials. Recently, Nicolai *et al*. [126] extended the interval and fuzzy FEMs to nonlinear heat conduction problems with uncertain parameters and verified the efficiency of the proposed methodology with two case studies from food

processing engineering. In [102], some of the above-mentioned algorithms developed by Nicolai *et al.* for stochastic heat conduction analysis were outlined and illustrated by some simple examples from thermal food process engineering.

10. Concluding remarks

This article reviewed the historical progress in the stochastic analysis of heat conduction and related thermal stresses. Existing papers related to this topic were classified into six groups according to the type of random or uncertain parameters considered in the analysis, and an extensive literature review was presented for each group. The overview indicates that among the studies for homogeneous bodies, many treated heat conduction problems only, but only a limited number also investigated the effects of random parameters on thermal stresses (or thermal deformation). Studies on nonhomogeneous bodies are far fewer than those targeting homogeneous bodies, although the former are increasing in number. With regard to parameters considered as stochastic quantities, a number of studies have analyzed problems in which the surface temperatures of an object or ambient temperatures are regarded as stochastic quantities, i.e., random heating problems. Furthermore, quite a few studies assumed the material properties of analysis objects to be stochastic quantities.

Some future research directions related to this topic are suggested as follows:

i.   In studies on the heat conduction and thermal stress analysis of bodies subjected to random heating, only stationary random processes have been targeted so far. Thus, when time functions included in thermal boundary conditions are nonstationary random processes, such as earthquake vibration, we need to divide the whole time interval into several intervals, conduct the analysis for a stationary process in each interval, and finally join the analysis results for the respective intervals. In an actual operation environment, structures may often be subjected to heat loads that are difficult to regard as stationary random processes. Therefore, the above analyses treated in the framework of the theory of a stationary random process need to be extended to nonstationary random processes.

ii. Uncertainties included in the material properties of "materials with microstructure," including particle-dispersed composite materials, are attributed to the variability in the microstructure (or microstructural morphology) as well as the variability in the material properties of the constituents. Hence, stochastic studies based on micromechanics, which consider various parameters at the microscale (e.g., the material properties of constituents and the shape and size of dispersed particles) as stochastic quantities, would be another potential research topic.

iii. All the existing stochastic studies on thermal stresses in solids have focused on the elastic range. Structures in which advanced heat-resistant materials are used are supposed to undergo extremely large temperature gradients; therefore, they may undergo partial plastic deformation. Thus, in order to extend the discussion from the elastic range to the plastic range, the same type of stochastic studies on the thermo-elastic-plastic behavior is another topic remaining to be addressed.

The stochastic analyses of this field lead directly to the reliability evaluation of high-temperature structures that require higher safety, such as space planes, hot gas turbines, and atomic reactors. Because these analyses are required in a variety of fields— e.g., food engineering and geophysical science—, they have wide applicability.

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