Provisional chapter

3-Algebras in String Theory

Matsuo Sato

Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/46480

1. Introduction

In this chapter, we review 3-algebras that appear as fundamental properties of string theory. 3-algebra is a generalization of Lie algebra; it is defined by a tri-linear bracket instead of by a bi-linear bracket, and satisfies fundamental identity, which is a generalization of Jacobi identity [1], [2], [3]. We consider 3-algebras equipped with invariant metrics in order to apply them to physics.

It has been expected that there exists M-theory, which unifies string theories. In M-theory, some structures of 3-algebras were found recently. First, it was found that by using $u(N) \oplus u(N)$ Hermitian 3-algebra, we can describe a low energy effective action of N coincident supermembranes [4], [5], [6], [7], [8], which are fundamental objects in M-theory.

With this as motivation, 3-algebras with invariant metrics were classified [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22]. Lie 3-algebras are defined in real vector spaces and tri-linear brackets of them are totally anti-symmetric in all the three entries. Lie 3-algebras with invariant metrics are classified into $_4$ algebra, and Lorentzian Lie 3-algebras, which have metrics with indefinite signatures. On the other hand, Hermitian 3-algebras are defined in Hermitian vector spaces and their tri-linear brackets are complex linear and anti-symmetric in the first two entries, whereas complex anti-linear in the third entry. Hermitian 3-algebras with invariant metrics are classified into $u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ Hermitian 3-algebras.

Moreover, recent studies have indicated that there also exist structures of 3-algebras in the Green-Schwartz supermembrane action, which defines full perturbative dynamics of a supermembrane. It had not been clear whether the total supermembrane action including fermions has structures of 3-algebras, whereas the bosonic part of the action can be described by using a tri-linear bracket, called Nambu bracket [23], [24], which is a generalization of Poisson bracket. If we fix to a light-cone gauge, the total action can be described by using

Poisson bracket, that is, only structures of Lie algebra are left in this gauge [25]. However, it was shown under an approximation that the total action can be described by Nambu bracket if we fix to a semi-light-cone gauge [26]. In this gauge, the eleven dimensional space-time of M-theory is manifest in the supermembrane action, whereas only ten dimensional part is manifest in the light-cone gauge.

The BFSS matrix theory is conjectured to describe an infinite momentum frame (IMF) limit of M-theory [27] and many evidences were found. The action of the BFSS matrix theory can be obtained by replacing Poisson bracket with a finite dimensional Lie algebra's bracket in the supermembrane action in the light-cone gauge. Because of this structure, only variables that represent the ten dimensional part of the eleven-dimensional space-time are manifest in the BFSS matrix theory. Recently, 3-algebra models of M-theory were proposed [26], [28], [29], by replacing Nambu bracket with finite dimensional 3-algebras' brackets in an action that is shown, by using an approximation, to be equivalent to the semi-light-cone supermembrane action. All the variables that represent the eleven dimensional space-time are manifest in these models. It was shown that if the DLCQ limit of the 3-algebra models of M-theory is taken, they reduce to the BFSS matrix theory [26], [28], as they should [30], [31], [32], [33], [34], [35].

2. Definition and classification of metric Hermitian 3-algebra

In this section, we will define and classify the Hermitian 3-algebras equipped with invariant metrics.

2.1. General structure of metric Hermitian 3-algebra

The metric Hermitian 3-algebra is a map $V \times V \times V \to V$ defined by $(x, y, z) \mapsto [x, y; z]$, where the 3-bracket is complex linear in the first two entries, whereas complex anti-linear in the last entry, equipped with a metric $< x, y >$, satisfying the following properties:

the fundamental identity

$$[[x, y; z], v; w] = [[x, v; w], y; z] + [x, [y, v; w]; z] - [x, y; [z, w; v]] \qquad \text{(id2)}$$

the metric invariance

$$< [x, v; w], y > - < x, [y, w; v] > = 0 \qquad \text{(id3)}$$

and the anti-symmetry

$$[x, y; z] = -[y, x; z] \qquad \text{(id4)}$$

for

http://dx.doi.org/10.5772/46480

$$x,\ y,\ z,\ v,\ w \in V \qquad \text{(id5)}$$

The Hermitian 3-algebra generates a symmetry, whose generators $D(x,\ y)$ are defined by

$$D(x,\ y)z :\ = [z,\ x;\ y] \qquad \text{(id6)}$$

From (▭), one can show that $D(x,\ y)$ form a Lie algebra,

$$[D(x,\ y),\ D(v,\ w)] = D(D(x,\ y)v,\ w) - D(v,\ D(y,\ x)w) \qquad \text{(id7)}$$

There is an one-to-one correspondence between the metric Hermitian 3-algebra and a class of metric complex super Lie algebras [19]. Such a class satisfies the following conditions among complex super Lie algebras $S = S_0 \oplus S_1$, where S_0 and S_1 are even and odd parts, respectively. S_1 is decomposed as $S_1 = V \oplus \overline{V}$, where V is an unitary representation of S_0: for $a \in S_0$, $u,\ v \in V$,

$$[a,\ u] \in V \qquad \text{(id8)}$$

and

$$< [a,\ u],\ v > + < u,\ [a^*,\ v] > = 0 \qquad \text{(id9)}$$

$\bar{v} \in \overline{V}$ is defined by

$$\bar{v} = < \ \ ,\ v > \qquad \text{(id10)}$$

The super Lie bracket satisfies

$$[V,\ V] = 0, \quad [\overline{V},\ \overline{V}] = 0 \qquad \text{(id11)}$$

From the metric Hermitian 3-algebra, we obtain the class of the metric complex super Lie algebra in the following way. The elements in S_0, V, and $\overline{V}$ are defined by (▭), (▭), and (▭), respectively. The algebra is defined by (▭) and

$$\begin{aligned}
&[D(x,\ y),\ z] :\ = D(x,\ y)z = [z,\ x;\ y] \\
&[D(x,\ y),\ \bar{z}] :\ = -\overline{D(y,\ x)z} = -\overline{[z,\ y;\ x]} \\
&[x,\ \bar{y}] :\ = D(x,\ y) \\
&[x,\ y] :\ = 0 \\
&[\bar{x},\ \bar{y}] :\ = 0
\end{aligned} \qquad \text{(id12)}$$

One can show that this algebra satisfies the super Jacobi identity and (▭)-(▭) as in [19].

Inversely, from the class of the metric complex super Lie algebra, we obtain the metric Hermitian 3-algebra by

$$[x,\ y;z]:=\alpha[[y,\ \bar{z}],\ x] \tag{id13}$$

where α is an arbitrary constant. One can also show that this algebra satisfies (▭)-(▭) for (▭) as in [19].

2.2. Classification of metric Hermitian 3-algebra

The classical Lie super algebras satisfying (▭)-(▭) are $A(m-1,\ n-1)$ and $C(n+1)$. The even parts of $A(m-1,\ n-1)$ and $C(n+1)$ are $u(m)\oplus u(n)$ and $sp(2n)\oplus u(1)$, respectively. Because the metric Hermitian 3-algebra one-to-one corresponds to this class of the super Lie algebra, the metric Hermitian 3-algebras are classified into $u(m)\oplus u(n)$ and $sp(2n)\oplus u(1)$ Hermitian 3-algebras.

First, we will construct the $u(m)\oplus u(n)$ Hermitian 3-algebra from $A(m-1,\ n-1)$, according to the relation in the previous subsection. $A(m-1,\ n-1)$ is simple and is obtained by dividing $sl(m,\ n)$ by its ideal. That is, $A(m-1,\ n-1)=sl(m,\ n)$ when $m\neq n$ and $A(n-1,\ n-1)=sl(n,\ n)/\lambda 1_{2n}$.

Real $sl(m,\ n)$ is defined by

$$\begin{pmatrix} h_1 & c \\ ic^\dagger & h_2 \end{pmatrix} \tag{id15}$$

where h_1 and h_2 are $m\times m$ and $n\times n$ anti-Hermite matrices and c is an $n\times m$ arbitrary complex matrix. Complex $sl(m,\ n)$ is a complexification of real $sl(m,\ n)$, given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{id16}$$

where α, β, γ, and δ are $m\times m$, $n\times m$, $m\times n$, and $n\times n$ complex matrices that satisfy

$$\mathrm{tr}\alpha=\mathrm{tr}\delta \tag{id17}$$

Complex $A(m-1,\ n-1)$ is decomposed as $A(m-1,\ n-1)=S_0\oplus V\oplus\overline{V}$, where

http://dx.doi.org/10.5772/46480

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in S_0$$
$$\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \in V \qquad \text{(id18)}$$
$$\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \in \bar{V}$$

(▭) is rewritten as $V \to \bar{V}$ defined by

$$B = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \mapsto B^\dagger = \begin{pmatrix} 0 & 0 \\ \beta^\dagger & 0 \end{pmatrix} \qquad \text{(id19)}$$

where $B \in V$ and $B^\dagger \in \bar{V}$. (▭) is rewritten as

$$[X, Y; Z] = \alpha[[Y, Z^\dagger], X] = \alpha \begin{pmatrix} 0 & yz^\dagger x - xz^\dagger y \\ 0 & 0 \end{pmatrix} \qquad \text{(id20)}$$

for

$$X = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in V$$
$$Y = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in V \qquad \text{(id21)}$$
$$Z = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in V$$

As a result, we obtain the $u(m) \oplus u(n)$ Hermitian 3-algebra,

$$[x, y; z] = \alpha\left(yz^\dagger x - xz^\dagger y\right) \qquad \text{(id22)}$$

where x, y, and z are arbitrary $n \times m$ complex matrices. This algebra was originally constructed in [8].

Inversely, from (), we can construct complex $A(m-1, n-1)$. (▭) is rewritten as

$$D(x, y) = \left(xy^\dagger, y^\dagger x\right) \in S_0 \qquad \text{(id23)}$$

(▭) and (▭) are rewritten as

$$
\begin{aligned}
&[(xy^{\dagger}, y^{\dagger}x), (x'y'^{\dagger}, y'^{\dagger}x')] = ([xy^{\dagger}, x'y'^{\dagger}], [y^{\dagger}x, y'^{\dagger}x'])\\
&[(xy^{\dagger}, y^{\dagger}x), z] = xy^{\dagger}z - zy^{\dagger}x\\
&[(xy^{\dagger}, y^{\dagger}x), w^{\dagger}] = y^{\dagger}xw^{\dagger} - w^{\dagger}xy^{\dagger}\\
&[x, y^{\dagger}] = (xy^{\dagger}, y^{\dagger}x)\\
&[x, y] = 0\\
&[x^{\dagger}, y^{\dagger}] = 0
\end{aligned}
\tag{id24}
$$

This algebra is summarized as

$$
\left[\begin{pmatrix} xy^{\dagger} & z \\ w^{\dagger} & y^{\dagger}x \end{pmatrix}, \begin{pmatrix} x'y'^{\dagger} & z' \\ w'^{\dagger} & y'^{\dagger}x' \end{pmatrix}\right]
\tag{id25}
$$

which forms complex $A(m-1, n-1)$.

Next, we will construct the $sp(2n)\oplus u(1)$ Hermitian 3-algebra from $C(n+1)$. Complex $C(n+1)$ is decomposed as $C(n+1) = S_0 \oplus V \oplus \overline{V}$. The elements are given by

$$
\begin{aligned}
&\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & -a^T \end{pmatrix} \in S_0\\
&\begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \in V\\
&\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_1 & y_2 \\ y_2^T & 0 & 0 & 0 \\ -y_1^T & 0 & 0 & 0 \end{pmatrix} \in \overline{V}
\end{aligned}
\tag{id26}
$$

where α is a complex number, a is an arbitrary $n \times n$ complex matrix, b and c are $n \times n$ complex symmetric matrices, and x_1, x_2, y_1 and y_2 are $n \times 1$ complex matrices. (▭) is rewritten as $V \to \overline{V}$ defined by $B \mapsto \overline{B} = UB^{*}U^{-1}$, where $B \in V$, $\overline{B} \in \overline{V}$ and

http://dx.doi.org/10.5772/46480

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{(id27)}$$

Explicitly,

$$B = \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \mapsto \bar{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_2^* & -x_1^* \\ -x_1^\dagger & 0 & 0 & 0 \\ -x_2^\dagger & 0 & 0 & 0 \end{pmatrix} \quad \text{(id28)}$$

(▭) is rewritten as

$$\begin{aligned} [X, Y; Z] \; &:= \; \alpha[[Y, \bar{Z}], X] \\ &= \alpha\left[\left[\begin{pmatrix} 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & y_2^T & 0 & 0 \\ 0 & -y_1^T & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z_2^* & -z_1^* \\ -z_1^\dagger & 0 & 0 & 0 \\ -z_2^\dagger & 0 & 0 & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix}\right] \\ &= \alpha \begin{pmatrix} 0 & 0 & w_1 & w_2 \\ 0 & 0 & 0 & 0 \\ 0 & w_2^T & 0 & 0 \\ 0 & -w_1^T & 0 & 0 \end{pmatrix} \end{aligned} \quad \text{(id29)}$$

for

$$X = \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \in V$$
$$Y = \begin{pmatrix} 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & y_2^T & 0 & 0 \\ 0 & -y_1^T & 0 & 0 \end{pmatrix} \in V \qquad \text{(id30)}$$
$$Z = \begin{pmatrix} 0 & 0 & z_1 & z_2 \\ 0 & 0 & 0 & 0 \\ 0 & z_2^T & 0 & 0 \\ 0 & -z_1^T & 0 & 0 \end{pmatrix} \in V$$

where w_1 and w_2 are given by

$$(w_1, w_2) = -\left(y_1 z_1^\dagger + y_2 z_2^\dagger\right)(x_1, x_2) + \left(x_1 z_1^\dagger + x_2 z_2^\dagger\right)(y_1, y_2) + \left(x_2 y_1^T - x_1 y_2^T\right)\left(z_2^*, -z_1^*\right) \qquad \text{(id31)}$$

As a result, we obtain the $sp(2n) \oplus u(1)$ Hermitian 3-algebra,

$$[x, y; z] = \alpha((y \odot \tilde{z})x + (\tilde{z} \odot x)y - (x \odot y)\tilde{z}) \qquad \text{(id32)}$$

for $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$, where x_1, x_2, y_1, y_2, z_1, and z_2 are n-vectors and

$$\begin{aligned} \tilde{z} &= \left(z_2^*, -z_1^*\right) \\ a \odot b &= a_1 \cdot b_2 - a_2 \cdot b_1 \end{aligned} \qquad \text{(id33)}$$

3. 3-algebra model of M-theory

In this section, we review the fact that the supermembrane action in a semi-light-cone gauge can be described by Nambu bracket, where structures of 3-algebra are manifest. The 3-algebra Models of M-theory are defined based on the semi-light-cone supermembrane action. We also review that the models reduce to the BFSS matrix theory in the DLCQ limit.

3.1. Supermembrane and 3-algebra model of M-theory

The fundamental degrees of freedom in M-theory are supermembranes. The action of the covariant supermembrane action in M-theory [36] is given by

$$S_{M2} = \int d^3\sigma\Big(\sqrt{-G} + \frac{i}{4}{}^{\alpha\beta\gamma}\bar{\Psi}\Gamma_{MN}\partial_\alpha\Psi(\Pi_\beta{}^M\Pi_\gamma{}^N + \frac{i}{2}\Pi_\beta{}^M\bar{\Psi}\Gamma^N\partial_\gamma\Psi - \frac{1}{12}\bar{\Psi}\Gamma^M\partial_\beta\Psi\bar{\Psi}\Gamma^N\partial_\gamma\Psi)\Big) \tag{id35}$$

where $M, N = 0, \cdots, 10$, $\alpha, \beta, \gamma = 0, 1, 2$, $G_{\alpha\beta} = \Pi_\alpha{}^M\Pi_{\beta M}$ and $\Pi_\alpha{}^M = \partial_\alpha X^M - \frac{i}{2}\bar{\Psi}\Gamma^M\partial_\alpha\Psi$. Ψ is a $SO(1, 10)$ Majorana fermion.

This action is invariant under dynamical supertransformations,

$$\begin{aligned} \delta\Psi &= \\ \delta X^M &= -i\bar{\Psi}\Gamma^M \end{aligned} \tag{id36}$$

These transformations form the = 1 supersymmetry algebra in eleven dimensions,

$$[\delta_1, \delta_2]X^M = -2i_1\Gamma^M{}_2 \tag{id37}$$

$$[\delta_1, \delta_2]\Psi = 0 \tag{id38}$$

The action is also invariant under the κ-symmetry transformations,

$$\begin{aligned} \delta\Psi &= (1+\Gamma)\kappa(\sigma) \\ \delta X^M &= i\bar{\Psi}\Gamma^M(1+\Gamma)\kappa(\sigma) \end{aligned} \tag{id39}$$

where

$$\Gamma = \frac{1}{3!\sqrt{-G}}{}^{\alpha\beta\gamma}\Pi_\alpha^L\,\Pi_\beta^M\Pi_\gamma^N\Gamma_{LMN} \tag{id40}$$

If we fix the κ-symmetry (▭) of the action by taking a semi-light-cone gauge [26]Advantages of a semi-light-cone gauges against a light-cone gauge are shown in [37], [38], [39]

$$\Gamma^{012}\Psi = -\Psi \tag{id42}$$

we obtain a semi-light-cone supermembrane action,

$$S_{M2} = \int d^3\sigma\Big(\sqrt{-G} + \frac{i}{4}{}^{\alpha\beta\gamma}\Big(\bar{\Psi}\Gamma_{\mu\nu}\partial_\alpha\Psi\Big(\Pi_\beta{}^\mu\Pi_\gamma{}^\nu + \frac{i}{2}\Pi_\beta{}^\mu\bar{\Psi}\Gamma^\nu\partial_\gamma\Psi - \frac{1}{12}\bar{\Psi}\Gamma^\mu\partial_\beta\Psi\bar{\Psi}\Gamma^\nu\partial_\gamma\Psi\Big) + \bar{\Psi}\Gamma_{IJ}\partial_\alpha\Psi\partial_\beta X^I\partial_\gamma X^J\Big)\Big) \tag{id43}$$

where $G_{\alpha\beta} = h_{\alpha\beta} + \Pi_\alpha{}^{\mu}\Pi_{\beta\mu}$, $\Pi_\alpha{}^{\mu} = \partial_\alpha X^{\mu} - \frac{i}{2}\bar{\Psi}\Gamma^{\mu}\partial_\alpha\Psi$, and $h_{\alpha\beta} = \partial_\alpha X^{I}\partial_\beta X_I$.

In [26], it is shown under an approximation up to the quadratic order in $\partial_\alpha X^{\mu}$ and $\partial_\alpha\Psi$ but exactly in X^{I}, that this action is equivalent to the continuum action of the 3-algebra model of M-theory,

$$\begin{aligned} S_{cl} \; = \; & \int d^3\sigma\sqrt{-g}\Big(-\frac{1}{12}\{X^I, X^J, X^K\}^2 - \frac{1}{2}\big(A_{\mu ab}\{\varphi^a, \varphi^b, X^I\}\big)^2 \\ & -\frac{1}{3}E^{\mu\nu\lambda}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\{\varphi^a, \varphi^c, \varphi^d\}\{\varphi^b, \varphi^e, \varphi^f\} + \frac{1}{2}\Lambda \\ & -\frac{i}{2}\bar{\Psi}\Gamma^{\mu}A_{\mu ab}\{\varphi^a, \varphi^b, \Psi\} + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}\{X^I, X^J, \Psi\}\Big) \end{aligned} \qquad \text{(id44)}$$

where $I, J, K = 3, \cdots, 10$ and $\{\varphi^a, \varphi^b, \varphi^c\} = {}^{\alpha\beta\gamma}\partial_\alpha\varphi^a\partial_\beta\varphi^b\partial_\gamma\varphi^c$ is the Nambu-Poisson bracket. An invariant symmetric bilinear form is defined by $\int d^3\sigma\sqrt{-g}\varphi^a\varphi^b$ for complete basis φ^a in three dimensions. Thus, this action is manifestly VPD covariant even when the world-volume metric is flat. X^I is a scalar and Ψ is a $SO(1,2)\times SO(8)$ Majorana-Weyl fermion satisfying (▭). $E^{\mu\nu\lambda}$ is a Levi-Civita symbol in three dimensions and Λ is a cosmological constant.

The continuum action of 3-algebra model of M-theory (▭) is invariant under 16 dynamical supersymmetry transformations,

$$\begin{aligned} & \delta X^I = i\Gamma^I\Psi \\ & \delta A_\mu(\sigma, \sigma') = \frac{i}{2}\Gamma_\mu\Gamma_I\big(X^I(\sigma)\Psi(\sigma') - X^I(\sigma')\Psi(\sigma)\big), \\ & \delta\Psi = -A_{\mu ab}\{\varphi^a, \varphi^b, X^I\}\Gamma^{\mu}\Gamma_I - \frac{1}{6}\{X^I, X^J, X^K\}\Gamma_{IJK} \end{aligned} \qquad \text{(id45)}$$

where $\Gamma_{012} = -$. These supersymmetries close into gauge transformations on-shell,

$$\begin{aligned} & [\delta_1, \delta_2]X^I = \Lambda_{cd}\{\varphi^c, \varphi^d, X^I\} \\ & [\delta_1, \delta_2]A_{\mu ab}\{\varphi^a, \varphi^b, \quad\} = \Lambda_{ab}\{\varphi^a, \varphi^b, A_{\mu cd}\{\varphi^c, \varphi^d, \quad\}\} \\ & \qquad - A_{\mu ab}\{\varphi^a, \varphi^b, \Lambda_{cd}\{\varphi^c, \varphi^d, \quad\}\} + 2i_2\Gamma^{\nu}{}_1O^A_{\mu\nu} \\ & [\delta_1, \delta_2]\Psi = \Lambda_{cd}\{\varphi^c, \varphi^d, \Psi\} + \Big(i_2\Gamma^{\mu}{}_1\Gamma_\mu - \frac{i}{4}{}_2\Gamma^{KL}{}_1\Gamma_{KL}\Big)O^{\Psi} \end{aligned} \qquad \text{(id46)}$$

where gauge parameters are given by $\Lambda_{ab} = 2i_2\Gamma^{\mu}{}_1A_{\mu ab} - i_2\Gamma_{JK1}X^J_a X^K_b$. $O^A_{\mu\nu} = 0$ and $O^{\Psi} = 0$ are equations of motions of $A_{\mu\nu}$ and Ψ, respectively, where

$$
\begin{aligned}
O^{A}_{\mu\nu} &= A_{\mu ab}\{\varphi^a, \varphi^b, A_{\nu cd}\{\varphi^c, \varphi^d, \quad\}\} - A_{\nu ab}\{\varphi^a, \varphi^b, A_{\mu cd}\{\varphi^c, \varphi^d, \quad\}\} \\
&\quad + E_{\mu\nu\lambda}\left(- \{X^I, A^{\lambda}_{ab}\{\varphi^a, \varphi^b, X_I\}, \quad\} + \frac{i}{2}\{\bar{\Psi}, \Gamma^{\lambda}\Psi, \quad\}\right) \\
O^{\Psi} &= -\Gamma^{\mu} A_{\mu ab}\{\varphi^a, \varphi^b, \Psi\} + \frac{1}{2}\Gamma_{IJ}\{X^I, X^J, \Psi\}
\end{aligned}
\qquad \text{(id47)}
$$

(▭) implies that a commutation relation between the dynamical supersymmetry transformations is

$$\delta_2\delta_1 - \delta_1\delta_2 = 0 \qquad \text{(id48)}$$

up to the equations of motions and the gauge transformations.

This action is invariant under a translation,

$$\delta X^I(\sigma) = \eta^I, \qquad \delta A^{\mu}(\sigma, \sigma') = \eta^{\mu}(\sigma) - \eta^{\mu}(\sigma') \qquad \text{(id49)}$$

where η^I are constants.

The action is also invariant under 16 kinematical supersymmetry transformations

$$\delta\Psi = \check{} \qquad \text{(id50)}$$

and the other fields are not transformed. ˇ is a constant and satisfy $\Gamma_{012}\check{} = \check{}.\check{}$ and should come from sixteen components of thirty-two $=1$ supersymmetry parameters in eleven dimensions, corresponding to eigen values $\pm$ 1 of Γ_{012}, respectively. This $=1$ supersymmetry consists of remaining 16 target-space supersymmetries and transmuted 16 κ-symmetries in the semi-light-cone gauge [26], [25], [40].

A commutation relation between the kinematical supersymmetry transformations is given by

$$\delta_2\delta_1 - \delta_1\delta_2 = 0 \qquad \text{(id51)}$$

A commutator of dynamical supersymmetry transformations and kinematical ones acts as

$$
\begin{aligned}
(\delta_2\delta_1 - \delta_1\delta_2)X^I(\sigma) &= i_1\Gamma^{I}{}_2 \equiv \eta_0^I \\
(\delta_2\delta_1 - \delta_1\delta_2)A^{\mu}(\sigma, \sigma') &= \frac{i}{2}{}_1\Gamma^{\mu}\Gamma_I\left(X^I(\sigma) - X^I(\sigma')\right)_2 \equiv \eta_0^{\mu}(\sigma) - \eta_0^{\mu}(\sigma')
\end{aligned}
\qquad \text{(id52)}
$$

where the commutator that acts on the other fields vanishes. Thus, the commutation relation is given by

$$\delta_2\delta_1 - \delta_1\delta_2 = \delta_\eta \tag{id53}$$

where δ_η is a translation.

If we change a basis of the supersymmetry transformations as

$$\begin{aligned} \delta^{'} &= \delta + \delta \\ \delta^{'} &= i(\delta - \delta) \end{aligned} \tag{id54}$$

we obtain

$$\begin{aligned} \delta_2^{'}\delta_1^{'} - \delta_1^{'}\delta_2^{'} &= \delta_\eta \\ \delta_2^{'}\delta_1^{'} - \delta_1^{'}\delta_2^{'} &= \delta_\eta \\ \delta_2^{'}\delta_1^{'} - \delta_1^{'}\delta_2^{'} &= 0 \end{aligned} \tag{id55}$$

These thirty-two supersymmetry transformations are summarised as $\Delta = \left(\delta^{'}, \delta^{'}\right)$ and (▭) implies the $= 1$ supersymmetry algebra in eleven dimensions,

$$\Delta_2\Delta_1 - \Delta_1\Delta_2 = \delta_\eta \tag{id56}$$

3.2. Lie 3-algebra models of M-theory

In this and next subsection, we perform the second quantization on the continuum action of the 3-algebra model of M-theory: By replacing the Nambu-Poisson bracket in the action (▭) with brackets of finite-dimensional 3-algebras, Lie and Hermitian 3-algebras, we obtain the Lie and Hermitian 3-algebra models of M-theory [26], [28], respectively. In this section, we review the Lie 3-algebra model.

If we replace the Nambu-Poisson bracket in the action (▭) with a completely antisymmetric real 3-algebra's bracket [21], [22],

$$\begin{aligned} &\int d^3\sigma\sqrt{-g} \rightarrow \langle \quad \rangle \\ &\left\{\varphi^a, \varphi^b, \varphi^c\right\} \rightarrow \left[T^a, T^b, T^c\right] \end{aligned} \tag{id58}$$

we obtain the Lie 3-algebra model of M-theory [26], [28],

http://dx.doi.org/10.5772/46480

$$
\begin{aligned}
S_0 \quad = \quad & < -\frac{1}{12}[X^I, X^J, X^K]^2 - \frac{1}{2}\left(A_{\mu ab}[T^a, T^b, X^I]\right)^2 \\
& -\frac{1}{3}E^{\mu\nu\lambda}A_{\mu ab}A_{\nu cd}A_{\lambda ef}[T^a, T^c, T^d][T^b, T^e, T^f] \\
& -\frac{i}{2}\bar{\Psi}\Gamma^\mu A_{\mu ab}[T^a, T^b, \Psi] + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}[X^I, X^J, \Psi] >
\end{aligned}
\qquad \text{(id59)}
$$

We have deleted the cosmological constant Λ, which corresponds to an operator ordering ambiguity, as usual as in the case of other matrix models [27], [41].

This model can be obtained formally by a dimensional reduction of the = 8 BLG model [4], [5], [6],

$$
\begin{aligned}
S_{=8BLG} \quad = \quad & \int d^3x < -\frac{1}{12}[X^I, X^J, X^K]^2 - \frac{1}{2}\left(D_\mu X^I\right)^2 - E^{\mu\nu\lambda}\Big(\frac{1}{2}A_{\mu ab}\partial_\nu A_{\lambda cd}T^a[T^b, T^c, T^d] \\
& + \frac{1}{3}A_{\mu ab}A_{\nu cd}A_{\lambda ef}[T^a, T^c, T^d][T^b, T^e, T^f]\Big) \\
& + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}[X^I, X^J, \Psi] >
\end{aligned}
\qquad \text{(id60)}
$$

The formal relations between the Lie (Hermitian) 3-algebra models of M-theory and the = 8 (= 6) BLG models are analogous to the relation among the = 4 super Yang-Mills in four dimensions, the BFSS matrix theory [27], and the IIB matrix model [41]. They are completely different theories although they are related to each others by dimensional reductions. In the same way, the 3-algebra models of M-theory and the BLG models are completely different theories.

The fields in the action (▭) are spanned by the Lie 3-algebra T^a as $X^I = X_a^I T^a$, $\Psi = \Psi_a T^a$ and $A^\mu = A_{ab}^\mu T^a \otimes T^b$, where $I = 3, \cdots, 10$ and $\mu = 0, 1, 2$. $< \ >$ represents a metric for the 3-algebra. Ψ is a Majorana spinor of SO(1,10) that satisfies $\Gamma_{012}\Psi = \Psi$. $E^{\mu\nu\lambda}$ is a Levi-Civita symbol in three-dimensions.

Finite dimensional Lie 3-algebras with an invariant metric is classified into four-dimensional Euclidean $_4$ algebra and the Lie 3-algebras with indefinite metrics in [9], [10], [11], [21], [22]. We do not choose $_4$ algebra because its degrees of freedom are just four. We need an algebra with arbitrary dimensions N, which is taken to infinity to define M-theory. Here we choose the most simple indefinite metric Lie 3-algebra, so called the Lorentzian Lie 3-algebra associated with $u(N)$ Lie algebra,

$$
\begin{aligned}
& [T^{-1}, T^a, T^b] = 0 \\
& [T^0, T^i, T^j] = [T^i, T^j] = f^{ij}{}_k T^k \\
& [T^i, T^j, T^k] = f^{ijk} T^{-1}
\end{aligned}
\qquad \text{(id61)}
$$

where $a = -1, 0, i$ ($i = 1, \cdots, N^2$). T^i are generators of $u(N)$. A metric is defined by a symmetric bilinear form,

$$\begin{aligned} < T^{-1}, T^0 > &= -1 \\ < T^i, T^j > &= h^{ij} \end{aligned} \tag{id62}$$

and the other components are 0. The action is decomposed as

$$\begin{aligned} S = \mathrm{Tr}\Big(&-\frac{1}{4}(x_0^K)^2[x^I, x^J]^2 + \frac{1}{2}(x_0^I[x_I, x^J])^2 - \frac{1}{2}(x_0^I b_\mu + [a_\mu, x^I])^2 - \frac{1}{2}E^{\mu\nu\lambda}b_\mu[a_\nu, a_\lambda] \\ &+ i\bar{\Psi}_0\Gamma^\mu b_\mu\psi - \frac{i}{2}\bar{\Psi}\Gamma^\mu[a_\mu, \psi] + \frac{i}{2}x_0^I\bar{\Psi}\Gamma_{IJ}[x^J, \psi] - \frac{i}{2}\bar{\Psi}_0\Gamma_{IJ}[x^I, x^J]\psi\Big) \end{aligned} \tag{id63}$$

where we have renamed $X_0^I \to x_0^I$, $X_i^I T^i \to x^I$, $\Psi_0 \to \psi_0$, $\Psi_i T^i \to \psi$, $2A_{\mu 0i}T^i \to a_\mu$, and $A_{\mu ij}[T^i, T^j] \to b_\mu$. a_μ correspond to the target coordinate matrices X^μ, whereas b_μ are auxiliary fields.

In this action, T^{-1} mode; X_{-1}^I, Ψ_{-1} or A_{-1a}^μ does not appear, that is they are unphysical modes. Therefore, the indefinite part of the metric (▭) does not exist in the action and the Lie 3-algebra model of M-theory is ghost-free like a model in [42]. This action can be obtained by a dimensional reduction of the three-dimensional = 8 BLG model [4], [5], [6] with the same 3-algebra. The BLG model possesses a ghost mode because of its kinetic terms with indefinite signature. On the other hand, the Lie 3-algebra model of M-theory does not possess a kinetic term because it is defined as a zero-dimensional field theory like the IIB matrix model [41].

This action is invariant under the translation

$$\delta x^I = \eta^I, \qquad \delta a^\mu = \eta^\mu \tag{id64}$$

where η^I and η^μ belong to $u(1)$. This implies that eigen values of x^I and a^μ represent an eleven-dimensional space-time.

The action is also invariant under 16 kinematical supersymmetry transformations

$$\delta\psi = \tag{id65}$$

and the other fields are not transformed. belong to $u(1)$ and satisfy $\Gamma_{012} = .$ and should come from sixteen components of thirty-two = 1 supersymmetry parameters in eleven dimensions, corresponding to eigen values ± 1 of Γ_{012}, respectively, as in the previous subsection.

A commutation relation between the kinematical supersymmetry transformations is given by

$$\delta_2\delta_1 - \delta_1\delta_2 = 0 \tag{id66}$$

The action is invariant under 16 dynamical supersymmetry transformations,

$$\begin{aligned}
&\delta X^I = i\Gamma^I\Psi \\
&\delta A_{\mu ab}[T^a, T^b, \quad] = i\Gamma_\mu\Gamma_I[X^I, \Psi, \quad] \\
&\delta\Psi = -A_{\mu ab}[T^a, T^b, X^I]\Gamma^\mu\Gamma_I - \frac{1}{6}[X^I, X^J, X^K]\Gamma_{IJK}
\end{aligned} \tag{id67}$$

where $\Gamma_{012} = -$. These supersymmetries close into gauge transformations on-shell,

$$\begin{aligned}
&[\delta_1, \delta_2]X^I = \Lambda_{cd}[T^c, T^d, X^I] \\
&[\delta_1, \delta_2]A_{\mu ab}[T^a, T^b, \quad] = \Lambda_{ab}[T^a, T^b, A_{\mu cd}[T^c, T^d, \quad]] \\
&\qquad - A_{\mu ab}[T^a, T^b, \Lambda_{cd}[T^c, T^d, \quad]] + 2i_2\Gamma^\nu{}_1 O^A_{\mu\nu} \\
&[\delta_1, \delta_2]\Psi = \Lambda_{cd}[T^c, T^d, \Psi] + \left(i_2\Gamma^\mu{}_1\Gamma_\mu - \frac{i}{4}{}_2\Gamma^{KL}{}_1\Gamma_{KL}\right)O^\Psi
\end{aligned} \tag{id68}$$

where gauge parameters are given by $\Lambda_{ab} = 2i_2\Gamma^\mu{}_1 A_{\mu ab} - i_2\Gamma_{JK1}X^J_a X^K_b$. $O^A_{\mu\nu} = 0$ and $O^\Psi = 0$ are equations of motions of $A_{\mu\nu}$ and Ψ, respectively, where

$$\begin{aligned}
O^A_{\mu\nu} &= A_{\mu ab}[T^a, T^b, A_{\nu cd}[T^c, T^d, \quad]] - A_{\nu ab}[T^a, T^b, A_{\mu cd}[T^c, T^d, \quad]] \\
&\quad + E_{\mu\nu\lambda}\left(-[X^I, A^\lambda_{ab}[T^a, T^b, X_I], \quad] + \frac{i}{2}[\bar{\Psi}, \Gamma^\lambda\Psi, \quad]\right) \\
O^\Psi &= -\Gamma^\mu A_{\mu ab}[T^a, T^b, \Psi] + \frac{1}{2}\Gamma_{IJ}[X^I, X^J, \Psi]
\end{aligned} \tag{id69}$$

(▭) implies that a commutation relation between the dynamical supersymmetry transformations is

$$\delta_2\delta_1 - \delta_1\delta_2 = 0 \tag{id70}$$

up to the equations of motions and the gauge transformations.

The 16 dynamical supersymmetry transformations (▭) are decomposed as

$$
\begin{aligned}
&\delta x^{I} = i\Gamma^{I}\psi \\
&\delta x_0^{I} = i\Gamma^{I}\psi_0 \\
&\delta x_{-1}^{I} = i\Gamma^{I}\psi_{-1} \\
&\delta\psi = -\left(b_\mu x_0^{I} + [a_\mu, x^{I}]\right)\Gamma^{\mu}\Gamma_I - \frac{1}{2}x_0^{I}[x^{J}, x^{K}]\Gamma_{IJK} \\
&\delta\psi_0 = 0 \\
&\delta\psi_{-1} = -\mathrm{Tr}\left(b_\mu x^{I}\right)\Gamma^{\mu}\Gamma_I - \frac{1}{6}\mathrm{Tr}\left([x^{I}, x^{J}]x^{K}\right)\Gamma_{IJK} \\
&\delta a_\mu = i\Gamma_\mu\Gamma_I\left(x_0^{I}\psi - \psi_0 x^{I}\right) \\
&\delta b_\mu = i\Gamma_\mu\Gamma_I[x^{I}, \psi] \\
&\delta A_{\mu\text{-}1i} = i\Gamma_\mu\Gamma_I\frac{1}{2}\left(x_{-1}^{I}\psi_i - \psi_{-1}x_i^{I}\right) \\
&\delta A_{\mu\text{-}10} = i\Gamma_\mu\Gamma_I\frac{1}{2}\left(x_{-1}^{I}\psi_0 - \psi_{-1}x_0^{I}\right)
\end{aligned}
\tag{id71}
$$

and thus a commutator of dynamical supersymmetry transformations and kinematical ones acts as

$$
\begin{aligned}
&(\delta_2\delta_1 - \delta_1\delta_2)x^{I} = \bar{i}_1\Gamma^{I}{}_2 \equiv \eta^{I} \\
&(\delta_2\delta_1 - \delta_1\delta_2)a^{\mu} = \bar{i}_1\Gamma^{\mu}\Gamma_I x_0^{I}{}_2 \equiv \eta^{\mu} \\
&(\delta_2\delta_1 - \delta_1\delta_2)A_{-1i}^{\mu}T^{i} = \frac{1}{2}\bar{i}_1\Gamma^{\mu}\Gamma_I x_{-1}^{I}{}_2
\end{aligned}
\tag{id72}
$$

where the commutator that acts on the other fields vanishes. Thus, the commutation relation for physical modes is given by

$$
\delta_2\delta_1 - \delta_1\delta_2 = \delta_\eta \tag{id73}
$$

where δ_η is a translation.

(▭), (▭), and (▭) imply the = 1 supersymmetry algebra in eleven dimensions as in the previous subsection.

3.3. Hermitian 3-algebra model of M-theory

In this subsection, we study the Hermitian 3-algebra models of M-theory [26]. Especially, we study mostly the model with the $u(N) \oplus u(N)$ Hermitian 3-algebra (▭).

The continuum action (▭) can be rewritten by using the triality of $SO(8)$ and the $SU(4) \times U(1)$ decomposition [8], [43], [44] as

$$
\begin{aligned}
S_{cl} = & \int d^3\sigma\sqrt{-g}\Big(-V - A_{\mu ba}\{Z^A, T^a, T^b\}A^{\mu}_{dc}\{Z_A, T^c, T^d\} \\
& + \frac{1}{3}E^{\mu\nu\lambda}A_{\mu ba}A_{\nu dc}A_{\lambda fe}\{T^a, T^c, T^d\}\{T^b, T^f, T^e\} \\
& + i\bar{\psi}^A\Gamma^{\mu}A_{\mu ba}\{\psi_A, T^a, T^b\} + \frac{i}{2}E_{ABCD}\bar{\psi}^A\{Z^C, Z^D, \psi^B\} - \frac{i}{2}E^{ABCD}Z_D\{\bar{\psi}_A, \psi_B, Z_C\} \\
& - i\bar{\psi}^A\{\psi_A, Z^B, Z_B\} + 2i\bar{\psi}^A\{\psi_B, Z^B, Z_A\}\Big)
\end{aligned}
\qquad \text{(id75)}
$$

where fields with a raised A index transform in the 4 of SU(4), whereas those with lowered one transform in the $\bar{4}$. $A_{\mu ba}$ ($\mu = 0, 1, 2$) is an anti-Hermitian gauge field, Z^A and Z_A are a complex scalar field and its complex conjugate, respectively. ψ_A is a fermion field that satisfies

$$
\Gamma^{012}\psi_A = -\psi_A \qquad \text{(id76)}
$$

and ψ^A is its complex conjugate. $E^{\mu\nu\lambda}$ and E^{ABCD} are Levi-Civita symbols in three dimensions and four dimensions, respectively. The potential terms are given by

$$
\begin{aligned}
V &= \frac{2}{3}\Upsilon^{CD}_B\Upsilon^B_{CD} \\
\Upsilon^{CD}_B &= \{Z^C, Z^D, Z_B\} - \frac{1}{2}\delta^C_B\{Z^E, Z^D, Z_E\} + \frac{1}{2}\delta^D_B\{Z^E, Z^C, Z_E\}
\end{aligned}
\qquad \text{(id77)}
$$

If we replace the Nambu-Poisson bracket with a Hermitian 3-algebra's bracket [19], [20],

$$
\begin{aligned}
&\int d^3\sigma\sqrt{-g} \to \langle \quad \rangle \\
&\{\varphi^a, \varphi^b, \varphi^c\} \to [T^a, T^b; \bar{T}^{\bar{c}}]
\end{aligned}
\qquad \text{(id78)}
$$

we obtain the Hermitian 3-algebra model of M-theory [26],

$$
\begin{aligned}
S = & < -V - A_{\mu\bar{b}a}[Z^A, T^a; \bar{T}^{\bar{b}}]\overline{A^{\mu}_{\bar{d}c}[Z_A, T^c; \bar{T}^{\bar{d}}]} + \frac{1}{3}E^{\mu\nu\lambda}A_{\mu\bar{b}a}A_{\nu\bar{d}c}A_{\lambda\bar{f}e}[T^a, T^c; \bar{T}^{\bar{d}}]\overline{[T^b, T^f; \bar{T}^{\bar{e}}]} \\
& + i\bar{\psi}^A\Gamma^{\mu}A_{\mu\bar{b}a}[\psi_A, T^a; \bar{T}^{\bar{b}}] + \frac{i}{2}E_{ABCD}\bar{\psi}^A[Z^C, Z^D; \bar{\psi}^B] - \frac{i}{2}E^{ABCD}\bar{Z}_D[\bar{\psi}_A, \psi_B; \bar{Z}_C] \\
& - i\bar{\psi}^A[\psi_A, Z^B; \bar{Z}_B] + 2i\bar{\psi}^A[\psi_B, Z^B; \bar{Z}_A] >
\end{aligned}
\qquad \text{(id79)}
$$

where the cosmological constant has been deleted for the same reason as before. The potential terms are given by

$$
\begin{aligned}
V &= \frac{2}{3}\Upsilon_B^{CD}\bar{\Upsilon}_{CD}^B \\
\Upsilon_B^{CD} &= [Z^C, Z^D; \bar{Z}_B] - \frac{1}{2}\delta_B^C[Z^E, Z^D; \bar{Z}_E] + \frac{1}{2}\delta_B^D[Z^E, Z^C; \bar{Z}_E]
\end{aligned}
\tag{id80}
$$

This matrix model can be obtained formally by a dimensional reduction of the $=6$ BLG action [8], which is equivalent to ABJ(M) action [7], [45]The authors of [46], [47], [48], [49] studied matrix models that can be obtained by a dimensional reduction of the ABJM and ABJ gauge theories on S^3. They showed that the models reproduce the original gauge theories on S^3 in planar limits.,

$$
\begin{aligned}
S_{=6BLG} = &\int d^3x < -V - D_\mu Z^A \overline{D^\mu Z_A} + E^{\mu\nu\lambda}\Big(\frac{1}{2}A_{\mu\bar{c}b}\partial_\nu A_{\lambda\bar{d}a}\bar{T}^{\bar{d}}[T^a, T^b; \bar{T}^{\bar{c}}] \\
&+ \frac{1}{3}A_{\mu\bar{b}a}A_{\nu\bar{d}c}A_{\lambda\bar{f}e}[T^a, T^c; \bar{T}^{\bar{d}}]\overline{[T^b, T^f; \bar{T}^{\bar{e}}]}\Big) \\
&- i\bar{\psi}^A\Gamma^\mu D_\mu\psi_A + \frac{i}{2}E_{ABCD}\bar{\psi}^A[Z^C, Z^D; \psi^B] - \frac{i}{2}E^{ABCD}\bar{Z}_D[\bar{\psi}_A, \psi_B; \bar{Z}_C] \\
&- i\bar{\psi}^A[\psi_A, Z^B; \bar{Z}_B] + 2i\bar{\psi}^A[\psi_B, Z^B; \bar{Z}_A] >
\end{aligned}
\tag{id82}
$$

The Hermitian 3-algebra models of M-theory are classified into the models with $u(m)\oplus u(n)$ Hermitian 3-algebra (▭) and $sp(2n)\oplus u(1)$ Hermitian 3-algebra (▭). In the following, we study the $u(N)\oplus u(N)$ Hermitian 3-algebra model. By substituting the $u(N)\oplus u(N)$ Hermitian 3-algebra (▭) to the action (▭), we obtain

$$
\begin{aligned}
S = &\mathrm{Tr}\Big(-\frac{(2\pi)^2}{k^2}V - (Z^A A_\mu^R - A_\mu^L Z^A)(Z^A A^{R\mu} - A^{L\mu}Z^A)^\dagger - \frac{k}{2\pi}\frac{i}{3}E^{\mu\nu\lambda}(A_\mu^R A_\nu^R A_\lambda^R - A_\mu^L A_\nu^L A_\lambda^L) \\
&- \bar{\psi}^A\Gamma^\mu(\psi_A A_\mu^R - A_\mu^L\psi_A) + \frac{2\pi}{k}(iE_{ABCD}\bar{\psi}^A Z^C\psi^{\dagger B}Z^D - iE^{ABCD}Z_D^\dagger\overline{\psi^\dagger}_A Z_C^\dagger\psi_B \\
&- i\bar{\psi}^A\psi_A Z_B^\dagger Z^B + i\bar{\psi}^A Z^B Z_B^\dagger\psi_A + 2i\bar{\psi}^A\psi_B Z_A^\dagger Z^B - 2i\bar{\psi}^A Z^B Z_A^\dagger\psi_B))
\end{aligned}
\tag{id83}
$$

where $A_\mu^R \equiv -\frac{k}{2\pi}iA_{\mu\bar{b}a}T^{\dagger\bar{b}}T^a$ and $A_\mu^L \equiv -\frac{k}{2\pi}iA_{\mu\bar{b}a}T^aT^{\dagger\bar{b}}$ are $N\times N$ Hermitian matrices. In the algebra, we have set $\alpha = \frac{2\pi}{k}$, where k is an integer representing the Chern-Simons level. We choose $k=1$ in order to obtain 16 dynamical supersymmetries. V is given by

$$
\begin{aligned}
V = &+\frac{1}{3}Z_A^\dagger Z^A Z_B^\dagger Z^B Z_C^\dagger Z^C + \frac{1}{3}Z^A Z_A^\dagger Z^B Z_B^\dagger Z^C Z_C^\dagger + \frac{4}{3}Z_A^\dagger Z^B Z_C^\dagger Z^A Z_B^\dagger Z^C \\
&- Z_A^\dagger Z^A Z_B^\dagger Z^C Z_C^\dagger Z^B - Z^A Z_A^\dagger Z^B Z_C^\dagger Z^C Z_B^\dagger
\end{aligned}
\tag{id84}
$$

By redefining fields as

$$\begin{aligned} Z^A &\to \left(\frac{k}{2\pi}\right)^{\frac{1}{3}} Z^A \\ A^\mu &\to \left(\frac{2\pi}{k}\right)^{\frac{1}{3}} A^\mu \\ \psi^A &\to \left(\frac{k}{2\pi}\right)^{\frac{1}{6}} \psi^A \end{aligned} \qquad \text{(id85)}$$

we obtain an action that is independent of Chern-Simons level:

$$\begin{aligned} S = & \mathrm{Tr}\Big(-V - (Z^A A^R_\mu - A^L_\mu Z^A)(Z^A A^{R\mu} - A^{L\mu} Z^A)^\dagger - \frac{i}{3} E^{\mu\nu\lambda}(A^R_\mu A^R_\nu A^R_\lambda - A^L_\mu A^L_\nu A^L_\lambda) \\ & -\bar{\psi}^A \Gamma^\mu (\psi_A A^R_\mu - A^L_\mu \psi_A) + iE_{ABCD}\bar{\psi}^A Z^C \psi^{\dagger B} Z^D - iE^{ABCD} Z^\dagger_D \overline{\psi^\dagger}_A Z^\dagger_C \psi_B \\ & -i\bar{\psi}^A \psi_A Z^\dagger_B Z^B + i\bar{\psi}^A Z^B Z^\dagger_B \psi_A + 2i\bar{\psi}^A \psi_B Z^\dagger_A Z^B - 2i\bar{\psi}^A Z^B Z^\dagger_A \psi_B \Big) \end{aligned} \qquad \text{(id86)}$$

as opposed to three-dimensional Chern-Simons actions.

If we rewrite the gauge fields in the action as $A^L_\mu = A_\mu + b_\mu$ and $A^R_\mu = A_\mu - b_\mu$, we obtain

$$\begin{aligned} S = & \mathrm{Tr}\Big(-V + ([A_\mu, Z^A] + \{b_\mu, Z^A\})([A^\mu, Z_A] - \{b^\mu, Z_A\}) + iE^{\mu\nu\lambda}\Big(\frac{2}{3} b_\mu b_\nu b_\lambda + 2A_\mu A_\nu b_\lambda\Big) \\ & +\bar{\psi}^A \Gamma^\mu ([A_\mu, \psi_A] + \{b_\mu, \psi_A\}) + iE_{ABCD}\bar{\psi}^A Z^C \psi^{\dagger B} Z^D - iE^{ABCD} Z^\dagger_D \overline{\psi^\dagger}_A Z^\dagger_C \psi_B \\ & -i\bar{\psi}^A \psi_A Z^\dagger_B Z^B + i\bar{\psi}^A Z^B Z^\dagger_B \psi_A + 2i\bar{\psi}^A \psi_B Z^\dagger_A Z^B - 2i\bar{\psi}^A Z^B Z^\dagger_A \psi_B \Big) \end{aligned} \qquad \text{(id87)}$$

where $[\ ,\ ]$ and $\{\ ,\ \}$ are the ordinary commutator and anticommutator, respectively. The $u(1)$ parts of A^μ decouple because A^μ appear only in commutators in the action. b^μ can be regarded as auxiliary fields, and thus A^μ correspond to matrices X^μ that represents three space-time coordinates in M-theory. Among $N \times N$ arbitrary complex matrices Z^A, we need to identify matrices X^I ($I = 3, \cdots 10$) representing the other space coordinates in M-theory, because the model possesses not $SO(8)$ but $SU(4) \times U(1)$ symmetry. Our identification is

$$\begin{aligned} Z^A &= iX^{A+2} - X^{A+6}, \\ X^I &= \hat{X}^I - ix^I 1 \end{aligned} \qquad \text{(id88)}$$

where $\hat{X}^I$ and x^I are $su(N)$ Hermitian matrices and real scalars, respectively. This is analogous to the identification when we compactify ABJM action, which describes N M2 branes, and obtain the action of N D2 branes [50], [7], [51]. We will see that this identification works also in our case. We should note that while the $su(N)$ part is Hermitian, the $u(1)$ part is anti-Hermitian. That is, an eigen-value distribution of X^μ, Z^A, and not X^I determine the space-time in the Hermitian model. In order to define light-cone coordinates, we need to perform Wick rotation: $a^0 \rightarrow -ia^0$. After the Wick rotation, we obtain

$$A^0 = \hat{A}^0 - ia^0 1 \tag{id89}$$

where $\hat{A}^0$ is a $su(N)$ Hermitian matrix.

3.4. DLCQ Limit of 3-algebra model of M-theory

It was shown that M-theory in a DLCQ limit reduces to the BFSS matrix theory with matrices of finite size [30], [31], [32], [33], [34], [35]. This fact is a strong criterion for a model of M-theory. In [26], [28], it was shown that the Lie and Hermitian 3-algebra models of M-theory reduce to the BFSS matrix theory with matrices of finite size in the DLCQ limit. In this subsection, we show an outline of the mechanism.

DLCQ limit of M-theory consists of a light-cone compactification, $x^- \approx x^- + 2\pi R$, where $x^\pm = \frac{1}{\sqrt{2}}(x^{10} \pm x^0)$, and Lorentz boost in x^{10} direction with an infinite momentum. After appropriate scalings of fields [26], [28], we define light-cone coordinate matrices as

$$\begin{aligned} X^0 &= \frac{1}{\sqrt{2}}(X^+ - X^-) \\ X^{10} &= \frac{1}{\sqrt{2}}(X^+ + X^-) \end{aligned} \tag{id91}$$

We integrate out b^μ by using their equations of motion.

A matrix compactification [52] on a circle with a radius R imposes the following conditions on X^- and the other matrices Y:

$$\begin{aligned} &X^- - (2\pi R)1 = U^\dagger X^- U \\ &Y = U^\dagger Y U \end{aligned} \tag{id92}$$

where U is a unitary matrix. In order to obtain a solution to (▭), we need to take $N \rightarrow \infty$ and consider matrices of infinite size [52]. A solution to (▭) is given by $X^- = \bar{X}^- + \tilde{X}^-$, $Y = \tilde{Y}$ and

$$U = \begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & 0 & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & 0 & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \otimes 1_{n\times n} \in U(N) \tag{id93}$$

Backgrounds $\overline{X}^-$ are

$$\overline{X}^- = -T^3\bar{x}_0^- T^0 - (2\pi R)\mathrm{diag}(\cdots, s-1, s, s+1, \cdots) \otimes 1_{n\times n} \tag{id94}$$

in the Lie 3-algebra case, whereas

$$\overline{X}^- = -i\left(T^3\bar{x}^-\right)1 - i(2\pi R)\mathrm{diag}(\cdots, s-1, s, s+1, \cdots) \otimes 1_{n\times n} \tag{id95}$$

in the Hermitian 3-algebra case. A fluctuation $\tilde{x}$ that represents $u(N)$ parts of $\widetilde{X}^-$ and $\widetilde{Y}$ is

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ \ddots & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & & & \ddots \\ \ddots & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & & \\ & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & \\ & & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) \\ & & & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \ddots \\ & \ddots & & & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \ddots \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{id96}$$

Each $\tilde{x}(s)$ is a $n \times n$ matrix, where s is an integer. That is, the (s, t)-th block is given by $\tilde{x}_{s,t} = \tilde{x}(s-t)$.

We make a Fourier transformation,

$$\tilde{x}(s) = \frac{1}{2\pi\widetilde{R}}\int_0^{2\pi\widetilde{R}} d\tau x(\tau) e^{is\frac{\tau}{\widetilde{R}}} \tag{id97}$$

where $x(\tau)$ is a $n \times n$ matrix in one-dimension and $R\widetilde{R} = 2\pi$. From (▭)-(▭), the following identities hold:

$$\begin{aligned} \sum_t \tilde{x}_{s,t}\widetilde{x'}_{t,u} &= \frac{1}{2\pi\widetilde{R}}\int_0^{2\pi\widetilde{R}} d\tau\, x(\tau)x'(\tau)e^{i(s-u)\frac{\tau}{\widetilde{R}}} \\ \mathrm{tr}\left(\sum_{s,t}\tilde{x}_{s,t}\widetilde{x'}_{t,s}\right) &= V\frac{1}{2\pi\widetilde{R}}\int_0^{2\pi\widetilde{R}} d\tau\, \mathrm{tr}\left(x(\tau)x'(\tau)\right) \\ [\bar{x}^-, \tilde{x}]_{s,t} &= \frac{1}{2\pi\widetilde{R}}\int_0^{2\pi\widetilde{R}} d\tau\, \partial_\tau x(\tau)e^{i(s-t)\frac{\tau}{\widetilde{R}}} \end{aligned} \tag{id98}$$

where tr is a trace over $n \times n$ matrices and $V = \sum_s 1$.

Next, we boost the system in x^{10} direction:

$$\widetilde{X}'^{+} = \frac{1}{T}\widetilde{X}^{+}$$
$$\widetilde{X}'^{-} = T\widetilde{X}^{-} \qquad \text{(id99)}$$

The DLCQ limit is achieved when $T \to \infty$, where the "novel Higgs mechanism" [50] is realized. In $T \to \infty$, the actions of the 3-algebra models of M-theory reduce to that of the BFSS matrix theory [27] with matrices of finite size,

$$S = \frac{1}{g^2}\int_{-\infty}^{\infty} d\tau \,\mathrm{tr}\left(\frac{1}{2}\left(D_0 x^P\right)^2 - \frac{1}{4}[x^P, x^Q]^2 + \frac{1}{2}\bar{\psi}\Gamma^0 D_0\psi - \frac{i}{2}\bar{\psi}\Gamma^P[x_P, \psi]\right) \qquad \text{(id100)}$$

where $P, Q = 1, 2, \cdots, 9$.

3.5. Supersymmetric deformation of Lie 3-algebra model of M-theory

A supersymmetric deformation of the Lie 3-algebra Model of M-theory was studied in [53] (see also [54], [55], [56]). If we add mass terms and a flux term,

$$S_m = \left\langle -\frac{1}{2}\mu^2\left(X^I\right)^2 - \frac{i}{2}\mu\bar{\Psi}\Gamma_{3456}\Psi + H_{IJKL}\,[X^I, X^J, X^K]X^L \right\rangle \qquad \text{(id102)}$$

such that

$$H_{IJKL} = \begin{cases} -\frac{\mu}{6}{}_{IJKL} & (I, J, K, L \ = 3, 4, 5, 6 \text{ or } 7, 8, 9, 10) \\ 0 & (\text{otherwise}) \end{cases} \qquad \text{(id103)}$$

to the action (▭), the total action $S_0 + S_m$ is invariant under dynamical 16 supersymmetries,

$$\delta X^I = i\Gamma^I\Psi$$
$$\delta A_{\mu ab}[T^a, T^b, \quad] = i\Gamma_\mu\Gamma_I[X^I, \Psi, \quad]$$
$$\delta\Psi = -\frac{1}{6}[X^I, X^J, X^K]\Gamma_{IJK} - A_{\mu ab}[T^a, T^b, X^I]\Gamma^\mu\Gamma_I + \mu\Gamma_{3456}X^I\Gamma_I \qquad \text{(id104)}$$

From this action, we obtain various interesting solutions, including fuzzy sphere solutions [53].

4. Conclusion

The metric Hermitian 3-algebra corresponds to a class of the super Lie algebra. By using this relation, the metric Hermitian 3-algebras are classified into $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$ Hermitian 3-algebras.

The Lie and Hermitian 3-algebra models of M-theory are obtained by second quantizations of the supermembrane action in a semi-light-cone gauge. The Lie 3-algebra model possesses manifest = 1 supersymmetry in eleven dimensions. In the DLCQ limit, both the models reduce to the BFSS matrix theory with matrices of finite size as they should.

Acknowledgements

We would like to thank T. Asakawa, K. Hashimoto, N. Kamiya, H. Kunitomo, T. Matsuo, S. Moriyama, K. Murakami, J. Nishimura, S. Sasa, F. Sugino, T. Tada, S. Terashima, S. Watamura, K. Yoshida, and especially H. Kawai and A. Tsuchiya for valuable discussions.

Author details

Matsuo Sato[1]

1 Hirosaki University,, Japan

References

[1] V. T. Filippov, n-Lie algebras, Sib. Mat. Zh. 26, No. 6, (1985) 126140.

[2] N. Kamiya, A structure theory of Freudenthal-Kantor triple systems, J. Algebra 110 (1987) 108.

[3] S. Okubo, N. Kamiya, Quasi-classical Lie superalgebras and Lie supertriple systems, Comm. Algebra 30 (2002) no. 8, 3825.

[4] J. Bagger, N. Lambert, Modeling Multiple M2's, Phys. Rev. D75 (2007) 045020.

[5] A. Gustavsson, Algebraic structures on parallel M2-branes, Nucl. Phys. B811 (2009) 66.

[6] J. Bagger, N. Lambert, Gauge Symmetry and Supersymmetry of Multiple M2-Branes, Phys. Rev. D77 (2008) 065008.

[7] O. Aharony, O. Bergman, D. L. Jafferis, J. Maldacena, N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals, JHEP 0810 (2008) 091.

[8] J. Bagger, N. Lambert, Three-Algebras and N=6 Chern-Simons Gauge Theories, Phys. Rev. D79 (2009) 025002.

[9] J. Figueroa-O'Farrill, G. Papadopoulos, Pluecker-type relations for orthogonal planes, J. Geom. Phys. 49 (2004) 294.

[10] G. Papadopoulos, M2-branes, 3-Lie Algebras and Plucker relations, JHEP 0805 (2008) 054.

[11] J. P. Gauntlett, J. B. Gutowski, Constraining Maximally Supersymmetric Membrane Actions, JHEP 0806 (2008) 053.

[12] D. Gaiotto, E. Witten, Janus Configurations, Chern-Simons Couplings, And The Theta-Angle in N=4 Super Yang-Mills Theory, arXiv:0804.2907[hep-th].

[13] K. Hosomichi, K-M. Lee, S. Lee, S. Lee, J. Park, N=5,6 Superconformal Chern-Simons Theories and M2-branes on Orbifolds, JHEP 0809 (2008) 002.

[14] M. Schnabl, Y. Tachikawa, Classification of N=6 superconformal theories of ABJM type, arXiv:0807.1102[hep-th].

[15] J. Gomis, G. Milanesi, J. G. Russo, Bagger-Lambert Theory for General Lie Algebras, JHEP 0806 (2008) 075.

[16] S. Benvenuti, D. Rodriguez-Gomez, E. Tonni, H. Verlinde, N=8 superconformal gauge theories and M2 branes, JHEP 0901 (2009) 078.

[17] P.-M. Ho, Y. Imamura, Y. Matsuo, M2 to D2 revisited, JHEP 0807 (2008) 003.

[18] M. A. Bandres, A. E. Lipstein, J. H. Schwarz, Ghost-Free Superconformal Action for Multiple M2-Branes, JHEP 0807 (2008) 117.

[19] P. de Medeiros, J. Figueroa-O'Farrill, E. Me'ndez-Escobar, P. Ritter, On the Lie-algebraic origin of metric 3-algebras, Commun. Math. Phys. 290 (2009) 871.

[20] S. A. Cherkis, V. Dotsenko, C. Saeman, On Superspace Actions for Multiple M2-Branes, Metric 3-Algebras and their Classification, Phys. Rev. D79 (2009) 086002.

[21] P.-M. Ho, Y. Matsuo, S. Shiba, Lorentzian Lie (3-)algebra and toroidal compactification of M/string theory, arXiv:0901.2003 [hep-th].

[22] P. de Medeiros, J. Figueroa-O'Farrill, E. Mendez-Escobar, P. Ritter, Metric 3-Lie algebras for unitary Bagger-Lambert theories, JHEP 0904 (2009) 037.

[23] Y. Nambu, Generalized Hamiltonian dynamics, Phys. Rev. D7 (1973) 2405.

[24] H. Awata, M. Li, D. Minic, T. Yoneya, On the Quantization of Nambu Brackets, JHEP 0102 (2001) 013.

[25] B. de Wit, J. Hoppe, H. Nicolai, On the Quantum Mechanics of Supermembranes, Nucl. Phys. B305 (1988) 545.

[26] M. Sato, Model of M-theory with Eleven Matrices, JHEP 1007 (2010) 026.

[27] T. Banks, W. Fischler, S.H. Shenker, L. Susskind, M Theory As A Matrix Model: A Conjecture, Phys. Rev. D55 (1997) 5112.

[28] M. Sato, Supersymmetry and the Discrete Light-Cone Quantization Limit of the Lie 3-algebra Model of M-theory, Phys. Rev. D85 (2012), 046003.

[29] M. Sato, Zariski Quantization as Second Quantization, arXiv:1202.1466 [hep-th].

[30] L. Susskind, Another Conjecture about M(atrix) Theory, hep-th/9704080.

[31] A. Sen, D0 Branes on T n and Matrix Theory, Adv. Theor. Math. Phys. 2 (1998) 51.

[32] N. Seiberg, Why is the Matrix Model Correct?, Phys. Rev. Lett. 79 (1997) 3577.

[33] J. Polchinski, M-Theory and the Light Cone, Prog. Theor. Phys. Suppl. 134 (1999) 158.

[34] J. Polchinski, String Theory Vol. 2: Superstring Theory and Beyond, Cambridge University Press, Cambridge, UK (1998).

[35] K. Becker, M. Becker, J. H. Schwarz, String Theory and M-theory, Cambridge University Press, Cambridge, UK (2007).

[36] E. Bergshoeff, E. Sezgin, P.K. Townsend, Supermembranes and Eleven-Dimensional Supergravity, Phys. Lett. B189 (1987) 75.

[37] S. Carlip, Loop Calculations For The Green-Schwarz Superstring, Phys. Lett. B186 (1987) 141.

[38] R.E. Kallosh, Quantization of Green-Schwarz Superstring, Phys. Lett. B195 (1987) 369.

[39] Y. Kazama, N. Yokoi, Superstring in the plane-wave background with RR flux as a conformal field theory, JHEP 0803 (2008) 057.

[40] T. Banks, N. Seiberg, S. Shenker, Branes from Matrices, Nucl. Phys. B490 (1997) 91.

[41] N. Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya, A Large-N Reduced Model as Superstring, Nucl. Phys. B498 (1997) 467.

[42] M. Sato, Covariant Formulation of M-Theory, Int. J. Mod. Phys. A24 (2009) 5019.

[43] H. Nishino, S. Rajpoot, Triality and Bagger-Lambert Theory, Phys. Lett. B671 (2009) 415.

[44] A. Gustavsson, S-J. Rey, Enhanced N=8 Supersymmetry of ABJM Theory on R(8) and R(8)/Z(2), arXiv:0906.3568 [hep-th].

[45] O. Aharony, O. Bergman, D. L. Jafferis, Fractional M2-branes, JHEP 0811 (2008) 043.

[46] M. Hanada, L. Mannelli, Y. Matsuo, Large-N reduced models of supersymmetric quiver, Chern-Simons gauge theories and ABJM, arXiv:0907.4937 [hep-th].

[47] G. Ishiki, S. Shimasaki, A. Tsuchiya, Large N reduction for Chern-Simons theory on S 3 , Phys. Rev. D80 (2009) 086004.

[48] H. Kawai, S. Shimasaki, A. Tsuchiya, Large N reduction on group manifolds, arXiv: 0912.1456 [hep-th].

[49] G. Ishiki, S. Shimasaki, A. Tsuchiya, A Novel Large-N Reduction on S 3 : Demonstration in Chern-Simons Theory, arXiv:1001.4917 [hep-th].

[50] Y. Pang, T. Wang, From N M2's to N D2's, Phys. Rev. D78 (2008) 125007.

[51] S. Mukhi, C. Papageorgakis, M2 to D2, JHEP 0805 (2008) 085.

[52] W. Taylor, D-brane field theory on compact spaces, Phys. Lett. B394 (1997) 283.

[53] J. DeBellis, C. Saemann, R. J. Szabo, Quantized Nambu-Poisson Manifolds in a 3-Lie Algebra Reduced Model, JHEP 1104 (2011) 075.

[54] M. M. Sheikh-Jabbari, Tiny Graviton Matrix Theory: DLCQ of IIB Plane-Wave String Theory, A Conjecture , JHEP 0409 (2004) 017.

[55] J. Gomis, A. J. Salim, F. Passerini, Matrix Theory of Type IIB Plane Wave from Membranes, JHEP 0808 (2008) 002.

[56] K. Hosomichi, K. Lee, S. Lee, Mass-Deformed Bagger-Lambert Theory and its BPS Objects, Phys.Rev. D78 (2008) 066015.