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Recent Research on Jensen's Inequality for Oparators

Jadranka Mićić1 and Josip Pečarić2

Additional information is available at the end of the chapter

1. Introduction

The self-adjoint operators on Hilbert spaces with their numerous applications play an important part in the operator theory. The bounds research for self-adjoint operators is a very useful area of this theory. There is no better inequality in bounds examination than Jensen's inequality. It is an extensively used inequality in various fields of mathematics.

Let I be a real interval of any type. A continuous function $f : I \to \mathbb{R}$ is said to be operator convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y) \tag{id1}$$

holds for each $\lambda \in [0, 1]$ and every pair of self-adjoint operators x and y (acting) on an infinite dimensional Hilbert space H with spectra in I (the ordering is defined by setting $x \leq y$ if $y - x$ is positive semi-definite).

Let f be an operator convex function defined on an interval I. Ch. Davis [1] provedThere is small typo in the proof. Davis states that ϕ by Stinespring's theorem can be written on the form $\phi(x) = P\rho(x)P$ where ρ is a * -homomorphism to $B(H)$ and$P$ is a projection on H. In fact, H may be embedded in a Hilbert space K on which ρ and P acts. The theorem then follows by the calculation $f(\phi(x)) = f(P\rho(x)P) \leq Pf(\rho(x))P = P\rho(f(x)P = \phi(f(x))$, where the pinching inequality, proved by Davis in the same paper, is applied. a Schwarz inequality

$$f(\phi(x)) \leq \phi(f(x)) \tag{id3}$$

where $\phi: \to B(K)$ is a unital completely positive linear mapping from a C^*-algebra to linear operators on a Hilbert space K, and x is a self-adjoint element in with spectrum in I. Subsequently M. D. Choi [2] noted that it is enough to assume that ϕ is unital and positive. In

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fact, the restriction of ϕ to the commutative C^*-algebra generated by x is automatically completely positive by a theorem of Stinespring.

F. Hansen and G. K. Pedersen [3] proved a Jensen type inequality

$$f\left(\sum_{i=1}^{n} a_i^* x_i a_i\right) \leq \sum_{i=1}^{n} a_i^* f\left(x_i\right) a_i \tag{id4}$$

for operator convex functions f defined on an interval $I = [0, \alpha)$ (with $\alpha \leq \infty$ and $f(0) \leq 0$) and self-adjoint operators $x_1, \cdots, x_n$ with spectra in I assuming that $\sum_{i=1}^{n} a_i^* a_i = \mathbf{1}$. The restriction on the interval and the requirement $f(0) \leq 0$ was subsequently removed by B. Mond and J. Pečarić in [4], cf. also [5].

The inequality ($\square$) is in fact just a reformulation of ($\square$) although this was not noticed at the time. It is nevertheless important to note that the proof given in [3] and thus the statement of the theorem, when restricted to $n \times n$ matrices, holds for the much richer class of $2n \times 2n$ matrix convex functions. Hansen and Pedersen used ($\square$) to obtain elementary operations on functions, which leave invariant the class of operator monotone functions. These results then served as the basis for a new proof of Löwner's theorem applying convexity theory and Krein-Milman's theorem.

B. Mond and J. Pečarić [6] proved the inequality

$$f\left(\sum_{i=1}^{n} w_i \phi_i\left(x_i\right)\right) \leq \sum_{i=1}^{n} w_i \phi_i\left(f\left(x_i\right)\right) \tag{id5}$$

for operator convex functions f defined on an interval I, where $\phi_i : B(H) \to B(K)$ are unital positive linear mappings, $x_1, \cdots, x_n$ are self-adjoint operators with spectra in I and $w_1, \cdots, w_n$ are are non-negative real numbers with sum one.

Also, B. Mond, J. Pečarić, T. Furuta et al. [6], [7], [8], [9], [10], [11] observed conversed of some special case of Jensen's inequality. So in [10] presented the following generalized converse of a Schwarz inequality ($\square$)

$$F[\phi(f(A)), g(\phi(A))] \leq \max_{m \leq t \leq M} F\left[f(m) + \frac{f(M) - f(m)}{M - m}(t - m), g(t)\right] 1_{\tilde{n}} \tag{id6}$$

for convex functions f defined on an interval $[m, M]$, $m < M$, where g is a real valued continuous function on $[m, M]$, $F(u, v)$ is a real valued function defined on $U \times V$, matrix non-decreasing in u, $U \supset f[m, M]$, $V \supset g[m, M]$, $\phi : H_n \to H_{\tilde{n}}$ is a unital positive linear mapping and A is a Hermitian matrix with spectrum contained in $[m, M]$.

There are a lot of new research on the classical Jensen inequality ($\square$) and its reverse inequalities. For example, J.I. Fujii et all. in [12], [13] expressed these inequalities by externally dividing points.

2. Classic results

In this section we present a form of Jensen's inequality which contains (▭), (▭) and (▭) as special cases. Since the inequality in (▭) was the motivating step for obtaining converses of Jensen's inequality using the so-called Mond-Pečarić method, we also give some results pertaining to converse inequalities in the new formulation.

We recall some definitions. Let T be a locally compact Hausdorff space and let be a C^*-algebra of operators on some Hilbert space H. We say that a field $(x_t)_{t \in T}$ of operators in is continuous if the function $t \mapsto x_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \mapsto \| x_t \|$ is integrable, then we can form *the Bochner integral* $\int_T x_t \, d\mu(t)$, which is the unique element in such that

$$\varphi\left(\int_T x_t \, d\mu(t)\right) = \int_T \varphi(x_t) \, d\mu(t)$$
()

for every linear functional φ in the norm dual *.

Assume furthermore that there is a field $(\phi_t)_{t \in T}$ of positive linear mappings $\phi_t : \to \mathcal{B}$ from to another *-algebra $\mathcal{B}$ of operators on a Hilbert space K. We recall that a linear mapping $\phi_t : \to \mathcal{B}$ is said to be a positive mapping if $\phi_t(x_t) \geq 0$ for all $x_t \geq 0$. We say that such a field is continuous if the function $t \mapsto \phi_t(x)$ is continuous for every $x \in$. Let the *-algebras include the identity operators and the function $t \mapsto \phi_t(1_H)$ be integrable with $\int_T \phi_t(1_H) \, d\mu(t) = k 1_K$ for some positive scalar k. Specially, if $\int_T \phi_t(1_H) \, d\mu(t) = 1_K$, we say that a *field* $(\phi_t)_{t \in T}$ is *unital*.

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H. We define bounds of an operator $x \in B(H)$ by

$$m_x = \inf_{\|\xi\|=1} \langle x\xi, \, \xi \rangle \quad \text{and} \quad M_x = \sup_{\|\xi\|=1} \langle x\xi, \, \xi \rangle$$
(id7)

for $\xi \in H$. If (x) denotes the spectrum of x, then $(x) \subseteq [m_x, M_x]$.

For an operator $x \in B(H)$ we define operators $\mid x \mid$, x^+, x^- by

$$\mid x \mid = (x^* x)^{1/2}, \qquad x^+ = (\mid x \mid + x)/2, \qquad x^- = (\mid x \mid - x)/2$$
()

Obviously, if x is self-adjoint, then $\mid x \mid = (x^2)^{1/2}$ and $x^+, x^- \geq 0$ (called positive and negative parts of $x = x^+ - x^-$).

2.1. Jensen's inequality with operator convexity

Firstly, we give a general formulation of Jensen's operator inequality for a unital field of positive linear mappings (see [14]).

Theorem 1 Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I and let and $\mathcal{B}$ be unital C^*-algebras acting on a Hilbert space H and K respectively. If $(\phi_t)_{t \in T}$ is a unital field of positive linear mappings $\phi_t : \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality

$$f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \le \int_T \phi_t(f(x_t))\, d\mu(t) \tag{id10}$$

holds for every bounded continuous field $(x_t)_{t \in T}$ of self-adjoint elements in with spectra contained in I.

We first note that the function $t \mapsto \phi_t(x_t) \in \mathcal{B}$ is continuous and bounded, hence integrable with respect to the bounded Radon measure μ. Furthermore, the integral is an element in the multiplier algebra $M(\mathcal{B})$ acting on K. We may organize the set $CB(T,)$ of bounded continuous functions on T with values in as a normed involutive algebra by applying the point-wise operations and setting

$$\| (y_t)_{t \in T} \| = \sup_{t \in T} \| y_t \| \qquad (y_t)_{t \in T} \in CB(T,) \tag{}$$

and it is not difficult to verify that the norm is already complete and satisfy the C^*-identity. In fact, this is a standard construction in C^*-algebra theory. It follows that $f\left((x_t)_{t \in T}\right) = (f(x_t))_{t \in T}$. We then consider the mapping

$$\pi \colon CB(T,) \to M(\mathcal{B}) \subseteq B(K) \tag{}$$

defined by setting

$$\pi\left((x_t)_{t \in T}\right) = \int_T \phi_t(x_t)\, d\mu(t) \tag{}$$

and note that it is a unital positive linear map. Setting $x = (x_t)_{t \in T} \in CB(T,)$, we use inequality ($\sqcap$) to obtain

$$f\left(\pi\left((x_t)_{t \in T}\right)\right) = f(\pi(x)) \le \pi(f(x)) = \pi\left(f\left((x_t)_{t \in T}\right)\right) = \pi\left((f(x_t))_{t \in T}\right) \tag{}$$

but this is just the statement of the theorem.

2.2. Converses of Jensen's inequality

In the present context we may obtain results of the Li-Mathias type cf. Chapter 3[15] and [16], [17].

Theorem 2 Let T be a locally compact Hausdorff space equipped with a bounded Radon measure μ. Let $(x_t)_{t \in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra with spectra in $[m, M]$, $m < M$. Furthermore, let $(\phi_t)_{t \in T}$ be a field of positive linear mappings $\phi_t : \to \mathcal{B}$ from to another unital C^*-algebra $\mathcal{B}$, such that the function $t \mapsto \phi_t(1_H)$ is integrable with $\int_T \phi_t(1_H) d\mu(t) = k1_K$ for some positive scalar k. Let m_x and M_x, $m_x \leq M_x$, be the bounds of the self-adjoint operator $x = \int_T \phi_t(x_t) d\mu(t)$ and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$, $F : U \times V \to \mathbb{R}$ be functions such that $(kf)([m, M]) \subset U$, $g([m_x, M_x]) \subset V$ and F is bounded. If F is operator monotone in the first variable, then

$$
\inf_{m_x \leq z \leq M_x} F\left[k \cdot h_1\left(\frac{1}{k}z\right), g(z)\right] 1_K \leq F\left[\int_T \phi_t(f(x_t)) d\mu(t), g\left(\int_T \phi_t(x_t) d\mu(t)\right)\right]
$$

$$
\leq \sup_{m_x \leq z \leq M_x} F\left[k \cdot h_2\left(\frac{1}{k}z\right), g(z)\right] 1_K
$$

(id13)

holds for every operator convex function h_1 on $[m, M]$ such that $h_1 \leq f$ and for every operator concave function h_2 on $[m, M]$ such that $h_2 \geq f$.

We prove only RHS of (□). Let h_2 be operator concave function on $[m, M]$ such that $f(z) \leq h_2(z)$ for every $z \in [m, M]$. By using the functional calculus, it follows that $f(x_t) \leq h_2(x_t)$ for every $t \in T$. Applying the positive linear mappings ϕ_t and integrating, we obtain

$$
\int_T \phi_t(f(x_t)) d\mu(t) \leq \int_T \phi_t(h_2(x_t)) d\mu(t)
$$

()

Furthermore, replacing ϕ_t by $\frac{1}{k}\phi_t$ in Theorem □, we obtain $\frac{1}{k}\int_T \phi_t(h_2(x_t)) d\mu(t) \leq h_2\left(\frac{1}{k}\int_T \phi_t(x_t) d\mu(t)\right)$, which gives $\int_T \phi_t(f(x_t)) d\mu(t) \leq k \cdot h_2\left(\frac{1}{k}\int_T \phi_t(x_t) d\mu(t)\right)$. Since $m_x 1_K \leq \int_T \phi_t(x_t) d\mu(t) \leq M_x 1_K$, then using operator monotonicity of $F(\cdot, v)$ we obtain

$$
F\left[\int_T \phi_t(f(x_t)) d\mu(t), g\left(\int_T \phi_t(x_t) d\mu(t)\right)\right]
$$

$$
\leq F\left[k \cdot h_2\left(\frac{1}{k}\int_T \phi_t(x_t) d\mu(t)\right), g\left(\int_T \phi_t(x_t) d\mu(t)\right)\right] \leq \sup_{m_x \leq z \leq M_x} F\left[k \cdot h_2\left(\frac{1}{k}z\right), g(z)\right] 1_K
$$

(id14)

Applying RHS of (▱) for a convex function f (or LHS of (▱) for a concave function f) we obtain the following generalization of (▱).

Theorem 3 Let $(x_t)_{t \in T}$, m_x, M_x and $(\phi_t)_{t \in T}$ be as in Theorem ▱. Let $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$, $F : U \times V \to \mathbb{R}$ be functions such that $(kf)([m, M]) \subset U$, $g([m_x, M_x]) \subset V$ and F is bounded. If F is operator monotone in the first variable and f is convex on the interval $[m, M]$, then

$$F\left[\int_T \phi_t(f(x_t)) d\mu(t),\ g\left(\int_T \phi_t(x_t) d\mu(t)\right)\right]$$
$$\leq \sup_{m_x \leq z \leq M_x} F\left[\frac{Mk - z}{M - m} f(m) + \frac{z - km}{M - m} f(M),\ g(z)\right] 1_K \tag{id16}$$

In the dual case (when f is concave) the opposite inequalities hold in (▱) with inf instead of sup .

We prove only the convex case. For convex f the inequality $f(z) \leq \frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M)$ holds for every $z \in [m, M]$. Thus, by putting $h_2(z) = \frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M)$ in (▱) we obtain (▱). Numerous applications of the previous theorem can be given (see [15]). Applying Theorem ▱ for the function $F(u, v) = u - \alpha v$ and $k = 1$, we obtain the following generalization of Theorem 2.4[15].

Corollary 4 Let $(x_t)_{t \in T}$, m_x, M_x be as in Theorem ▱ and $(\phi_t)_{t \in T}$ be a unital field of positive linear mappings $\phi_t : \to \mathcal{B}$. If $f : [m, M] \to \mathbb{R}$ is convex on the interval $[m, M]$, $m < M$, and $g : [m, M] \to \mathbb{R}$, then for any $\alpha \in \mathbb{R}$

$$\int_T \phi_t(f(x_t)) d\mu(t) \leq \alpha\ g\left(\int_T \phi_t(x_t) d\mu(t)\right) + C1_K \tag{id18}$$

where

$$C = \max_{m_x \leq z \leq M_x} \left\{ \frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M) - \alpha g(z) \right\}$$
$$\leq \max_{m \leq z \leq M} \left\{ \frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M) - \alpha g(z) \right\} \tag{()}$$

If furthermore αg is strictly convex differentiable, then the constant $C \equiv C(m, M, f, g, \alpha)$ can be written more precisely as

$$C = \frac{M - z_0}{M - m} f(m) + \frac{z_0 - m}{M - m} f(M) - \alpha g(z_0) \tag{()}$$

where

$$z_0 = \begin{cases} g^{'-1}\left(\dfrac{f(M) - f(m)}{\alpha(M - m)} \right) & \text{if } \alpha g'(m_x) \leq \dfrac{f(M) - f(m)}{M - m} \leq \alpha g'(M_x) \\ m_x & \text{if } \alpha g'(m_x) \geq \dfrac{f(M) - f(m)}{M - m} \\ M_x & \text{if } \alpha g'(M_x) \leq \dfrac{f(M) - f(m)}{M - m} \end{cases}$$
()

In the dual case (when f is concave and αg is strictly concave differentiable) the opposite inequalities hold in (▭) with min instead of max with the opposite condition while determining z_0.

3. Inequalities with conditions on spectra

In this section we present Jensens's operator inequality for real valued continuous convex functions with conditions on the spectra of the operators. A discrete version of this result is given in [18]. Also, we obtain generalized converses of Jensen's inequality under the same conditions.

Operator convexity plays an essential role in (▭). In fact, the inequality (▭) will be false if we replace an operator convex function by a general convex function. For example, M.D. Choi in Remark 2.6[2] considered the function $f(t) = t^4$ which is convex but not operator convex. He demonstrated that it is sufficient to put $\dim H = 3$, so we have the matrix case as follows.

Let $\Phi : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ be the contraction mapping $\Phi\big((a_{ij})_{1\leq i, j\leq 3}\big) = (a_{ij})_{1\leq i, j\leq 2}$. If $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$,

then $\Phi(A)^4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neg \leq \begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix} = \Phi(A^4)$ and no relation between $\Phi(A)^4$ and $\Phi(A^4)$ under the operator order.

Example 5 It appears that the inequality (▭) will be false if we replace the operator convex function by a general convex function. We give a small example for the matrix cases and $T = \{1, 2\}$. We define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ by $\Phi_1\big((a_{ij})_{1\leq i, j\leq 3}\big) = \frac{1}{2}(a_{ij})_{1\leq i, j\leq 2}$, $\Phi_2 = \Phi_1$. Then $\Phi_1(I_3) + \Phi_2(I_3) = I_2$.
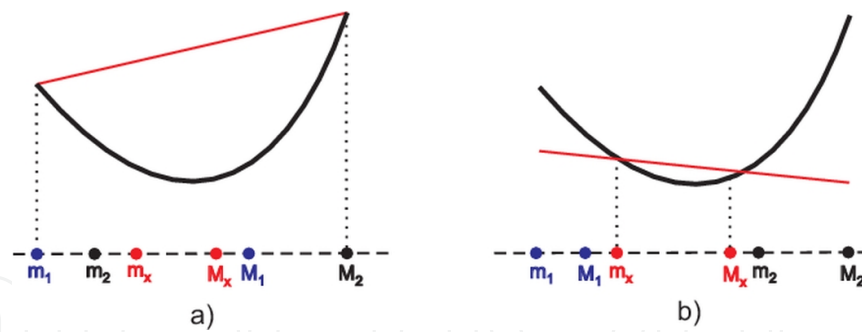
I)

• If

$$X_1 = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
()

then

Figure 1. Spectral conditions for a convex function f

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} \neg \leq \begin{pmatrix} 80 & 40 \\ 40 & 24 \end{pmatrix} = \Phi_1(X_1^4) + \Phi_2(X_2^4)$$

()

Given the above, there is no relation between $(\Phi_1(X_1) + \Phi_2(X_2))^4$ and $\Phi_1(X_1^4) + \Phi_2(X_2^4)$ under the operator order. We observe that in the above case the following stands $X = \Phi_1(X_1) + \Phi_2(X_2) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $[m_x, M_x] = [0, 2],$ $[m_1, M_1] \subset [-1.60388, 4.49396],$ $[m_2, M_2] = [0, 2],$ i.e.

$$(m_x, M_x) \subset [m_1, M_1] \cup [m_2, M_2]$$

()

(see Fig. 1.a).

II)

- If

$$X_1 = \begin{pmatrix} -14 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 15 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

()

then

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix} < \begin{pmatrix} 89660 & -247 \\ -247 & 51 \end{pmatrix} = \Phi_1(X_1^4) + \Phi_2(X_2^4)$$

()

So we have that an inequality of type (⊟) now is valid. In the above case the following stands $X = \Phi_1(X_1) + \Phi_2(X_2) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$ and $[m_x, M_x] = [0, 0.5],$ $[m_1, M_1] \subset [-14.077, -0.328566], [m_2, M_2] = [2, 15],$ i.e.

$$\left(m_x, M_x\right) \cap \left[m_1, M_1\right] = \varnothing \quad \text{and} \quad \left(m_x, M_x\right) \cap \left[m_2, M_2\right] = \varnothing \tag{}$$

(see Fig. 1.b).

3.1. Jensen's inequality without operator convexity

It is no coincidence that the inequality (□) is valid in Example □-II). In the following theorem we prove a general result when Jensen's operator inequality (□) holds for convex functions.

Theorem 6 Let $\left(x_t\right)_{t \in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let m_t and M_t, $m_t \leq M_t$, be the bounds of x_t, $t \in T$. Let $\left(\phi_t\right)_{t \in T}$ be a unital field of positive linear mappings $\phi_t : \rightarrow \mathcal{B}$ from to another unital C^* - algebra $\mathcal{B}$. If

$$\left(m_x, M_x\right) \cap \left[m_t, M_t\right] = \varnothing, \qquad t \in T \tag{}$$

where m_x and M_x, $m_x \leq M_x$, are the bounds of the self-adjoint operator $x = \int_T \phi_t(x_t)\, d\mu(t)$, then

$$f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \leq \int_T \phi_t\left(f\left(x_t\right)\right) d\mu(t) \tag{id25}$$

holds for every continuous convex function $f : I \rightarrow \mathbb{R}$ provided that the interval I contains all m_t, M_t.

If $f : I \rightarrow \mathbb{R}$ is concave, then the reverse inequality is valid in (□).

We prove only the case when f is a convex function. If we denote $m = \inf_{t \in T} \{m_t\}$ and $M = \sup_{t \in T} \{M_t\}$, then $[m, M] \subseteq I$ and $m 1_H \leq A_t \leq M 1_H$, $t \in T$. It follows $m 1_K \leq \int_T \phi_t(x_t)\, d\mu(t) \leq M 1_K$. Therefore $[m_x, M_x] \subseteq [m, M] \subseteq I$.

a) Let $m_x < M_x$. Since f is convex on $\left[m_x, M_x\right]$, then

$$f(z) \leq \frac{M_x - z}{M_x - m_x} f\left(m_x\right) + \frac{z - m_x}{M_x - m_x} f\left(M_x\right), \quad z \in \left[m_x, M_x\right] \tag{id26}$$

but since f is convex on $\left[m_t, M_t\right]$ and since $\left(m_x, M_x\right) \cap \left[m_t, M_t\right] = \varnothing$, then

$$f(z) \geq \frac{M_x - z}{M_x - m_x} f\left(m_x\right) + \frac{z - m_x}{M_x - m_x} f\left(M_x\right), \quad z \in \left[m_t, M_t\right], \quad t \in T \tag{id27}$$

Since $m_x 1_K \leq \int_T \phi_t(x_t)\,d\mu(t) \leq M_x 1_K,$ then by using functional calculus, it follows from ($\square$)

$$f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \leq \frac{M_x 1_K - \int_T \phi_t(x_t)\,d\mu(t)}{M_x - m_x} f(m_x) + \frac{\int_T \phi_t(x_t)\,d\mu(t) - m_x 1_K}{M_x - m_x} f(M_x) \tag{id28}$$

On the other hand, since $m_t 1_H \leq x_t \leq M_t 1_H,\ t \in T$, then by using functional calculus, it follows from ($\square$)

$$f(x_t) \geq \frac{M_x 1_H - x_t}{M_x - m_x} f(m_x) + \frac{x_t - m_x 1_H}{M_x - m_x} f(M_x), \qquad t \in T \tag{}$$

Applying a positive linear mapping ϕ_t and summing, we obtain

$$\int_T \phi_t(f(x_t))\,d\mu(t) \geq \frac{M_x 1_K - \int_T \phi_t(x_t)\,d\mu(t)}{M_x - m_x} f(m_x) + \frac{\int_T \phi_t(x_t)\,d\mu(t) - m_x 1_K}{M_x - m_x} f(M_x) \tag{id29}$$

since $\int_T \phi_t(1_H)\,d\mu(t) = 1_K$. Combining the two inequalities ($\square$) and ($\square$), we have the desired inequality ($\square$).

b) Let $m_x = M_x$. Since f is convex on $[m, M]$, we have

$$f(z) \geq f(m_x) + l(m_x)(z - m_x) \quad \text{for every } z \in [m, M] \tag{id30}$$

where l is the subdifferential of f. Since $m 1_H \leq x_t \leq M 1_H,\ t \in T$, then by using functional calculus, applying a positive linear mapping ϕ_t and summing, we obtain from ($\square$)

$$\int_T \phi_t(f(x_t))\,d\mu(t) \geq f(m_x) 1_K + l(m_x)\left(\int_T \phi_t(x_t)\,d\mu(t) - m_x 1_K\right) \tag{id31}$$

Since $m_x 1_K = \int_T \phi_t(x_t)\,d\mu(t)$, it follows

$$\int_T \phi_t(f(x_t))\,d\mu(t) \geq f(m_x) 1_K = f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \tag{id32}$$

which is the desired inequality ($\square$). Putting $\phi_t(y) = a_t y$ for every $y \in$, where $a_t \geq 0$ is a real number, we obtain the following obvious corollary of Theorem $\square$.

Corollary 7 Let $(x_t)_{t \in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let m_t and M_t, $m_t \leq M_t$, be the bounds of $x_t,\ t \in T$. Let $(a_t)_{t \in T}$ be a continuous field of nonnegative real numbers such that $\int_T a_t\,d\mu(t) = 1$. If

$$(m_x, M_x) \cap [m_t, M_t] = \varnothing, \qquad t \in T \tag{}$$

where m_x and M_x, $m_x \le M_x$, are the bounds of the self-adjoint operator $x = \int_T a_t x_t \, d\mu(t)$, then

$$f\left(\int_T a_t x_t \, d\mu(t)\right) \le \int_T a_t f(x_t) \, d\mu(t) \tag{id34}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_t, M_t.

3.2. Converses of Jensen's inequality with conditions on spectra

Using the condition on spectra we obtain the following extension of Theorem □.

Theorem 8 Let $(x_t)_{t \in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Furthermore, let $(\phi_t)_{t \in T}$ be a field of positive linear mappings $\phi_t : \to \mathcal{B}$ from to another unital C^*-algebra $\mathcal{B}$, such that the function $t \mapsto \phi_t(1_H)$ is integrable with $\int_T \phi_t(1_H) \, d\mu(t) = k 1_K$ for some positive scalar k. Let m_t and M_t, $m_t \le M_t$, be the bounds of x_t, $t \in T$, $m = \inf_{t \in T} \{m_t\}$, $M = \sup_{t \in T} \{M_t\}$, and m_x and M_x, $m_x < M_x$, be the bounds of $x = \int_T \phi_t(x_t) \, d\mu(t)$. If

$$(m_x, M_x) \cap [m_t, M_t] = \varnothing, \qquad t \in T \tag{}$$

and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$, $F : U \times V \to \mathbb{R}$ are functions such that $(kf)([m, M]) \subset U$, $g([m_x, M_x]) \subset V$, f is convex, F is bounded and operator monotone in the first variable, then

$$\inf_{m_x \le z \le M_x} F\left[\frac{M_x k - z}{M_x - m_x} f(m_x) + \frac{z - k m_x}{M_x - m_x} f(M_x), \, g(z)\right] 1_K$$

$$F\left[\int_T \phi_t(f(x_t)) \, d\mu(t), \, g\left(\int_T \phi_t(x_t) \, d\mu(t)\right)\right] \tag{id37}$$

$$\le \sup_{m_x \le z \le M_x} F\left[\frac{Mk - z}{M - m} f(m) + \frac{z - km}{M - m} f(M), \, g(z)\right] 1_K$$

In the dual case (when f is concave) the opposite inequalities hold in (□) by replacing inf and sup with sup and inf, respectively.

We prove only LHS of (□). It follows from (□) (compare it to (□))

$$\int_T \phi_t(f(x_t))\,d\mu(t) \geq \frac{M_x k 1_K - \int_T \phi_t(x_t)\,d\mu(t)}{M_x - m_x} f(m_x) + \frac{\int_T \phi_t(x_t)\,d\mu(t) - m_x k 1_K}{M_x - m_x} f(M_x) \tag{}$$

since $\int_T \phi_t(1_H)\,d\mu(t) = k 1_K$. By using operator monotonicity of $F(\,\cdot\,, v)$ we obtain

$$F\left[\int_T \phi_t(f(x_t))d\mu(t),\, g\!\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right] \geq F\left[\frac{M_x k 1_K - \int_T \phi_t(x_t)\,d\mu(t)}{M_x - m_x} f(m_x) + \frac{\int_T \phi_t(x_t)\,d\mu(t) - m_x k 1_K}{M_x - m_x} f(M_x),\, g\!\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right]$$

$$m_x z M_x\ F[M_x\ k\text{-}z M_x\text{-}m_x f(m_x)\text{+}z\text{-}k m_x M_x\text{-}m_x f(M_x), g(z)]\ 1_K$$

$$\tag{}$$

Putting $F(u, v) = u - \alpha v$ or $F(u, v) = v^{-1/2} u v^{-1/2}$ in Theorem □, we obtain the next corollary.

Corollary 9 Let $(x_t)_{t \in T}$, m_t, M_t, m_x, M_x, m, M, $(\phi_t)_{t \in T}$ be as in Theorem □ and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$ be continuous functions. If

$$(m_x, M_x) \cap [m_t, M_t] = \varnothing, \qquad t \in T \tag{}$$

and f is convex, then for any $\alpha \in \mathbb{R}$

$$\min_{m_x \leq z \leq M_x} \left\{ \frac{M_x k - z}{M_x - m_x} f(m_x) + \frac{z - k m_x}{M_x - m_x} f(M_x) - g(z) \right\} 1_K + \alpha g\!\left(\int_T \phi_t(x_t)\,d\mu(t)\right)$$

$$\leq \int_T \phi_t(f(x_t))\,d\mu(t) \tag{id39}$$

$$\leq \alpha g\!\left(\int_T \phi_t(x_t)\,d\mu(t)\right) + \max_{m_x \leq z \leq M_x} \left\{ \frac{M k - z}{M - m} f(m) + \frac{z - k m}{M - m} f(M) - g(z) \right\} 1_K$$

If additionally $g > 0$ on $[m_x, M_x]$, then

$$\min_{m_x \leq z \leq M_x} \left\{ \frac{\dfrac{M_x k - z}{M_x - m_x} f(m_x) + \dfrac{z - k m_x}{M_x - m_x} f(M_x)}{g(z)} \right\} g\!\left(\int_T \phi_t(x_t)\,d\mu(t)\right)$$

$$\leq \int_T \phi_t(f(x_t))\,d\mu(t) \leq \max_{m_x \leq z \leq M_x} \left\{ \frac{\dfrac{M k - z}{M - m} f(m) + \dfrac{z - k m}{M - m} f(M)}{g(z)} \right\} g\!\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \tag{id40}$$

In the dual case (when f is concave) the opposite inequalities hold in (□) by replacing min and max with max and min, respectively. If additionally $g > 0$ on $[m_x, M_x]$, then the oppo-

site inequalities also hold in (□) by replacing min and max with max and min, respective-
ly.

4. Refined Jensen's inequality

In this section we present a refinement of Jensen's inequality for real valued continuous con-
vex functions given in Theorem □. A discrete version of this result is given in [19].

To obtain our result we need the following two lemmas.

Lemma 10 Let f be a convex function on an interval I, $m, M \in I$ and $p_1, p_2 \in [0, 1]$ such
that $p_1 + p_2 = 1$. Then

$$\min\{p_1, p_2\}\left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right] \le p_1 f(m) + p_2 f(M) - f(p_1 m + p_2 M) \tag{id42}$$

These results follows from Theorem 1, p. 717[20].

Lemma 11 Let x be a bounded self-adjoint elements in a unital C^*-algebra of operators on
some Hilbert space H. If the spectrum of x is in $[m, M]$, for some scalars $m < M$, then

$$
\begin{aligned}
f(x) &\le \frac{M1_H - x}{M - m} f(m) + \frac{x - m1_H}{M - m} f(M) - \delta_f \tilde{x} \\
(\text{resp.} \quad f(x) &\ge \frac{M1_H - x}{M - m} f(m) + \frac{x - m1_H}{M - m} f(M) + \delta_f \tilde{x} \Big)
\end{aligned}
\tag{id44}
$$

holds for every continuous convex (resp. concave) function $f : [m, M] \to \mathbb{R}$, where

$$
\delta_f = f(m) + f(M) - 2f\left(\frac{m+M}{2}\right) \quad \left(\text{resp. } \delta_f = 2f\left(\frac{m+M}{2}\right) - f(m) - f(M)\right)
$$
$$
\text{and} \quad \tilde{x} = \frac{1}{2}1_H - \frac{1}{M - m}\left|x - \frac{m+M}{2}1_H\right| \tag{}
$$

We prove only the convex case. It follows from (□) that

$$
\begin{aligned}
f(p_1 m + p_2 M) &\le p_1 f(m) + p_2 f(M) \\
&\quad - \min\{p_1, p_2\}\left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right)
\end{aligned}
\tag{id45}
$$

for every $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 = 1$. For any $z \in [m, M]$ we can write

$$f(z) = f\left(\frac{M-z}{M-m}m + \frac{z-m}{M-m}M\right)$$

()

Then by using (□) for $p_1 = \frac{M-z}{M-m}$ and $p_2 = \frac{z-m}{M-m}$ we obtain

$$
\begin{aligned}
f(z) \leq \; & \frac{M-z}{M-m}f(m) + \frac{z-m}{M-m}f(M) \\
& - \left(\frac{1}{2} - \frac{1}{M-m}\left|z - \frac{m+M}{2}\right|\right)\left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right)
\end{aligned}
$$

(id46)

since

$$\min\left\{\frac{M-z}{M-m}, \frac{z-m}{M-m}\right\} = \frac{1}{2} - \frac{1}{M-m}\left|z - \frac{m+M}{2}\right|$$

()

Finally we use the continuous functional calculus for a self-adjoint operator x: $f, g \in (I)$, $Sp(x) \subseteq I$ and $f \leq g$ on I implies $f(x) \leq g(x)$; and $h(z) = |z|$ implies $h(x) = |x|$. Then by using (□) we obtain the desired inequality (□).

Theorem 12 Let $(x_t)_{t \in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let m_t and M_t, $m_t \leq M_t$, be the bounds of x_t, $t \in T$. Let $(\phi_t)_{t \in T}$ be a unital field of positive linear mappings $\phi_t : \; \to \mathcal{B}$ from to another unital C^*- algebra $\mathcal{B}$. Let

$$(m_x, M_x) \cap [m_t, M_t] = \varnothing, \quad t \in T, \qquad \text{and} \qquad m < M$$

()

where m_x and M_x, $m_x \leq M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\,d\mu(t)$ and

$$m = \sup\{M_t : M_t \leq m_x, \; t \in T\}, \quad M = \inf\{m_t : m_t \geq M_x, \; t \in T\}$$

()

If $f : I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval I contains all m_t, M_t, then

$$f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \leq \int_T \phi_t(f(x_t))\,d\mu(t) - \delta_f \tilde{x} \leq \int_T \phi_t(f(x_t))\,d\mu(t)$$

(id48)

(resp.

$$f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \geq \int_T \phi_t(f(x_t))\,d\mu(t) - \delta_f \tilde{x} \geq \int_T \phi_t(f(x_t))\,d\mu(t))$$

(id49)

holds, where

$$\delta_f \equiv \delta_f(\overline{m}, \overline{M}) = f(\overline{m}) + f(\overline{M}) - 2f\left(\frac{\overline{m}+\overline{M}}{2}\right)$$

$$\text{(resp.} \quad \delta_f \equiv \delta_f(\overline{m}, \overline{M}) = 2f\left(\frac{\overline{m}+\overline{M}}{2}\right) - f(\overline{m}) - f(\overline{M})) \tag{id50}$$

$$\tilde{x} \equiv \tilde{x}_x(\overline{m}, \overline{M}) = \frac{1}{2}1_K - \frac{1}{M-\overline{m}}\left| x - \frac{\overline{m}+\overline{M}}{2}1_K \right|$$

and $\overline{m} \in [m, m_A], \overline{M} \in [M_A, M], \overline{m} < \overline{M}$, are arbitrary numbers.

We prove only the convex case. Since $x = \int_T \phi_t(x_t) d\mu(t) \in \mathcal{B}$ is the self-adjoint elements such that $\overline{m}1_K \le m_x 1_K \le \int_T \phi_t(x_t) d\mu(t) \le M_x 1_K \le \overline{M}1_K$ and f is convex on $[\overline{m}, \overline{M}] \subseteq I$, then by Lemma ▢ we obtain

$$f\left(\int_T \phi_t(x_t) d\mu(t)\right) \le \frac{\overline{M}1_K - \int_T \phi_t(x_t) d\mu(t)}{M-\overline{m}}f(\overline{m}) + \frac{\int_T \phi_t(x_t) d\mu(t) - \overline{m}1_K}{M-\overline{m}}f(\overline{M}) - \delta_f \tilde{x} \tag{id51}$$

where δ_f and $\tilde{x}$ are defined by (▢).

But since f is convex on $[m_t, M_t]$ and $(m_x, M_x) \cap [m_t, M_t] = \varnothing$ implies $(\overline{m}, \overline{M}) \cap [m_t, M_t] = \varnothing$, then

$$f(x_t) \ge \frac{\overline{M}1_H - x_t}{M-\overline{m}}f(\overline{m}) + \frac{x_t - \overline{m}1_H}{M-\overline{m}}f(\overline{M}), \quad t \in T \tag{}$$

Applying a positive linear mapping ϕ_t, integrating and adding $-\delta_f \tilde{x}$, we obtain

$$\int_T \phi_t(f(x_t)) d\mu(t) - \delta_f \tilde{x} \ge \frac{\overline{M}1_K - \int_T \phi_t(x_t) d\mu(t)}{M-\overline{m}}f(\overline{m}) + \frac{\int_T \phi_t(x_t) d\mu(t) - \overline{m}1_K}{M-\overline{m}}f(\overline{M}) - \delta_f \tilde{x} \tag{id52}$$

since $\int_T \phi_t(1_H) d\mu(t) = 1_K$. Combining the two inequalities (▢) and (▢), we have LHS of (▢). Since $\delta_f \ge 0$ and $\tilde{x} \ge 0$, then we have RHS of (▢).

If $m < M$ and $m_x = M_x$, then the inequality (▢) holds, but $\delta_f(m_x, M_x)\tilde{x}(m_x, M_x)$ is not defined (see Example ▢ I) and II)).

Example 13 We give examples for the matrix cases and $T = \{1, 2\}$. Then we have refined inequalities given in Fig. 2.

We put $f(t) = t^4$ which is convex but not operator convex in (▢). Also, we define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_1((a_{ij})_{1 \le i, j \le 3}) = \frac{1}{2}(a_{ij})_{1 \le i, j \le 2'}$ $\Phi_2 = \Phi_1$ (then $\Phi_1(I_3) + \Phi_2(I_3) = I_2$).

$$f(\phi_1(x_1)) + f(\phi_2(x_2)) \le \phi_1(f(x_1)) + \phi_2(f(x_2)) - \delta_f \tilde{x},$$

where

$$\delta_f = f(\overline{m}) + f(\overline{M}) - 2f((\overline{M} + \overline{m})/2),$$

$$\tilde{x} = \frac{1}{2}1_K - \frac{1}{\overline{M} - \overline{m}} \left| \phi_1(x_1) + \phi_2(x_2) - \frac{\overline{M} + \overline{m}}{2} 1_K \right|$$
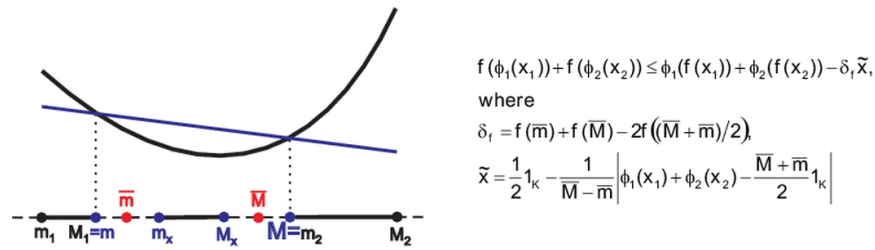
Figure 2. Refinement for two operators and a convex function f

I) First, we observe an example when $\delta_f \widetilde{X}$ is equal to the difference RHS and LHS of Jensen's inequality. If $X_1 = -3I_3$ and $X_2 = 2I_3$, then $X = \Phi_1(X_1) + \Phi_2(X_2) = -0.5I_2$, so $m = -3$, $M = 2$. We also put $\overline{m} = -3$ and $\overline{M} = 2$. We obtain

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = 0.0625I_2 < 48.5I_2 = \Phi_1(X_1^4) + \Phi_2(X_2^4) \tag{}$$

and its improvement

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = 0.0625I_2 = \Phi_1(X_1^4) + \Phi_2(X_2^4) - 48.4375I_2 \tag{}$$

since $\delta_f = 96.875$, $\widetilde{X} = 0.5I_2$. We remark that in this case $m_x = M_x = -1/2$ and $\widetilde{X}(m_x, M_x)$ is not defined.

II) Next, we observe an example when $\delta_f \widetilde{X}$ is not equal to the difference RHS and LHS of Jensen's inequality and $m_x = M_x$. If

$$X_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad \text{then } X = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } m = -1, \ M = 2 \tag{}$$

In this case $\tilde{x}(m_x, M_x)$ is not defined, since $m_x = M_x = 1/2$. We have

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} < \begin{pmatrix} \dfrac{17}{2} & 0 \\ 0 & \dfrac{97}{2} \end{pmatrix} = \Phi_1(X_1^4) + \Phi_2(X_2^4) \tag{}$$

and putting $\overline{m} = -1$, $\overline{M} = 2$ we obtain $\delta_f = 135/8$, $\widetilde{X} = I_2/2$ which give the following improvement

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} < \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 641 \end{pmatrix} = \Phi_1(X_1^4) + \Phi_2(X_2^4) - \frac{135}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{}$$

III) Next, we observe an example with matrices that are not special. If

$$X_1 = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \quad \text{then} \quad X = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

so $m_1 = -4.8662$, $M_1 = -0.3446$, $m_2 = 1.3446$, $M_2 = 5.8662$, $m = -0.3446$, $M = 1.3446$ and we put $\overline{m} = m$, $\overline{M} = M$ (rounded to four decimal places). We have

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} < \begin{pmatrix} \dfrac{1283}{2} & -255 \\ -255 & \dfrac{237}{2} \end{pmatrix} = \Phi_1(X_1^4) + \Phi_2(X_2^4)$$

and its improvement

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$< \begin{pmatrix} 639.9213 & -255 \\ -255 & 117.8559 \end{pmatrix} = \Phi_1(X_1^4) + \Phi_2(X_2^4) - \begin{pmatrix} 1.5787 & 0 \\ 0 & 0.6441 \end{pmatrix}$$

(rounded to four decimal places), since $\delta_f = 3.1574$, $\widetilde{X} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2040 \end{pmatrix}$. But, if we put $\overline{m} = m_x = 0$, $\overline{M} = M_x = 0.5$, then $\widetilde{X} = 0$, so we do not have an improvement of Jensen's inequality. Also, if we put $\overline{m} = 0$, $\overline{M} = 1$, then $\widetilde{X} = 0.5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\delta_f = 7/8$ and $\delta_f \widetilde{X} = 0.4375\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is worse than the above improvement.

Putting $\Phi_t(y) = a_t y$ for every $y \in$, where $a_t \geq 0$ is a real number, we obtain the following obvious corollary of Theorem □.

Corollary 14 Let $(x_t)_{t \in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let m_t and M_t, $m_t \leq M_t$, be the bounds of x_t, $t \in T$. Let $(a_t)_{t \in T}$ be a continuous field of nonnegative real numbers such that $\int_T a_t \, d\mu(t) = 1$. Let

$$(m_x, M_x) \cap [m_t, M_t] = \varnothing, \quad t \in T, \quad \text{and} \quad m < M$$

where m_x and M_x, $m_x \leq M_x$, are the bounds of the operator $x = \int_T \phi_t(x_t) \, d\mu(t)$ and

$$m = \sup \{M_t : M_t \leq m_x, \, t \in T\}, \quad M = \inf \{m_t : m_t \geq M_x, \, t \in T\}$$

If $f : I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval I contains all m_t, M_t, then

$$
\begin{aligned}
f\left(\int_T a_t x_t \, d\mu(t)\right) &\leq \int_T a_t f(x_t) \, d\mu(t) - \delta_f \tilde{\tilde{x}} \leq \int_T a_t f(x_t) \, d\mu(t) \\
\left(\text{resp.} \quad f\left(\int_T a_t x_t \, d\mu(t)\right)\right. &\geq \left.\int_T a_t f(x_t) \, d\mu(t) + \delta_f \tilde{\tilde{x}} \geq \int_T a_t f(x_t) \, d\mu(t)\right)
\end{aligned}
$$

()

holds, where δ_f is defined by ($\square$), $\tilde{\tilde{x}} = \frac{1}{2}1_H - \frac{1}{M - \bar{m}} \left| \int_T a_t x_t \, d\mu(t) - \frac{\bar{m} + \bar{M}}{2}1_H \right|$ and $\bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

5. Extension Jensen's inequality

In this section we present an extension of Jensen's operator inequality for n - tuples of self-adjoint operators, unital n - tuples of positive linear mappings and real valued continuous convex functions with conditions on the spectra of the operators.

In a discrete version of Theorem $\square$ we prove that Jensen's operator inequality holds for every continuous convex function and for every n - tuple of self-adjoint operators $(A_1, \ldots, A_n)$, for every n - tuple of positive linear mappings $(\Phi_1, \ldots, \Phi_n)$ in the case when the interval with bounds of the operator $A = \sum_{i=1}^n \Phi_i(A_i)$ has no intersection points with the interval with bounds of the operator A_i for each $i = 1, \ldots, n$, i.e. when $(m_A, M_A) \cap [m_i, M_i] = \varnothing$ for $i = 1, \ldots, n$, where m_A and M_A, $m_A \leq M_A$, are the bounds of A, and m_i and M_i, $m_i \leq M_i$, are the bounds of A_i, $i = 1, \ldots, n$. It is interesting to consider the case when $(m_A, M_A) \cap [m_i, M_i] = \varnothing$ is valid for several $i \in \{1, \ldots, n\}$, but not for all $i = 1, \ldots, n$. We study it in the following theorem (see [21]).

Theorem 15 Let $(A_1, \ldots, A_n)$ be an n - tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an n - tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. For $1 \leq n_1 < n$, we denote $m = \min\{m_1, \ldots, m_{n_1}\}$, $M = \max\{M_1, \ldots, M_{n_1}\}$ and $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha \, 1_K$, $\sum_{i=n_1+1}^n \Phi_i(1_H) = \beta \, 1_K$, where $\alpha, \beta > 0$, $\alpha + \beta = 1$. If

$$
(m, M) \cap [m_i, M_i] = \varnothing, \qquad i = n_1 + 1, \ldots, n
$$

()

and one of two equalities

$$
\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^n \Phi_i(A_i) = \sum_{i=1}^n \Phi_i(A_i)
$$

()

is valid, then

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\big(f(A_i)\big) \le \sum_{i=1}^{n}\Phi_i\big(f(A_i)\big) \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\big(f(A_i)\big) \tag{id57}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i, $i = 1, \dots, n$. If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in ($\square$).

We prove only the case when f is a convex function. Let us denote

$$A = \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i), \qquad B = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i), \qquad C = \sum_{i=1}^{n}\Phi_i(A_i) \tag{}$$

It is easy to verify that $A = B$ or $B = C$ or $A = C$ implies $A = B = C$.

a) Let $m < M$. Since f is convex on $[m, M]$ and $[m_i, M_i] \subseteq [m, M]$ for $i = 1, \dots, n_1$, then

$$f(z) \le \frac{M - z}{M - m}f(m) + \frac{z - m}{M - m}f(M), \quad z \in [m_i, M_i] \text{ for } i = 1, \dots, n_1 \tag{id58}$$

but since f is convex on all $[m_i, M_i]$ and $(m, M) \cap [m_i, M_i] = \varnothing$ for $i = n_1 + 1, \dots, n$, then

$$f(z) \ge \frac{M - z}{M - m}f(m) + \frac{z - m}{M - m}f(M), \quad z \in [m_i, M_i] \text{ for } i = n_1 + 1, \dots, n \tag{id59}$$

Since $m_i 1_H \le A_i \le M_i 1_H$, $i = 1, \dots, n_1$, it follows from ($\square$)

$$f(A_i) \le \frac{M 1_H - A_i}{M - m}f(m) + \frac{A_i - m 1_H}{M - m}f(M), \qquad i = 1, \dots, n_1 \tag{}$$

Applying a positive linear mapping Φ_i and summing, we obtain

$$\sum_{i=1}^{n_1}\Phi_i\big(f(A_i)\big) \le \frac{M\alpha 1_K - \sum_{i=1}^{n_1}\Phi_i(A_i)}{M - m}f(m) + \frac{\sum_{i=1}^{n_1}\Phi_i(A_i) - m\alpha 1_K}{M - m}f(M) \tag{}$$

since $\sum_{i=1}^{n_1}\Phi_i(1_H) = \alpha 1_K$. It follows

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\big(f(A_i)\big) \le \frac{M 1_K - A}{M - m}f(m) + \frac{A - m 1_K}{M - m}f(M) \tag{id60}$$

Similarly to ($\square$) in the case $m_i 1_H \le A_i \le M_i 1_H$, $i = n_1 + 1, \dots, n$, it follows from ($\square$)

$$\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \geq \frac{M1_K - B}{M - m}f(m) + \frac{B - m1_K}{M - m}f(M) \tag{id61}$$

Combining (□) and (□) and taking into account that $A = B$, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \tag{id62}$$

It follows

$$
\begin{aligned}
\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) &= \sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \frac{\beta}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) && \text{(by } \alpha + \beta = 1) \\
&\leq \sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) && \text{(by ())} \\
&= \sum_{i=1}^{n}\Phi_i(f(A_i)) && \text{(id63)} \\
&\leq \frac{\alpha}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) + \sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) && \text{(by ())} \\
&= \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) && \text{(by } \alpha + \beta = 1)
\end{aligned}
$$

which gives the desired double inequality (□).

b) Let $m = M$. Since $[m_i, M_i] \subseteq [m, M]$ for $i = 1, \ldots, n_1$, then $A_i = m1_H$ and $f(A_i) = f(m)1_H$ for $i = 1, \ldots, n_1$. It follows

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) = m1_K \qquad \text{and} \qquad \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) = f(m)1_K \tag{id64}$$

On the other hand, since f is convex on I, we have

$$f(z) \geq f(m) + l(m)(z - m) \quad \text{for every } z \in I \tag{id65}$$

where l is the subdifferential of f. Replacing z by A_i for $i = n_1 + 1, \ldots, n$, applying Φ_i and summing, we obtain from (□) and (□)

$$
\begin{aligned}
\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) &\geq f(m)1_K + l(m)\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i) - m1_K\right) \\
&= f(m)1_K = \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))
\end{aligned}
\tag{()}
$$

So ($\square$) holds again. The remaining part of the proof is the same as in the case a).

Remark 16 We obtain the equivalent inequality to the one in Theorem $\square$ in the case when $\sum_{i=1}^{n} \Phi_i(1_H) = \gamma 1_K$, for some positive scalar γ. If $\alpha + \beta = \gamma$ and one of two equalities

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i) = \frac{1}{\gamma}\sum_{i=1}^{n} \Phi_i(A_i) \tag{}$$

is valid, then

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\gamma}\sum_{i=1}^{n} \Phi_i(f(A_i)) \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \tag{}$$

holds for every continuous convex function f.

Remark 17 Let the assumptions of Theorem $\square$ be valid.

1. We observe that the following inequality

$$f\left(\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i)\right) \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \tag{id68}$$

holds for every continuous convex function $f : I \to \mathbb{R}$.

Indeed, by the assumptions of Theorem $\square$ we have

$$m\alpha 1_H \le \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le M\alpha 1_H \quad \text{and} \quad \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i) \tag{}$$

which implies

$$m1_H \le \sum_{i=n_1+1}^{n} \frac{1}{\beta}\Phi_i(f(A_i)) \le M1_H \tag{}$$

Also $(m, M) \cap [m_i, M_i] = \varnothing$ for $i = n_1 + 1, \ldots, n$ and $\sum_{i=n_1+1}^{n} \frac{1}{\beta}\Phi_i(1_H) = 1_K$ hold. So we can apply Theorem $\square$ on operators $A_{n_1+1}, \ldots, A_n$ and mappings $\frac{1}{\beta}\Phi_i$ and obtain the desired inequality.

2. We denote by m_C and M_C the bounds of $C = \sum_{i=1}^{n} \Phi_i(A_i)$. If $(m_C, M_C) \cap [m_i, M_i] = \varnothing$, $i = 1, \ldots, n_1$ or f is an operator convex function on $[m, M]$, then the double inequality ($\square$) can be extended from the left side if we use Jensen's operator inequality (see Theorem 2.1[16])

$$
\begin{aligned}
f\Big(\sum_{i=1}^{n}\Phi_i(A_i)\Big) &= f\Big(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)\Big) \\
&\leq \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \leq \sum_{i=1}^{n}\Phi_i(f(A_i)) \leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))
\end{aligned}
\tag{}
$$

Example 18 If neither assumptions $(m_C, M_C) \cap [m_i, M_i] = \varnothing$, $i = 1, \ldots, n_1$, nor f is operator convex in Remark □ - 2. is satisfied and if $1 < n_1 < n$, then (□) can not be extended by Jensen's operator inequality, since it is not valid. Indeed, for $n_1 = 2$ we define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ by $\Phi_1((a_{ij})_{1 \leq i, j \leq 3}) = \frac{\alpha}{2}(a_{ij})_{1 \leq i, j \leq 2}$, $\Phi_2 = \Phi_1$. Then $\Phi_1(I_3) + \Phi_2(I_3) = \alpha I_2$. If

$$
A_1 = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\tag{}
$$

then

$$
\Big(\frac{1}{\alpha}\Phi_1(A_1) + \frac{1}{\alpha}\Phi_2(A_2)\Big)^4 = \frac{1}{\alpha^4}\begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} \neg \leq \frac{1}{\alpha}\begin{pmatrix} 80 & 40 \\ 40 & 24 \end{pmatrix} = \frac{1}{\alpha}\Phi_1(A_1^4) + \frac{1}{\alpha}\Phi_2(A_2^4)
\tag{}
$$

for every $\alpha \in (0, 1)$. We observe that $f(t) = t^4$ is not operator convex and $(m_C, M_C) \cap [m_i, M_i] \neq \varnothing$, since $C = A = \frac{1}{\alpha}\Phi_1(A_1) + \frac{1}{\alpha}\Phi_2(A_2) = \frac{1}{\alpha}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $[m_C, M_C] = [0, 2/\alpha]$, $[m_1, M_1] \subset [-1.60388, 4.49396]$ and $[m_2, M_2] = [0, 2]$.

With respect to Remark □, we obtain the following obvious corollary of Theorem □.

Corollary 19 Let $(A_1, \ldots, A_n)$ be an n - tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \leq M_i$, $i = 1, \ldots, n$. For some $1 \leq n_1 < n$, we denote $m = \min\{m_1, \ldots, m_{n_1}\}$, $M = \max\{M_1, \ldots, M_{n_1}\}$. Let $(p_1, \ldots, p_n)$ be an n - tuple of non-negative numbers, such that $0 < \sum_{i=1}^{n_1} p_i = {}_1 < {} = \sum_{i=1}^{n} p_i$. If

$$
(m, M) \cap [m_i, M_i] = \varnothing, \qquad i = n_1 + 1, \ldots, n
\tag{}
$$

and one of two equalities

$$
\frac{1}{_1}\sum_{i=1}^{n_1} p_i A_i = \frac{1}{}\sum_{i=1}^{n} p_i A_i = \frac{1}{-_1}\sum_{i=n_1+1}^{n} p_i A_i
\tag{}
$$

is valid, then

$$\frac{1}{1}\sum_{i=1}^{n_1} p_i f(A_i) \le \frac{1}{1}\sum_{i=1}^{n} p_i f(A_i) \le \frac{1}{-_1}\sum_{i=n_1+1}^{n} p_i f(A_i) \tag{id71}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i, $i = 1$, ..., n.

If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in ($\square$).

As a special case of Corollary $\square$ we can obtain a discrete version of Corollary $\square$ as follows.

Corollary 20 (Discrete version of Corollary $\square$) Let $(A_1, ..., A_n)$ be an n - tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \le M_i$, $i = 1$, ..., n. Let $(\alpha_1, ..., \alpha_n)$ be an n - tuple of nonnegative real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. If

$$(m_A, M_A) \cap [m_i, M_i] = \varnothing, \qquad i = 1, ..., n \tag{id73}$$

where m_A and M_A, $m_A \le M_A$, are the bounds of $A = \sum_{i=1}^{n} \alpha_i A_i$, then

$$f\left(\sum_{i=1}^{n} \alpha_i A_i\right) \le \sum_{i=1}^{n} \alpha_i f(A_i) \tag{id74}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i.

We prove only the convex case. We define $(n+1)$ - tuple of operators $(B_1, ..., B_{n+1})$, $B_i \in B(H)$, by $B_1 = A = \sum_{i=1}^{n} \alpha_i A_i$ and $B_i = A_{i-1}$, $i = 2$, ..., $n+1$. Then $m_{B_1} = m_A$, $M_{B_1} = M_A$ are the bounds of B_1 and $m_{B_i} = m_{i-1}$, $M_{B_i} = M_{i-1}$ are the ones of B_i, $i = 2$, ..., $n+1$. Also, we define $(n+1)$ - tuple of non-negative numbers $(p_1, ..., p_{n+1})$ by $p_1 = 1$ and $p_i = \alpha_{i-1}$, $i = 2$, ..., $n+1$. Then $\sum_{i=1}^{n+1} p_i = 2$ and by using ($\square$) we have

$$(m_{B_1}, M_{B_1}) \cap [m_{B_i}, M_{B_i}] = \varnothing, \qquad i = 2, ..., n+1 \tag{id75}$$

Since

$$\sum_{i=1}^{n+1} p_i B_i = B_1 + \sum_{i=2}^{n+1} p_i B_i = \sum_{i=1}^{n} \alpha_i A_i + \sum_{i=1}^{n} \alpha_i A_i = 2B_1 \tag{}$$

then

$$p_1 B_1 = \frac{1}{2}\sum_{i=1}^{n+1} p_i B_i = \sum_{i=2}^{n+1} p_i B_i \tag{id76}$$

Taking into account (□) and (□), we can apply Corollary □ for $n_1 = 1$ and B_i, p_i as above, and we get

$$p_1 f(B_1) \leq \frac{1}{2} \sum_{i=1}^{n+1} p_i f(B_i) \leq \sum_{i=2}^{n+1} p_i f(B_i) \tag{}$$

which gives the desired inequality (□).

6. Extension of the refined Jensen's inequality

There is an extensive literature devoted to Jensen's inequality concerning different refinements and extensive results, see, for example [22], [23], [24], [25], [26], [27], [28], [29].

In this section we present an extension of the refined Jensen's inequality obtained in Section □ and a refinement of the same inequality obtained in Section □.

Theorem 21 Let $(A_1, ..., A_n)$ be an n - tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \leq M_i$, $i = 1, ..., n$. Let $(\Phi_1, ..., \Phi_n)$ be an n - tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta 1_K$, where $1 \leq n_1 < n$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Let $m_L = \min\{m_1, ..., m_{n_1}\}$, $M_R = \max\{M_1, ..., M_{n_1}\}$ and

$$\begin{aligned} m &= \max\{M_i : M_i \leq m_L, i \in \{n_1 + 1, ..., n\}\} \\ M &= \min\{m_i : m_i \geq M_R, i \in \{n_1 + 1, ..., n\}\} \end{aligned} \tag{}$$

If

$$(m_L, M_R) \cap [m_i, M_i] = \varnothing, \quad i = n_1 + 1, ..., n, \quad \text{and} \quad m < M \tag{}$$

and one of two equalities

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) = \sum_{i=1}^{n} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i) \tag{}$$

is valid, then

$$\begin{aligned} \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) &\leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta \delta_f \widetilde{A} \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) \\ &\leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha \delta_f \widetilde{A} \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \end{aligned} \tag{id78}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all $m_i, M_i, i = 1, ..., n$, where

$$\delta_f \equiv \delta_f(\overline{m}, \overline{M}) = f(\overline{m}) + f(\overline{M}) - 2f\left(\frac{\overline{m} + \overline{M}}{2}\right)$$

$$\widetilde{A} \equiv \widetilde{A}_{A,\Phi,n_1,\alpha}(\overline{m}, \overline{M}) = \frac{1}{2}1_K - \frac{1}{\alpha(M-m)}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\overline{m} + \overline{M}}{2}1_H\right|\right)$$

(id79)

and $\overline{m} \in [m, m_L], \overline{M} \in [M_R, M], \overline{m} < \overline{M}$, are arbitrary numbers. If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in ($\square$).

We prove only the convex case. Let us denote

$$A = \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i), \qquad B = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i), \qquad C = \sum_{i=1}^{n}\Phi_i(A_i)$$

()

It is easy to verify that $A = B$ or $B = C$ or $A = C$ implies $A = B = C$.

Since f is convex on $[\overline{m}, \overline{M}]$ and $(A_i) \subseteq [m_i, M_i] \subseteq [\overline{m}, \overline{M}]$ for $i = 1, ..., n_1$, it follows from Lemma $\square$ that

$$f(A_i) \le \frac{\overline{M}1_H - A_i}{\overline{M} - \overline{m}}f(\overline{m}) + \frac{A_i - \overline{m}1_H}{\overline{M} - \overline{m}}f(\overline{M}) - \delta_f \widetilde{A}_i, \qquad i = 1, ..., n_1$$

()

holds, where $\delta_f = f(\overline{m}) + f(\overline{M}) - 2f\left(\frac{\overline{m}+\overline{M}}{2}\right)$ and $\widetilde{A}_i = \frac{1}{2}1_H - \frac{1}{\overline{M}-\overline{m}}\left|A_i - \frac{\overline{m}+\overline{M}}{2}1_H\right|$. Applying a positive linear mapping Φ_i and summing, we obtain

$$\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{\overline{M}\alpha 1_K - \sum_{i=1}^{n_1}\Phi_i(A_i)}{\overline{M}-\overline{m}}f(\overline{m}) + \frac{\sum_{i=1}^{n_1}\Phi_i(A_i) - \overline{m}\alpha 1_K}{\overline{M}-\overline{m}}f(\overline{M})$$
$$- \delta_f\left(\frac{\alpha}{2}1_K - \frac{1}{\overline{M}-\overline{m}}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\overline{m}+\overline{M}}{2}1_H\right|\right)\right)$$

()

since $\sum_{i=1}^{n_1}\Phi_i(1_H) = \alpha 1_K$. It follows that

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{\overline{M}1_K - A}{\overline{M}-\overline{m}}f(\overline{m}) + \frac{A - \overline{m}1_K}{\overline{M}-\overline{m}}f(\overline{M}) - \delta_f\widetilde{A}$$

(id80)

where $\widetilde{A} = \frac{1}{2}1_K - \frac{1}{\alpha(M-\overline{m})}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\overline{m}+\overline{M}}{2}1_H\right|\right)$.

Additionally, since f is convex on all $[m_i, M_i]$ and $(\overline{m}, \overline{M}) \cap [m_i, M_i] = \varnothing$, $i = n_1 + 1, \ldots, n$, then

$$f(A_i) \geq \frac{\overline{M}1_H - A_i}{\overline{M} - \overline{m}} f(\overline{m}) + \frac{A_i - \overline{m}1_H}{\overline{M} - \overline{m}} f(\overline{M}), \qquad i = n_1 + 1, \ldots, n \qquad \text{(id81)}$$

It follows

$$\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \geq \frac{\overline{M}1_K - B}{\overline{M} - \overline{m}} f(\overline{m}) + \frac{B - \overline{m}1_K}{\overline{M} - \overline{m}} f(\overline{M}) - \delta_f \widetilde{A} \qquad \text{(id82)}$$

Combining ($\square$) and ($\square$) and taking into account that $A = B$, we obtain

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \qquad \text{(id83)}$$
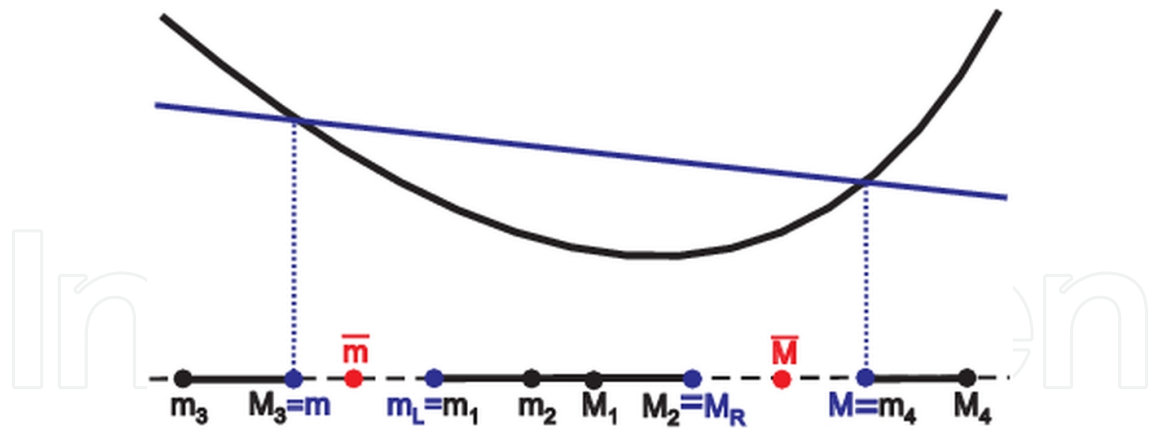
Next, we obtain

$$
\begin{aligned}
&\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \\
&= \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \frac{\beta}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \quad \text{(by } \alpha + \beta = 1) \\
&\leq \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \beta \delta_f \widetilde{A} \qquad \text{(by ())} \\
&\leq \frac{\alpha}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha \delta_f \widetilde{A} + \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \beta \delta_f \widetilde{A} \qquad \text{(by ())} \\
&= \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \quad \text{(by } \alpha + \beta = 1)
\end{aligned}
\qquad ()
$$

which gives the following double inequality

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) - \beta \delta_f \widetilde{A} \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \qquad \text{(id84)}$$

Adding $\beta \delta_f \widetilde{A}$ in the above inequalities, we get

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta \delta_f \widetilde{A} \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha \delta_f \widetilde{A} \qquad \text{(id85)}$$

Figure 3. An example a convex function and the bounds of four operators

Now, we remark that $\delta_f \geq 0$ and $\widetilde{A} \geq 0$. (Indeed, since f is convex, then $f((\overline{m} + \overline{M})/2) \leq (f(\overline{m}) + f(\overline{M}))/2$, which implies that $\delta_f \geq 0$. Also, since

$$(A_i) \subseteq [\overline{m}, \overline{M}] \quad \Rightarrow \quad \left| A_i - \frac{\overline{M} + \overline{m}}{2} 1_H \right| \leq \frac{\overline{M} - \overline{m}}{2} 1_H, \qquad i = 1, \ldots, n_1 \tag{}$$

then

$$\sum_{i=1}^{n_1} \Phi_i \left(\left| A_i - \frac{\overline{M} + \overline{m}}{2} 1_H \right| \right) \leq \frac{\overline{M} - \overline{m}}{2} \alpha 1_K \tag{}$$

which gives

$$0 \leq \frac{1}{2} 1_K - \frac{1}{\alpha(\overline{M} - \overline{m})} \sum_{i=1}^{n_1} \Phi_i \left(\left| A_i - \frac{\overline{M} + \overline{m}}{2} 1_H \right| \right) = \widetilde{A} \; \Bigg) \tag{}$$

Consequently, the following inequalities

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta \delta_f \widetilde{A}$$
$$\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha \delta_f \widetilde{A} \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \tag{}$$

hold, which with (□) proves the desired series inequalities (□). 1.05

Example 22 We observe the matrix case of Theorem □ for $f(t) = t^4$, which is the convex function but not operator convex, $n = 4$, $n_1 = 2$ and the bounds of matrices as in Fig. 3.

We show an example such that

$$\frac{1}{\alpha}\left(\Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right)\right) < \frac{1}{\alpha}\left(\Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right)\right) + \beta\delta_f\widetilde{A}$$
$$< \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right) + \Phi_3\left(A_3^4\right) + \Phi_4\left(A_4^4\right) \qquad\qquad \text{(id88)}$$
$$< \frac{1}{\beta}\left(\Phi_3\left(A_3^4\right) + \Phi_4\left(A_4^4\right)\right) - \alpha\delta_f\widetilde{A} < \frac{1}{\beta}\left(\Phi_3\left(A_3^4\right) + \Phi_4\left(A_4^4\right)\right)$$

holds, where $\delta_f = \overline{M}^4 + \overline{m}^4 - (\overline{M} + \overline{m})^48$ and

$$\widetilde{A} = \frac{1}{2}I_2 - \frac{1}{\alpha(M-\overline{m})}\left(\Phi_1\left(\left|\ A_1 - \frac{\overline{M}+\overline{m}}{2}I_h\ \right|\right) + \Phi_2\left(\left|\ A_2 - \frac{\overline{M}+\overline{m}}{2}I_3\ \right|\right)\right) \qquad\qquad ()$$

We define mappings $\Phi_i : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_i\left((a_{jk})_{1\le j,k\le 3}\right) = \frac{1}{4}(a_{jk})_{1\le j,k\le 2}$, $i = 1, \ldots, 4$. Then $\sum_{i=1}^4 \Phi_i(I_3) = I_2$ and $\alpha = \beta = \frac{1}{2}$.

Let

$$A_1 = 2\begin{pmatrix} 2 & 9/8 & 1 \\ 9/8 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix},\ A_2 = 3\begin{pmatrix} 2 & 9/8 & 0 \\ 9/8 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},\ A_3 = -3\begin{pmatrix} 4 & 1/2 & 1 \\ 1/2 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix},\ A_4 = 12\begin{pmatrix} 5/3 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad ()$$

Then $m_1 = 1.28607$, $M_1 = 7.70771$, $m_2 = 0.53777$, $M_2 = 5.46221$, $m_3 = -14.15050$, $M_3 = -4.71071$, $m_4 = 12.91724$, $M_4 = 36.$, so $m_L = m_2$, $M_R = M_1$, $m = M_3$ and $M = m_4$ (rounded to five decimal places). Also,

$$\frac{1}{\alpha}\left(\Phi_1(A_1) + \Phi_2(A_2)\right) = \frac{1}{\beta}\left(\Phi_3(A_3) + \Phi_4(A_4)\right) = \begin{pmatrix} 4 & 9/4 \\ 9/4 & 3 \end{pmatrix} \qquad\qquad ()$$

and

$$A_f \equiv \frac{1}{\alpha}\left(\Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right)\right) = \begin{pmatrix} 989.00391 & 663.46875 \\ 663.46875 & 526.12891 \end{pmatrix}$$
$$C_f \equiv \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right) + \Phi_3\left(A_3^4\right) + \Phi_4\left(A_4^4\right) = \begin{pmatrix} 68093.14258 & 48477.98437 \\ 48477.98437 & 51335.39258 \end{pmatrix} \qquad ()$$
$$B_f \equiv \frac{1}{\beta}\left(\Phi_3\left(A_3^4\right) + \Phi_4\left(A_4^4\right)\right) = \begin{pmatrix} 135197.28125 & 96292.5 \\ 96292.5 & 102144.65625 \end{pmatrix}$$

Then

$$A_f < C_f < B_f \qquad\qquad \text{(id89)}$$

holds (which is consistent with (□)).

We will choose three pairs of numbers $(\overline{m}, \overline{M})$, $\overline{m} \in [-4.71071, 0.53777]$, $\overline{M} \in [7.70771, 12.91724]$ as follows

i) $\overline{m} = m_L = 0.53777$, $\overline{M} = M_R = 7.70771$, then

$$\widetilde{\varDelta}_1 = \beta\delta_f \widetilde{A} = 0.5 \cdot 2951.69249 \cdot \begin{pmatrix} 0.15678 & 0.09030 \\ 0.09030 & 0.15943 \end{pmatrix} = \begin{pmatrix} 231.38908 & 133.26139 \\ 133.26139 & 235.29515 \end{pmatrix}$$

ii) $\overline{m} = m = -4.71071$, $\overline{M} = M = 12.91724$, then

$$\widetilde{\varDelta}_2 = \beta\delta_f \widetilde{A} = 0.5 \cdot 27766.07963 \cdot \begin{pmatrix} 0.36022 & 0.03573 \\ 0.03573 & 0.36155 \end{pmatrix} = \begin{pmatrix} 5000.89860 & 496.04498 \\ 496.04498 & 5019.50711 \end{pmatrix}$$

iii) $\overline{m} = -1$, $\overline{M} = 10$, then

$$\widetilde{\varDelta}_3 = \beta\delta_f \widetilde{A} = 0.5 \cdot 9180.875 \cdot \begin{pmatrix} 0.28203 & 0.08975 \\ 0.08975 & 0.27557 \end{pmatrix} = \begin{pmatrix} 1294.66 & 411.999 \\ 411.999 & 1265. \end{pmatrix}$$

New, we obtain the following improvement of (□) (see (□))

Table 1.

Using Theorem □ we get the following result.

Corollary 23 Let the assumptions of Theorem □ hold. Then

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \gamma_1\delta_f \widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \tag{id91}$$

and

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \gamma_2\delta_f \widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \tag{id92}$$

holds for every γ_1, γ_2 in the close interval joining α and β, where δ_f and $\widetilde{A}$ are defined by (□).

Adding $\alpha\delta_f \widetilde{A}$ in (□) and noticing $\delta_f \widetilde{A} \ge 0$, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \alpha\delta_f \widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \tag{id93}$$

Taking into account the above inequality and the left hand side of (□) we obtain (□).

Similarly, subtracting $\beta\delta_f\,\widetilde{A}$ in ($\square$) we obtain ($\square$).

Remark 24 We can obtain extensions of inequalities which are given in Remark $\square$ and $\square$. Also, we can obtain a special case of Theorem $\square$ with the convex combination of operators A_i putting $\Phi_i(B) = \alpha_i B$, for $i = 1, ..., n$, similarly as in Corollary $\square$. Finally, applying this result, we can give another proof of Corollary $\square$. The interested reader can see the details in [30].

Author details

Jadranka Mićić[1] and Josip Pečarić[2]

Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Ivana Lučića 5, 10000 Zagreb,, Croatia

Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 30, 10000 Zagreb,, Croatia

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