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# Gauge Theory, Combinatorics, and Matrix Models

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## 1. Introduction

Quantum field theory is the most universal method in physics, applied to all the area from condensed-matter physics to high-energy physics. The standard tool to deal with quantum field theory is the perturbation method, which is quite useful if we know the vacuum of the system, namely the starting point of our analysis. On the other hand, sometimes the vacuum itself is not obvious due to the quantum nature of the system. In that case, since the perturbative method is not available any longer, we have to treat the theory in a non-perturbative way.

Supersymmetric gauge theory plays an important role in study on the non-perturbative aspects of quantum field theory. The milestone paper by Seiberg and Witten proposed a *solution* to  $\mathcal{N} = 2$  supersymmetric gauge theory [1], [2], which completely describes the low energy effective behavior of the theory. Their solution can be written down by an auxiliary complex curve, called *Seiberg-Witten curve*, but its meaning was not yet clear and the origin was still mysterious. Since the establishment of Seiberg-Witten theory, tremendous number of works are devoted to understand the Seiberg-Witten's solution, not only by physicists but also mathematicians. In this sense the solution was not a *solution* at that time, but just a *starting point* of the exploration.

One of the most remarkable progress in  $\mathcal{N} = 2$  theories referring to Seiberg-Witten theory is then the exact derivation of the gauge theory partition function by performing the integral over the instanton moduli space [3]. The partition function is written down by multiple partitions, thus we can discuss it in a combinatorial way. It was mathematically proved that the partition function correctly reproduces the Seiberg-Witten solution. This means Seiberg-Witten theory was mathematically established at that time.

The recent progress on the four dimensional  $\mathcal{N} = 2$  supersymmetric gauge theory has revealed a remarkable relation to the two dimensional conformal field theory [4]. This relation pro-

vides the explicit interpretation for the partition function of the four dimensional gauge theory as the conformal block of the two dimensional Liouville field theory. It is naturally regarded as a consequence of the M-brane compactifications [5], [6], and also reproduces the results of Seiberg-Witten theory. It shows how Seiberg-Witten curve characterizes the corresponding four dimensional gauge theory, and thus we can obtain a novel viewpoint of Seiberg-Witten theory.

Based on the connection between the two and four dimensional theories, established results on the two dimensional side can be reconsidered from the viewpoint of the four dimensional theory, and vice versa. One of the useful applications is the matrix model description of the supersymmetric gauge theory [7], [8], [9], [10]. This is based on the fact that the conformal block on the sphere can be also regarded as the matrix integral, which is called Dotsenko-Fateev integral representation [11], [12]. In this direction some extensions of the matrix model description are performed by starting with the two dimensional conformal field theory.

Another type of the matrix model is also investigated so far [13], [14], [15], [16], [17]. This is apparently different from the Dotsenko-Fateev type matrix models, but both of them correctly reproduce the results of the four dimensional gauge theory, e.g. Seiberg-Witten curve. While these studies mainly focus on rederiving the gauge theory results, the present author reveals the new kind of Seiberg-Witten curve by studying the corresponding new matrix model [16], [17]. Such a matrix models is directly derived from the combinatorial representation of the partition function by considering its asymptotic behavior. This treatment is quite analogous to the matrix integral representation of the combinatorial object, for example, the longest increasing subsequences in random permutations [18], the non-equilibrium stochastic model, so-called TASEP [19], and so on (see also [20]). Their remarkable connection to the Tracy-Widom distribution [21] can be understood from the viewpoint of the random matrix theory through the Robinson-Schensted-Knuth (RSK) correspondence (see e.g. [22]).

In this article we review such a universal relation between combinatorics and the matrix model, and discuss its relation to the gauge theory. The gauge theory consequence can be naturally extracted from such a matrix model description. Actually the spectral curve of the matrix model can be interpreted as Seiberg-Witten curve for  $\mathfrak{su}(2)$  supersymmetric gauge theory. This identification suggests some aspects of the gauge theory are also described by the significant universality of the matrix model.

This article is organized as follows. In section 2 we introduce statistical models defined in a combinatorial manner. These models are based on the Plancherel measure on a combinatorial object, and its origin from the gauge theory perspective is also discussed. In section 3 it is shown that the matrix model is derived from the combinatorial model by considering its asymptotic limit. There are various matrix integral representations, corresponding to some deformations of the combinatorial model. In section 4 we investigate the large matrix size limit of the matrix model. It is pointed out that the algebraic curve is quite useful to study one-point function. Its relation to Seiberg-Witten theory is also discussed. Section 5 is devoted to conclusion.



**Figure 1.** Graphical representation of a partition  $\lambda=(5,4,3,1,1)$  and its transposed partition  $\lambda^T=(5,3,2,2,1)$  by the associated Young diagrams. There are 5 non-zero entries in both of them,  $\ell(\lambda)=\lambda^T_1=5$  and  $\ell(\lambda^T)=\lambda_1=5$ .

## 2. Combinatorial partition function

In this section we introduce several kinds of combinatorial models. Their partition functions are defined as summation over partitions with a certain weight function, which is called *Plancherel measure*. It is also shown that such a combinatorial partition function is obtained by performing the path integral for supersymmetric gauge theories.

### 2.1. Random partition model

Let us first recall a partition of a positive integer  $n$ : it is a way of writing  $n$  as a sum of positive integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}) \quad (\text{id2})$$

satisfying the following conditions,

$$n = \sum_{i=1}^{\ell(\lambda)} \lambda_i \equiv |\lambda|, \quad \lambda_1 \geq \lambda_2 \geq \dots \lambda_{\ell(\lambda)} > 0 \quad (\text{id3})$$

Here  $\ell(\lambda)$  is the number of non-zero entries in  $\lambda$ . Now it is convenient to define  $\lambda_i = 0$  for  $i > \ell(\lambda)$ . Fig. 1 shows *Young diagram*, which graphically describes a partition  $\lambda = (5, 4, 2, 1, 1)$  with  $\ell(\lambda) = 5$ .

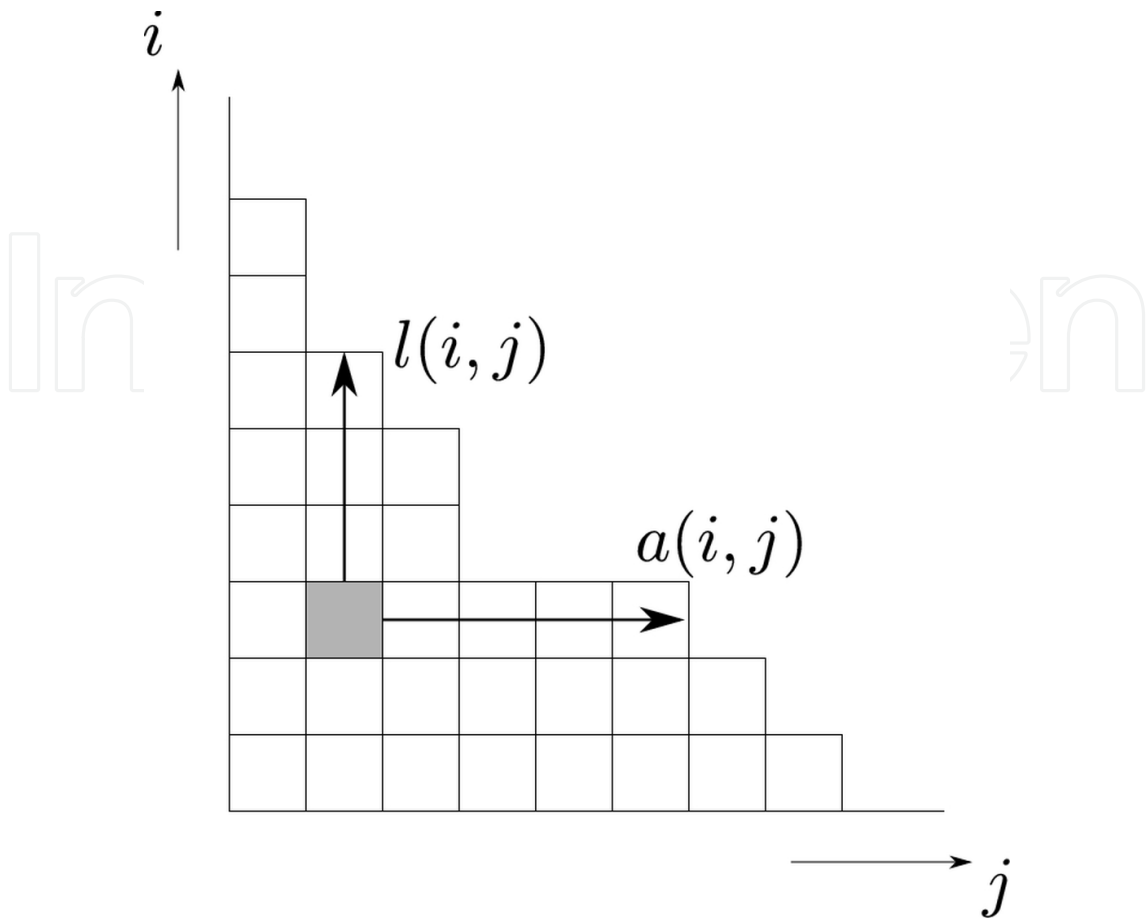
It is known that the partition is quite useful for representation theory. We can obtain an irreducible representation of symmetric group  $S_n$ , which is in one-to-one correspondence with a partition  $\lambda$  with  $|\lambda| = n$ . For such a finite group, one can define a natural measure, which is called *Plancherel measure*,

$$\mu_n(\lambda) = \frac{(\dim \lambda)^2}{n!} \quad (\text{id5})$$

This measure is normalized as

$$\sum_{\lambda \text{ s.t. } |\lambda|=n} \mu_n(\lambda) = 1 \quad (\text{id6})$$

It is also interpreted as Fourier transform of Haar measure on the group. This measure has another useful representation, which is described in a combinatorial way,



**Figure 2.** Combinatorics of Young diagram. Definitions of hook, arm and leg lengths are shown in (). For the shaded box in this figure,  $a(2,3)=4$ ,  $l(2,3)=3$ , and  $h(2,3)=8$ .

$$\mu_n(\lambda) = n! \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)^2} \quad (\text{id7})$$

This  $h(i, j)$  is called *hook length*, which is defined with *arm length* and *leg length*,

$$\begin{aligned} h(i, j) &= a(i, j) + l(i, j) + 1, \\ a(i, j) &= \lambda_i - j, \\ l(i, j) &= \tilde{\lambda}_j - i \end{aligned} \quad (\text{id8})$$

Here  $\tilde{\lambda}$  stands for the transposed partition. Thus the height of a partition  $\lambda$  can be explicitly written as  $\ell(\lambda) = \tilde{\lambda}_1$ .

With this combinatorial measure, we now introduce the following partition function,

$$Z_{\text{U}(1)} = \sum_{\lambda} \left( \frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)^2} \quad (\text{id10})$$

This model is often called *random partition model*. Here  $\Lambda$  is regarded as a parameter like a *chemical potential*, or a *fugacity*, and  $\hbar$  stands for the size of boxes.

Note that a deformed model, which includes higher Casimir potentials, is also investigated in detail [23],

$$Z_{\text{higher}} = \sum_{\lambda} \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)^2} \prod_{k=1} e^{-g_k C_k(\lambda)} \quad (\text{id11})$$

In this case the chemical potential term is absorbed by the linear potential term. There is an interesting interpretation of this deformation in terms of topological string, gauge theory and so on [24], [25].

In order to compute the U(1) partition function it is useful to rewrite it in a “canonical form” instead of the “grand canonical form” which is originally shown in (□),

$$Z_{\text{U(1)}} = \sum_{n=0} \sum_{\lambda \text{ s.t. } |\lambda|=n} \left( \frac{\Lambda}{\hbar} \right)^{2n} \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)^2} \quad (\text{id12})$$

Due to the normalization condition (□), this partition function can be computed as

$$Z_{\text{U(1)}} = \exp \left( \frac{\Lambda}{\hbar} \right)^2 \quad (\text{id13})$$

Although this is explicitly solvable, its universal property and explicit connections to other models are not yet obvious. We will show, in section □ and section □, the matrix model description plays an important role in discussing such an interesting aspect of the combinatorial model.

Now let us remark one interesting observation, which is partially related to the following discussion. The combinatorial partition function (□) has another field theoretical representation using the free boson field [26]. We now consider the following coherent state,

$$|\psi\rangle = \exp \left( \frac{\Lambda}{\hbar} a_{-1} \right) |0\rangle \quad (\text{id14})$$

Here we introduce Heisenberg algebra, satisfying the commutation relation,  $[a_n, a_m] = n\delta_{n+m,0}$ , and the vacuum  $|0\rangle$  annihilated by any positive modes,  $a_n |0\rangle = 0$  for  $n > 0$ . Then it is easy to show the norm of this state gives rise to the partition function,

$$Z_{\text{U(1)}} = \langle \psi | \psi \rangle \quad (\text{id15})$$

Similar kinds of observation is also performed for generalized combinatorial models introduced in section □ [26], [27], [28].

Let us then introduce some generalizations of the  $U(1)$  model. First is what we call  $\beta$ -deformed model including an arbitrary parameter  $\beta \in \mathbb{R}$ ,

$$Z_{U(1)}^{(\beta)} = \sum_{\lambda} \left( \frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{(i,j) \in \lambda} \frac{1}{h_{\beta}(i, j) h^{\beta}(i, j)} \quad (\text{id16})$$

Here we involve the deformed hook lengths,

$$h_{\beta}(i, j) = a(i, j) + \beta l(i, j) + 1, \quad h^{\beta}(i, j) = a(i, j) + \beta l(i, j) + \beta \quad (\text{id17})$$

This generalized model corresponds to Jack polynomial, which is a kind of symmetric polynomial obtained by introducing a free parameter to Schur polynomial [29]. This Jack polynomial is applied to several physical theories: quantum integrable model called Calogero-Sutherland model [30], [31], quantum Hall effect [32], [33], [34] and so on.

Second is a further generalized model involving two free parameters,

$$Z_{U(1)}^{(q,t)} = \sum_{\lambda} \left( \frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{(i,j) \in \lambda} \frac{(1-q)(1-q^{-1})}{(1-q^{a(i,j)+1} t^{l(i,j)})(1-q^{-a(i,j)} t^{-l(i,j)-1})} \quad (\text{id18})$$

This is just a  $q$ -analog of the previous combinatorial model. One can see this is reduced to the  $\beta$ -deformed model ( $\Rightarrow$ ) in the limit of  $q \rightarrow 1$  with fixing  $t = q^{\beta}$ . This generalization is also related to the symmetric polynomial, which is called *Macdonald polynomial* [29]. This symmetric polynomial is used to study Ruijsenaars-Schneider model [35], and the stochastic process based on this function has been recently proposed [36].

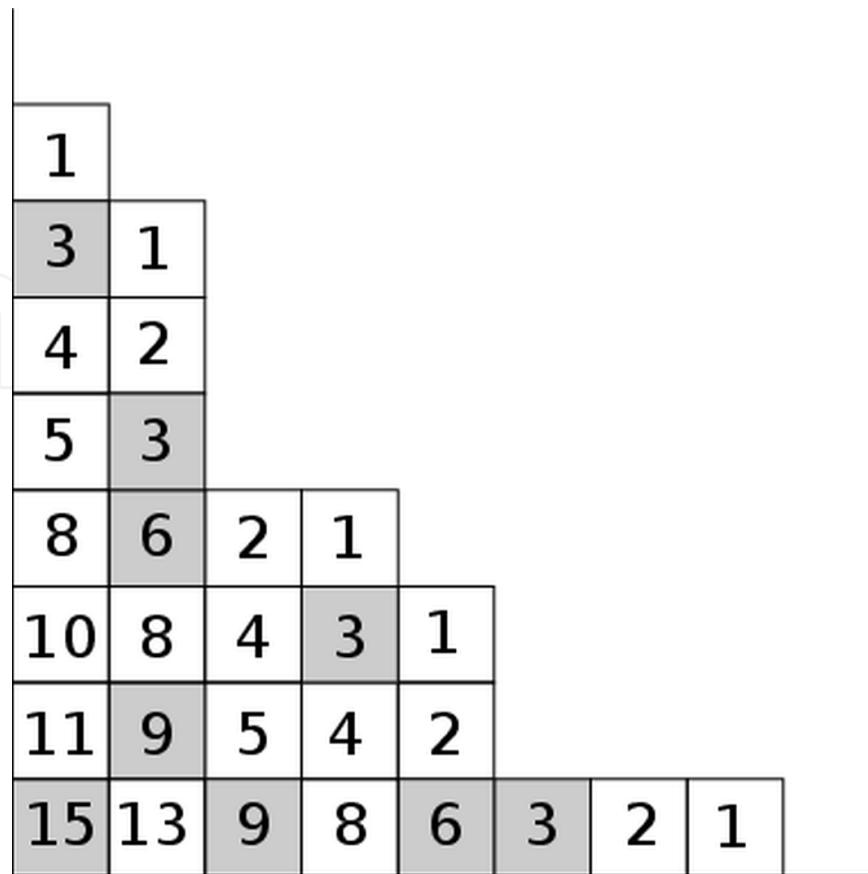
Next is  $\mathbb{Z}_r$ -generalization of the model, which is defined as

$$Z_{\text{orbifold}, U(1)} = \sum_{\lambda} \left( \frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{\Gamma\text{-inv} \subset \lambda} \frac{1}{h(i, j)^2} \quad (\text{id20})$$

Here the product is taken only for the  $\Gamma$ -invariant sector as shown in Fig.  $\square$ ,

$$h(i, j) = a(i, j) + l(i, j) + 1 \equiv 0 \pmod{r} \quad (\text{id21})$$

This restriction is considered in order to study the four dimensional supersymmetric gauge theory on orbifold  $\mathbb{R}^4 / \mathbb{Z}_r \cong 2 / \mathbb{Z}_r$  [37], [38], [16], thus we call this *orbifold partition function*. This also corresponds to a certain symmetric polynomial [39] (see also [40]), which is related to the Calogero-Sutherland model involving spin degrees of freedom. We can further generalize this model ( $\Rightarrow$ ) to the  $\beta$ - or the  $q$ -deformed  $\mathbb{Z}_r$ -orbifold model, and the generic toric orbifold model [17].



**Figure 3.**  $\Gamma$ -invariant sector for  $U(1)$  theory with  $\lambda=(8,5,5,4,2,2,2,1)$ . Numbers in boxes stand for their hook lengths  $h(i,j)=\lambda_i-j+\lambda^{\vee}_j-i+1$ . Shaded boxes are invariant under the action of  $\Gamma=\mathbb{Z}_3$ .

Let us comment on a relation between the orbifold partition function and the  $q$ -deformed model. Taking the limit of  $q \rightarrow 1$ , the latter is reduced to the  $U(1)$  model because the  $q$ -integer is just replaced by the usual integer in such a limit,

$$[x]_q \equiv \frac{1-q^{-x}}{1-q^{-1}} x \quad (\text{id22})$$

This can be easily shown by l'Hopital's rule and so on. On the other hand, parametrizing  $q \rightarrow \omega_r q$  with  $\omega_r = \exp(2\pi i/r)$  being the primitive  $r$ -th root of unity, we have

$$\frac{1-(\omega_r q)^{-x}}{1-(\omega_r q)^{-1}} \begin{cases} x & (x \equiv 0, \mod r) \\ 1 & (x \not\equiv 0, \mod r) \end{cases} \quad (\text{id23})$$

Therefore the orbifold partition function  $(\square)$  is derived from the  $q$ -deformed one  $(\square)$  by taking this root of unity limit. This prescription is useful to study its asymptotic behavior.



## 2.2. Gauge theory partition function

The path integral in quantum field theory involves some kinds of divergence, which are due to infinite degrees of freedom in the theory. On the other hand, we can exactly perform the path integral for several highly supersymmetric theories. We now show that the gauge theory partition function can be described in a combinatorial way, and yields some extended versions of the model we have introduced in section  $\square$ .

The main part of the gauge theory path integral is just evaluation of the moduli space volume for a topological excitation, for example, a vortex in two dimensional theory and an instanton in four dimensional theory. Here we concentrate on the four dimensional case. See [41], [42], [43] for the two dimensional vortex partition function. The most useful method to deal with the instanton is ADHM construction [44]. According to this, the instanton moduli space for  $k$ -instanton in  $SU(n)$  gauge theory on  $\mathbb{R}^4$ , is written as a kind of hyper-Kähler quotient,

$$\mathcal{M}_{n,k} = \{(B_1, B_2, I, J) \mid \mu_{\mathbb{R}} = 0, \mu_0\} / U(k) \quad (\text{id25})$$

$$B_{1,2} \in \text{Hom}(k, k), \quad I \in \text{Hom}(n, k), \quad J \in \text{Hom}(k, n) \quad (\text{id26})$$

$$\begin{aligned} \mu_{\mathbb{R}} &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \\ \mu_{\mathbb{C}} &= [B_1, B_2] + IJ \end{aligned} \quad (\text{id27})$$

The  $k \times k$  matrix condition  $\mu_{\mathbb{R}} = \mu_0$ , and parameters  $(B_1, B_2, I, J)$  satisfying this condition are called ADHM equation and ADHM data. Note that they are identified under the following  $U(k)$  transformation,

$$(B_1, B_2, I, J) \sim (g B_1 g^{-1}, g B_2 g^{-1}, g I, J g^{-1}), \quad g \in U(k) \quad (\text{id28})$$

Thus all we have to do is to estimate the volume of this parameter space. However it is well known that there are some singularities in this moduli space, so that one has to regularize it in order to obtain a meaningful result. Its regularized volume had been derived by applying the localization formula to the moduli space integral [45], and it was then shown that the partition function correctly reproduces Seiberg-Witten theory [3].

We then consider the action of isometries on  $2 \cong \mathbb{R}^4$  for the ADHM data. If we assign  $(z_1, z_2) \rightarrow (e^{i_1} z_1, e^{i_2} z_2)$  for the spatial coordinate of 2, and  $U(1)^{n-1}$  rotation coming from the gauge symmetry  $SU(n)$ , ADHM data transform as

$$(B_1, B_2, I, J) \quad (T_1 B_1, T_2 B_2, I T_a^{-1}, T_1 T_2 T_a J) \quad (\text{id29})$$

where we define the torus actions as  $T_a = \text{diag}(e^{i a_1}, \dots, e^{i a_n}) \in U(1)^{n-1}$ ,  $T_\alpha = e^{i \alpha} \in U(1)^2$ . Note that these toric actions are based on the maximal torus of the gauge theory symmetry,  $U(1)^2 \times U(1)^{n-1} \subset SO(4) \times SU(n)$ . We have to consider the fixed point of these isometries up to gauge transformation  $g \in U(k)$  to perform the localization formula.

The localization formula in the instanton moduli space is based on the vector field  $\xi^*$ , which is associated with  $\xi \in U(1)^2 \times U(1)^{n-1}$ . It generates the one-parameter flow  $e^{t\xi}$  on the moduli space  $\mathcal{M}$ , corresponding to the isometries. The vector field is represented by the element of the maximal torus of the gauge theory symmetry under the  $\Omega$ -background deformation. The gauge theory action is invariant under the deformed BRST transformation, whose generator satisfies  $\xi^* = \{Q^*, Q^*\}/2$ . Thus this generator can be interpreted as the equivariant derivative  $d_\xi = d + i_{\xi^*}$  where  $i_{\xi^*}$  stands for the contraction with the vector field  $\xi^*$ . The localization formula is given by

$$\int_{\mathcal{M}} \alpha(\xi) = (-2\pi)^{n/2} \sum_{x_0} \frac{\alpha_0(\xi)(x_0)}{\det^{1/2} \mathcal{L}_{x_0}} \quad (\text{id30})$$

where  $\alpha(\xi)$  is an equivariant form, which is related to the gauge theory action.  $\alpha_0(\xi)$  is zero degree part and  $\mathcal{L}_{x_0} : T_{x_0} \mathcal{M} \rightarrow T_{x_0} \mathcal{M}$  is the map generated by the vector field  $\xi^*$  at the fixed points  $x_0$ . These fixed points are defined as  $\xi^*(x_0) = 0$  up to  $U(k)$  transformation of the instanton moduli space.

Let us then study the fixed point in the moduli space. The fixed point condition for them are obtained from the infinitesimal version of  $(\square)$  and  $(\square)$  as

$$(\phi_i - \phi_j + \alpha) B_{\alpha, ij} = 0, \quad (\phi_i - a_l) I_{il} = 0, \quad (-\phi_i + a_l + ) J_{li} = 0 \quad (\text{id31})$$

where the element of  $U(k)$  gauge transformation is diagonalized as  $e^{i\phi} = \text{diag}(e^{i\phi_1}, \dots, e^{i\phi_k}) \in U(k)$  with  $\phi = \phi_1 + \phi_2$ . We can show that an eigenvalue of  $\phi$  turns out to be

$$a_l + (j-1)_1 + (i-1)_2 \quad (\text{id32})$$

and the corresponding eigenvector is given by

$$B_1^{j-1} B_2^{i-1} I_l \quad (\text{id33})$$

Since  $\phi$  is a finite dimensional matrix, we can obtain  $k_l$  independent vectors from  $(\sqsupset)$  with  $k_1 + \dots + k_n = k$ . This means that the solution of this condition can be characterized by  $n$ -tuple Young diagrams, or partitions  $\vec{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$  [46]. Thus the characters of the vector spaces are yielding

$$V = \sum_{l=1}^n \sum_{(i,j) \in \lambda^{(l)}} T_{a_l} T_1^{-j+1} T_2^{-i+1}, \quad W = \sum_{l=1}^n T_{a_l} \quad (\text{id34})$$

and that of the tangent space at the fixed point under the isometries can be represented in terms of the  $n$ -tuple partition as

$$\begin{aligned} \chi_{\vec{\lambda}} &= -V^* V (1 - T_1)(1 - T_2) + W^* V + V^* W T_1 T_2 \\ &= \sum_{l,m} \sum_{(i,j) \in \lambda^{(l)}} \left( T_{a_{ml}} T_1^{-\lambda_j^{(l)}+i} T_2^{\lambda_i^{(m)}-j+1} + T_{a_{lm}} T_1^{\lambda_j^{(l)}-i+1} T_2^{-\lambda_i^{(m)}+j} \right) \end{aligned} \quad (\text{id35})$$

Here  $\vec{\lambda}$  is a conjugated partition. Therefore the instanton partition function is obtained by reading the weight function from the character [3], [26],

$$\begin{aligned} Z_{\text{SU}(n)} &= \sum_{\vec{\lambda}} \Lambda^{2n|\vec{\lambda}|} Z_{\vec{\lambda}} \\ Z_{\vec{\lambda}} &= \prod_{l,m} \prod_{(i,j) \in \lambda^{(l)}} \frac{1}{a_{ml} + 2(\lambda_i^{(m)} - j + 1) - 1(\chi_j^{(l)} - i)} \frac{1}{a_{lm} - 2(\lambda_i^{(m)} - j) + 1(\chi_j^{(l)} - i + 1)} \end{aligned} \quad (\text{id36})$$

This is regarded as a generalized model of  $(\sqsupset)$  or  $(\sqsubset)$ . Furthermore by lifting it to the five dimensional theory on  $\mathbb{R}^4 \times S^1$ , one can obtain a generalized version of the  $q$ -deformed partition function  $(\sqsupset)$ . Actually it is easy to see these  $\text{SU}(n)$  models are reduced to the  $\text{U}(1)$  models in the case of  $n = 1$ . Note, if we take into account other matter contributions in addition to the vector multiplet, this partition function involves the associated combinatorial factors. We can extract various properties of the gauge theory from these partition functions, especially its asymptotic behavior.

### 3. Matrix model description

In this section we discuss the matrix model description of the combinatorial partition function. The matrix integral representation can be treated in a standard manner, which is developed in the random matrix theory [47].

#### 3.1. Matrix integral

Let us consider the following  $N \times N$  matrix integral,

$$Z_{\text{matrix}} = \int X e^{-\frac{1}{\hbar} \text{Tr } V(X)} \quad (\text{id38})$$

Here  $X$  is an hermitian matrix, and  $X$  is the associated matrix measure. This matrix can be diagonalized by a unitary transformation,  $gXg^{-1} = \text{diag}(x_1, \dots, x_N)$  with  $g \in U(N)$ , and the integrand is invariant under this transformation,  $\text{Tr } V(X) = \text{Tr } V(gXg^{-1}) = \sum_{i=1}^N V(x_i)$ . On the other hand, we have to take care of the matrix measure in (□): the non-trivial Jacobian is arising from the matrix diagonalization (see, e.g. [47]),

$$X = x U \Delta(x)^2 \quad (\text{id39})$$

The Jacobian part is called *Vandermonde determinant*, which is written as

$$\Delta(x) = \prod_{i < j}^N (x_i - x_j) \quad (\text{id40})$$

and  $U$  is the Haar measure, which is invariant under unitary transformation,  $(gU) = U$ . The diagonal part is simply given by  $x \equiv \prod_{i=1}^N dx_i$ . Therefore, by integrating out the off-diagonal part, the matrix integral (□) is reduced to the integral over the matrix eigenvalues,

$$Z_{\text{matrix}} = \int x \Delta(x)^2 e^{-\frac{1}{\hbar} \sum_{i=1}^N V(x_i)} \quad (\text{id41})$$

This expression is up to a constant factor, associated with the volume of the unitary group,  $\text{vol}(U(N))$ , coming from the off-diagonal integral.

When we consider a real symmetric or a quaternionic self-dual matrix, it can be diagonalized by orthogonal/symplectic transformation. In these cases, the Jacobian part is slightly modified,

$$Z_{\text{matrix}} = \int x \Delta(x)^{2\beta} e^{-\frac{1}{\hbar} \sum_{i=1}^N V(x_i)} \quad (\text{id42})$$

The power of the Vandermonde determinant is given by  $\beta = \frac{1}{2}, 1, 2$  for symmetric, hermitian and self-dual, respectively. This notation is different from the standard one:  $2\beta \rightarrow \beta = 1, 2, 4$  for symmetric, hermitian and self-dual matrices. They correspond to orthogonal, unitary, symplectic ensembles in random matrix theory, and the model with a generic  $\beta \in \mathbb{R}$  is called  *$\beta$ -ensemble matrix model*.

### 3.2. U(1) partition function

We would like to show an essential connection between the combinatorial partition function and the matrix model. By considering the thermodynamical limit of the partition function, it can be represented as a matrix integral discussed above.

Let us start with the most fundamental partition function ( $\square$ ). The main part of its partition function is the product all over the boxes in the partition  $\lambda$ . After some calculations, we can show this combinatorial factor is rewritten as

$$\prod_{(i,j) \in \lambda} \frac{1}{h(i,j)} = \prod_{i < j}^N (\lambda_i - \lambda_j + j - i) \prod_{i=1}^N \frac{1}{\Gamma(\lambda_i + N - i + 1)} \quad (\text{id45})$$

where  $N$  is an arbitrary integer satisfying  $N > \ell(\lambda)$ . This can be also represented in an infinite product form,

$$\prod_{(i,j) \in \lambda} \frac{1}{h(i,j)} = \prod_{i < j}^{\infty} \frac{\lambda_i - \lambda_j + j - i}{j - i} \quad (\text{id46})$$

These expressions correspond to an embedding of the finite dimensional symmetric group  $S_N$  into the infinite dimensional one  $S_{\infty}$ .

By introducing a new set of variables  $\xi_i = \lambda_i + N - i + 1$ , we have another representation of the partition function,

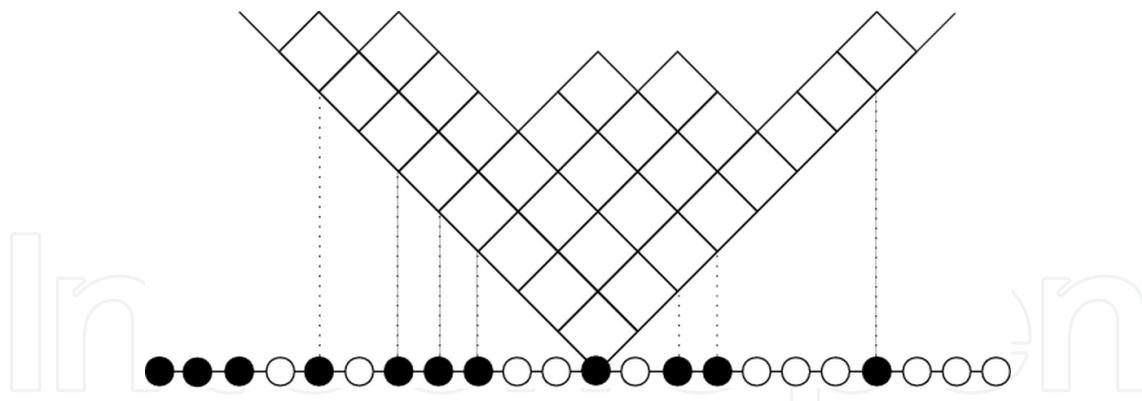
$$Z_{U(1)} = \sum_{\lambda} \left( \frac{\Lambda}{\hbar} \right)^{2 \sum_{i=1}^N \xi_i - N(N+1)} \prod_{i < j}^N (\xi_i - \xi_j)^2 \prod_{i=1}^N \frac{1}{\Gamma(\xi_i)^2} \quad (\text{id47})$$

These new variables satisfy  $\xi_1 > \xi_2 > \dots > \xi_{\ell(\lambda)}$  while the original ones satisfy  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)}$ . This means  $\{\xi_i\}$  and  $\{\lambda_i\}$  are interpreted as fermionic and bosonic degrees of freedom. Fig.  $\square$  shows the correspondence between the bosonic and fermionic variables. The bosonic excitation is regarded as density fluctuation of the fermionic particles around the Fermi energy. This is just the bosonization method, which is often used to study quantum one-dimensional systems (For example, see [48]). Especially we concentrate only on either of the Fermi points. Thus it yields the chiral conformal field theory.

We would like to show that the matrix integral form is obtained from the expression ( $\square$ ). First we rewrite the summation over partitions as

$$\sum_{\lambda} = \sum_{\lambda_1 \geq \dots \geq \lambda_N} = \sum_{\xi_1 > \dots > \xi_N} = \frac{1}{N!} \sum_{\xi_1, \dots, \xi_N} \quad (\text{id49})$$

Then, introducing another variable defined as  $x_i = \hbar \xi_i$ , it can be regarded as a continuous variable in the large  $N$  limit,



**Figure 4.** Shape of Young diagram can be represented by introducing one-dimensional exclusive particles. Positions of particles would be interpreted as eigenvalues of the matrix.

$$N \rightarrow \infty, \quad \hbar \rightarrow 0, \quad \hbar N = (1) \quad (\text{id50})$$

This is called 't Hooft limit. The measure for this variable is given by

$$dx_i \approx \hbar \sim \frac{1}{N} \quad (\text{id51})$$

Therefore the partition function ( $\Xi$ ) is rewritten as the following matrix integral,

$$Z_{U(1)} \approx \int dx \Delta(x)^2 e^{-\frac{1}{\hbar} \sum_{i=1}^N V(x_i)} \quad (\text{id52})$$

Here the matrix potential is derived from the asymptotic behavior of the  $\Gamma$ -function,

$$\hbar \log \Gamma(x/\hbar) \sim x \log x - x, \quad \hbar \rightarrow 0 \quad (\text{id53})$$

Since this variable can take a negative value, the potential term should be simply extended to the region of  $x < 0$ . Thus, taking into account the fugacity parameter  $\Lambda$ , the matrix potential is given by

$$V(x) = 2 \left[ x \log \left| \frac{x}{\Lambda} \right| - x \right] \quad (\text{id54})$$

This is the simplest version of the  $^1$  matrix model [24]. If we start with the partition function including the higher Casimir operators ( $\Xi$ ), the associated integral expression just yields the  $^1$  matrix model.

Let us comment on other possibilities to obtain the matrix model. It is shown that the matrix integral form can be derived without taking the large  $N$  limit [23]. Anyway one can see that it is reduced to the model we discussed above in the large  $N$  limit. There is another kind of the matrix model derived from the combinatorial partition function by *poissonizing* the prob-

ability measure. In this case, only the linear potential is arising in the matrix potential term. Such a matrix model is called Bessel-type matrix model, where its short range fluctuation is described by the Bessel kernel.

Next we shall derive the matrix model corresponding to the  $\beta$ -deformed  $U(1)$  model ( $\square$ ). The combinatorial part of the partition function is similarly given by

$$\prod_{(i,j) \in \lambda} \frac{1}{h_\beta(i, j) h^\beta(i, j)} = \Gamma(\beta)^N \prod_{i < j}^N \frac{\Gamma(\lambda_i - \lambda_j + \beta(j - i) + \beta)}{\Gamma(\lambda_i - \lambda_j + \beta(j - i))} \frac{\Gamma(\lambda_i - \lambda_j + \beta(j - i) + 1)}{\Gamma(\lambda_i - \lambda_j + \beta(j - i) + 1 - \beta)} \quad (\text{id55})$$

$$\times \prod_{i=1}^N \frac{1}{\Gamma(\lambda_i + \beta(N - i) + \beta)} \frac{1}{\Gamma(\lambda_i + \beta(N - i) + 1)}$$

In this case we shall introduce the following variables,  $\xi_i^{(\beta)} = \lambda_i + \beta(N - i) + 1$  or  $\xi_i^{(\beta)} = \lambda_i + \beta(N - i) + \beta$ , satisfying  $\xi_i^{(\beta)} - \xi_{i+1}^{(\beta)} \geq \beta$ . This means the parameter  $\beta$  characterizes how they are exclusive. They satisfy the generalized fractional exclusive statistics for  $\beta \neq 1$  [49] (see also [40]). They are reduced to fermions and bosons for  $\beta = 1$  and  $\beta = 0$ , respectively. Then, rescaling the variables,  $x_i = \hbar \xi_i^{(\beta)}$ , the combinatorial part ( $\square$ ) in the 't Hooft limit yields

$$\prod_{(i,j) \in \lambda} \frac{1}{h_\beta(i, j) h^\beta(i, j)} \Delta(x)^{2\beta} e^{-\frac{1}{\hbar} \sum_{i=1}^N V(x_i)} \quad (\text{id56})$$

Here we use  $\Gamma(\alpha + \beta) / \Gamma(\alpha) \sim \alpha^\beta$  with  $\alpha \rightarrow \infty$ . The matrix potential obtained here is the same as ( $\square$ ). Therefore the matrix model associated with the  $\beta$ -deformed partition function is given by

$$Z_{U(1)}^{(\beta)} \approx \int x \Delta(x)^{2\beta} e^{-\frac{1}{\hbar} \sum_{i=1}^N V(x_i)} \quad (\text{id57})$$

This is just the  $\beta$ -ensemble matrix model shown in ( $\square$ ).

We can consider the matrix model description of the  $(q, t)$ -deformed partition function. In this case the combinatorial part of ( $\square$ ) is written as

$$\prod_{(i,j) \in \lambda} \frac{1 - q}{1 - q^{a(i,j)+1} t^{l(i,j)}} = (1 - q)^{|\lambda|} \prod_{i < j}^N \frac{(q^{\lambda_i - \lambda_j + 1} t^{j-i-1}; q)_\infty}{(q^{\lambda_i - \lambda_j + 1} t^{j-i}; q)_\infty} \prod_{i=1}^N \frac{(q^{\lambda_i + 1} t^{N-i}; q)_\infty}{(q; q)_\infty} \quad (\text{id58})$$

$$\prod_{(i,j) \in \lambda} \frac{1 - q^{-1}}{1 - q^{-a(i,j)} t^{-l(i,j)-1}} = (1 - q^{-1})^{|\lambda|} \prod_{i < j}^N \frac{(q^{-\lambda_i + \lambda_j + 1} t^{-j+i-1}; q)_\infty}{(q^{-\lambda_i + \lambda_j + 1} t^{-j+i}; q)_\infty} \prod_{i=1}^N \frac{(qt^{-1}; q)_\infty}{(q^{-\lambda_i + 1} t^{-N+i-1}; q)_\infty}$$

Here  $(x; q)_n = \prod_{m=0}^{n-1} (1 - xq^m)$  is the  $q$ -Pochhammer symbol. When we parametrize  $q = e^{-\hbar R}$  and  $t = q^\beta$ , a set of the variables  $\{\xi_i^{(\beta)}\}$  plays an important role in considering the large  $N$

limit as well as the  $\beta$ -deformed model. Thus, rescaling these as  $x_i = \hbar \xi_i^{(\beta)}$  and taking the 't Hooft limit, we obtain the integral expression of the  $q$ -deformed partition function,

$$Z_{U(1)}^{(q,t)} \approx \int x (\Delta_R(x))^{2\beta} e^{-\frac{1}{\hbar} \sum_{i=1}^N V_R(x_i)} \quad (\text{id59})$$

The matrix measure and potential are given by

$$\Delta_R(x) = \prod_{i < j}^N \frac{2}{R} \sinh \frac{R}{2} (x_i - x_j) \quad (\text{id60})$$

$$V_R(x) = -\frac{1}{R} [\text{Li}_2(e^{Rx}) - \text{Li}_2(e^{-Rx})] \quad (\text{id61})$$

We will discuss how to obtain these expressions below. We can see they are reduced to the standard ones in the limit of  $R \rightarrow 0$ ,

$$\Delta_R(x) \rightarrow \Delta(x), \quad V_R(x) \rightarrow V(x) \quad (\text{id62})$$

Note that this hyperbolic-type matrix measure is also investigated in the Chern-Simons matrix model [50], which is extensively involved with the recent progress on the three dimensional supersymmetric gauge theory via the localization method [51].

Let us comment on useful formulas to derive the integral expression ( $\Rightarrow$ ). The measure part is relevant to the asymptotic form of the following function,

$$\frac{(x; q)_\infty}{(tx; q)_\infty} \frac{(x; q)_\infty}{(tx; q)_\infty} \Big|_{q \rightarrow 1} = (1-x)^\beta, \quad x \rightarrow 0 \quad (\text{id63})$$

This essentially corresponds to the  $q \rightarrow 1$  limit of the  $q$ -Vandermonde determinant. This expression is up to logarithmic term, which can be regarded as the zero mode contribution of the free boson field. See [17], [52] for details.,

$$\Delta_{q,t}^2(x) = \prod_{i \neq j}^N \frac{(x_i/x_j; q)_\infty}{(tx_i/x_j; q)_\infty} \quad (\text{id65})$$

Then, to investigate the matrix potential term, we now introduce the quantum dilogarithm function,

$$g(x; q) = \prod_{n=1}^{\infty} \left( 1 - \frac{1}{x} q^n \right) \quad (\text{id66})$$



Its asymptotic expansion is given by (see, e.g. [23])

$$\log g(x; q = e^{-\hbar R}) = -\frac{1}{\hbar R} \sum_{m=0}^{\infty} \text{Li}_{2-m}(x^{-1}) \frac{B_m}{m!} (\hbar R)^m \quad (\text{id67})$$

where  $B_m$  is the  $m$ -th Bernoulli number, and  $\text{Li}_m(x) = \sum_{k=1}^{\infty} x^k / k^m$  is the polylogarithm function. The potential term is coming from the leading term of this expression.

### 3.3. SU( $n$ ) partition function

Generalizing the result shown in section  $\square$ , we deal with the combinatorial partition function for SU( $n$ ) gauge theory ( $\square$ ). Its matrix model description is evolved in [13].

The combinatorial factor of the SU( $n$ ) partition function ( $\square$ ) can be represented as

$$Z_{\vec{\lambda}} = \frac{1}{2^{n|\vec{\lambda}|}} \prod_{(l,i) \neq (m,j)} \frac{\Gamma(\lambda_i^{(l)} - \lambda_j^{(m)} + \beta(j-i) + b_{lm} + \beta)}{\Gamma(\lambda_i^{(l)} - \lambda_j^{(m)} + \beta(j-i) + b_{lm})} \frac{\Gamma(\beta(j-i) + b_{lk})}{\Gamma(\beta(j-i) + b_{lk} + \beta)} \quad (\text{id69})$$

where we define parameters as  $\beta = -1/2$ ,  $b_{lm} = a_{lm}/2$ . This is an infinite product expression of the partition function. Anyway in this case one can see it is useful to introduce  $n$  kinds of fermionic variables, corresponding to the  $n$ -tuplet partition,

$$\xi_i^{(l)} = \lambda_i^{(l)} + \beta(N-i) + 1 + b_l \quad (\text{id70})$$

Then, assuming  $b_{lm} \gg 1$ , let us introduce a set of variables,

$$(\zeta_1, \zeta_2, \dots, \zeta_{nN}) = (\xi_1^{(n)}, \dots, \xi_N^{(n)}, \xi_1^{(n-1)}, \dots, \xi_N^{(2)}, \xi_1^{(1)}, \dots, \xi_N^{(1)}) \quad (\text{id71})$$

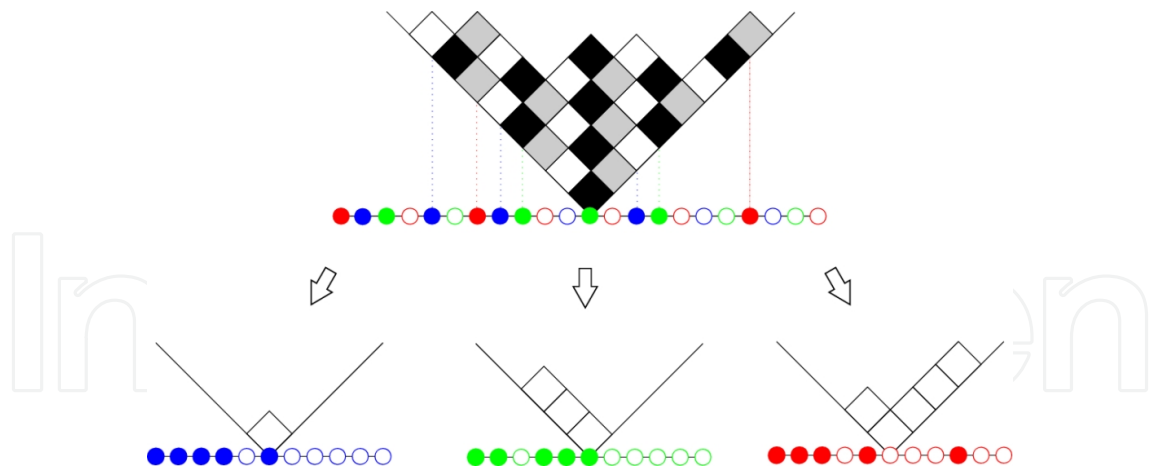
satisfying  $\zeta_1 > \zeta_2 > \dots > \zeta_{nN}$ . The combinatorial factor ( $\square$ ) is rewritten with these variables as

$$Z_{\vec{\lambda}} = \frac{1}{2^{n|\vec{\lambda}|}} \prod_{i < j}^{nN} \frac{\Gamma(\zeta_i - \zeta_j + \beta)}{\Gamma(\zeta_i - \zeta_j)} \prod_{i=1}^{nN} \prod_{l=1}^n \frac{\Gamma(-\zeta_i + b_l + 1)}{\Gamma(\zeta_i - b_l - 1 + \beta)} \quad (\text{id72})$$

From this expression we can obtain the matrix model description for SU( $n$ ) gauge theory partition function, by rescaling  $x_i = \hbar \zeta_i$  with reparametrizing  $\hbar = 2$ ,

$$Z_{\text{SU}(n)} \approx \int \mathcal{X} \Delta(x)^{2\beta} e^{-\frac{1}{\hbar} \sum_{i=1}^{nN} V_{\text{SU}(n)}(x_i)} \quad (\text{id73})$$

In this case the matrix potential is given by



**Figure 5.** The decomposition of the partition for  $\mathbb{Z}/3$ . First suppose the standard correspondence between the one-dimensional particles and the original partition, and then rearrange them with respect to mod  $r$ .

$$V_{\text{SU}(n)}(x) = 2 \sum_{l=1}^n \left[ (x - a_l) \log \left| \frac{x - a_l}{\Lambda} \right| - (x - a_l) \right] \quad (\text{id74})$$

Note that this matrix model is regarded as the  $U(1)$  matrix model with external fields  $a_l$ . We will discuss how to extract the gauge theory consequences from this matrix model in section  $\square$ .

### 3.4. Orbifold partition function

The matrix model description for the random partition model is also possible for the orbifold theory. We would like to derive another kind of the matrix model from the combinatorial orbifold partition function ( $\square$ ). We now concentrate on the  $U(1)$  orbifold partition function for simplicity. See [16], [17] for details of the  $SU(n)$  theory.

To obtain the matrix integral representation of the combinatorial partition function, we have to find the associated one-dimensional particle description of the combinatorial factor. In this case, although the combinatorial weight itself is the same as the standard  $U(1)$  model, there is restriction on its product region. Thus it is useful to introduce another basis obtained by dividing the partition as follows,

$$\{r(\lambda_i^{(u)} + N^{(u)} - i) + u \mid i = 1, \dots, N^{(u)}, u = 0, \dots, r - 1\} = \{\lambda_i + N - i \mid i = 1, \dots, N\} \quad (\text{id77})$$

Fig.  $\square$  shows the meaning of this procedure graphically. We now assume  $N^{(u)} = N$  for all  $u$ . With these one-dimensional particles, we now utilize the relation between the orbifold partition function and the  $q$ -deformed model as discussed in section  $\square$ . Its calculation is quite straightforward, but a little bit complicated. See [16], [17] for details.

After some computations, we finally obtain the matrix model for the  $\beta$ -deformed orbifold partition function,

$$Z_{\text{orbifold}, U(1)}^{(\beta)} \approx \tilde{\mathcal{K}} \left( \Delta_{\text{orb}}^{(\beta)}(x) \right)^2 e^{-\frac{1}{\hbar} \sum_{u=0}^{r-1} \sum_{i=1}^N V(x_i^{(u)})} \quad (\text{id78})$$

In this case, we have a multi-matrix integral representation, since we introduce  $r$  kinds of partitions from the original partition. The matrix measure and the matrix potential are given as follows,

$$\vec{x} = \prod_{u=0}^{r-1} \prod_{i=1}^N dx_i^{(u)} \quad (\text{id79})$$

$$\left( \Delta_{\text{orb}}^{(\beta)}(x) \right)^2 = \prod_{u=0}^{r-1} \prod_{i < j}^N (x_i^{(u)} - x_j^{(u)})^{2(\beta-1)/r+2} \prod_{u < v}^{r-1} \prod_{i, j}^N (x_i^{(u)} - x_j^{(v)})^{2(\beta-1)/r} \quad (\text{id80})$$

$$V(x) = \frac{2}{r} \left[ x \log \left| \frac{x}{\Lambda} \right| - x \right] \quad (\text{id81})$$

The matrix measure consists of two parts, interaction between eigenvalues from the same matrix and that between eigenvalues from different matrices. Note that in the case of  $\beta = 1$ , because the interaction part in the matrix measure between different matrices is vanishing, this multi-matrix model is simply reduced to the one-matrix model.

## 4. Large N analysis

One of the most important aspects of the matrix model is universality arising in the large  $N$  limit. The universality class described by the matrix model covers huge kinds of the statistical models, in particular its characteristic fluctuation rather than the eigenvalue density function. In the large  $N$  limit, which is regarded as a justification to apply a kind of the mean field approximation, analysis of the matrix model is extremely reduced to the saddle point equation and a simple fluctuation around it.

### 4.1. Saddle point equation and spectral curve

Let us first define the prepotential, which is also interpreted as the effective action for the eigenvalues, from the matrix integral representation

( $\square$ ),

$$-\frac{1}{\hbar^2} \mathcal{F}(\{x_i\}) = -\frac{1}{\hbar} \sum_{i=1}^N V(x_i) + 2 \sum_{i < j}^N \log(x_i - x_j) \quad (\text{id83})$$

This is essentially the genus zero part of the prepotential. In the large  $N$  limit, in particular 't Hooft limit ( $\square$ ) with  $N\hbar \equiv t$ , we shall investigate the saddle point equation for the matrix integral. We can obtain the condition for criticality by differentiating the prepotential,

$$V'(x_i) = 2\hbar \sum_{j(\neq i)}^N \frac{1}{x_j - x_i}, \quad \text{for all } i \quad (\text{id84})$$

This is also given by the extremal condition of the effective potential defined as

$$V_{\text{eff}}(x_i) = V(x_i) - 2\hbar \sum_{j(\neq i)}^N \log(x_i - x_j) \quad (\text{id85})$$

This potential involves a logarithmic Coulomb repulsion between eigenvalues. If the 't Hooft coupling is small, the potential term dominates the Coulomb interaction and eigenvalues concentrate on extrema of the potential  $V'(x) = 0$ . On the other hand, as the coupling gets bigger, the eigenvalue distribution is extended.

To deal with such a situation, we now define the density of eigenvalues,

$$\rho(x) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i) \quad (\text{id86})$$

where  $x_i$  is the solution of the criticality condition ( $\square$ ). In the large  $N$  limit, it is natural to think this eigenvalue distribution is smeared, and becomes a continuous function. Furthermore, we assume the eigenvalues are distributed around the critical points of the potential  $V(x)$  as linear segments. Thus we generically denote the  $l$ -th segment for  $\rho(x)$  as  ${}_l$ , and the total number of eigenvalues  $N$  splits into  $n$  integers for these segments,

$$N = \sum_{l=1}^n N_l \quad (\text{id87})$$

where  $N_l$  is the number of eigenvalues in the interval  ${}_l$ . The density of eigenvalues  $\rho(x)$  takes non-zero value only on the segment  ${}_l$ , and is normalized as

$$\int_{{}_l} dx \rho(x) = \frac{N_l}{N} \equiv \nu_l \quad (\text{id88})$$

where we call it *filling fraction*. According to these fractions, we can introduce the partial 't Hooft parameters,  $t_l = N_l \hbar$ . Note there are  $n$  't Hooft couplings and filling fractions, but only  $n - 1$  fractions are independent since they have to satisfy  $\sum_{l=1}^n \nu_l = 1$  while all the 't Hooft couplings are independent.

We then introduce the resolvent for this model as an auxiliary function, a kind of Green function. By taking the large  $N$  limit, it can be given by the integral representation,

$$\omega(x) = t \int dy \frac{\rho(y)}{x - y} \quad (\text{id89})$$

This means that the density of states is regarded as the Hilbert transformation of this resolvent function. Indeed the density of states is associated with the discontinuities of the resolvent,

$$\rho(x) = -\frac{1}{2\pi i t} (\omega(x + i) - \omega(x - i)) \quad (\text{id90})$$

Thus all we have to do is to determine the resolvent instead of the density of states with satisfying the asymptotic behavior,

$$\omega(x) \sim \frac{1}{x}, \quad x \rightarrow \infty \quad (\text{id91})$$

Writing down the prepotential with the density of states,

$$\mathcal{F}(\{x_i\}) = t \int dx \rho(x) V(x) - t^2 P \int dx dy \rho(x) \rho(y) \log(x - y) \quad (\text{id92})$$

the criticality condition is given by

$$\frac{1}{2t} V'(x) = P \int dy \frac{\rho(y)}{x - y} \quad (\text{id93})$$

Here P stands for the principal value. Thus this saddle point equation can be also written in the following convenient form to discuss its analytic property,

$$V'(x) = \omega(x + i) + \omega(x - i) \quad (\text{id94})$$

On the other hand, we have another convenient form to treat the saddle point equation, which is called *loop equation*, given by

$$y^2(x) - V'(x)^2 + R(x) = 0 \quad (\text{id95})$$

where we denote

$$y(x) = V'(x) - 2\omega(x) = -2\omega_{\text{sing}}(x) \quad (\text{id96})$$

$$R(x) = \frac{4t}{N} \sum_{i=1}^N \frac{V'(x) - V'(x_i)}{x - x_i} \quad (\text{id97})$$

It is obtained from the saddle point equation by multiplying  $1/(x - x_i)$  and taking their summation and the large  $N$  limit. This representation ( $\Rightarrow$ ) is more appropriate to reveal its geometric meaning. Indeed this algebraic curve is interpreted as the hyperelliptic curve which is given by resolving the singular form,

$$y^2(x) - V'(x)^2 = 0 \quad (\text{id98})$$

The genus of the Riemann surface is directly related to the number of cuts of the corresponding resolvent. The filling fraction, or the partial 't Hooft coupling, is simply given by the contour integral on the hyperelliptic curve

$$t_l = \frac{1}{2\pi i} \oint_{\gamma_l} dx \, \omega_{\text{sing}}(x) = -\frac{1}{4\pi i} \oint_{\gamma_l} dx \, y(x) \quad (\text{id99})$$

## 4.2. Relation to Seiberg-Witten theory

We now discuss the relation between Seiberg-Witten curve and the matrix model. In the first place, the matrix model captures the asymptotic behavior of the combinatorial representation of the partition function. The energy functional, which is derived from the asymptotics of the partition function [26], in terms of the profile function

$$\mathcal{E}_\Lambda(f) = \frac{1}{4} \text{Pf}_{y < x} dx dy \, f''(x) f''(y) (x - y)^2 \left( \log \left( \frac{x - y}{\Lambda} \right) - \frac{3}{2} \right) \quad (\text{id101})$$

can be rewritten as

$$E_\Lambda() = -\text{Pf}_{x \neq y} dx dy \, \frac{(x)(y)}{(x - y)^2} - 2 \int dx \, (x) \log \prod_{l=1}^N \left( \frac{x - a_l}{\Lambda} \right) \quad (\text{id102})$$

up to the perturbative contribution

$$\frac{1}{2} \sum_{l,m} (a_l - a_m)^2 \log \left( \frac{a_l - a_m}{\Lambda} \right) \quad (\text{id103})$$

by identifying

$$f(x) - \sum_{l=1}^n |x - a_l| = (x) \quad (\text{id104})$$

Then integrating ( $\Rightarrow$ ) by parts, we have

$$E_{\Lambda}() = -\text{Pf}_{x \neq y} dx dy \rho'(x) \rho'(y) \log(x - y) + 2 \int dx \rho'(x) \sum_{l=1}^n \left[ (x - a_l) \log \left( \frac{x - a_l}{\Lambda} \right) - (x - a_l) \right] \quad (\text{id105})$$

This is just the matrix model discussed in section 4 if we identify  $\rho'(x) = \rho(x)$ . Therefore analysis of this matrix model is equivalent to that of [53]. But in this section we reconsider the result of the gauge theory from the viewpoint of the matrix model.

We can introduce a regular function on the complex plane, except at the infinity,

$$P_n(x) = \Lambda^n (e^{y/2} + e^{-y/2}) \equiv \Lambda^n \left( w + \frac{1}{w} \right) \quad (\text{id106})$$

It is because the saddle point equation (4) yields the following equation,

$$e^{y(x+i)/2} + e^{-y(x+i)/2} = e^{y(x-i)/2} + e^{-y(x-i)/2} \quad (\text{id107})$$

This entire function turns out to be a monic polynomial  $P_n(x) = x^n + \dots$ , because it is an analytic function with the following asymptotic behavior,

$$\Lambda^n e^{y/2} = \Lambda^n e^{-\omega(x)} \prod_{l=1}^n \left( \frac{x - a_l}{\Lambda} \right) x^n, \quad x \rightarrow \infty \quad (\text{id108})$$

Here  $w$  should be the smaller root with the boundary condition as

$$w \sim \frac{\Lambda^n}{x^n}, \quad x \rightarrow \infty \quad (\text{id109})$$

thus we now identify

$$w = e^{-y/2} \quad (\text{id110})$$

Therefore from the hyperelliptic curve (4) we can relate Seiberg-Witten curve to the spectral curve of the matrix model,

$$\begin{aligned} dS &= \frac{1}{2\pi i} x \frac{dw}{w} \\ &= -\frac{1}{2\pi i} \log w \, dx \\ &= \frac{1}{4\pi i} y(x) dz \end{aligned} \quad (\text{id111})$$

Note that it is shown in [25], [54] we have to take the vanishing fraction limit to obtain the Coulomb moduli from the matrix model contour integral. This is the essential difference between the profile function method and the matrix model description.

### 4.3. Eigenvalue distribution

We now demonstrate that the eigenvalue distribution function is indeed derived from the spectral curve of the matrix model. The spectral curve ( $\square$ ) in the case of  $n = 1$  with setting  $\Lambda = 1$  and  $P_{n=1}(x) = x$  is written as

$$x = w + \frac{1}{w} \quad (\text{id113})$$

From this relation the singular part of the resolvent can be extracted as

$$\omega_{\text{sing}}(x) = \text{arccosh}\left(\frac{x}{2}\right) \quad (\text{id114})$$

This has a branch cut only on  $x \in [-2, 2]$ , namely a one-cut solution. Thus the eigenvalue distribution function is written as follows at least on  $x \in [-2, 2]$ ,

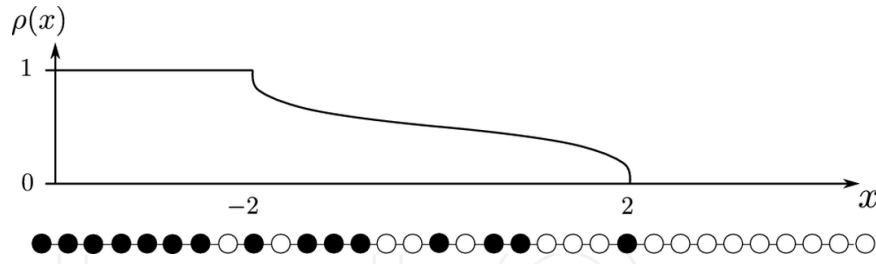
$$\rho(x) = \frac{1}{\pi} \arccos\left(\frac{x}{2}\right) \quad (\text{id115})$$

Note that this function has a non-zero value at the left boundary of the cut,  $\rho(-2) = 1$ , while at the right boundary we have  $\rho(2) = 0$ . Equivalently we now choose the cut of  $\arccos$  function in this way. This seems a little bit strange because the eigenvalue density has to vanish except for on the cut. On the other hand, recalling the meaning of the eigenvalues, i.e. positions of one-dimensional particles, as shown in Fig.  $\square$ , this situation is quite reasonable. The region below the Fermi level is filled of the particles, and thus the density has to be a non-zero constant in such a region. This is just a property of the Fermi distribution function. ( $1/N$  correction could be interpreted as a finite temperature effect.) Therefore the total eigenvalue distribution function is given by

$$\rho(x) = \begin{cases} 1 & x < -2 \\ \frac{1}{\pi} \arccos\left(\frac{x}{2}\right) & |x| < 2 \\ 0 & x > 2 \end{cases} \quad (\text{id116})$$

Remark the eigenvalue density ( $\square$ ) is quite similar to the Wigner's semi-circle distribution function, especially its behavior around the edge,





**Figure 6.** The eigenvalue distribution function for the U(1) model.

$$\rho_{\text{circ}}(x) = \frac{1}{\pi} \sqrt{1 - \left(\frac{x}{2}\right)^2} \frac{1}{\pi} \sqrt{2 - x}, \quad x \geq 2 \quad (\text{id118})$$

The fluctuation at the spectral edge of the random matrix obeys Tracy-Widom distribution [21], thus it is natural that the edge fluctuation of the combinatorial model is also described by Tracy-Widom distribution. This remarkable fact was actually shown by [55]. Evolving such a similarity to the gaussian random matrix theory, the kernel of this model is also given by the following sine kernel,

$$K(x, y) = \frac{\sin \rho_0 \pi (x - y)}{\pi (x - y)} \quad (\text{id119})$$

where  $\rho_0$  is the averaged density of eigenvalues. This means the U(1) combinatorial model belongs to the GUE random matrix universal class [47]. Then all the correlation functions can be written as a determinant of this kernel,

$$\rho(x_1, \dots, x_k) = \det [K(x_i, x_j)]_{1 \leq i, j \leq k} \quad (\text{id120})$$

Let us then remark a relation to the profile function of the Young diagram. It was shown that the shape of the Young diagram goes to the following form in the thermodynamical limit [56], [57], [58],

$$\Omega(x) = \begin{cases} \frac{2}{\pi} \left( x \arcsin \frac{x}{2} + \sqrt{4 - x^2} \right) & |x| < 2 \\ 1 & |x| > 2 \end{cases} \quad (\text{id121})$$

Rather than this profile function itself, the derivative of this function is more relevant to our study,

$$\Omega'(x) = \begin{cases} -1 & x < -2 \\ \frac{2}{\pi} \arcsin \left( \frac{x}{2} \right) & |x| < 2 \\ 1 & x > 2 \end{cases} \quad (\text{id122})$$

One can see the eigenvalue density ( $\rho$ ) is directly related to this derivative function ( $\Omega'$ ) as

$$\rho(x) = \frac{1 - \Omega'(x)}{2} \quad (\text{id123})$$

This relation is easily obtained from the correspondence between the Young diagram and the one-dimensional particle as shown in Fig. 4.

## 5. Conclusion

In this article we have investigated the combinatorial statistical model through its matrix model description. Starting from the  $U(1)$  model, which is motivated by representation theory, we have dealt with its  $\beta$ -deformation and  $q$ -deformation. We have shown that its non-Abelian generalization, including external field parameters, is obtained as the four dimensional supersymmetric gauge theory partition function. We have also referred to the orbifold partition function, and its relation to the  $q$ -deformed model through the root of unity limit.

We have then shown the matrix integral representation is derived from such a combinatorial partition function by considering its asymptotic behavior in the large  $N$  limit. Due to variety of the combinatorial model, we can obtain the  $\beta$ -ensemble matrix model, the hyperbolic matrix model, and those with external fields. Furthermore from the orbifold partition function the multi-matrix model is derived.

Based on the matrix model description, we have study the asymptotic behavior of the combinatorial models in the large  $N$  limit. In this limit we can extract various important properties of the matrix model by analysing the saddle point equation. Introducing the resolvent as an auxiliary function, we have obtained the algebraic curve for the matrix model, which is called the spectral curve. We have shown it can be interpreted as Seiberg-Witten curve, and then the eigenvalue distribution function is also obtained from this algebraic curve.

Let us comment on some possibilities of generalization and perspective. As discussed in this article we can obtain various interesting results from Macdonald polynomial by taking the corresponding limit. It is interesting to research its matrix model consequence from the exotic limit of Macdonald polynomial. For example, the  $q \rightarrow 0$  limit of Macdonald polynomial, which is called Hall-Littlewood polynomial, is not investigated with respect to its connection with the matrix model. We also would like to study properties of the  $BC$ -type polynomial [59], which is associated with the corresponding root system. Recalling the meaning of the  $q$ -deformation in terms of the gauge theory, namely lifting up to the five dimensional theory  $\mathbb{R}^4 \times S^1$  by taking into account all the Kaluza-Klein modes, it seems interesting to study the six dimensional theory on  $\mathbb{R}^4 \times T^2$ . In this case it is natural to obtain the elliptic generalization of the matrix model. It can not be interpreted as matrix integral representation any longer, however the large  $N$  analysis could be anyway performed in the standard manner. We would like to expect further development beyond this work.

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