Provisional chapter

Operator Means and Applications

Pattrawut Chansangiam

Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/46479

1. Introduction

The theory of scalar means was developed since the ancient Greek by the Pythagoreans until the last century by many famous mathematicians. See the development of this subject in a survey article [1]. In Pythagorean school, various means are defined via the method of proportions (in fact, they are solutions of certain algebraic equations). The theory of matrix and operator means started from the presence of the notion of parallel sum as a tool for analyzing multi-port electrical networks in engineering; see [2]. Three classical means, namely, arithmetic mean, harmonic mean and geometric mean for matrices and operators are then considered, e.g., in [3], [4], [5], [6], [7]. These means play crucial roles in matrix and operator theory as tools for studying monotonicity and concavity of many interesting maps between algebras of operators; see the original idea in [3]. Another important mean in mathematics, namely the power mean, is considered in [8]. The parallel sum is characterized by certain properties in [9]. The parallel sum and these means share some common properties. This leads naturally to the definitions of the so-called connection and mean in a seminal paper [10]. This class of means cover many in-practice operator means. A major result of Kubo-Ando states that there are one-to-one correspondences between connections, operator monotone functions on the non-negative reals and finite Borel measures on the extended half-line. The mean theoretic approach has many applications in operator inequalities (see more information in Section 8), matrix and operator equations (see e.g. [11], [12]) and operator entropy. The concept of operator entropy plays an important role in mathematical physics. The *relative operator entropy* is defined in [13] for invertible positive operators A, B by

$$S(A \mid B) = A^{1/2} \log \left(A^{-1/2} B A^{-1/2}\right) A^{1/2}. \tag{id1}$$

In fact, this formula comes from the Kubo-Ando theory–$S(\cdot \mid \cdot)$ is the connection corresponds to the operator monotone function $t \mapsto \log t$. See more information in Chapter IV[14] and its references.

In this chapter, we treat the theory of operator means by weakening the original definition of connection in such a way that the same theory is obtained. Moreover, there is a one-to-one correspondence between connections and finite Borel measures on the unit interval. Each connection can be regarded as a weighed series of weighed harmonic means. Hence, every mean in Kubo-Ando's sense corresponds to a probability Borel measure on the unit interval. Various characterizations of means are obtained; one of them is a usual property of scalar mean, namely, the betweenness property. We provide some new properties of abstract operator connections, involving operator monotonicity and concavity, which include specific operator means as special cases.

For benefits of readers, we provide the development of the theory of operator means. In Section 2, we setup basic notations and state some background about operator monotone functions which play important roles in the theory of operator means. In Section 3, we consider the parallel sum together with its physical interpretation in electrical circuits. The arithmetic mean, the geometric mean and the harmonic mean of positive operators are investigated and characterized in Section 4. The original definition of connection is improved in Section 5 in such a way that the same theory is obtained. In Section 6, several characterizations and examples of Kubo-Ando means are given. We provide some new properties of general operator connections, related to operator monotonicity and concavity, in Section 7. Many operator versions of classical inequalities are obtained via the mean-theoretic approach in Section 8.

2. Preliminaries

Throughout, let $B(\mathcal{H})$ be the von Neumann algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. Let $B(\mathcal{H})^{sa}$ be the real vector space of self-adjoint operators on $\mathcal{H}$. Equip $B(\mathcal{H})$ with a natural partial order as follows. For $A, B \in B(\mathcal{H})^{sa}$, we write $A \quad B$ if $B - A$ is a positive operator. The notation $T \in B(\mathcal{H})^{+}$ or $T \quad 0$ means that T is a positive operator. The case that $T \quad 0$ and T is invertible is denoted by $T > 0$ or $T \in B(\mathcal{H})^{++}$. Unless otherwise stated, every limit in $B(\mathcal{H})$ is taken in the strong-operator topology. Write $A_n \to A$ to indicate that A_n converges strongly to A. If A_n is a sequence in $B(\mathcal{H})^{sa}$, the expression $A_n \downarrow A$ means that A_n is a decreasing sequence and $A_n \to A$. Similarly, $A_n \uparrow A$ tells us that A_n is increasing and $A_n \to A$. We always reserve A, B, C, D for positive operators. The set of non-negative real numbers is denoted by $\mathbb{R}^{+}$.

Remark 0.1 It is important to note that if A_n is a decreasing sequence in $B(\mathcal{H})^{sa}$ such that $A_n \quad A$, then $A_n \to A$ if and only if $\langle A_n x, x \rangle \to \langle Ax, x \rangle$ for all $x \in \mathcal{H}$. Note first that this sequence is convergent by the order completeness of $B(\mathcal{H})$. For the sufficiency, if $x \in \mathcal{H}$, then

$$\| (A_n - A)^{1/2}x \| ^2 = \langle (A_n - A)^{1/2}x, (A_n - A)^{1/2}x \rangle = \langle (A_n - A)x, x \rangle \to 0 \qquad ()$$

and hence $\| (A_n - A)x \| \to 0$.

The spectrum of $T \in B(\mathcal{H})$ is defined by

$$\mathrm{Sp}\,(T) = \{\lambda \in T - \lambda I \text{ is not invertible}\}. \qquad ()$$

Then $\mathrm{Sp}\,(T)$ is a nonempty compact Hausdorff space. Denote by $C(\,\mathrm{Sp}\,(T))$ the C*-algebra of continuous functions from $\mathrm{Sp}\,(T)$ to .*Let*T B(H)*beanormaloperatorand*z: Sp(T) the inclusion. Then there exists a unique unital * -homomorphism $\phi : C(\,\mathrm{Sp}\,(T)) \to B(\mathcal{H})$ such that $\phi(z) = T$, i.e.,

- ϕ is linear
- $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in C(\,\mathrm{Sp}\,(T))$
- $\phi(\bar{f}) = (\phi(f))^*$ for all $f \in C(\,\mathrm{Sp}\,(T))$
- $\phi(1) = I$.

Moreover, ϕ is isometric. We call the unique isometric * -homomorphism which sends $f \in C(\,\mathrm{Sp}\,(T))$ to $\phi(f) \in B(\mathcal{H})$ the *continuous functional calculus* of T. We write $f(T)$ for $\phi(f)$.

Example 0.2

- If $f(t) = a_0 + a_1 t + \cdots + a_n t^n$, then $f(T) = a_0 I + a_1 T + \cdots + a_n T^n$.
- If $f(t) = \bar{t}$, then $f(T) = \phi(f) = \phi(\bar{z}) = \phi(z)^* = T^*$
- If $f(t) = t^{1/2}$ for $t \in \mathbb{R}^+$ and $T \ \ 0$, then we define $T^{1/2} = f(T)$. Equivalently, $T^{1/2}$ is the unique positive square root of T.
- If $f(t) = t^{-1/2}$ for $t > 0$ and $T > 0$, then we define $T^{-1/2} = f(T)$. Equivalently, $T^{-1/2} = (T^{1/2})^{-1} = (T^{-1})^{1/2}$.

A continuous real-valued function f on an interval I is called an *operator monotone function* if one of the following equivalent conditions holds:

- $A \ \ B \Rightarrow f(A) \ \ f(B)$ for all Hermitian matrices A, B of all orders whose spectrums are contained in I;
- $A \ \ B \Rightarrow f(A) \ \ f(B)$ for all Hermitian operators $A, B \in B(\mathcal{H})$ whose spectrums are contained in I and for an infinite dimensional Hilbert space $\mathcal{H}$;
- $A \ \ B \Rightarrow f(A) \ \ f(B)$ for all Hermitian operators $A, B \in B(\mathcal{H})$ whose spectrums are contained in I and for all Hilbert spaces $\mathcal{H}$.

This concept is introduced in [15]; see also [14], [16], [17], [18]. Every operator monotone function is always continuously differentiable and monotone increasing. Here are examples of operator monotone functions:

- $t \mapsto \alpha t + \beta$ on $\mathbb{R}$, for α 0 and $\beta \in \mathbb{R}$,
- $t \mapsto -t^{-1}$ on $(0, \infty)$,
- $t \mapsto (c - t)^{-1}$ on (a, b), for $c \notin (a, b)$,
- $t \mapsto \log t$ on $(0, \infty)$,
- $t \mapsto (t-1)/\log t$ on $\mathbb{R}^+$, where $0 \mapsto 0$ and $1 \mapsto 1$.

The next result is called the Löwner-Heinz's inequality [15].

Theorem 0.3 For $A, B \in B(\mathcal{H})^+$ and $r \in [0, 1]$, if A B, then A^r B^r. That is the map $t \mapsto t^r$ is an operator monotone function on $\mathbb{R}^+$ for any $r \in [0, 1]$.

A key result about operator monotone functions is that there is a one-to-one correspondence between nonnegative operator monotone functions on $\mathbb{R}^+$ and finite Borel measures on $[0, \infty]$ via integral representations. We give a variation of this result in the next proposition.

Proposition 0.4 A continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone if and only if there exists a finite Borel measure μ on $[0, 1]$ such that

$$f(x) = \int_{[0,1]} 1 \,!_t\; x \, d\mu(t), \qquad x \in \mathbb{R}^+. \tag{id22}$$

Here, the weighed harmonic mean $!_t$ is defined for $a, b > 0$ by

$$a \,!_t\; b = [(1-t)a^{-1} + tb^{-1}]^{-1} \tag{id23}$$

and extended to a, b 0 by continuity. Moreover, the measure μ is unique. Hence, there is a one-to-one correspondence between operator monotone functions on the non-negative reals and finite Borel measures on the unit interval.

Recall that a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone if and only if there exists a unique finite Borel measure ν on $[0, \infty]$ such that

$$f(x) = \int_{[0,\infty]} \phi_x(\lambda) \, d\nu(\lambda), \qquad x \in \mathbb{R}^+ \qquad ()$$

where

$$\phi_x(\lambda) = \frac{x(\lambda+1)}{x+\lambda} \text{ for } \lambda > 0, \quad \phi_x(0) = 1, \quad \phi_x(\infty) = x. \qquad ()$$

Consider the Borel measurable function $\psi : [0, 1] \to [0, \infty], \ t \mapsto \frac{t}{1-t}$. Then, for each $x \in \mathbb{R}^+$,

$$\begin{aligned} \int_{[0,\infty]} \phi_x(\lambda)\, d\nu(\lambda) \quad &= \int_{[0,1]} \phi_x \ \psi(t)\, d\nu\psi(t) \\ &= \int_{[0,1]} \frac{x}{x - xt + t}\, d\nu\psi(t) \\ &= \int_{[0,1]} 1\, !_t \quad x\, d\nu\psi(t). \end{aligned} \qquad ()$$

Now, set $\mu = \nu\psi$. Since ψ is bijective, there is a one-to-one corresponsence between the finite Borel measures on $[0, \infty]$ of the form ν and the finite Borel measures on $[0, 1]$ of the form $\nu\psi$. The map $f \ \mapsto \mu$ is clearly well-defined and bijective.

3. Parallel sum: A notion from electrical networks

In connections with electrical engineering, Anderson and Duffin [2] defined the *parallel sum* of two positive definite matrices A and B by

$$A : B = (A^{-1} + B^{-1})^{-1}. \qquad \text{(id24)}$$

The impedance of an electrical network can be represented by a positive (semi)definite matrix. If A and B are impedance matrices of multi-port networks, then the parallel sum $A : B$ indicates the total impedance of two electrical networks connected in parallel. This notion plays a crucial role for analyzing multi-port electrical networks because many physical interpretations of electrical circuits can be viewed in a form involving parallel sums. This is a starting point of the study of matrix and operator means. This notion can be extended to invertible positive operators by the same formula.

Lemma 0.5 Let $A, B, C, D, A_n, B_n \in B(\mathcal{H})^{++}$ for all $n \in \mathbb{N}$.

- If $A_n \downarrow A$, then $A_n^{-1} \uparrow A^{-1}$. If $A_n \uparrow A$, then $A_n^{-1} \downarrow A^{-1}$.
- If $A \ \ C$ and $B \ \ D$, then $A : B \ \ C : D$.
- If $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n : B_n \downarrow A : B$.
- If $A_n \downarrow A$ and $B_n \downarrow B$, then $\lim A_n : B_n$ exists and does not depend on the choices of A_n, B_n.

(1) Assume $A_n \downarrow A$. Then A_n^{-1} is increasing and, for each $x \in \mathcal{H}$,

$$\langle (A_n^{-1} - A^{-1})x, x \rangle = \langle (A - A_n)A^{-1}x, A_n^{-1}x \rangle \quad \| (A - A_n)A^{-1}x \| \ \ \| A_n^{-1} \| \ \ \| x \| \ \to 0. \qquad ()$$

(2) Follow from (1).

(3) Let $A_n, B_n \in B(\mathcal{H})^{++}$ be such that $A_n \downarrow A$ and $B_n \downarrow A$ where $A, B > 0$. Then $A_n^{-1} \uparrow A^{-1}$ and $B_n^{-1} \uparrow B^{-1}$. So, $A_n^{-1} + B_n^{-1}$ is an increasing sequence in $B(\mathcal{H})^+$ such that

$$A_n^{-1} + B_n^{-1} \to A^{-1} + B^{-1}, \qquad ()$$

i.e. $A_n^{-1} + B_n^{-1} \uparrow A^{-1} + B^{-1}$. By (1), we thus have $(A_n^{-1} + B_n^{-1})^{-1} \downarrow (A^{-1} + B^{-1})^{-1}$.

(4) Let $A_n, B_n \in B(\mathcal{H})^{++}$ be such that $A_n \downarrow A$ and $B_n \downarrow B$. Then, by (2), $A_n : B_n$ is a decreasing sequence of positive operators. The order completeness of $B(\mathcal{H})$ guaruntees the existence of the strong limit of $A_n : B_n$. Let $A_n^{'}$ and $B_n^{'}$ be another sequences such that $A_n^{'} \downarrow A$ and $B_n^{'} \downarrow B$. Note that for each $n, m \in \mathbb{N}$, we have $A_n \quad A_n + A_m^{'} - A$ and $B_n \quad B_n + B_m^{'} - B$. Then

$$A_n : B_n \quad \left(A_n + A_m^{'} - A\right) : \left(B_n + B_m^{'} - B\right). \qquad ()$$

Note that as $n \to \infty$, $A_n + A_m^{'} - A \to A_m^{'}$ and $B_n + B_m^{'} - B \to B_m^{'}$. We have that as $n \to \infty$,

$$\left(A_n + A_m^{'} - A\right) : \left(B_n + B_m^{'} - B\right) \to A_m^{'} : B_m^{'}. \qquad ()$$

Hence, $\lim_{n\to\infty} A_n : B_n \quad A_m^{'} : B_m^{'}$ and $\lim_{n\to\infty} A_n : B_n \quad \lim_{m\to\infty} A_m^{'} : B_m^{'}$. By symmetry, $\lim_{n\to\infty} A_n : B_n \quad \lim_{m\to\infty} A_m^{'} : B_m^{'}$.

We define the *parallel sum* of $A, B \quad 0$ to be

$$A : B = \lim_{\downarrow 0} (A + I) : (B + I) \qquad \text{(id30)}$$

where the limit is taken in the strong-operator topology.

Lemma 0.6 For each $x \in \mathcal{H}$,

$$\langle (A : B)x, x\rangle = \inf \{\langle Ay, y\rangle + \langle Bz, z\rangle : y, z \in \mathcal{H}, y + z = x\}. \qquad \text{(id32)}$$

First, assume that A, B are invertible. Then for all $x, y \in \mathcal{H}$,

$$\begin{aligned} &\langle Ay, y\rangle + \langle B(x - y), x - y\rangle - \langle (A : B)x, x\rangle \\ &= \langle Ay, y\rangle + \langle Bx, x\rangle - 2Re\langle Bx, y\rangle + \langle By, y\rangle - \langle \left(B - B(A + B)^{-1}B\right)x, x\rangle \\ &= \langle (A + B)y, y\rangle - 2Re\langle Bx, y\rangle + \langle (A + B)^{-1}Bx, Bx\rangle \\ &= \| (A + B)^{1/2}y \|^2 - 2Re\langle Bx, y\rangle + \| (A + B)^{-1/2}Bx \|^2 \\ &\quad 0. \end{aligned} \qquad ()$$

With $y = (A+B)^{-1}Bx$, we have

$$\langle Ay,\, y\rangle + \langle B(x-y),\, x-y\rangle - \langle (A:B)x,\, x\rangle = 0. \qquad ()$$

Hence, we have the claim for $A, B > 0$. For $A, B \;\; 0$, consider $A + I$ and $B + I$ where $\downarrow 0$.

Remark 0.7 This lemma has a physical interpretation, called the *Maxwell's minimum power principle*. Recall that a positive operator represents the impedance of a electrical network while the power dissipation of network with impedance A and current x is the inner product $\langle Ax,\, x\rangle$. Consider two electrical networks connected in parallel. For a given current input x, the current will divide $x = y + z$, where y and z are currents of each network, in such a way that the power dissipation is minimum.

Theorem 0.8 The parallel sum satisfies

- monotonicity: $A_1 \;\; A_2$, $B_1 \;\; B_2 \Rightarrow A_1 : B_1 \;\; A_2 : B_2$.
- transformer inequality: $S^*(A:B)S \;\; (S^*AS):(S^*BS)$ for every $S \in B(\mathcal{H})$.
- continuity from above: if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n : B_n \downarrow A : B$.

(1) The monotonicity follows from the formula (▭) and Lemma ▭(2).

(2) For each $x, y, z \in \mathcal{H}$ such that $x = y + z$, by the previous lemma,

$$\begin{aligned} \langle S^*(A:B)Sx,\, x\rangle &= \langle (A:B)Sx,\, Sx\rangle \\ &\;\; \langle ASy,\, Sy\rangle + \langle S^*BSz,\, z\rangle \\ &= \langle S^*ASy,\, y\rangle + \langle S^*BSz,\, z\rangle. \end{aligned} \qquad ()$$

Again, the previous lemma assures $S^*(A:B)S \;\; (S^*AS):(S^*BS)$.

(3) Let A_n and B_n be decreasing sequences in $B(\mathcal{H})^+$ such that $A_n \downarrow A$ and $B_n \downarrow B$. Then $A_n : B_n$ is decreasing and $A : B \;\; A_n : B_n$ for all $n \in \mathbb{N}$. We have that, by the joint monotonicity of parallel sum, for all $\;\; > 0$

$$A_n : B_n \;\; (A_n + I):(B_n + I). \qquad ()$$

Since $A_n{+}I \downarrow A + I$ and $B_n{+}I \downarrow B + I$, by Lemma 3.1.4(3) we have $A_n : B_n \downarrow A : B$.

Remark 0.9 The positive operator S^*AS represents the impedance of a network connected to a transformer. The transformer inequality means that the impedance of parallel connection with transformer first is greater than that with transformer last.

Proposition 0.10 The set of positive operators on $\mathcal{H}$ is a partially ordered commutative semigroup with respect to the parallel sum.

For $A, B, C > 0$, we have $(A : B) : C = A : (B : C)$ and $A : B = B : A$. The continuity from above in Theorem ▭ implies that $(A : B) : C = A : (B : C)$ and $A : B = B : A$ for all $A, B, C \; 0$. The monotonicity of the parallel sum means that the positive operators form a partially ordered semigroup.

Theorem 0.11 For $A, B, C, D \; 0$, we have the series-parallel inequality

$$(A + B) : (C + D) \;\; A : C + B : D. \tag{id41}$$

In other words, the parallel sum is concave.

For each $x, y, z \in \mathcal{H}$ such that $x = y + z$, we have by the previous lemma that

$$\begin{aligned} \langle (A : C + B : D)x, x\rangle &= \langle (A : C)x, x\rangle + \langle (B : D)x, x\rangle \\ & \;\; \langle Ay, y\rangle + \langle Cz, z\rangle + \langle By, y\rangle + \langle Dz, z\rangle \\ &= \langle (A + B)y, y\rangle + \langle (C + D)z, z\rangle. \end{aligned} \tag{}$$

Applying the previous lemma yields $(A + B) : (C + D) \;\; A : C + B : D$.

Remark 0.12 The ordinary sum of operators represents the total impedance of two networks with series connection while the parallel sum indicates the total impedance of two networks with parallel connection. So, the series-parallel inequality means that the impedance of a series-parallel connection is greater than that of a parallel-series connection.

4. Classical means: arithmetic, harmonic and geometric means

Some desired properties of any object that is called a "mean" M on $B(\mathcal{H})^+$ should have are given here.

- *positivity*: $A, B \; 0 \Rightarrow M(A, B) \; 0$;
- *monotonicity*: $A \; A', B \; B' \Rightarrow M(A, B) \; M(A', B')$;
- *positive homogeneity*: $M(kA, kB) = kM(A, B)$ for $k \in \mathbb{R}^+$;
- *transformer inequality*: $X^* M(A, B) X \;\; M(X^* AX, X^* BX)$ for $X \in B(\mathcal{H})$;
- *congruence invariance*: $X^* M(A, B) X = M(X^* AX, X^* BX)$ for invertible $X \in B(\mathcal{H})$;
- *concavity*: $M(tA + (1 - t)B, tA' + (1 - t)B') \;\; tM(A, A') + (1 - t)M(B, B')$ for $t \in [0, 1]$;
- *continuity from above*: if $A_n \downarrow A$ and $B_n \downarrow B$, then $M(A_n, B_n) \downarrow M(A, B)$;
- *betweenness*: if $A \; B$, then $A \; M(A, B) \; B$;
- *fixed point property*: $M(A, A) = A$.

In order to study matrix or operator means in general, the first step is to consider three classical means in mathematics, namely, arithmetic, geometric and harmonic means.

The *arithmetic mean* of $A, B \in B(\mathcal{H})^+$ is defined by

$$A \triangledown B = \frac{1}{2}(A + B). \tag{id52}$$

Then the arithmetic mean satisfies the properties (A1)–(A9). In fact, the properties (A5) and (A6) can be replaced by a stronger condition:

$X^*M(A, B)X = M(X^*AX, X^*BX)$ for all $X \in B(\mathcal{H})$.

Moreover, the arithmetic mean satisfies

affinity: $M(kA + C, kB + C) = kM(A, B) + C$ for $k \in \mathbb{R}^+$.

Define the *harmonic mean* of positive operators $A, B \in B(\mathcal{H})^+$ by

$$A \,!\, B = 2(A : B) = \lim_{\downarrow 0} 2(A^{-1} + B^{-1})^{-1} \tag{id53}$$

where $A \equiv A + I$ and $B \equiv B + I$. Then the harmonic mean satisfies the properties (A1)–(A9).

The geometric mean of matrices is defined in [7] and studied in details in [3]. A usage of congruence transformations for treating geometric means is given in [19]. For a given invertible operator $C \in B(\mathcal{H})$, define

$$\Gamma_C : B(\mathcal{H})^{sa} \to B(\mathcal{H})^{sa}, \; A \mapsto C^*AC. \tag{}$$

Then each Γ_C is a linear isomorphism with inverse $\Gamma_{C^{-1}}$ and is called a *congruence transformation*. The set of congruence transformations is a group under multiplication. Each congruence transformation preserves positivity, invertibility and, hence, strictly positivity. In fact, Γ_C maps $B(\mathcal{H})^+$ and $B(\mathcal{H})^{++}$ onto themselves. Note also that Γ_C is order-preserving.

Define the *geometric mean* of $A, B > 0$ by

$$A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} = \Gamma_{A^{1/2}}\Gamma^{1/2}_{A^{-1/2}}(B). \tag{id54}$$

Then $A \# B > 0$ for $A, B > 0$. This formula comes from two natural requirements: This definition should coincide with the usual geometric mean in $\mathbb{R}^+$: $A \# B = (AB)^{1/2}$ provided that $AB = BA$. The second condition is that, for any invertible $T \in B(\mathcal{H})$,

$$T^*(A \# B)T = (T^*AT) \# (T^*BT). \tag{id55}$$

The next theorem characterizes the geometric mean of A and B in term of the solution of a certain operator equation.

Theorem 0.13 For each $A,\ B>0$, the Riccati equation $\Gamma_X\left(A^{-1}\right):\ = XA^{-1}X=B$ has a unique positive solution, namely, $X=A\ \#\ B$.

The direct computation shows that $(A\ \#\ B)A^{-1}(A\ \#\ B)=B$. Suppose there is another positive solution $Y\ \ 0$. Then

$$\left(A^{-1/2}XA^{-1/2}\right)^2=A^{-1/2}XA^{-1}XA^{-1/2}=A^{-1/2}YA^{-1}YA^{-1/2}=\left(A^{-1/2}YA^{-1/2}\right)^2. \qquad ()$$

The uniqueness of positive square roots implies that $A^{-1/2}XA^{-1/2}=A^{-1/2}YA^{-1/2}$, i.e., $X=Y$.

Theorem 0.14 (Maximum property of geometric mean) For $A,\ B>0$,

$$A\ \#\ B=\max\left\{X\ \ 0\ :\ XA^{-1}X\ \ B\right\} \qquad \text{(id58)}$$

where the maximum is taken with respect to the positive semidefinite ordering.

If $XA^{-1}X\ \ B$, then

$$\left(A^{-1/2}XA^{-1/2}\right)^2=A^{-1/2}XA^{-1}XA^{-1/2}\ \ A^{-1/2}BA^{-1/2} \qquad ()$$

and $A^{-1/2}XA^{-1/2}\ \ \left(A^{-1/2}BA^{-1/2}\right)^{1/2}$ i.e. $X\ \ A\ \#\ B$ by Theorem ▭.

Recall the fact that if $f:[a,\ b]\to$ *iscontinuousand* An A*with* Sp(An) [a,b]*foralln* N, *then*Sp(A) [a,b]*and*f(An) f(A).

Lemma 0.15 Let $A,\ B,\ C,\ D,\ A_n,\ B_n\in B(\mathcal{H})^{++}$ for all $n\in\mathbb{N}$.

- If $A\ \ C$ and $B\ \ D$, then $A\ \#\ B\ \ C\ \#\ D$.
- If $A_n\downarrow A$ and $B_n\downarrow B$, then $A_n\ \#\ B_n\downarrow A\ \#\ B$.
- If $A_n\downarrow A$ and $B_n\downarrow B$, then $\lim A_n\ \#\ B_n$ exists and does not depend on the choices of $A_n,\ B_n$.

(1) The extremal characterization allows us to prove only that $(A\ \#\ B)C^{-1}(A\ \#\ B)\ \ D$. Indeed,

$$\begin{aligned}(A\ \#\ B)C^{-1}(A\ \#\ B)\ &=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}C^{-1}A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}\\ &\ \ A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}A^{-1}A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}\\ &=B\\ &\ \ D.\end{aligned} \qquad ()$$

(2) Assume $A_n \downarrow A$ and $B_n \downarrow B$. Then $A_n \ \# \ B_n$ is a decreasing sequence of strictly positive operators which is bounded below by 0. The order completeness of $B(\mathcal{H})$ implies that this sequence converges strongly to a positive operator. Since $A_n^{-1} \quad A^{-1}$, the Löwner-Heinz's inequality assures that $A_n^{-1/2} \quad A^{-1/2}$ and hence $\| A_n^{-1/2} \| \quad \| A^{-1/2} \|$ for all $n \in \mathbb{N}$. Note also that $\| B_n \| \quad \| B_1 \|$ for all $n \in \mathbb{N}$. Recall that the multiplication is jointly continuous in the strong-operator topology if the first variable is bounded in norm. So, $A_n^{-1/2} B_n A_n^{-1/2}$ converges strongly to $A^{-1/2} B A^{-1/2}$. It follows that

$$\left(A_n^{-1/2} B_n A_n^{-1/2}\right)^{1/2} \to \left(A^{-1/2} B A^{-1/2}\right)^{1/2}. \qquad ()$$

Since $A_n^{1/2}$ is norm-bounded by $\| A^{1/2} \|$ by Löwner-Heinz's inequality, we conclude that

$$A_n^{1/2}\left(A_n^{-1/2} B_n A_n^{-1/2}\right)^{1/2} A_n^{1/2} \to A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^{1/2} A^{1/2}. \qquad ()$$

The proof of (3) is just the same as the case of harmonic mean.

We define the *geometric mean* of $A, B \quad 0$ by

$$A \ \# \ B = \lim_{\downarrow 0} (A + I) \ \# \ (B + I). \qquad \text{(id63)}$$

Then $A \ \# \ B \quad 0$ for any $A, B \quad 0$.

Theorem 0.16 The geometric mean enjoys the following properties

- monotonicity: $A_1 \quad A_2, B_1 \quad B_2 \Rightarrow A_1 \ \# \ B_1 \quad A_2 \ \# \ B_2$.
- continuity from above: $A_n \downarrow A, B_n \downarrow B \Rightarrow A_n \ \# \ B_n \downarrow A \ \# \ B$.
- fixed point property: $A \ \# \ A = A$.
- self-duality: $(A \ \# \ B)^{-1} = A^{-1} \ \# \ B^{-1}$.
- symmetry: $A \ \# \ B = B \ \# \ A$.
- congruence invariance: $\Gamma_C(A) \ \# \ \Gamma_C(B) = \Gamma_C(A \ \# \ B)$ for all invertible C.

(1) Use the formula (▭) and Lemma ▭ (1).

(2) Follows from Lemma ▭ and the definition of the geometric mean.

(3) The unique positive solution to the equation $X A^{-1} X = A$ is $X = A$.

(4) The unique positive solution to the equation $X^{-1} A^{-1} X^{-1} = B$ is $X^{-1} = A \ \# \ B$. But this equstion is equivalent to $XAX = B^{-1}$. So, $A^{-1} \ \# \ B^{-1} = X = (A \ \# \ B)^{-1}$.

(5) The equation $XA^{-1}X = B$ has the same solution to the equation $XB^{-1}X = A$ by taking inverse in both sides.

(6) We have

$$\begin{aligned} \Gamma_C(A \# B)(\Gamma_C(A))^{-1}\Gamma_C(A \# B) &= \Gamma_C(A \# B)\Gamma_{C^{-1}}(A^{-1})\Gamma_C(A \# B) \\ &= \Gamma_C((A \# B)A^{-1}(A \# B)) \\ &= \Gamma_C(B). \end{aligned} \quad ()$$

Then apply Theorem ▭.

The congruence invariance asserts that Γ_C is an isomorphism on $B(\mathcal{H})^{++}$ with respect to the operation of taking the geometric mean.

Lemma 0.17 For $A > 0$ and $B \; 0$, the operator

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \quad ()$$

is positive if and only if $B - C^*A^{-1}C$ is positive, i.e., $B \; C^*A^{-1}C$.

By setting

$$X = \begin{pmatrix} I & -A^{-1}C \\ 0 & I \end{pmatrix}, \quad ()$$

we compute

$$\begin{aligned} \Gamma_X\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} &= \begin{pmatrix} I & 0 \\ -C^*A^{-1} & I \end{pmatrix}\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}\begin{pmatrix} I & -A^{-1}C \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ 0 & B - C^*A^{-1}C \end{pmatrix}. \end{aligned} \quad ()$$

Since Γ_G preserves positivity, we obtain the desired result.

Theorem 0.18 The geometric mean $A \# B$ of $A, B \in B(\mathcal{H})^+$ is the largest operator $X \in B(\mathcal{H})^{sa}$ for which the operator

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \quad \text{(id73)}$$

is positive.

By continuity argumeny, we may assume that $A,\ B > 0$. If $X = A\ \#\ B$, then the operator (▭) is positive by Lemma ▭. Let $X \in B(\mathcal{H})^{sa}$ be such that the operator (▭) is positive. Then Lemma ▭ again implies that $X A^{-1} X \quad B$ and

$$\left(A^{-1/2} X A^{-1/2}\right)^2 = A^{-1/2} X A^{-1} X A^{-1/2} \quad A^{-1/2} B A^{-1/2}. \qquad ()$$

The Löwner-Heinz's inequality forces $A^{-1/2} X A^{-1/2} \quad (A^{-1/2} B A^{-1/2})^{1/2}$. Now, applying $\Gamma_{A^{1/2}}$ yields $X \quad A\ \#\ B$.

Remark 0.19 The arithmetic mean and the harmonic mean can be easily defined for multivariable positive operators. The case of geometric mean is not easy, even for the case of matrices. Many authors tried to defined geometric means for multivariable positive semidefinite matrices but there is no satisfactory definition until 2004 in [20].

5. Operator connections

We see that the arithmetic, harmonic and geometric means share the properties (A1)–(A9) in common. A mean in general should have algebraic, order and topological properties. Kubo and Ando [10] proposed the following definition:

Definition 0.20 A *connection* on $B(\mathcal{H})^+$ is a binary operation σ on $B(\mathcal{H})^+$ satisfying the following axioms for all $A,\ A',\ B,\ B',\ C \in B(\mathcal{H})^+$:

- *monotonicity*: $A \quad A',\ B \quad B' \Rightarrow A \sigma B \quad A' \sigma B'$
- *transformer inequality*: $C(A \sigma B)C \quad (CAC) \sigma (CBC)$
- *joint continuity from above*: if $A_n,\ B_n \in B(\mathcal{H})^+$ satisfy $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \sigma B_n \downarrow A \sigma B$.

The term "connection" comes from the study of electrical network connections.

Example 0.21 The following are examples of connections:

- the *left trivial mean* $(A,\ B) \mapsto A$ and the *right trivial mean* $(A,\ B) \mapsto B$
- the sum $(A,\ B) \mapsto A + B$
- the parallel sum
- arithmetic, geometric and harmonic means
- the *weighed arithmetic mean* with weight $\alpha \in [0,\ 1]$ which is defined for each $A,\ B \quad 0$ by $A \triangledown_\alpha \ B = (1 - \alpha)A + \alpha B$
- the *weighed harmonic mean* with weight $\alpha \in [0,\ 1]$ which is defined for each $A,\ B > 0$ by $A\, !_\alpha \ B = [(1 - \alpha)A^{-1} + \alpha B^{-1}]^{-1}$ and extended to the case $A,\ B \quad 0$ by continuity.

From now on, assume $\dim \mathcal{H} = \infty$. Consider the following property:

- *separate continuity from above*: if $A_n, B_n \in B(\mathcal{H})^+$ satisfy $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \,\sigma\, B \downarrow A \,\sigma\, B$ and $A \,\sigma\, B_n \downarrow A \,\sigma\, B$.

The condition (M3') is clearly weaker than (M3). The next theorem asserts that we can improve the definition of Kubo-Ando by replacing (M3) with (M3') and still get the same theory. This theorem also provides an easier way for checking a binary opertion to be a connection.

Theorem 0.22 If a binary operation σ on $B(\mathcal{H})^+$ satisfies (M1), (M2) and (M3'), then σ satisfies (M3), that is, σ is a connection.

Denote by $OM(\mathbb{R}^+)$ the set of operator monotone functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. If a binary operation σ has a property (A), we write $\sigma \in BO(A)$. The following properties for a binary operation σ and a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ play important roles:

- : If a projection $P \in B(\mathcal{H})^+$ commutes with $A, B \in B(\mathcal{H})^+$, then

$$P(A \,\sigma\, B) = (PA) \,\sigma\, (PB) = (A \,\sigma\, B)P; \qquad ()$$

- : $f(t)I = I \,\sigma\, (tI)$ for any $t \in \mathbb{R}^+$.

Proposition 0.23 The transformer inequality (M2) implies

- *Congruence invariance*: For $A, B \;\; 0$ and $C > 0$, $C(A\sigma B)C = (CAC) \,\sigma\, (CBC)$;
- *Positive homogeneity*: For $A, B \;\; 0$ and $\alpha \in (0, \infty)$, $\alpha(A \,\sigma\, B) = (\alpha A) \,\sigma\, (\alpha B)$.

For $A, B \;\; 0$ and $C > 0$, we have

$$C^{-1}[(CAC) \,\sigma\, (CBC)]C^{-1} \;\; (C^{-1}CACC^{-1}) \,\sigma\, (C^{-1}CBCC^{-1}) = A \,\sigma\, B \qquad ()$$

and hence $(CAC) \,\sigma\, (CBC) \;\; C(A \,\sigma\, B)C$. The positive homogeneity comes from the congruence invariance by setting $C = \sqrt{\alpha} I$.

Lemma 0.24 Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. If σ satisfies the positive homogeneity, (M3') and (F), then f is continuous.

To show that f is right continuous at each $t \in \mathbb{R}^+$, consider a sequence t_n in $\mathbb{R}^+$ such that $t_n \downarrow t$. Then by (M3')

$$f(t_n)I = I \,\sigma\, t_n I \downarrow I \,\sigma\, tI = f(t)I, \qquad ()$$

i.e. $f(t_n) \downarrow f(t)$. To show that f is left continuous at each $t > 0$, consider a sequence $t_n > 0$ such that $t_n \uparrow t$. Then $t_n^{-1} \downarrow t^{-1}$ and

$$\begin{aligned}\lim t_n^{-1} f(t_n) I &= \lim t_n^{-1}(I \,\sigma\, t_n I) = \lim \left(t_n^{-1} I\right) \sigma\, I = \left(t^{-1} I\right) \sigma\, I \\ &= t^{-1}(I \,\sigma\, tI) = t^{-1} f(t) I\end{aligned} \quad ()$$

Since f is increasing, $t_n^{-1} f(t_n)$ is decreasing. So, $t \mapsto t^{-1} f(t)$ and f are left continuous.

Lemma 0.25 Let σ be a binary operation on $B(\mathcal{H})^+$ satisfying (M3') and (P). If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function such that σ and f satisfy (F), then $f(A) = I \,\sigma\, A$ for any $A \in B(\mathcal{H})^+$.

First consider $A \in B(\mathcal{H})^+$ in the form $\sum_{i=1}^m \lambda_i P_i$ where $\{P_i\}_{i=1}^m$ is an orthogonal family of projections with sum I and $\lambda_i > 0$ for all $i = 1, \cdots, m$. Since each P_i commutes with A, we have by the property (P) that

$$\begin{aligned}I \,\sigma\, A &= \sum P_i(I \,\sigma\, A) = \sum P_i \,\sigma\, P_i A = \sum P_i \,\sigma\, \lambda_i P_i \\ &= \sum P_i(I \,\sigma\, \lambda_i I) = \sum f(\lambda_i) P_i = f(A).\end{aligned} \quad ()$$

Now, consider $A \in B(\mathcal{H})^+$. Then there is a sequence A_n of strictly positive operators in the above form such that $A_n \downarrow A$. Then $I \,\sigma\, A_n \downarrow I \,\sigma\, A$ and $f(A_n)$ converges strongly to $f(A)$. Hence, $I \,\sigma\, A = \lim I \,\sigma\, A_n = \lim f(A_n) = f(A)$.

Proof of Theorem ▭: Let $\sigma \in BO(M1, M2, M3')$. As in [10], the conditions (M1) and (M2) imply that σ satisfies (P) and there is a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ subject to (F). If $0 \;\; t_1 \;\; t_2$, then by (M1)

$$f(t_1) I = I \,\sigma\, (t_1 I) \;\; I \,\sigma\, (t_2 I) = f(t_2) I, \quad ()$$

i.e. $f(t_1) \;\; f(t_2)$. The assumption (M3') is enough to guarantee that f is continuous by Lemma ▭. Then Lemma ▭ results in $f(A) = I\sigma A$ for all $A \;\; 0$. Now, (M1) and the fact that $\dim \mathcal{H} = \infty$ yield that f is operator monotone. If there is another $g \in OM(\mathbb{R}^+)$ satisfying (F), then $f(t) I = I \,\sigma\, tI = g(t) I$ for each $t \;\; 0$, i.e. $f = g$. Thus, we establish a well-defined map $\sigma \in BO(M1, M2, M3') \mapsto f \in OM(\mathbb{R}^+)$ such that σ and f satisfy (F).

Now, given $f \in OM(\mathbb{R}^+)$, we construct σ from the integral representation (▭) in Proposition ▭. Define a binary operation $\sigma : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \to B(\mathcal{H})^+$ by

$$A \,\sigma\, B = \int_{[0,1]} A \,!_t\, B \, d\mu(t) \quad \text{(id95)}$$

where the integral is taken in the sense of Bochner. Consider $A, B \in B(\mathcal{H})^+$ and set $F_t = A \,!_t\, B$ for each $t \in [0, 1]$. Since $A \;\; \| A \| I$ and $B \;\; \| B \| I$, we get

$$A\,!_t\;\;B \quad \| A \| \, I \; !_t \quad \| B \| \, I \quad = \quad \frac{\| A \| \;\; \| B \|}{t \, \| A \| \; + (1 - t) \, \| B \|} I. \tag{}$$

By Banach-Steinhaus' theorem, there is an $M > 0$ such that $\| F_t \| \;\; M$ for all $t \in [0, 1]$. Hence,

$$\int_{[0,1]} \| F_t \| \; d\mu(t) \;\; \int_{[0,1]} M \; d\mu(t) < \infty. \tag{}$$

So, F_t is Bochner integrable. Since $F_t \;\; 0$ for all $t \in [0, 1]$, $\int_{[0,1]} F_t \, d\mu(t) \;\; 0$. Thus, $A \, \sigma \, B$ is a well-defined element in $B(\mathcal{H})^+$. The monotonicity (M1) and the transformer inequality (M2) come from passing the monotonicity and the transformer inequality of the weighed harmonic mean through the Bochner integral. To show (M3'), let $A_n \downarrow A$ and $B_n \downarrow B$. Then $A_n \, !_t \;\; B \downarrow A \, !_t \;\; B$ for $t \in [0, 1]$ by the monotonicity and the separate continuity from above of the weighed harmonic mean. Let $\xi \in H$. Define a bounded linear map $\Phi : B(\mathcal{H}) \to$ *by*(T) = T , .*Foreach* n N, *set*Tn(t) = An !t B*andput*T(t) = A !t B.*Thenforeach* n N{}, Tn *isBochnerintegrableandSince*Tn(t) T (t), *wehavethat*Tn(t) ,T(t) ,*as*n *foreach* t [0,1] .*Weobtainfromthedominatedconvergencetheoremthat**So*, An B A B.*Similarly*, A Bn A B .*Thus*, *satisfies*$(M3')$.*Itiseasytoseethat*f(t) I = I (t I) *for*t 0.*Thisshowsthatthemap*f*issurjective*.To show the injectivity of this map, let $\sigma_1, \sigma_2 \in BO(M1, M2, M3')$ be such that $\sigma_i \mapsto f$ where, for each $t \;\; 0$, $I \, \sigma_i \, (tI) = f(t)I$, $i = 1, 2$. Since σ_i satisfies the property (P), we have $I \, \sigma_i \, A = f(A)$ for $A \;\; 0$ by Lemma ▭. Since σ_i satisfies the congruence invariance, we have that for $A > 0$ and $B \;\; 0$,

$$A \, \sigma_i \, B = A^{1/2} \big(I \, \sigma_i \, A^{-1/2} B A^{-1/2} \big) A^{1/2} = A^{1/2} f \big(A^{-1/2} B A^{-1/2} \big) A^{1/2}, \quad i = 1, 2. \tag{}$$

For each $A, \; B \;\; 0$, we obtain by (M3') that

$$\begin{aligned} A \, \sigma_1 \, B &= \lim_{\downarrow 0} A \, \sigma_1 \, B \\ &= \lim_{\downarrow 0} A^{1/2} \big(I \, \sigma_1 \, A^{-1/2} B A^{-1/2} \big) A^{1/2} \\ &= \lim_{\downarrow 0} A^{1/2} f \big(A^{-1/2} B A^{-1/2} \big) A^{1/2} \\ &= \lim_{\downarrow 0} A^{1/2} \big(I \, \sigma_2 \, A^{-1/2} B A^{-1/2} \big) A^{1/2} \\ &= \lim_{\downarrow 0} A \, \sigma_2 \, B \\ &= A \, \sigma_2 \, B, \end{aligned} \tag{}$$

where $A \equiv A + I$. That is $\sigma_1 = \sigma_2$. Therefore, there is a bijection between $OM(\mathbb{R}^+)$ and $BO(M1, M2, M3')$. Every element in $BO(M1, M2, M3')$ admits an integral representation (▭). Since the weighed harmonic mean possesses the joint continuity (M3), so is any element in $BO(M1, M2, M3')$. □

http://dx.doi.org/10.5772/46479

The next theorem is a fundamental result of [10].

Theorem 0.26 There is a one-to-one correspondence between connections σ and operator monotone functions f on the non-negative reals satisfying

$$f(t)I = I\,\sigma\,(tI), \quad t \in \mathbb{R}^{+}. \tag{id97}$$

There is a one-to-one correspondence between connections σ and finite Borel measures ν on $[0, \infty]$ satisfying

$$A\,\sigma\,B \quad = \int_{[0,\infty]} \frac{t+1}{t}(tA\ :\ B)\,d\nu(t), \quad A,\ B\ \ 0. \tag{id98}$$

Moreover, the map $\sigma \mapsto f$ is an affine order-isomorphism between connections and non-negative operator monotone functions on $\mathbb{R}^{+}$. Here, the order-isomorphism means that when $\sigma_i \mapsto f_i$ for $i = 1, 2$, $A\,\sigma_1 B\ \ A\,\sigma_2 B$ for all $A,\ B \in B(\mathcal{H})^{+}$ if and only if $f_1\ \ f_2$.

Each connection σ on $B(\mathcal{H})^{+}$ produces a unique scalar function on $\mathbb{R}^{+}$, denoted by the same notation, satisfying

$$(s\,\sigma\,t)I = (sI)\,\sigma\,(tI), \quad s,\ t \in \mathbb{R}^{+}. \tag{id99}$$

Let $s,\ t \in \mathbb{R}^{+}$. If $s > 0$, then $s\,\sigma\,t = sf(t/s)$. If $t > 0$, then $s\,\sigma\,t = tf(s/t)$.

Theorem 0.27 There is a one-to-one correspondence between connections and finite Borel measures on the unit interval. In fact, every connection takes the form

$$A\,\sigma\,B \quad = \int_{[0,1]} A\,!_t\ \ B\,d\mu(t), \quad A,\ B\ \ 0 \tag{id101}$$

for some finite Borel measure μ on $[0, 1]$. Moreover, the map $\mu \mapsto \sigma$ is affine and order-preserving. Here, the order-presering means that when $\mu_i \mapsto \sigma_i$ (i=1,2), if $\mu_1(E)\ \ \mu_2(E)$ for all Borel sets E in $[0, 1]$, then $A\,\sigma_1\,B\ \ A\,\sigma_2\,B$ for all $A,\ B \in B(\mathcal{H})^{+}$.

The proof of the first part is contained in the proof of Theorem ▭. This map is affine because of the linearity of the map $\mu \mapsto \int f\,d\mu$ on the set of finite positive measures and the bijective correspondence between connections and Borel measures. It is straight forward to show that this map is order-preserving.

Remark 0.28 Let us consider operator connections from electrical circuit viewpoint. A general connection represents a formulation of making a new impedance from two given impedances. The integral representation (▭) shows that such a formulation can be described as a weighed series connection of (infinite) weighed harmonic means. From this point of view, the theory of operator connections can be regarded as a mathematical theory of electrical circuits.

Definition 0.29 Let σ be a connection. The operator monotone function f in (▭) is called the *representing function* of σ. If μ is the measure corresponds to σ in Theorem ▭, the measure $\mu\psi^{-1}$ that takes a Borel set E in $[0, \infty]$ to $\mu(\psi^{-1}(E))$ is called the *representing measure* of σ in the Kubo-Ando's theory. Here, $\psi : [0, 1] \to [0, \infty]$ is a homeomorphism $t \mapsto t/(1-t)$.

Since every connection σ has an integral representation (▭), properties of weighed harmonic means reflect properties of a general connection. Hence, every connection σ satisfies the following properties for all $A, B \;\; 0$, $T \in B(\mathcal{H})$ and invertible $X \in B(\mathcal{H})$:

- *transformer inequality*: $T^*(A \,\sigma\, B)T \;\; (T^*AT)\,\sigma\,(T^*BT)$;
- *congruence invariance*: $X^*(A \,\sigma\, B)X = (X^*AX)\,\sigma\,(X^*BX)$;
- *concavity*: $(tA + (1-t)B)\,\sigma\,(tA' + (1-t)B') \;\; t(A \,\sigma\, A') + (1-t)(B \,\sigma\, B')$ for $t \in [0, 1]$.

Moreover, if $A, B > 0$,

$$A \,\sigma\, B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2} \qquad \text{(id107)}$$

and, in general, for each $A, B \;\; 0$,

$$A \,\sigma\, B = \lim_{\downarrow 0} A \,\sigma\, B \qquad \text{(id108)}$$

where $A \equiv A + I$ and $B \equiv B + I$. These properties are useful tools for deriving operator inequalities involving connections. The formulas (▭) and (▭) give a way for computing the formula of connection from its representing function.

Example 0.30

- The left- and the right-trivial means have representing functions given by $t \mapsto 1$ and $t \mapsto t$, respectively. The representing measures of the left- and the right-trivial means are given respectively by δ_0 and δ_∞ where δ_x is the Dirac measure at x. So, the α-weighed arithmetic mean has the representing function $t \mapsto (1-\alpha) + \alpha t$ and it has $(1-\alpha)\delta_0 + \alpha\delta_\infty$ as the representing measure.
- The geometric mean has the representing function $t \mapsto t^{1/2}$.
- The harmonic mean has the representing function $t \mapsto 2t/(1+t)$ while $t \mapsto t/(1+t)$ corrsponds to the parallel sum.

Remark 0.31 The map $\sigma \mapsto \mu$, where μ is the representing measure of σ, is not order-preserving in general. Indeed, the representing measure of ▿ is given by $\mu = (\delta_0 + \delta_\infty)/2$ while the representing measure of ! is given by δ_1. We have ! ▿ but $\delta_1 \mu$.

http://dx.doi.org/10.5772/46479

6. Operator means

According to [1], a (scalar) mean is a binary operation M on $(0, \infty)$ such that $M(s, t)$ lies between s and t for any $s, t > 0$. For a connection, this property is equivalent to various properties in the next theorem.

Theorem 0.32 The following are equivalent for a connection σ on $B(\mathcal{H})^+$:

- σ satisfies the *betweenness property*, i.e. $A \quad B \Rightarrow A \quad A \sigma B \quad B$.
- σ satisfies the *fixed point property*, i.e. $A \sigma A = A$ for all $A \in B(\mathcal{H})^+$.
- σ is normalized, i.e. $I \sigma I = I$.
- the representing function f of σ is normalized, i.e. $f(1) = 1$.
- the representing measure μ of σ is normalized, i.e. μ is a probability measure.

Clearly, (i) ⇒ (iii) ⇒ (iv). The implication (iii) ⇒ (ii) follows from the congruence invariance and the continuity from above of σ. The monotonicity of σ is used to prove (ii) ⇒ (i). Since

$$I \sigma I = \int_{[0,1]} I \,!_t\, I \, d\mu(t) = \mu([0, 1]) I, \qquad ()$$

we obtain that (iv) ⇒ (v) ⇒ (iii).

Definition 0.33 A *mean* is a connection satisfying one, and thus all, of the properties in the previous theorem.

Hence, every mean in Kubo-Ando's sense satisfies the desired properties (A1)–(A9) in Section 3. As a consequence of Theorem ▭, a convex combination of means is a mean.

Theorem 0.34 Given a Hilbert space $\mathcal{H}$, there exist affine bijections between any pair of the following objects:

- the means on $B(\mathcal{H})^+$,
- the operator monotone functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(1) = 1$,
- the probability Borel measures on $[0, 1]$.

Moreover, these correspondences between (i) and (ii) are order isomorphic. Hence, there exists an affine order isomorphism between the means on the positive operators acting on different Hilbert spaces.

Follow from Theorems ▭ and ▭.

Example 0.35 The left- and right-trivial means, weighed arithmetic means, the geometric mean and the harmonic mean are means. The parallel sum is not a mean since its representing function is not normalized.

Example 0.36 The function $t \mapsto t^{\alpha}$ is an operator monotone function on $\mathbb{R}^+$ for each $\alpha \in [0, 1]$ by the Löwner-Heinz's inequality. So it produces a mean, denoted by $\#_\alpha$, on $B(\mathcal{H})^+$. By the direct computation,

$$s \#_\alpha \; t = s^{1-\alpha} t^{\alpha}, \tag{id127}$$

i.e. $\#_\alpha$ is the α-weighed geometric mean on $\mathbb{R}^+$. So the α-weighed geometric mean on $\mathbb{R}^+$ is really a Kubo-Ando mean. The *α-weighed geometric mean* on $B(\mathcal{H})^+$ is defined to be the mean corresponding to that mean on $\mathbb{R}^+$. Since t^{α} has an integral expression

$$t^{\alpha} = \frac{\sin \alpha\pi}{\pi} \int_0^{\infty} \frac{t\lambda^{\alpha-1}}{t+\lambda}\, dm(\lambda) \tag{id128}$$

(see [14]) where m denotes the Lebesgue measure, the representing measure of $\#_\alpha$ is given by

$$d\mu(\lambda) = \frac{\sin \alpha\pi}{\pi} \frac{\lambda^{\alpha-1}}{\lambda+1}\, dm(\lambda). \tag{id129}$$

Example 0.37 Consider the operator monotone function

$$t \mapsto \frac{t}{(1-\alpha)t+\alpha}, \quad t \; 0, \alpha \in [0, 1]. \tag{}$$

The direct computation shows that

$$s \,!_\alpha\; t = \begin{cases} ((1-\alpha)s^{-1} + \alpha t^{-1})^{-1}, & s, t > 0; \\ 0, & \text{otherwise,} \end{cases} \tag{id131}$$

which is the α-weighed harmonic mean. We define the *α-weighed harmonic mean* on $B(\mathcal{H})^+$ to be the mean corresponding to this operator monotone function.

Example 0.38 Consider the operator monotone function $f(t) = (t-1)/\log t$ for $t > 0, t \neq 1$, $f(0) \equiv 0$ and $f(1) \equiv 1$. Then it gives rise to a mean, denoted by λ, on $B(\mathcal{H})^+$. By the direct computation,

$$s \,\lambda\, t = \begin{cases} \dfrac{s-t}{\log s - \log t}, & s > 0, t > 0, s \neq t; \\ s, & s = t \\ 0, & \text{otherwise,} \end{cases} \tag{id133}$$

i.e. λ is the logarithmic mean on $\mathbb{R}^+$. So the logarithmic mean on $\mathbb{R}^+$ is really a mean in Kubo-Ando's sense. The *logarithmic mean* on $B(\mathcal{H})^+$ is defined to be the mean corresponding to this operator monotone function.

Example 0.39 The map $t \mapsto \left(t^r + t^{1-r}\right)/2$ is operator monotone for any $r \in [0, 1]$. This function produces a mean on $B(\mathcal{H})^+$. The computation shows that

$$(s,\ t) \mapsto \frac{s^r t^{1-r} + s^{1-r} t^r}{2}. \qquad ()$$

However, the corresponding mean on $B(\mathcal{H})^+$ is not given by the formula

$$(A,\ B) \mapsto \frac{A^r B^{1-r} + A^{1-r} B^r}{2} \qquad \text{(id135)}$$

since it is not a binary operation on $B(\mathcal{H})^+$. In fact, the formula (▭) is considered in [21], called the *Heinz mean* of A and B.

Example 0.40 For each $p \in [-1, 1]$ and $\alpha \in [0, 1]$, the map

$$t \mapsto [(1-\alpha) + \alpha t^p]^{1/p} \qquad ()$$

is an operator monotone function on $\mathbb{R}^+$. Here, the case $p = 0$ is understood that we take limit as $p \to 0$. Then

$$s \#_{p,\alpha} \ t = [(1-\alpha)s^p + \alpha t^p]^{1/p}. \qquad \text{(id137)}$$

The corresponding mean on $B(\mathcal{H})^+$ is called the *quasi-arithmetic power mean* with parameter (p, α), defined for $A > 0$ and B 0 by

$$A \#_{p,\alpha} \ B = A^{1/2}\left[(1-\alpha)I + \alpha\left(A^{-1/2} B A^{-1/2}\right)^p\right]^{1/p} A^{1/2}. \qquad \text{(id138)}$$

The class of quasi-arithmetic power means contain many kinds of means: The mean $\#_{1,\alpha}$ is the α-weighed arithmetic mean. The case $\#_{0,\alpha}$ is the α-weighed geometric mean. The case $\#_{-1,\alpha}$ is the α-weighed harmonic mean. The mean $\#_{p,1/2}$ is the *power mean* or *binomial mean* of order p. These means satisfy the property that

$$A \#_{p,\alpha} \ B = B \#_{p,1-\alpha} \ A. \qquad \text{(id139)}$$

Moreover, they are interpolated in the sense that for all $p, q, \alpha \in [0, 1]$,

$$(A \#_{r,p} \ B) \#_{r,\alpha} \ (A \#_{r,q} \ B) \ = \ A \#_{r,(1-\alpha)p+\alpha q} \ B. \qquad \text{(id140)}$$

Example 0.41 If σ_1, σ_2 are means such that $\sigma_1 \ \sigma_2$, then there is a family of means that interpolates between σ_1 and σ_2, namely, $(1-\alpha)\sigma_1 + \alpha\sigma_2$ for all $\alpha \in [0, 1]$. Note that the map $\alpha \mapsto (1-\alpha)\sigma_1 + \alpha\sigma_2$ is increasing. For instance, the *Heron mean* with weight $\alpha \in [0, 1]$ is defined to be $h_\alpha = (1-\alpha)\ \# \ + \alpha \ \triangledown$. This family is the linear interpolations between the geometric mean and the arithmetic mean. The representing function of h_α is given by

$$t \mapsto (1-\alpha)t^{1/2} + \frac{\alpha}{2}(1+t). \qquad ()$$

The case $\alpha = 2/3$ is called the *Heronian mean* in the literature.

7. Applications to operator monotonicity and concavity

In this section, we generalize the matrix and operator monotonicity and concavity in the literature (see e.g. [3], [22]) in such a way that the geometric mean, the harmonic mean or specific operator means are replaced by general connections. Recall the following terminology. A continuous function $f : I \to \mathbb{R}$ is called an *operator concave function* if

$$f(tA + (1-t)B) \ \ tf(A) + (1-t)f(B) \qquad ()$$

for any $t \in [0, 1]$ and Hermitian operators $A, B \in B(\mathcal{H})$ whose spectrums are contained in the interval I and for all Hilbert spaces $\mathcal{H}$. A well-known result is that a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone if and only if it is operator concave. Hence, the maps $t \mapsto t^r$ and $t \mapsto \log t$ are operator concave for $r \in [0, 1]$. Let $\mathcal{H}$ and be Hilbert spaces. A map $\Phi : B(\mathcal{H}) \to B()$ is said to be *positive* if $\Phi(A) \ \ 0$ whenever $A \ \ 0$. It is called *unital* if $\Phi(I) = I$. We say that a positive map Φ is *strictly positive* if $\Phi(A) > 0$ when $A > 0$. A map Ψ from a convex subset of $B(\mathcal{H})^{sa}$ to $B()^{sa}$ is called *concave* if for each $A, B \in$ and $t \in [0, 1]$,

$$\Psi(tA + (1-t)B) \ \ t\Psi(A) + (1-t)\Psi(B). \qquad ()$$

A map $\Psi : B(\mathcal{H})^{sa} \to B()^{sa}$ is called *monotone* if $A \ \ B$ assures $\Psi(A) \ \ \Psi(B)$. So, in particular, the map $A \mapsto A^r$ is monotone and concave on $B(\mathcal{H})^+$ for each $r \in [0, 1]$. The map $A \mapsto \log A$ is monotone and concave on $B(\mathcal{H})^{++}$.

Note first that, from the previous section, the quasi-arithmetic power mean $(A, B) \mapsto A \#_{p,\alpha} \ B$ is monotone and concave for any $p \in [-1, 1]$ and $\alpha \in [0, 1]$. In particular, the following are monotone and concave:

http://dx.doi.org/10.5772/46479

- any weighed arithmetic mean,
- any weighed geometric mean,
- any weighed harmonic mean,
- the logarithmic mean,
- any weighed power mean of order $p \in [-1, 1]$.

Recall the following lemma from [22].

Lemma 0.42 (Choi's inequality) If $\Phi : B(\mathcal{H}) \to B()$ is linear, strictly positive and unital, then for every $A > 0$, $\Phi(A)^{-1} \;\; \Phi(A^{-1})$.

Proposition 0.43 If $\Phi : B(\mathcal{H}) \to B()$ is linear and strictly positive, then for any $A, B > 0$

$$\Phi(A)\Phi(B)^{-1}\Phi(A) \;\; \Phi(AB^{-1}A). \tag{id149}$$

For each $X \in B(\mathcal{H})$, set $\Psi(X) = \Phi(A)^{-1/2}\Phi(A^{1/2}XA^{1/2})\Phi(A)^{-1/2}$. Then Ψ is a unital strictly positive linear map. So, by Choi's inequality, $\Psi(A)^{-1} \;\; \Psi(A^{-1})$ for all $A > 0$. For each $A, B > 0$, we have by Lemma ▭ that

$$\begin{aligned} \Phi(A)^{1/2}\Phi(B)^{-1}\Phi(A)^{1/2} &= \Psi(A^{-1/2}BA^{-1/2})^{-1} \\ & \;\; \Psi((A^{-1/2}BA^{-1/2})^{-1}) \\ &= \Phi(A)^{-1/2}\Phi(AB^{-1}A)\Phi(A)^{-1/2}. \end{aligned} \tag{}$$

So, we have the claim.

Theorem 0.44 If $\Phi : B(\mathcal{H}) \to B()$ is a positive linear map which is norm-continuous, then for any connection σ on $B()^{+}$ and for each $A, B > 0$,

$$\Phi(A \,\sigma\, B) \;\; \Phi(A) \,\sigma\, \Phi(B). \tag{id151}$$

If, addition, Φ is strongly continuous, then (▭) holds for any $A, B \;\; 0$.

First, consider $A, B > 0$. Assume that Φ is strictly positive. For each $X \in B(\mathcal{H})$, set

$$\Psi(X) = \Phi(B)^{-1/2}\Phi(B^{1/2}XB^{1/2})\Phi(B)^{-1/2}. \tag{}$$

Then Ψ is a unital strictly positive linear map. So, by Choi's inequality, $\Psi(C)^{-1} \;\; \Psi(C^{-1})$ for all $C > 0$. For each $t \in [0, 1]$, put $X_t = B^{-1/2}(A\,!_t\;\; B)B^{-1/2} > 0$. We obtain from the previous proposition that

$$\begin{aligned}\Phi(A\,!_t\ \ B) &= \Phi(B)^{1/2}\Psi(X_t)\Phi(B)^{1/2} \\ &\ \ \Phi(B)^{1/2}[\Psi(X_t^{-1})]^{-1}\Phi(B)^{1/2} \\ &= \Phi(B)[\Phi(B((1-t)A^{-1}+tB^{-1})B)]^{-1}\Phi(B) \\ &= \Phi(B)[(1-t)\Phi(BA^{-1}B)+t\Phi(B)]^{-1}\Phi(B) \\ &\ \ \Phi(B)[(1-t)\Phi(B)\Phi(A)^{-1}\Phi(B)+t\Phi(B)]^{-1}\Phi(B) \\ &= \Phi(A)\,!_t\ \ \Phi(B).\end{aligned} \qquad ()$$

For general case of Φ, consider the family $\Phi\,(A) = \Phi(A) + \ I$ where $\ > 0$. Since the map $(A,\ B) \mapsto A\,!_t\ \ B = [(1-t)A^{-1} + tB^{-1}]^{-1}$ is norm-continuous, we arrive at

$$\Phi(A\,!_t\ \ B)\ \ \Phi(A)\,!_t\ \ \Phi(B). \qquad ()$$

For each connection σ, since Φ is a bounded linear operator, we have

$$\begin{aligned}\Phi(A\,\sigma\,B) &= \Phi\left(\int_{[0,1]} A\,!_t\ \ B\,d\mu(t)\right) = \int_{[0,1]}\Phi(A\,!_t\ \ B)\,d\mu(t) \\ &\ \ \int_{[0,1]}\Phi(A)\,!_t\ \ \Phi(B)\,d\mu(t) = \Phi(A)\,\sigma\,\Phi(B).\end{aligned} \qquad ()$$

Suppose further that Φ is strongly continuous. Then, for each $A,\ B\ \ 0$,

$$\begin{aligned}\Phi(A\,\sigma\,B) &= \Phi(\lim_{\ \downarrow 0}(A+\ I)\,\sigma\,(B+\ I)) = \lim_{\ \downarrow 0}\Phi((A+\ I)\,\sigma\,(B+\ I)) \\ &\ \ \lim_{\ \downarrow 0}\Phi(A+\ I)\,\sigma\,\Phi(B+\ I) = \Phi(A)\,\sigma\,\Phi(B).\end{aligned} \qquad ()$$

The proof is complete.

As a special case, if $\Phi : M_n(\to M_n($ is a positive linear map, then for any connection σ and for any positive semidefinite matrices $A,\ B \in M_n($, we have

$$\Phi(A\sigma B)\ \ \Phi(A)\,\sigma\,\Phi(B). \qquad ()$$

In particular, $\Phi(A)\,\#_{p,\alpha}\ \ \Phi(B)\ \ \Phi(A)\,\#_{p,\alpha}\ \ \Phi(B)$ for any $p \in [-1,\ 1]$ and $\alpha \in [0,\ 1]$.

Theorem 0.45 If $\Phi_1,\ \Phi_2 : B(\mathcal{H})^+ \to B()^+$ are concave, then the map

$$(A_1,\ A_2) \mapsto \Phi_1(A_1)\,\sigma\,\Phi_2(A_2) \qquad \text{(id153)}$$

is concave for any connection σ on $B()^+$.

Let $A_1,\ A_1',\ A_2,\ A_2'\ \ 0$ and $t \in [0,\ 1]$. The concavity of Φ_1 and Φ_2 means that for $i = 1,\ 2$

http://dx.doi.org/10.5772/46479

$$\Phi_i\left(tA_i + (1-t)A_i^{'}\right) \quad t\Phi_i(A_i) + (1-t)\Phi_i\left(A_i^{'}\right). \tag{}$$

It follows from the monotonicity and concavity of σ that

$$\begin{aligned} \Phi_1\left(tA_1 + (1-t)A_1^{'}\right) & \quad \sigma\, \Phi_2\left(tA_2 + (1-t)A_2^{'}\right) \\ & \left[t\Phi_1(A_1) + (1-t)\Phi_1\left(A_1^{'}\right)\right]\sigma\left[t\Phi_2(A_2) + (1-t)\Phi_2\left(A_2^{'}\right)\right] \\ & t[\Phi_1(A_1)\,\sigma\,\Phi_2(A_2)] + (1-t)[\Phi_1(A_1)\,\sigma\,\Phi_2(A_2)]. \end{aligned} \tag{}$$

This shows the concavity of the map $(A_1, A_2) \mapsto \Phi_1(A_1)\,\sigma\,\Phi_2(A_2)$.

In particular, if Φ_1 and Φ_2 are concave, then so is $(A, B) \mapsto \Phi_1(A)\,\#_{p,\alpha}\;\Phi_2(B)$ for $p \in [-1, 1]$ and $\alpha \in [0, 1]$.

Corollary 0.46 Let σ be a connection. Then, for any operator monotone functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, the map $(A, B) \mapsto f(A)\,\sigma\, g(B)$ is concave. In particular,

- the map $(A, B) \mapsto A^r\,\sigma\,B^s$ is concave on $B(\mathcal{H})^+$ for any $r, s \in [0, 1]$,
- the map $(A, B) \mapsto (\log A)\,\sigma\,(\log B)$ is concave on $B(\mathcal{H})^{++}$.

Theorem 0.47 If $\Phi_1, \Phi_2 : B(\mathcal{H})^+ \to B()^+$ are monotone, then the map

$$(A_1, A_2) \mapsto \Phi_1(A_1)\,\sigma\,\Phi_2(A_2) \tag{id158}$$

is monotone for any connection σ on $B()^+$.

Let $A_1 \;\; A_1^{'}$ and $A_2 \;\; A_2^{'}$. Then $\Phi_1(A_1) \;\; \Phi_1\left(A_1^{'}\right)$ and $\Phi_2(A_2) \;\; \Phi_2\left(A_2^{'}\right)$ by the monotonicity of Φ_1 and Φ_2. Now, the monotonicity of σ forces $\Phi_1(A_1)\,\sigma\,\Phi_2(A_2) \;\; \Phi_1\left(A_1^{'}\right)\sigma\,\Phi_2\left(A_2^{'}\right)$.

In particular, if Φ_1 and Φ_2 are monotone, then so is $(A, B) \mapsto \Phi_1(A)\,\#_{p,\alpha}\;\Phi_2(B)$ for $p \in [-1, 1]$ and $\alpha \in [0, 1]$.

Corollary 0.48 Let σ be a connection. Then, for any operator monotone functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, the map $(A, B) \mapsto f(A)\,\sigma\,g(B)$ is monotone. In particular,

- the map $(A, B) \mapsto A^r\,\sigma\,B^s$ is monotone on $B(\mathcal{H})^+$ for any $r, s \in [0, 1]$,
- the map $(A, B) \mapsto (\log A)\,\sigma\,(\log B)$ is monotone on $B(\mathcal{H})^{++}$.

Corollary 0.49 Let σ be a connection on $B(\mathcal{H})^+$. If $\Phi_1, \Phi_2 : B(\mathcal{H})^+ \to B(\mathcal{H})^+$ is monotone and strongly continuous, then the map

$$(A, B) \mapsto \Phi_1(A)\,\sigma\,\Phi_2(B) \tag{id163}$$

is a connection on $B(\mathcal{H})^{+}$. Hence, the map

$$(A,\ B) \mapsto f(A)\,\sigma\, g(B) \tag{id164}$$

is a connection for any operator monotone functions $f,\ g : \mathbb{R}^{+} \to \mathbb{R}^{+}$.

The monotonicity of this map follows from the previous result. It is easy to see that this map satisfies the transformer inequality. Since Φ_1 and Φ_2 strongly continuous, this binary operation satisfies the (separate or joint) continuity from above. The last statement follows from the fact that if $A_n \downarrow A$, then $\mathrm{Sp}\,(A_n) \subseteq [0,\ \|A_1\|\,]$ for all n and hence $f(A_n) \to f(A)$.

8. Applications to operator inequalities

In this section, we apply Kubo-Ando's theory in order to get simple proofs of many classical inequalities in the context of operators.

Theorem 0.50 (AM-LM-GM-HM inequalities) For $A,\ B\ \ 0$, we have

$$A\ !\ B\ \ A\ \#\ B\ \ A\,\lambda\, B\ \ A\ \triangledown\ B. \tag{id166}$$

It is easy to see that, for each $t > 0,\ t \neq 1$,

$$\frac{2t}{1+t}\ \ t^{1/2}\ \ \frac{t-1}{\log t}\ \ \frac{1+t}{2}. \tag{}$$

Now, we apply the order isomorphism which converts inequalities of operator monotone functions to inequalities of the associated operator connections.

Theorem 0.51 (Weighed AM-GM-HM inequalities) For $A,\ B\ \ 0$ and $\alpha \in [0,\ 1]$, we have

$$A\,!_{\alpha}\ \ B\ \ A\,\#_{\alpha}\ \ B\ \ A\,\triangledown_{\alpha}\ \ B. \tag{id168}$$

Apply the order isomorphism to the following inequalities:

$$\frac{t}{(1-\alpha)t+\alpha}\ \ t^{\alpha}\ \ 1-\alpha+\alpha t,\quad\ t\ \ 0. \tag{}$$

The next two theorems are given in [23].

Theorem 0.52 For each $i = 1,\ \cdots,\ n$, let $A_i,\ B_i \in B(\mathcal{H})^{+}$. Then for each connection σ

$$\sum_{i=1}^{n} \left(A_i \,\sigma\, B_i\right) \quad \sum_{i=1}^{n} A_i \,\sigma \sum_{i=1}^{n} B_i. \tag{id170}$$

Use the concavity of σ together with the induction.

By replacing σ with appropriate connections, we get some interesting inequalities.

(1) Cauchy-Schwarz's inequality: For $A_i,\ B_i \in B(\mathcal{H})^{sa}$,

$$\sum_{i=1}^{n} A_i^2 \ \#\ B_i^2 \quad \sum_{i=1}^{n} A_i^2 \ \#\ \sum_{i=1}^{n} B_i^2. \tag{id171}$$

(2) Hölder's inequality: For $A_i,\ B_i \in B(\mathcal{H})^+$ and $p,\ q > 0$ such that $1/p + 1/q = 1$,

$$\sum_{i=1}^{n} A_i^p \,\#_{1/p}\ B_i^q \quad \sum_{i=1}^{n} A_i^p \,\#_{1/p}\ \sum_{i=1}^{n} B_i^q. \tag{id172}$$

(3) Minkowski's inequality: For $A_i,\ B_i \in B(\mathcal{H})^{++}$,

$$\Big(\sum_{i=1}^{n} (A_i + B_i)^{-1}\Big)^{-1} \quad \Big(\sum_{i=1}^{n} A_i^{-1}\Big)^{-1} + \Big(\sum_{i=1}^{n} B_i^{-1}\Big)^{-1}. \tag{id173}$$

Theorem 0.53 Let $A_i,\ B_i \in B(\mathcal{H})^+$, $i = 1,\ \cdots,\ n$, be such that

$$A_1 - A_2 - \cdots - A_n \ \ 0 \quad \text{and} \quad B_1 - B_2 - \cdots - B_n \ \ 0. \tag{}$$

Then

$$A_1 \,\sigma\, B_1 - \sum_{i=2}^{n} A_i \,\sigma\, B_i \quad \Big(A_1 - \sum_{i=2}^{n} A_i\Big) \,\sigma\, \Big(B_1 - \sum_{i=2}^{n} B_i\Big). \tag{id175}$$

Substitute A_1 to $A_1 - A_2 - \cdots - A_n$ and B_1 to $B_1 - B_2 - \cdots - B_n$ in ($\square$).

Here are consequences.

(1) Aczél's inequality: For $A_i,\ B_i \in B(\mathcal{H})^{sa}$, if

$$A_1^2 - A_2^2 - \cdots - A_n^2 \ \ 0 \quad \text{and} \quad B_1^2 - B_2^2 - \cdots - B_n^2 \ \ 0, \tag{}$$

then

$$A_1^2 \ \# \ B_1^2 - \sum_{i=2}^{n} A_i^2 \ \# \ B_i^2 \quad \left(A_1^2 - \sum_{i=2}^{n} A_i^2\right) \ \# \ \left(B_1^2 - \sum_{i=2}^{n} B_i^2\right). \qquad \text{(id176)}$$

(2) Popoviciu's inequality: For A_i, $B_i \in B(\mathcal{H})^+$ and p, $q > 0$ such that $1/p + 1/q = 1$, if p, $q > 0$ are such that $1/p + 1/q = 1$ and

$$A_1^p - A_2^p - \cdots - A_n^p \quad 0 \qquad \text{and} \qquad B_1^q - B_2^q - \cdots - B_n^q \quad 0, \qquad ()$$

then

$$A_1^p \#_{1/p} \ B_1^q - \sum_{i=2}^{n} A_i^p \#_{1/p} \ B_i^q \quad \left(A_1^p - \sum_{i=2}^{n} A_i^p\right) \#_{1/p} \ \left(B_1^q - \sum_{i=2}^{n} B_i^q\right). \qquad \text{(id177)}$$

(3) Bellman's inequality: For A_i, $B_i \in B(\mathcal{H})^{++}$, if

$$A_1^{-1} - A_2^{-1} - \cdots - A_n^{-1} > 0 \qquad \text{and} \qquad B_1^{-1} - B_2^{-1} - \cdots - B_n^{-1} > 0, \qquad ()$$

then

$$\left[\left(A_1^{-1} + B_1^{-1}\right) - \sum_{i=2}^{n} (A_i + B_i)^{-1}\right]^{-1} \quad \left(A_1^{-1} - \sum_{i=2}^{n} A_i^{-1}\right)^{-1} + \left(B_1^{-1} - \sum_{i=2}^{n} B_i^{-1}\right)^{-1}. \qquad \text{(id178)}$$

The mean-theoretic approach can be used to prove the famous Furuta's inequality as follows. We cite [24] for the proof.

Theorem 0.54 (Furuta's inequality) For $A \quad B \quad 0$, we have

$$\begin{array}{ll} (B^r A^p B^r)^{1/q} & B^{(p+2r)/q} \\ A^{(p+2r)/q} & (A^r B^p A^r)^{1/q} \end{array} \qquad \text{(id180)}$$

where $r \quad 0$, $p \quad 0$, $q \quad 1$ and $(1+2r)q \quad p+2r$.

By the continuity argument, assume that A, $B > 0$. Note that (▭) and (▭) are equivalent. Indeed, if (▭) holds, then (▭) comes from applying (▭) to $A^{-1} \quad B^{-1}$ and taking inverse on both sides. To prove (▭), first consider the case $0 \quad p \quad 1$. We have $B^{p+2r} = B^r B^p B^r \quad B^r A^p B^r$ and the Löwner-Heinz's inequality (LH) implies the desired result. Now, consider the case $p \quad 1$ and $q = (p+2r)/(1+2r)$, since (▭) for $q > (p+2r)/(1+2r)$ can be obtained by (LH). Let $f(t) = t^{1/q}$ and let σ be the associated connection (in fact, $\sigma = \#_{1/q}$). Must show that, for any $r \quad 0$,

$$B^{-2r}\,\sigma\,A^{p}\quad B. \tag{id181}$$

For $0\quad r\quad \frac{1}{2}$, we have by (LH) that $A^{2r}\quad B^{2r}$ and

$$B^{-2r}\,\sigma\,A^{p}\quad A^{-2r}\,\sigma\,A^{p}=A^{-2r(1-1/q)}A^{p/q}=A\quad B=B^{-2r}\,\sigma\,B^{p}. \tag{}$$

Now, set $s=2r+\frac{1}{2}$ and $q_1=(p+2s)/(1+2s)\quad 1$. Let $f_1(t)=t^{1/q_1}$ and consider the associated connection σ_1. The previous step, the monotonicity and the congruence invariance of connections imply that

$$\begin{aligned} B^{-2s}\,\sigma_1\,A^{p} &= B^{-r}\left[B^{-(2r+1)}\,\sigma_1\left(B^{r}A^{p}B^{r}\right)\right]B^{-r} \\ & B^{-r}\left[\left(B^{r}A^{p}B^{r}\right)^{-1/q_1}\sigma_1\left(B^{r}A^{p}B^{r}\right)\right]B^{-r} \\ &= B^{-r}\left(B^{r}A^{p}B^{r}\right)^{1/q}B^{-r} \\ & B^{-r}B^{1+2r}B^{-r} \\ &= B. \end{aligned} \tag{}$$

Note that the above result holds for $A,\,B\quad 0$ via the continuity of a connection. The desired equation (▭) holds for all $r\quad 0$ by repeating this process.

Acknowledgement

The author thanks referees for article processing.

Author details

Pattrawut Chansangiam[1]

1 King Mongkut's Institute of Technology Ladkrabang,, Thailand

References

[1] Anderson, W. & Duffin, R. (1969). Series and parallel addition of matrices, *Journal of Mathematical Analysis and Applications*, Vol. 26, 576–594.

[2] Anderson, W. & Morley, T. & Trapp, G. (1990). Positive solutions to X=A-BX -1 B, *Linear Algebra and its Applications*, Vol. 134, 53–62.

[3] Ando, T. (1979). Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Linear Algebra and its Applications*, Vol. 26, 203–241.

[4] Ando, T. (1983). On the arithmetic-geometric-harmonic mean inequality for positive definite matrices, *Linear Algebra and its Applications*, Vol. 52-53, 31–37.

[5] Ando, T. & Li, C. & Mathias, R. (2004). Geometric Means, *Linear Algebra and its Applications*, Vol. 385, 305–334.

[6] Bhagwat, K. & Subramanian, A. (1978). Inequalities between means of positive operators, *Mathematical Proceedings of the Cambridge Philosophical Society*, Vol. 83, 393–401.

[7] Bhatia, R. (1997). *Matrix Analysis*, Springer-Verlag New York Inc., New York.

[8] Bhatia, R. (2006). Interpolating the arithmetic-geometric mean inequality and its operator version, *Linear Algebra and its Applications*, Vol. 413, 355–363.

[9] Bhatia, R. (2007). *Positive Definite Matrices*, Princeton University Press, New Jersey.

[10] Donoghue, W. (1974). *Monotone matrix functions and analytic continuation*, Springer-Verlag New York Inc., New York.

[11] Fujii, J. (1978). Arithmetico-geometric mean of operators, *Mathematica Japonica*, Vol. 23, 667–669.

[12] Fujii, J. (1979). On geometric and harmonic means of positive operators, *Mathematica Japonica*, Vol. 24, No. 2, 203–207.

[13] Fujii, J. (1992). Operator means and the relative operator entropy, *Operator Theory: Advances and Applications*, Vol. 59, 161–172.

[14] Furuta, T. (1989). A proof via operator means of an order preserving inequality, *Linear Algebra and its Applications*, Vol. 113, 129–130.

[15] Hiai, F. (2010). Matrix analysis: matrix monotone functions, matrix means, and majorizations, *Interdisciplinary Information Sciences*, Vol. 16, No. 2, 139–248,

[16] Hiai, F. & Yanagi, K. (1995). *Hilbert spaces and linear operators*, Makino Pub. Ltd.

[17] Kubo, F. & Ando, T. (1980). Means of positive linear operators, *Mathematische Annalen*, Vol. 246, 205–224.

[18] Lawson, J. & Lim, Y. (2001). The geometric mean, matrices, metrices and more, *The American Mathematical Monthly*, Vol. 108, 797–812.

[19] Lim, Y. (2008). On Ando–Li–Mathias geometric mean equations, *Linear Algebra and its Applications*, Vol. 428, 1767–1777.

[20] Löwner, C. (1934). Über monotone matrix funktionen. *Mathematische Zeitschrift*, Vol. 38, 177–216.

[21] Mond, B. & Pec ˇaric ´, J. & S ˇunde, J. & Varos ˇanec, S. (1997). Operator versions of some classical inequalities, *Linear Algebra and its Applications*, Vol. 264, 117–126.

[22] Nishio, K. & Ando, T. (1976). Characterizations of operations derived from network connections. *Journal of Mathematical Analysis and Applications*, Vol. 53, 539–549.

[23] Pusz, W. & Woronowicz, S. (1975). Functional calculus for sesquilinear forms and the purification map, *Reports on Mathematical Physics*, Vol. 8, 159–170.

[24] Toader, G. & Toader, S. (2005). Greek means and the arithmetic-geometric mean, RGMIA Monographs, Victoria University, (ONLINE: http://rgmia.vu.edu.au/monographs).

IntechOpen