Chapter 20

Fuzzy Spheres Decays and Black Hole Thermodynamics

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Additional information is available at the end of the chapter

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1. Introduction

A black hole is, by definition, a region in spacetime in which the gravitational field is so strong that it precludes even light from escaping to infinity. A black hole is formed when a body of mass M contracts to a size less than the so called gravitational radius $r_g = 2GM/c^2$. (G is the Newton's gravitational constant, and c is the speed of light). The velocity required to leave the boundary of the black hole and move away to infinity (the escape velocity) equals the speed of light. In this way, one easily concludes that neither signals nor particles can escape from the region inside the black hole since the speed of light is the limiting propagation velocity for physical signals. From the fact that no signals can escape from a black hole, while physical objects and radiation can fall into it, the surface bounding the black hole in spacetime (called the event horizon) is a lightlike surface.

The term "black hole" was introduced by Wheeler in 1967 although the theoretical study of these objects has quite a long history. The very possibility of the existence of such objects was first discussed by Michell and Laplace within the framework of the Newtonian theory at the end of the 18th century [1–3]. In general relativity context, the problem arose within a year after the theory had been developed, i.e., after Schwarzschild (1916) obtained the first exact (spherically symmetric) solution of Einstein's equations in vacuum.

In particular, black hole developments in the last forty years have shown that black holes have thermodynamics properties like entropy and temperature, and as a consequence of the instability of the vacuum in strong gravitational fields, they are sources of quantum radiation [4–6]. String theory and loop quantum gravity, lately, showed that the origin of the black hole thermodynamics must be related with the quantum structure of the spacetime, bringing together the developments in black hole physics and the improvement of our understanding on the nature of the spacetime in quantum gravity regime [7, 8]. In this way, it is believed that black holes may play a major role in our attempts to shed some light on the quantum nature of the spacetime such as the role played by atoms in the early development of quantum mechanics.

However the understanding of black hole thermodynamics in the semiclassical and furthermore in quantum regime has been a very difficult, and still unsolved problem. To explain the situation, it is known that, in statistical physics, entropy counts the number of accessible microstates that a system can occupy, where all states are presumed to occur with equal externally observable classical parameters: mass, electric charge, and angular momentum. All other information about the matter which formed a black hole "disappears" behind its event horizon, and therefore the nature of these microstates is obscure. Thus, the origin of the black hole entropy is not clear. Furthermore, in order to justify the name "entropy", one must to explain also why the sum of the entropy of a black hole and the entropy of its vicinity is a non-decreasing function of time. In other words, why black holes obey the so called "Generalized Second Law of thermodynamics (GSL)".

The situation becomes even worse if black hole evaporation is considered. Since black holes evaporate, one could expect, from black hole radiation, any information about the state which collapsed into the black hole. However, Hawking showed, through semiclassical arguments, that black hole radiation is thermal, and therefore does not carry any information about its initial state. In this situation, the matter that formed the black hole, which initially was in a pure state has evolved into a mixed state. This fact bring us a contradiction with quantum mechanics, where a pure state can only evolve into another pure state because of the unitarity of the evolution operator [4, 5, 9, 10].

In this context, a new phenomenon arises as one way to solve the drawbacks between black hole physics and quantum mechanics. This phenomenon is related with quantum gravity, and consists in a topology change of the spacetime, where a new topologically disconnected region arises inside the black hole, and information can be stored and preserved there. This scenario can be produced by the gravitational collapse, which would lead to a region of Planckian densities and curvature where quantum gravitational effects becomes important. Topology change must occur deep inside the black hole horizon, in a way that, it is entirely invisible to observers outside the black hole, which see the usual Hawking evaporation. In this situation, a complete state specification of the (now topologically non-trivial) universe requires a wavefunction which has a component on the new topologically disconnected region too. In this way, observers without access to this new region, have incomplete information about the universe as a whole.

In this chapter, the black hole thermodynamics will be addressed in the context of topology change, as conceived for some classes of quantum spaces, called fuzzy spheres. It will be argued that a model based on the topology change of these fuzzy manifolds can be used to shed some light on the origin of the black hole entropy, including why black hole evaporation process obeys the GSL. In this sense, the selection rules will be addressed for the black hole area transitions in a black hole evaporation process driven by topology change. Moreover, the information loss problem will be discussed, including the possibility of some information about the black hole initial state could be recovered by an observer in our universe, where one can perform measures.

This chapter is organized as follows. In the second section, the black hole thermodynamics will be addressed, introducing the laws of black hole mechanics and the Hawking effect. In the third section, the fuzzy sphere model and the topology change process for fuzzy spaces will be addressed. In the forth section, the relation between fuzzy spaces topology change and black hole thermodynamics will be addressed, where the selection rules for the black hole area transitions in the evaporation process are obtained. In the fifth section, it will be investigated the obedience to GSL by the black hole evaporation in the fuzzy topology change approach. The sixth section is devoted to conclusions.

2. Black hole thermodynamics

Over the last forty years, black holes have been shown to have a number of surprising properties. These discoveries have revealed unforeseen relations between the otherwise distinct areas of general relativity, quantum physics and statistical mechanics. This interplay, in turn, led to a number of deep puzzles at the very foundations of physics. Some have been resolved while others remain open. The starting point of these fascinating developments remounts to the early 1970s, where a set of relations among neighboring black hole solutions were found, culminating in Bardeen, Carter, and Hawking's "Four Laws of Black Hole Mechanics" [11]. These laws dictate the behavior of black holes in equilibrium, under small perturbations away from equilibrium, and in fully dynamical situations. While they are consequences of classical general relativity alone, they have a close similarity with the laws of thermodynamics. The origin of this seemingly strange coincidence lies in quantum physics.

2.1. The laws of black hole mechanics

1 - **(Zeroth law)** If the energy-momentum tensor $T_{\mu\nu}$ obeys the dominant energy condition, then the surface gravity κ is constant on the future event horizon H^+.

Although the surface gravity κ is defined locally on the horizon, it turns out that it is always constant over the horizon of a stationary black hole. This constancy is reminiscent of the Zeroth Law of Thermodynamics which states that the temperature is uniform everywhere in a system in thermal equilibrium. The zeroth law of black hole mechanics can be proved without field equations or energy conditions [12–15] assuming that the horizon is a Killing horizon (i.e. there is a Killing field tangent to the null generators of the horizon) and that the black hole is either (i) static (i.e. stationary and time reflection symmetric) or (ii) axisymmetric and "$t - \phi$" reflection symmetric. Alternatively it can be proved (Hawking) assuming only stationarity together with Einstein's field equations with the dominant energy condition for matter [16, 17]. [1]

2 - **(First law)** If a stationary black hole of mass M, charge Q and angular momentum J, with future event horizon of surface gravity κ, electric potential Φ_H and angular velocity Ω_H, is perturbed such that it settles down to another black hole with mass $M + dM$, charge $Q + dQ$ and angular momentum $J + dJ$, then,

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ \; . \tag{1}$$

The first law relates nearby stationary black hole solutions of the Einstein's equation, and has been derived in many ways. If stationary matter (other than the electromagnetic field) is present outside the black hole, then there are additional matter terms on the right hand side of the equation (1). The surface gravity κ evidently plays the hole of temperature. Although the quantities κ, Ω_H, and Φ_H are all defined locally on the horizon, they are always constant

[1] Assuming also hyperbolic field equations for matter, and analyticity of the spacetime, Hawking also shows that the event horizon must be a Killing horizon, and that the spacetime must be either static or axisymmetric.

over the horizon of a stationary black hole (modulo some assumptions; see for example the assumptions above for κ).

3 - **(Second law)** If $T_{\mu\nu}$ satisfies the weak energy condition, and assuming that the cosmic censorship hypothesis is true then the area of the future event horizon of an asymptotically flat spacetime is a non-decreasing function of time:

$$\delta A \geq 0\,. \tag{2}$$

4 - **(Third law)** It is impossible by any procedure to reduce the surface gravity κ to zero in a finite number of steps. [2]

Bardeen, Carter, and Hawking noted that these laws closely parallel the ordinary laws of thermodynamics, with the horizon area playing the role of entropy and the surface gravity playing the role of temperature. But can one truly considers that this analogy is something more than a mere formal coincidence?

The physical temperature of a black hole in classical general relativity is absolute zero. The existence of the event horizon prevents the black hole to emit anything, and it can be regarded as a perfect absorber, with absolute zero temperature. In this way, the identification of the surface gravity of a black hole with a temperature, in the context of the classical theory, is completely non-physical, and so the identification of the event horizon area with an entropy.

This was the general view of most physicists before the discovery of the Hawking effect. The most notable exception was Bekenstein, who put forward the idea of a real physical connection between entropy and area of the event horizon, even before the work of Hawking. He also suggested a generalized second law for black holes:

Generalized Second Law (GSL) :

"The sum of the black holes entropy and the entropy of matter outside black holes would never decrease"

$$\delta\Big(S_{out} + \frac{1}{8\pi\alpha}A\Big) \geq 0\,. \tag{3}$$

The existence of black holes is not compatible with the ordinary second law of thermodynamics. If matter can fall into a black hole and disappear, the entropy of matter for the external observer decreases. However, the area of the event horizon increases. Bekenstein suggested that the generalized entropy $S' = S_{out} + A/8\pi\alpha$ does not decrease.

A more detailed inspection of this suggestion shows that it is not consistent. One can considers a black hole immersed in a thermal bath at a temperature lower that $\alpha c^2\kappa/G$. Since the black hole will absorb part of the radiation without emitting anything, one has a heat flow from a cold thermal radiation to a "hotter" black hole. This would disagree with the generalized second law because the loss of entropy from the thermal radiation would be greater than the increase in black hole entropy. An additional physical input is required to pass from a formal to a physical analogy.

Some insights can be gained analyzing the dimension of the constant α. A simple look unravels that, since S has the dimension of Boltzmann's constant k_B. In this way, αk_B must

[2] As in ordinary thermodynamics, there are a number of formulations of the third law, which are not strictly equivalent. The version given here is analogous to the Nernst form of the third law of thermodynamics.

have dimensions of length squared. With the physical constants that one has in classical general relativity (i.e., Newton's constant G and the velocity of light c) it is not possible to form a constant (to be identified with αk_B) with dimensions of length squared. The Planck's constant $\hbar$ is necessary for that.

From G, c and $\hbar$, one can form the Planck length

$$l_P = \sqrt{\frac{G\hbar}{c^3}} \,. \tag{4}$$

With this fundamental length available ($\alpha k_B \propto l_P^2$), one can go further in the analogy and write

$$T \propto \frac{\hbar}{k_B c}\kappa \,, \quad S \propto \frac{k_B c^3}{G\hbar} A \,. \tag{5}$$

The lesson of this brief discussion is that the input required to properly establish a physical analogy between black holes and thermodynamics involves considering quantum effects. This will be the topic of the next section.

2.2. The Hawking effect

In 1974, Hawking demonstrated that all black holes emit blackbody radiation [4]. The original derivation of the Hawking effect made direct use of the formalism for calculating particle creation in a curved spacetime. Hawking considered a classical spacetime describing gravitational collapse to a Schwarzschild black hole. A free (i.e., linear) quantum field propagating in this background spacetime was considered, which is initially in its vacuum state prior to the collapse, and the particle content of the field at infinity at late times was computed. This calculation involves taking the positive frequency mode function corresponding to a particle state at late times, propagating it backwards in time, and determining its positive and negative frequency parts in the asymptotic past. Hawking's calculation revealed that at late times, the expected number of particles at infinity corresponds to emission from a perfect black body (of finite size)

$$\langle N_\omega^{(\Im^+)} \rangle = \frac{\Gamma_\omega}{e^{2\pi\omega/\kappa} - 1} \tag{6}$$

at the Hawking temperature

$$T = \hbar \frac{\kappa}{2\pi k_B} \,, \tag{7}$$

where Γ_ω is a scattering coefficient sometimes called the gray-body factor since it indicates the emissivity of the black hole which is not that of a perfect blackbody.

It should be noted that this result relies only on the analysis of quantum fields in the region exterior to the black hole, and it does not make use of any gravitational field equations. In this way, Hawking's calculation has two main ingredients: the first is that the quantum mechanical vacuum is filled with virtual particle-antiparticle pairs that fluctuate briefly into and out of existence. Energy is conserved, so one member of each pair must have negative energy. [3] Normally, negative energy is forbidden - in a stable quantum field theory, the vacuum must

[3] To avoid a common confusion, note that either the particle or the antiparticle can be the negative-energy partner.

be the lowest energy state - but energy has a quantum mechanical uncertainty of order $\hbar/t$, so a virtual pair of energy $\pm E$ can exist for a time of order $\hbar/E$. The second ingredient is the observation that, in general relativity, energy - and, in particular, the sign of energy - can be frame dependent. The easiest way to see this is to note that the Hamiltonian is the generator of time translations, and thus depends on one's choice of a time coordinate. [4]

In this way, one can conclude that a black hole may radiate its energy at a temperature given by equation (7). This result makes the relation between the laws of black hole mechanics and the laws of thermodynamics to be more than a mere analogy.

From the results above, one can attribute an entropy to black hole which is given by:

$$S_{BH} = \frac{A}{4l_P^2} \, . \tag{8}$$

Another interesting result is a consequence of the Stephan law. For a black hole with a horizon area A:

$$\frac{dE}{dt} \backsimeq -\sigma A T_H^4 \, , \tag{9}$$

where $\sigma = \pi^2 k_B^4 / 60\hbar^3 c^2$.

Moreover, from the fact that

$$E = Mc^2, \quad k_B T_H \sim \frac{\hbar c^3}{GM}, \tag{10}$$

and, for a Schwarzchild black hole:

$$A = 4\pi \left(\frac{MG}{c^2} \right)^2, \tag{11}$$

another result is that

$$\frac{dE}{dt} \sim \frac{\hbar c^4}{G^2 M^2} \, . \tag{12}$$

In this way, a Schwarzchild black hole possess a finite lifetime which is given by:

$$\tau \sim \left(\frac{G^2}{\hbar c^4} \right) M^3 \, . \tag{13}$$

[4] One must therefore be careful about what one means by positive and negative energy for a virtual pair. In particular, consider the Schwarzschild scenario. Outside the event horizon, t is the usual time coordinate, measuring the proper time of an observer at infinity. Inside the horizon, though, components of the metric change sign, and r becomes a time coordinate, while t becomes a spatial coordinate: an observer moving forward in time is one moving in the direction of decreasing r, and not necessarily increasing t. Hence an ingoing virtual particle that has negative energy relative to an external observer may have positive energy relative to an observer inside the horizon. The uncertainty principle can thus be circumvented: if the negative-energy member of a virtual pair crosses the horizon, it need no longer vanish in a time $\hbar/E$, and its positive-energy partner may escape to infinity [18].

3. Open questions

The results described in the previous sections provide a remarkably compelling case that stationary black holes are localized thermal equilibrium states of the quantum gravitational field, and that the laws of black hole mechanics are simply the ordinary laws of thermodynamics applied to a system containing a black hole. Although no results on black hole thermodynamics have been subject to any experimental or observational tests, the theoretical foundation of black hole thermodynamics appears to be sufficiently firm to provide a solid basis for further research and speculation on the nature of quantum gravitational phenomena. In this section, it will be briefly discussed two key unresolved issues in black hole thermodynamics which may shed considerable further light upon quantum gravitational physics.

3.1. The origin of black hole entropy

From the results above, it is clear that black holes are really thermodynamical systems with an actual temperature and entropy. What remains to be understood is the meaning of this entropy in terms of statistical mechanics.

It is known that in statistical physics, entropy counts the number of accessible microstates that a system can occupy, where all states are presumed to occur with equal probability. In the thermodynamical description, entropy S is related to the number of all consistent microscopic states N as

$$S = k_B ln N \, . \tag{14}$$

In analogy, it is expected that there might be a microscopic description of the black hole thermodynamics, too. However, it is also known that black holes can be completely characterized by only three externally observable classical parameters: mass, electric charge, and angular momentum. All other information about the matter which formed a black hole “disappears” behind its event horizon, and therefore the nature of these microstates is obscure. Then, what is the origin of the black hole entropy?

It is widely believed that the black hole entropy might be related to a number of microscopic states. Since the microscopic description seems to require a quantum theory of gravity, detailed investigations of the black hole entropy should contribute a lot toward construction of the theory of quantum gravity. This is one among the several reasons why the origin of the black hole entropy needs to be understood at the fundamental level. Another question is related with GSL. In order to justify the name “entropy”, one must to explain also why $S = S_{bh} + S_{out}$ is a non-decreasing function of time, in other words, why black holes obey a GSL.

A strong motivation to investigate the black hole entropy is the so-called information loss problem, which will be addressed in the next section.

3.2. The information loss problem

Black holes can be completely characterized by only three externally observable classical parameters: mass, electric charge, and angular momentum. All other information about the matter which formed a black hole “disappears” behind the black hole event horizon, and therefore the nature of these microstates is obscure. Since black holes evaporate, one could expect, from the Hawking radiation, any information about the state which collapsed into the black hole. However, Hawking showed that this radiation is thermal, and therefore does not carry any information about the black hole initial state. That is to say, no information can escape from inside of the black hole horizon. In this situation, the matter that formed the black hole, which initially was in a pure state has evolved into a mixed state. But, it contradict our basic knowledge about quantum mechanics. There, a pure state can only evolve into another pure state because of the unitarity of the evolution operator $U(U^{\dagger}U = 1)$. This problem is known as the black hole information loss paradox.

Some possibilities to solve this paradox have already been proposed. Among these possibilities, one has

i - The evolution is indeed non-unitary and the basics concepts of quantum mechanics must be revised [5, 19]. However, these proposal has difficulties related with violation of energy conservation and the absence of an empty vacuum as the ground state [20–23].

ii - The black hole radiation is not thermal and carries information. The problem is that a new physics is necessary, which is radically different from the one is known: concepts as locality and causality must be left, since matter behind the horizon has to influence matter outside the horizon [24–26].

iii - The information is stored in a stable black hole remnant [27]. The main problem with remnants is that, since the initial black hole could have been arbitrarily massive, the remnant must be capable of carrying an arbitrarily large amount of information (about M^2/M^2_{Planck} bits, if the initial mass was M). This means that there must be an infinite number of species of stable remnants, all with mass comparable to M_{Planck} . Black hole remnants have appeared in several noncommutative approaches of black holes, including one where the fuzzy sphere model is used [28–30].

iv - Information could be stored in a topological disconnected region which arises inside of the black hole [31]. Gravitational collapse leads to a region of Planckian densities and curvature where quantum gravitational effects can lead to a topology change process where a new topologically disconnected region (a baby universe) appears. Information about the black hole initial state can be stored there. It is possible, but not necessary, that information returns via quantum gravity tunneling after some long timescale. In this point, the baby universe ceases to exist, and the black hole evolution as seen by an observer outside the black hole is unitary. Topology change had been claim to be non-unitary and therefore to suffer from the same problems of the first proposal. The other objection against topology change is the violation of cluster decomposition(locality) [31, 32].

In this chapter, the topology change approach will be addressed. The main idea present here is to see the black hole event horizon as a fuzzy sphere, and using its known quantum symmetries properties, find out a topology change process to black holes, which is free of the problems related with unitarity and locality. If this is possible, a solution to the information loss paradox will be gotten. Moreover, this model, based on the topology change of a quantum manifold, is used to explain the origin of the black hole thermodynamics. It will be argued that this model can explain the origin of the black hole entropy, and why black holes obey a generalized second law of thermodynamics. In the following sections, all fundamental constants will be considered equals to one

4. Fuzzy spaces topology change and black hole thermodynamics

4.1. The fuzzy sphere model

Fuzzy spheres consist in one of the most simplest example of noncommutative spaces and appear as vacuum solutions in Euclidean gravity [33–35]. It is obtained when one quantizes the usual sphere S^2 replacing the commutative algebra of functions on this manifold by the noncommutative algebra of matrices.

It is known that any function defined on the usual sphere can be expanded in terms of the spherical harmonics Y_{lm} as

$$f(x)\mid_{S^2} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_{lm}(x) \ , \tag{15}$$

where c_{lm} are complex coefficients, and the product of these functions is commutative.

The introduction of the noncommutative geometry is performed as one quantizes the coordinates x^μ $(\mu = 1, 2, 3)$ on S^2, through the transformation

$$x^\mu \rightarrow \hat{x}^\mu = \kappa \hat{J}^\mu \ , \tag{16}$$

with

$$\hat{x}^\mu \hat{x}_\mu = r^2 \mathbf{1} \, , \tag{17}$$

where $\hat{J}^\mu$ form the n-dimensional irreducible representation of the $SU(2)$ algebra, whereas r is the fuzzy sphere radius, and

$$\kappa = \frac{r}{\sqrt{N^2 - 1}} \ . \tag{18}$$

In this way, the coordinates on the fuzzy sphere S_F^2 satisfy the following commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i \chi r^{-1} \varepsilon^{\mu\nu\alpha} \hat{x}^\alpha \ , \tag{19}$$

where χ has a dimension of $(length)^2$, and plays here a role analogous to that played by Planck's constant in quantum mechanics. The fact that the coordinates $\hat{x}^\mu$ do not commute anymore implies that the points on the sphere are "smeared out", and one has to substitute the idea of points for the idea of elementary (Planck) cells.

A consequence of the process above is the introduction of a cutoff N on the expansion (15), in a way that it becomes

$$f(x)\mid_{S_F^2} = \sum_{l=0}^{N} \sum_{m=-l}^{l} c_{lm} \hat{Y}_{lm}(x) \ , \tag{20}$$

where now $\hat{Y}_{lm}(x)$ are matrices. In this way the function $f(x)$ is replaced by a matrix $(N+1) \times (N+1)$ in a way that its product becomes noncommutative. The commutative limit is given by $\chi \rightarrow 0$ or $N \rightarrow \infty$.

In the context of black hole physics, the use of fuzzy spheres is mostly motivated by the Bekenstein's limit [6], which says that the black hole entropy is finite and proportional to the event horizon area. Since fuzzy spheres, are obtained from quantization of a compact space, they are described by finite dimensional matrices, in a way that the number of independent states defined on the fuzzy sphere is limited, and the entropy associated with these states is finite, in agreement with the Bekenstein's limit [28, 34, 36–39].

4.2. Fuzzy spheres and Hopf algebras

Another important feature of fuzzy spheres is its close relationship with Hopf algebras, which allow us to define a linear operation (the coproduct of a Hopf algebra) on S^2_F and compose two fuzzy spheres preserving algebraic properties intact. This operation produces a topology change process where a fuzzy sphere splits into two others [40], and can be used as a good mathematical model to black hole topology change [36, 37].

In order to define a Hopf algebra, one has that a bi-algebra is a vector space A over a field of scalars F which is both an algebra and a coalgebra in a compatible way. The algebra structure is given by F-linear maps $m : A \otimes A \to A$ (the product) and $\eta : F \to A$ (the unit map), where $xy := m(x, y)$ and $\eta(1) = 1A$. The coalgebra structure is likewise given by linear maps $\Delta : A \to A \otimes A$ (the coproduct) and $\epsilon : A \to F$ (the counit map). One writes $\iota : A \to A$, or sometimes ιA , to denote the identity map on A. The required properties are

- Associativity: $m(m \otimes \iota) = m(\iota \otimes m) : A \otimes A \otimes A \to A$;
- Unity: $m(\eta \otimes \iota) = m(\iota \otimes \eta) = \iota : A \to A$;
- Coassociativity: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta : A \to A \otimes A \otimes A$;
- Counity: $(\epsilon \otimes \iota)\Delta = (\iota \otimes \epsilon)\Delta = \iota : A \to A$;

A Hopf algebra is defined as a bi-algebra H together with a (necessarily unique) convolution inverse S for the identity map $\iota = \iota H$; the map S is called the antipode of H.

Fuzzy spaces possess quantum groups properties related with a Hopf algebra [40].[5] To describe the fuzzy sphere topology change, one has that under the quantization procedure (16), functions defined on S^2 are replaced by matrices on S^2_F [33]. In this way, let a matrix $\hat{M}$ describing a wave function on S^2_F, the Hopf coproduct $\Delta : S^2_F(j) \to S^2_F(K) \otimes S^2_F(L)$ acts on $\hat{M}$ as

$$\Delta_{(K,L)}(\hat{M}) = \sum_{\mu_1,\mu_2,m_1,m_2} C_{K,L,J;\mu_1,\mu_2} C_{K,L,J;m_1,m_2} \times M_{\mu_1+\mu_2,m_1+m_2} e^{\mu_1 m_1}(K) \otimes e^{\mu_2 m_2}(L) \ , \tag{21}$$

where C's are the Clebsh-Gordan coefficients and $e^{\mu_i m_j}$'s are basis for a matrix space defined on the fuzzy sphere [40].

The coproduct Δ has the following properties:

$$\Delta_{(K,L)}(M^\dagger) = \Delta_{(K,L)}(M)^\dagger \ , \tag{22}$$

$$\Delta_{(K,L)}(MN) = \Delta_{(K,L)}(M)\Delta_{(K,L)}(N) \ , \tag{23}$$

$$Tr\Delta_{(K,L)}(M) = Tr(M) \ ,$$

[5] Actually, fuzzy spaces possess algebraic properties more general than a Hopf algebra. It is due to the fact that a coproduct of two different algebras is possible, whereas in an ordinary Hopf algebra only the coproduct of an algebra by itself is possible.

and

$$\langle \Delta_{(K,L)}(M^{\dagger}), \Delta_{(K,L)}(N^{\dagger}) \rangle = \langle M, N \rangle \tag{24}$$

In this way, the coproduct Δ preserves the Hermitian conjugation, the matrix product, the matrix trace, and the matrix inner product. These properties of the coproduct Δ assure that (21) is a unitary process, and preserves the algebraic properties of the operators defined on the fuzzy sphere [36, 37, 40].

4.3. Fuzzy spaces topology change and black hole thermodynamics

The basic assumptions in this chapter stay in the meaning of the eq. (21). This equation says that a wavefunction $\hat{M} \in S_F^2(J)$ splits into a superposition of wavefunctions on $S_F^2(K) \otimes S_F^2(L)$. In this way, the information in $\hat{M}$ is divided between two regions of the spacetime, i.e, the two fuzzy spheres with spins K and L respectively. The following consequences are imminent:

(i) If one use the fuzzy sphere Hilbert space as the ones of the black hole, the maximum of information about the black hole that an outside observer can obtain would be encoded in wave functions defined on the fuzzy sphere Hilbert space.

(ii) One will find out, through the Hopf coproduct Δ, a topology change process for the black hole. In this process the information about the black hole initial state, will be divided into two spacetime regions. One of them is a fuzzy sphere with spin K, which will be considered as the original world and name it "the main world". The other one is a fuzzy sphere with spin L which will be named "the baby world".

(iii) The process is unitary, in this way there is no information loss. However, since the baby world arises in the black hole interior, an observer in the main world can not access the degrees of freedom there. In this way, from his standpoint, the black hole will appear to evolve from a pure to a mixed state described by a density matrix $\hat{\rho}$. It enable us to define an entropy, measured by the observer in the main world, associated to the black hole horizon.

Now, in order to analyze how the topology change process drives the black hole evaporation, it will be necessary to investigate how the fuzzy topology change drives the black hole area transitions. It will be admitted that the selection rules for the black hole area transitions are the ones for the topology change. These rules are obtained from the eq. (21), when one traces over the degrees of freedom in the baby universe.

The splitting process (21) for a matrix $\hat{M} = \mid J, m\rangle\langle J, m' \mid$ with $L = 1/2$, and $K = J - 1/2$ is given by:

$$\begin{aligned}\Delta(\mid J, m\rangle\langle J, m' \mid) \quad &= \\ &\frac{\sqrt{(K+m_K+1)(K+m'_K+1)}}{2K+1} \mid K, m-1/2\rangle\langle K, m'-1/2 \mid \\ &+ \frac{\sqrt{(K-m_K+1)(K-m'_K+1)}}{2K+1} \mid K, m+1/2\rangle\langle K, m'+1/2 \mid . \end{aligned} \tag{25}$$

In that point, it still necessary to ensure that the splitting process above can be performed repeatedly, under identical circumstances, with statistically independent results to outside observers in different regions of spacetime. That is to say, it is necessary to ensure that locality is preserved in this process. In ordinary quantum field theory these requirements are insured by the Cluster Decomposition Theorem.

Cluster decomposition theorem

The vacuum expectation value of a product of many operators - each of them being either in different regions A and B, where A and B are very separated - asymptotically equals to the expectation value of the product of the operators in A, times a similar factor from the region B. Consequently, sufficiently separated regions behave independently. If $A_1, ..., A_n$ are n operators each localized in a bounded region and one picks some subset of the n operators to translate $\mathbf{x_i}$ into $\mathbf{x'_i} = \mathbf{x_i} + \rho\mathbf{a}$,

$$\begin{aligned}\lim_{\rho\to\infty} \langle M_0, A_1(x_1)A_2(x_2), ..., A_{j-1}(x_{j-1})A_j(x'_j), ...A_n(x'_n), M_0\rangle \quad (26)\\ = \langle M_0, A_1(x_1)A_2(x_2), ..., A_{j-1}(x_{j-1}), M_0\rangle \\ \times\langle M_0, A_j(x'_j), ...A_n(x'_n), M_0\rangle \ ,\end{aligned}$$

where M_0 represents the vacuum state.

If one admits that (26) is valid for an outside observer is easy to see, from Eq. (24) that

$$\begin{aligned}\lim_{\rho\to\infty} \langle \Delta(M_0), \Delta(A_1(x_1))\Delta(A_2(x_2)), ..., \Delta(A_{j-1}(x_{j-1}))\Delta(A_j(x'_j)), ..., \Delta(A_n(x'_n)), \Delta(M_0)\rangle \\ = \langle \Delta(M_0), \Delta(A_1(x_1))\Delta(A_2(x_2)), ..., \Delta(A_{j-1}(x_{j-1})), \Delta(M_0)\rangle \\ \times\langle \Delta(M_0), \Delta(A_j(x'_j)), ..., \Delta(A_n(x'_n)), \Delta(M_0)\rangle \ , \quad (27)\end{aligned}$$

where $\Delta(A_j(x_j))$ and $\Delta(A_j(x'_j))$ represent the splitting process in different points of spacetime. The splitting process, then, occurs in a way that cluster decomposition is preserved and locality is not violated.

From the equation (25), and from the fact that the splitting process (21) obeys cluster decomposition, in a way that different steps $J \to J - 1/2$, in the black hole evaporation, are independent events, the probability amplitude for a n-steps transition is given by

$$a_{Jn} = \left(\frac{2J+1}{2J-n+1}\right). \quad (28)$$

Now, in order to analyze the black hole area transitions, it will be introduced a canonical ensemble in which our system (the BH) can occupy different area microstates. The idea of using these types of ensembles goes back to Krasnov [41–43] and is, somehow, a necessity in the Loop Quantum Gravity formalism as the count of states is naturally done by using the horizon area instead of BH mass [44, 45]. In this framework, the probability amplitude for the BH evaporate is given by

$$a_{Jn} = e^{-\beta\delta A_{Jn}} \ , \quad (29)$$

where β is a temperature-like parameter dual to the black hole area [46–48].

The probability amplitude (28) will be identified with (29), in a way that the value of the black hole area in the J-state will be written as

$$A_J = \beta^{-1}\ln(2J+1) \ . \quad (30)$$

Moreover, the density matrix describing the black hole quantum states can be written as

$$\hat{\rho} = (1/Z) \sum_{J=0}^{Dim(S_F^2)} e^{-\beta A_J} \mid J\rangle\langle J \mid = (1/Z)e^{-\beta\hat{A}} \ , \tag{31}$$

where $Z = Tre^{-\beta\hat{A}}$ is the partition function.

The matrix $\hat{\rho}$ in the eq. (31) satisfy the Bloch equation

$$i\frac{\partial\hat{\rho}}{\partial\Theta} = -\frac{\hat{A}}{8\pi}\hat{\rho} \ , \tag{32}$$

where β has been replaced by $-i\Theta/8\pi$.

The equation (32) will govern the transitions between black hole area states. It must be used when working in the Euclidean continuation of the black hole, supplementing the Wheeler-DeWitt equation, where Θ plays the role of a sort of "dimensionless internal time" associated with the horizon [46–48]. Moreover, $\Theta = i\Theta_E$, where Θ_E is the Euclidian angle. Regularity of the Euclidean manifold at the horizon imposes a fixed Euclidean angle given by $\Theta_E = 2\pi$. In this way, at the horizon $\beta = 1/4$.

From the equation (30), and the results above, the entropy $S = -Tr(\hat{\rho}ln\hat{\rho}) = ln(2J+1)$, associated for an outside observer to black hole is given by

$$S = \frac{A}{4} \quad , \tag{33}$$

which corresponds to the Bekenstein-Hawking formula.

The logarithmic dependence of the black hole area spectrum on J, in the expression (30), tell us that the decrease in the horizon area is continuous at large values of J, and discrete to small values of J, when the black hole approaches the Planck scale. The black hole area spectrum is showed in the figure (1).

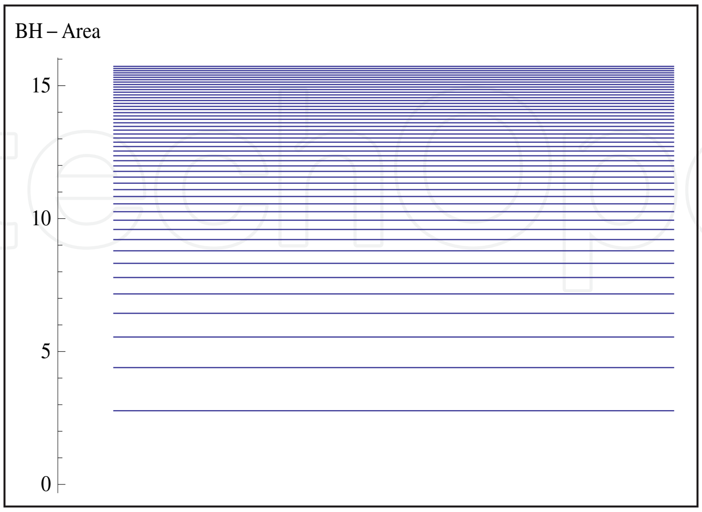

Figure 1. The black hole area spectrum for topology change approach

In this way, if one models a black hole horizon by a fuzzy sphere and consider its quantum symmetry properties, a topological change process which can be used to solve the black hole information loss paradox is obtained. In this process a black hole event horizon, modeled by a fuzzy sphere with spin J , splits into two others. The fuzzy sphere splitting can be used to describe a black hole evaporation process in which information about the black hole initial state is divided between two topologically disconnected regions: the main and the baby world. Nor unitarity or locality is broken in the evolution of the whole system. On the other hand, an observer in the main world sees the topology change process occurs in a non-unitary way, due to the impossibility of access the degrees of freedom in the baby world.

It is possible, but not necessary, that information returns via quantum gravity tunneling at the final stages of black hole evaporation. In this point, the baby universe ceases to exist, and the black hole evolution as seen by an observer outside the black hole is unitary. In the next section it will seen how the selection rules, inherited from the topology change process, will bring essential consequences to the way how entropy is emitted by black holes. One has that Hawking's radiation is known semi-classically to be continuous. However, the Hawking quanta of energy are not able to hover at a fixed distance from the horizon since the geometry of the horizon has to fluctuate, once quantum gravitational effects are included. Thus, one suspects a modification of the black hole radiation when quantum geometrical effects are taken into account. As will be seen in the next section, a possible modification on the description of the black hole emission process occurs at the final stages of black hole evaporation, where its area spectrum becomes discrete.

4.4. Entropy emitted during the evaporation process

It is known that the entropy of a system measures one's lack of information about its actual internal configuration. Suppose that everything one knows about the internal configuration of the system is that it may be found in any of a number of states, with probability p_n for the nth state. Then the entropy associated with the system is given by Shannon's well-known relation $S = -\sum p_n ln p_n$ [49–52].

The probability for a black hole to emit a specific quantum should be given by the expression (28), in which one must yet include a gray-body factor Γ (representing a scattering of the quantum off the spacetime curvature surround the black hole). Thus, the probability p_n to the black hole goes n steps down in the area ladder is proportional to $\Gamma(n)\ e^{-\frac{\delta A_{Jn}}{4}}$. Moreover, the discrete area spectrum (30) implies a discrete line emission from a quantum black hole.

To gain some insight into the physical problem, it will be considered a simple toy model suggested by Hod [53, 54]. To begin with, it is well known that, for massless fields, $\Gamma(M\omega)$ approaches 0 in the low-frequency limit, and has a high-frequency limit of 1. A rough approximation of this effect can be archived by introducing a low frequency cutoff at some $\omega = \omega_c$ [55]. That is, $\Gamma(\bar{\omega}) = 0$ for $\bar{\omega} < \bar{\omega}_c$, and $\Gamma(\bar{\omega}) = 1$ otherwise, where $\bar{\omega} = M\omega$ [56–58].

The ratio $R = | \dot{S}_{rad}/\dot{S}_{BH} |$ of entropy emission rate from the quantum black hole to the rate of black hole entropy decrease is given by:

$$R = \left| \frac{\sum_{i=1}^{N_s} \sum_{n=1}^{2J} C\Gamma(n) e^{-\frac{\delta A_{Jn}}{4}} ln\left[C\Gamma(n) e^{-\frac{\delta A_{Jn}}{4}}\right]}{\sum_{i=1}^{N_s} \sum_{n=1}^{2J} C\Gamma(n) e^{-\frac{\delta A_{Jn}}{4}} \left(\frac{\delta A_{Jn}}{4}\right)} \right| , \tag{34}$$

where C is a normalization factor, defined by the normalization condition:

$$\sum_{i=1}^{N_s}\sum_{n=1}^{2J} C\Gamma(n)e^{-\frac{\delta A_{Jn}}{4}} = 1 \ . \tag{35}$$

For the effective number of particle species emitted (N_s), it will be taken into account the various massless modes emitted. Here, N_s will be considered as

$$N_s = \begin{cases} 2J+1 \text{ for } & 2J+1 < 112\,, \\ \quad 112 \text{ for } & 2J+1 \geq 112\,. \end{cases}$$

In this way N_s is upper limited by the number of modes of massless particles in nature which make the dominant contribution to the black-hole spectrum (the 1/2, 3/2, 5/2 neutrino modes, the 1 and 2 photon modes, and the 2 and 3 graviton modes [53, 54, 56–58]), and by the size of the fuzzy sphere Hilbert space.

In the figure (2), R has been plotted down taking $\bar{\omega}_c \simeq 0.2$ (the location of the peak in the total power spectrum [56–58]). With this frequency cutoff, the minimal non-null value to the quantum number J, in order to have $\Gamma \neq 0$, is $J = 6.0$. In this point, the black hole must evaporate completely.

From the graphic for R, one has that the non-unitary evolution of the black hole geometry in the main world, due to the topology change process, imposes obedience to a "second law of thermodynamics" on the black hole evolution process, since R is ever larger than (or equal) to unity. The value of R approaches the value of 1.3 at the large J limit in agreement with known Zurek's semiclassical results [55]

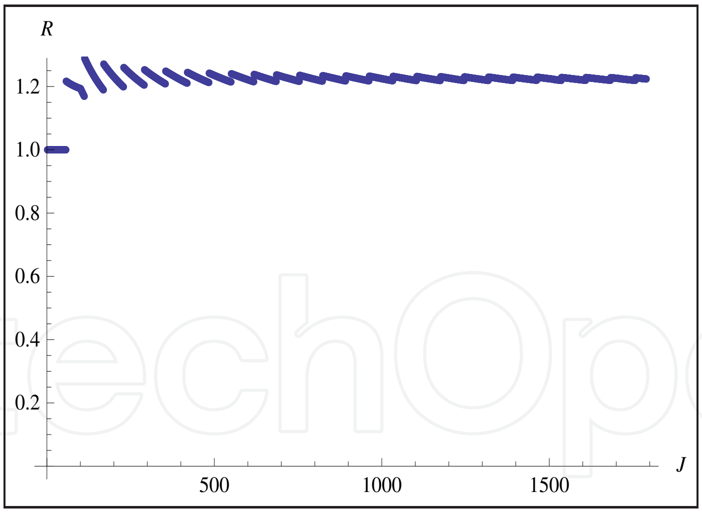

Figure 2. The ratio of entropy emission rate from the quantum black hole to the rate of black hole entropy decrease.

It is important to notice that the entropy emitted from the black hole decreases as the area spacing increases. The entropy of the radiation should be maximal in the semiclassical limit where the black hole can be in any area state, and the various transitions have almost the same probabilities. On the other hand, in the quantum limit, only special values are allowed to the black hole area, and then only special transitions are allowed. In this way, the entropy of the radiation emitted by the black hole becomes smaller. The striking consequence of this is the possibility that, since the black hole radiation becomes less and less entropic as the evaporation process takes place, some information about the black hole initial state could leak out from its interior and be accessible to an observer in our universe, where measures can be performed. The possibility of information leakage from a black hole with a discrete area spectrum is already pointed out by Hod [53, 54]

5. Remarks and conclusions

It has been argued that a model based on the topology change of a quantum manifold can be used to shed some light on the problems of the origin of black hole entropy and information loss. In this approach, black hole entropy is generated because of the non-unitary evolutions of the main world geometry due to the topology change process in black hole interior. To do this, the process of emission of quanta of radiation by black holes has been putted in connection with topology changes in the quantum manifold which is assumed to describe the horizon (the fuzzy sphere).

Nor unitarity or locality is broken in the evolution of the whole system in topology change approach in a way that no information is lost. Moreover, from the topology change model, the selection rules for the black hole area transitions have been obtained. In this way, an expression for the probability amplitudes of black hole transitions has been derived. From them, an understanding of the Bekenstein-Hawking formula for black hole entropy is provided. The topology change approach gives us a relation of states to points that brings together the black hole entropy and our standard concept of entropy as the logarithm of the number of microstates.

Through the study of the black hole evaporation process, an area spectrum, which is continuous in the semiclassical limit, and becomes discrete as the black hole approaches the Planck scale has been obtained. In order to investigate the influences of the area spectrum shape to the black hole emission, the ratio R between the rates of entropy emission and black hole entropy decrease has been calculated. At first, R is found to be larger than 1, showing that the considered mechanism is able to produce a generalized second law. Then, R approach 1 as the black hole shrinks to the Planck scale, and the area spectrum becomes discrete. These results point to a possible information leak out from black hole, since its radiation becomes less and less entropic as the black hole evaporates, and could alleviate the information problem for an observer outside the black hole. Since the possible information leakage would occurs more strongly in the quantum gravity limit, it would not require radical modifications in the laws of physics above the Planck scale. The task of found an appropriate quantum mechanism for information leakage remains.

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