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# Nonlinear Phenomena and Stability Analysis for Discrete Control Systems

Yoshifumi Okuyama  
Tottori University, Emeritus  
Japan

## 1. Introduction

Almost all feedback control systems are realized using discretized (discrete-time and discrete-value, i.e., digital) signals. However, the analysis and design method of discretized/quantized (nonlinear) control systems has not been established (Desoer et al., 1975; Elia et al., 2001; Harris et al., 1983; Kalman, 1956; Katz, 1981). This article analyzes the nonlinear phenomena and stability of discretized control systems in a frequency domain<sup>1</sup> (Okuyama, 2006; 2007; 2008). In these studies, it is assumed that the discretization is executed on the input and output sides of a nonlinear element at equal spaces, and the sampling period is chosen of such a size suitable for the discretization in the space. Based on the premise, the discretized (point-to-point) nonlinear characteristic is examined from two viewpoints, i.e., global and local. By partitioning the discretized nonlinear characteristic into two sections and by defining a sectorial area over a specified threshold, the concept of the robust stability condition for nonlinear discrete-time systems is applied to the discretized (hereafter, simply written as discrete) nonlinear control system in question. As a result, the nonlinear phenomena of discrete control systems are clarified, and the stability of discrete nonlinear feedback systems is elucidated.

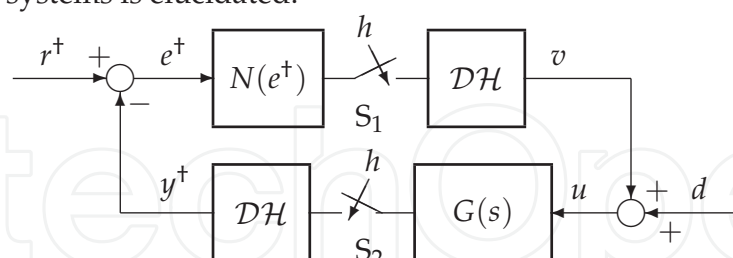


Fig. 1. Nonlinear sampled-data control system.

## 2. Discrete nonlinear control system

The discrete nonlinear control system to be considered here is represented by a sampled-data control system with two samplers,  $S_1$ ,  $S_2$  and the continuous nonlinear characteristic  $N(\cdot)$  as

<sup>1</sup> In the time domain analysis (e.g., Lyapunov function method), it is difficult to find a Lyapunov function for the discretized (severe nonlinear characteristic) feedback system. The frequency domain analysis will be important in cases where physical systems with uncertainty in the system-order are considered.

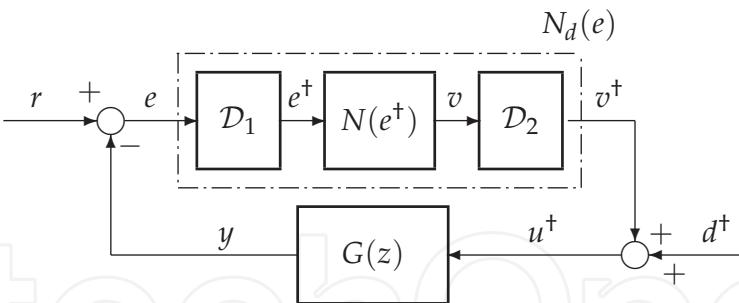


Fig. 2. Discrete nonlinear control system.

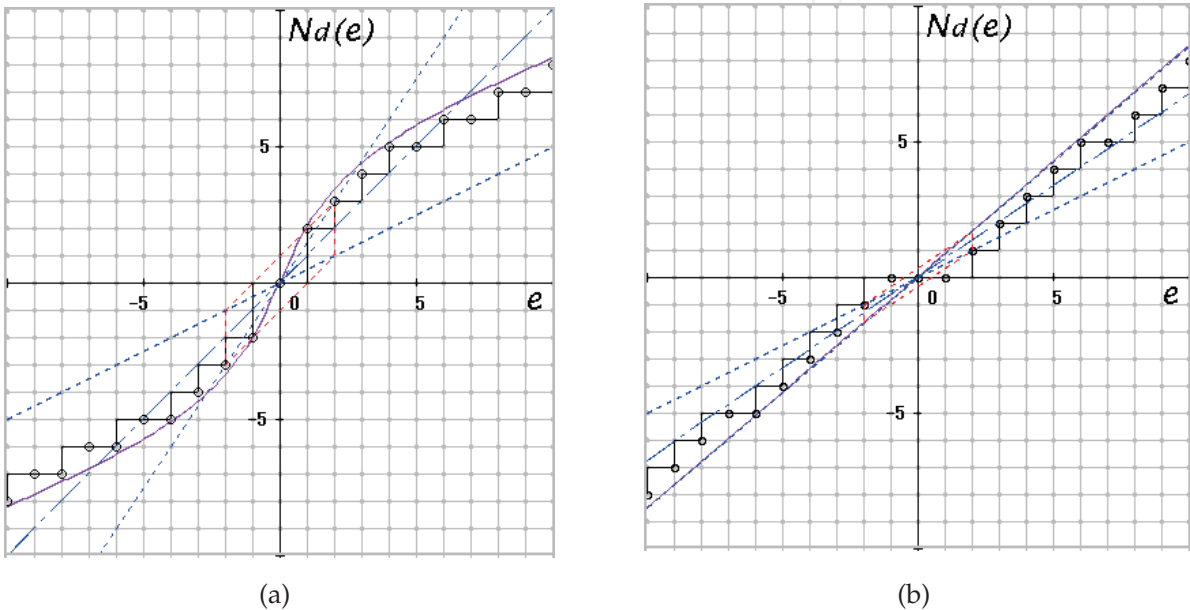


Fig. 3. Discretized nonlinear characteristics.

shown in Fig. 1. Here,  $\mathcal{DH}$  denotes the discretization and zero-order-hold, which are usually performed in A/D(D/A) conversion, and  $G(s)$  is the transfer function of the linear controlled system. It is assumed that the two samplers with a sampling period  $h$  operate synchronously.

The sampled-data control system can be equivalently transformed into a discrete control system as shown in Fig. 2. Here,  $G(z)$  is the  $z$ -transform of  $G(s)$  together with zero-order-hold, and  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are the discretizing units on the input and output sides of the nonlinear element, respectively. The relationship between  $e$  and  $v^+ = N_d(e)$  in the figure becomes a stepwise nonlinear characteristic on integer grid coordinates as shown in Fig. 3 (a). Here, a round-down discretization, which is usually executed on a computer, is applied. Therefore, the relationship between  $e^+$  and  $u^+$  is indicated by small circles (i.e. a point-to-point transition) on the stepwise nonlinear characteristic. Even if continuous characteristic  $N(\cdot)$  is linear, the discretized characteristic  $v^+$  becomes nonlinear on integer grid coordinates as shown in Fig. 3 (b) (Okuyama, 2009).

In Fig. 2, each symbol  $e, u, y, \dots$  indicates the sequence  $e(k), u(k), y(k), \dots, (k = 0, 1, 2, \dots)$  in discrete time, but for continuous value. On the other hand, each symbol  $e^+, u^+, \dots$  indicates a

discrete value that can be assigned to an integer number, e.g.,

$$\begin{aligned} e^{\dagger} &\in \{\dots, -3\gamma, -2\gamma, -\gamma, 0, \gamma, 2\gamma, 3\gamma, \dots\}, \\ u^{\dagger} &\in \{\dots, -3\gamma, -2\gamma, -\gamma, 0, \gamma, 2\gamma, 3\gamma, \dots\}, \end{aligned}$$

where  $\gamma$  is the resolution of each variable. Here, it is assumed that the input and output signals of the nonlinear characteristic have the same resolution in the discretization. In the figure,  $e^{\dagger}$  and  $u^{\dagger}$  also represent the sequence  $e^{\dagger}(k)$  and  $u^{\dagger}(k)$ . Without loss of generality, hereafter,  $\gamma = 1.0$  is assumed. Thus, the input and output variables of the nonlinear element can be considered in the set of integer numbers, i.e.,

$$\begin{aligned} e^{\dagger}(k), u^{\dagger}(k) &\in Z \\ Z &\stackrel{\text{def}}{=} \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}. \end{aligned}$$

### 3. Equivalent discrete-time system

In this study, the stepwise and point-to-point nonlinear characteristic is partitioned into the following two sections:

$$\begin{aligned} N_d(e) &= K(e + v(e)), \quad 0 < K < \infty, \\ |v(e)| &\leq \bar{v} < \infty, \end{aligned} \quad (1)$$

for  $|e| < \varepsilon$ , and

$$\begin{aligned} N_d(e) &= K(e + n(e)), \quad 0 < K < \infty, \\ |n(e)| &\leq \alpha|e|, \quad 0 < \alpha \leq 1, \end{aligned} \quad (2)$$

for  $|e| \geq \varepsilon$ , where  $v(e)$  and  $n(e)$  are nonlinear terms relative to nominal linear gain  $K$ . Equation (1) represents a bounded nonlinearity which exists in a finite region. On the other hand, (2) represents a sectorial nonlinearity of which the equivalent linear gain exists in a limited range. Therefore, when we consider the robust stability “in a global sense”, it is sufficient to consider the nonlinear term  $n(e)$ . Here,  $\varepsilon$  is a threshold of the input signal  $e$ . As a matter of course, (1) and (2) must be satisfied with respect to the discretized value  $e = e^{\dagger}$  because  $e^{\dagger} \in e$ .

Based on the above consideration, the following new sequences  $e_m^{*\dagger}(k)$  and  $w_m^{*\dagger}(k)$  are defined:

$$e_m^{*\dagger}(k) = e_m^{\dagger}(k) + q \cdot \frac{\Delta e^{\dagger}(k)}{h}, \quad (3)$$

$$w_m^{*\dagger}(k) = w_m^{\dagger}(k) - \alpha q \cdot \frac{\Delta e^{\dagger}(k)}{h}. \quad (4)$$

where  $q$  is a non-negative number,  $e_m^{\dagger}(k)$  and  $w_m^{\dagger}(k)$  are neutral points of sequences  $e^{\dagger}(k)$  and  $w^{\dagger}(k)$ ,

$$e_m^{\dagger}(k) = \frac{e^{\dagger}(k) + e^{\dagger}(k-1)}{2}, \quad (5)$$

$$w_m^{\dagger}(k) = \frac{w^{\dagger}(k) + w^{\dagger}(k-1)}{2}, \quad (6)$$

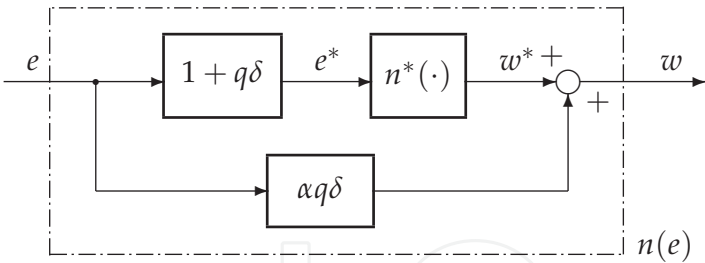


Fig. 4. Nonlinear subsystem.

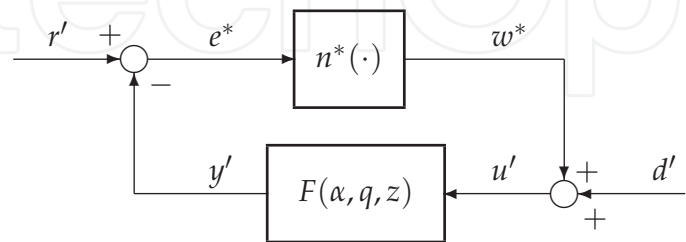


Fig. 5. Equivalent feedback system.

and  $\Delta e^+(k)$  is the backward difference of sequence  $e^+(k)$ ,

$$\Delta e^+(k) = e^+(k) - e^+(k - 1).$$

(7)

The relationship between equations (3) and (4) in regard to the continuous values is shown by the block diagram in Fig. 4. In this figure,  $\delta$  is defined as

$$\delta(z) := \frac{2}{h} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}.$$

(8)

Equation (8) corresponds to the bilinear transformation between  $z$  and  $\delta$ . Thus, the loop transfer function from  $w^*$  to  $e^*$  can be given by  $F(\alpha, q, z)$ , as shown in Fig. 5, where

$$F(\alpha, q, z) = \frac{(1 + q\delta(z))KG(z)}{1 + (1 + \alpha q\delta(z))KG(z)},$$

(9)

and  $r', d'$  are transformed exogenous inputs. Here, the variables such as  $w^*, u'$  and  $y'$  written in Fig. 5 indicate the  $z$ -transformed ones.

In this study, the following assumption is provided on the basis of the relatively fast sampling and the slow response of the controlled system.

**[Assumption]** The absolute value of the backward difference of sequence  $e(k)$  is not more than  $\gamma$ , i.e.,

$$|\Delta e(k)| = |e(k) - e(k - 1)| \leq \gamma.$$

(10)

If the condition (10) is satisfied,  $\Delta e^+(k)$  defined by (7) is exactly  $\pm\gamma$  or 0 because of the discretization. That is, the absolute value of the backward difference can be given as

$$|\Delta e^+(k)| = |e^+(k) - e^+(k - 1)| = \gamma \text{ or } 0. \quad \square$$

This assumption will be satisfied in the following examples.

#### 4. Norm conditions

In this section, some lemmas for norm conditions are presented. Here, in regard to (2), the following new nonlinear function is defined.

$$f(e) := n(e) + \alpha e. \quad (11)$$

When considering the discretized output of the nonlinear term,  $w^\dagger = n(e^\dagger)$ , the following expression can be given:

$$f(e^\dagger(k)) = w^\dagger(k) + \alpha e^\dagger(k). \quad (12)$$

From inequality (2), it can be seen that the function (12) belongs to the first and third quadrants. Considering the equivalent linear characteristic, the following inequality can be defined:

$$0 \leq \beta(k) := \frac{f(e^\dagger(k))}{e^\dagger(k)} \leq 2\alpha. \quad (13)$$

When this type of nonlinearity  $\beta(k)$  is used, inequality (2) can be expressed as

$$w^\dagger(k) = n(e^\dagger(k)) = (\beta(k) - \alpha)e^\dagger(k). \quad (14)$$

For the neutral points of  $e^\dagger(k)$  and  $w^\dagger(k)$ , the following expression is given from (12):

$$\frac{1}{2}(f(e^\dagger(k)) + f(e^\dagger(k-1))) = w_m^\dagger(k) + \alpha e_m^\dagger(k). \quad (15)$$

Moreover, equation (14) is rewritten as

$$w_m^\dagger(k) = (\beta(k) - \alpha)e_m^\dagger(k).$$

Since  $|e_m^\dagger(k)| \leq |e_m(k)|$ , the following inequality is satisfied when a round-down discretization is executed:

$$|w_m^\dagger(k)| \leq \alpha |e_m^\dagger(k)| \leq \alpha |e_m(k)|. \quad (16)$$

Based on the above premise, the following norm inequalities are examined (Okuyama et al., 1999; Okuyama, 2006).

**[Lemma-1]** The following inequality holds for a positive integer  $p$ :

$$\|w_m^\dagger(k)\|_{2,p} \leq \alpha \|e_m^\dagger(k)\|_{2,p} \leq \alpha \|e_m(k)\|_{2,p}. \quad (17)$$

Here,  $\|\cdot\|_{2,p}$  denotes the Euclidean norm, which can be defined by

$$\|x(k)\|_{2,p} := \left( \sum_{k=1}^p x^2(k) \right)^{1/2}.$$

(Proof) The proof is clear from inequality (16).  $\square$

**[Lemma-2]** If the following inequality is satisfied in regard to the inner product of the neutral points of (12) and the backward difference (7):

$$\langle w_m^\dagger(k) + \alpha e_m^\dagger(k), \Delta e^\dagger(k) \rangle_p \geq 0, \quad (18)$$

the following inequality can be obtained:

$$\|w_m^{*+}(k)\|_{2,p} \leq \alpha \|e_m^{*+}(k)\|_{2,p} \quad (19)$$

for any  $q \geq 0$ . Here,  $\langle \cdot, \cdot \rangle_p$  denotes the inner product, which can be defined as

$$\langle x_1(k), x_2(k) \rangle_p = \sum_{k=1}^p x_1(k)x_2(k).$$

(Proof) The following equation is obtained from (3) and (4):

$$\begin{aligned} \alpha^2 \|e_m^{*+}(k)\|_{2,p}^2 - \|w_m^{*+}(k)\|_{2,p}^2 &= \alpha^2 \left\| e_m^+(k) + q \frac{\Delta e^+(k)}{h} \right\|_{2,p}^2 - \left\| w_m^+(k) - \alpha q \frac{\Delta e^+(k)}{h} \right\|_{2,p}^2 \\ &= \alpha^2 \|e_m^+(k)\|_{2,p}^2 - \|w_m^+(k)\|_{2,p}^2 + \frac{2\alpha q}{h} \cdot \langle w_m^+(k) + \alpha e_m^+(k), \Delta e^+(k) \rangle_p. \end{aligned} \quad (20)$$

Thus, (19) is satisfied by using the left inequality of (17).  $\square$

In regard to the input of  $n^*(\cdot)$ , the following inequality can be obtained from (20) and the second inequality of (17) as follows:

$$\|w_m^{*+}(k)\|_{2,p} \leq \alpha \|e_m^*(k)\|_{2,p}, \quad (21)$$

when inequality (18) is satisfied.

## 5. Sum of trapezoidal areas

The left side of inequality (18) can be expressed as a sum of trapezoidal areas.

**[Lemma-3]** For any step  $p$ , the following equation is satisfied:

$$\sigma(p) := \langle w_m^+(k) + \alpha e_m^+(k), \Delta e^+(k) \rangle_p = \frac{1}{2} \sum_{k=1}^p (f(e^+(k)) + f(e^+(k-1))) \Delta e^+(k). \quad (22)$$

(Proof) The proof is clear from (15).  $\square$

In general, the sum of trapezoidal areas holds the following property.

**[Lemma-4]** If inequality (10) is satisfied in regard to the discretization of the control system, the sum of trapezoidal areas becomes non-negative for any  $p$ , that is,

$$\sigma(p) \geq 0. \quad (23)$$

(Proof) Since  $f(e^+(k))$  belongs to the first and third quadrants, the area of each trapezoid

$$\tau(k) := \frac{1}{2} (f(e^+(k)) + f(e^+(k-1))) \Delta e^+(k) \quad (24)$$

is non-negative when  $e(k)$  increases (decreases) in the first (third) quadrant. On the other hand, the trapezoidal area  $\tau(k)$  is non-positive when  $e(k)$  decreases (increases) in the first (third) quadrant.

Strictly speaking, when  $(e(k) \geq 0 \text{ and } \Delta e(k) \geq 0)$  or  $(e(k) \leq 0 \text{ and } \Delta e(k) \leq 0)$ ,  $\tau(k)$  is non-negative for any  $k$ . On the other hand, when  $(e(k) \geq 0 \text{ and } \Delta e(k) \leq 0)$  or  $(e(k) \leq 0 \text{ and } \Delta e(k) \geq 0)$ ,  $\tau(k)$  is non-positive for any  $k$ .

0 and  $\Delta e(k) \geq 0$ ),  $\tau(k)$  is non-positive for any  $k$ . Here,  $\Delta e(k) \geq 0$  corresponds to  $\Delta e^\dagger(k) = \gamma$  or 0 (and  $\Delta e(k) \leq 0$  corresponds to  $\Delta e^\dagger(k) = -\gamma$  or 0) for the discretized signal, when inequality (10) is satisfied.

The sum of trapezoidal area is given from (22) as:

$$\sigma(p) = \sum_{k=1}^p \tau(k). \quad (25)$$

Therefore, the following result is derived based on the above. The sum of trapezoidal areas becomes non-negative,  $\sigma(p) \geq 0$ , regardless of whether  $e(k)$  (and  $e^\dagger(k)$ ) increases or decreases. Since the discretized output traces the same points on the stepwise nonlinear characteristic, the sum of trapezoidal areas is canceled when  $e(k)$  (and  $e^\dagger(k)$ ) decreases (increases) from a certain point  $(e^\dagger(k), f(e^\dagger(k)))$  in the first (third) quadrant. (Here, without loss of generality, the response of discretized point  $(e^\dagger(k), f(e^\dagger(k)))$  is assumed to commence at the origin.) Thus, the proof is concluded.  $\square$

## 6. Stability in a global sense

By applying a small gain theorem to the loop transfer characteristic (9), the following robust stability condition of the discrete nonlinear control system can be derived.

**[Theorem]** If there exists a  $q \geq 0$  in which the sector parameter  $\alpha$  in regard to nonlinear term  $n(\cdot)$  satisfies the following inequality, the discrete-time control system with sector nonlinearity (2) is robust stable in an  $\ell_2$  sense:

$$\alpha < \eta(q, \omega) := \frac{-q\Omega V + \sqrt{q^2\Omega^2 V^2 + (U^2 + V^2)\{(1+U)^2 + V^2\}}}{U^2 + V^2}, \quad (26)$$

$\forall \omega \in [0, \omega_c], \quad \omega_c : \text{cutoff frequency}$

when the linearized system with nominal gain  $K$  is stable. Here,  $\Omega(\omega)$  is the distorted frequency of angular frequency  $\omega$  and is given by

$$\delta(e^{j\omega h}) = j\Omega(\omega) = j\frac{2}{h} \tan\left(\frac{\omega h}{2}\right), \quad j = \sqrt{-1}. \quad (27)$$

In addition,  $U(\omega)$  and  $V(\omega)$  are the real and the imaginary parts of  $KG(e^{j\omega h})$ , respectively.

(Proof) Based on the loop characteristic in Fig. 5, the following inequality can be given in regard to  $z = e^{j\omega h}$ :

$$\|e_m^*(z)\|_{2,p} \leq c_1 \|r'_m(z)\|_{2,p} + c_2 \|d'_m(z)\|_{2,p} + \sup_{z=1} |F(\alpha, q, z)| \cdot \|w_m^{*+}(z)\|_{2,p}.$$

Here,  $r'_m(z)$  and  $d'_m(z)$  denote the  $z$ -transformation for the neutral points of sequences  $r'(k)$  and  $d'(k)$ , respectively. Moreover,  $c_1$  and  $c_2$  are positive constants.

By applying inequality (21), the following expression is obtained:

$$\left(1 - \alpha \cdot \sup_{z=1} |F(\alpha, q, z)|\right) \|e_m^*(z)\|_{2,p} \leq c_1 \|r'_m(z)\|_{2,p} + c_2 \|d'_m(z)\|_{2,p}. \quad (28)$$



Therefore, if the following inequality (i.e., the small gain theorem with respect to  $\ell_2$  gains) is valid,

$$|F(\alpha, q, e^{j\omega h})| \leq 1/\alpha, \quad (29)$$

the sequences  $e_m^*(k)$ ,  $e_m(k)$ ,  $e(k)$  and  $y(k)$  in the feedback system are restricted in finite values when exogenous inputs  $r(k)$ ,  $d(k)$  are finite and  $p \rightarrow \infty$ .

By substituting (9) into inequality (29), the following is obtained:

$$\left| \frac{(1 + jq\Omega(\omega))KG(e^{j\omega h})}{1 + (1 + j\alpha q\Omega(\omega))KG(e^{j\omega h})} \right| < \frac{1}{\alpha}. \quad (30)$$

From the square of both sides of inequality (30),

$$\alpha^2(1 + q^2\Omega^2)(U^2 + V^2) < (1 + U - \alpha q\Omega V)^2 + (V + \alpha q\Omega U)^2$$

Then,

$$\alpha^2(U^2 + V^2) + 2\alpha q\Omega V - \{(1 + U^2) + V^2\} < 0. \quad (31)$$

Consequently, as a solution of inequality (31),

$$\alpha < \frac{-q\Omega V + \sqrt{q^2\Omega^2 V^2 + (U^2 + V^2)\{(1 + U)^2 + V^2\}}}{U^2 + V^2}$$

can be given.  $\square$

Since inequality (26) in Theorem-1 is for all  $\omega$  (and  $\Omega$ ) considered and a certain  $q$ , the condition is rewritten as the following max-min problem.

**[Corollary]** If the following inequality is satisfied, the discrete-time control system with sector nonlinearity (2) is robust stable:

$$\alpha < \eta(q_0, \omega_0) = \max_q \min_{\omega} \eta(q, \omega), \quad (32)$$

when the linearized system with nominal gain is stable.  $\square$

In this study, a non-conservative sufficient condition for the stability of discrete-time and discrete-value control systems is derived by applying the concept of robust stability in our previous paper (Okuyama et al., 2002a). The stability condition is, however, not satisfied for the entire area of the input of nonlinearity  $N(e)$  because of the stepwise and point-to-point characteristic. Even if the response seems to be asymptotic, there may remain a fluctuation (a sustained oscillation in the discrete time) or an offset. Of course, a divergent response that reaches the sustained oscillation may occur. These responses are typical nonlinear phenomena. The theorem (and corollary) derived here should be considered as the robust stability condition in a global sense. In addition, it is valid based on an assumption in the relationship between the sampling period and the system dynamics. However, this result will be useful in designing a discrete (digital, packet transmission) control system in practice.

Naturally, the stability condition becomes that of continuous-time and continuous-value nonlinear control systems, when the sampling period  $h$  and the resolution  $\gamma$  approach zero. Inequality (26) in Theorem-1 corresponds to Popov's criterion for discrete-time systems and contains the circle criterion for nonlinear time-varying (discrete-time) systems in a special case. The relationship between them will be described in the next section.

## 7. Relation to Popov's criterion

Inequality (30) can be rewritten as follows:

$$\left| \frac{\alpha H(\alpha, q, e^{j\omega h})}{1 + \alpha H(\alpha, q, e^{j\omega h})} \right| < 1, \quad (33)$$

where

$$H(\alpha, q, e^{j\omega h}) = \frac{(1 + jq\Omega(\omega))KG(e^{j\omega h})}{1 + (1 - \alpha)KG(e^{j\omega h})}.$$

From (33), the following inequality is obtained:

$$2\alpha \cdot \Re\{H(\alpha, q, e^{j\omega h})\} + 1 > 0. \quad (34)$$

Therefore, the following robust stability condition can be given:

$$\Re \left\{ \frac{1 + (1 + \alpha)KG(e^{j\omega h}) + 2jq\Omega(\omega)KG(e^{j\omega h})}{1 + (1 - \alpha)KG(e^{j\omega h})} \right\} > 0, \quad (35)$$

which is equivalent to inequality (26). When  $\alpha = 1$  is chosen, (35) can be written as follows:

$$\frac{1}{K_m} + \Re\{(1 + jq\Omega(\omega))G(e^{j\omega h})\}, \quad (36)$$

where  $K_m = 2K$ . In this case, the allowable sector of nonlinear characteristic  $N(\cdot)$  is given as

$$0 \leq N(e)e \leq K_m e^2, \quad e \neq 0. \quad (37)$$

When  $h$  approaches zero (or  $\omega$  is a low frequency), inequalities (36) and (37) are equivalent to an expression of Popov's criterion for continuous-time systems.

In case of  $q = 0$ , the left side of (26) becomes the inverse the absolute value of complementary sensitivity function  $T(j\omega)$ .

$$\eta(0, \omega) = \frac{\sqrt{(1 + U^2)) + V^2}}{\sqrt{U^2 + V^2}} = \frac{1}{|T(j\omega)|} > \alpha. \quad (38)$$

On the other hand, from (35)

$$\Re \left\{ \frac{1 + (1 + \alpha)KG(e^{j\omega h})}{1 + (1 - \alpha)KG(e^{j\omega h})} \right\} > 0 \quad (39)$$

is obtained. Inequalities (38) and (39) correspond to the circle criterion for nonlinear time-varying systems.

## 8. Validity of Aizerman's conjecture

In the following case, Theorem-1 becomes equal to the robust stability condition of the linear interval gain that corresponds to Aizerman's conjecture which was extended into discrete-time systems (Okuyama et al., 1998).

**[Theorem-2]** If the right side of (32) is satisfied at the saddle point,

$$\left( \frac{\partial \eta(q, \omega)}{\partial q} \right)_{q=q_0, \omega=\omega_0} = 0, \quad (40)$$

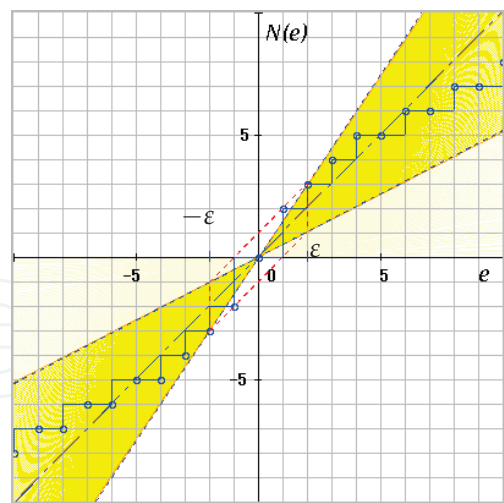


Fig. 6. Discretized nonlinear characteristic and stable sector for Example-1.

inequality (26) of Theorem-1 becomes equal to the robust stability condition provided for a linear time-invariant discrete-time system.  
(Proof) This theorem can easily be proven by using the right side of (26). Then,

$$\frac{\partial \eta(q, \omega)}{\partial q} = \frac{-\eta(q, \omega) \Omega(\omega) V(\omega)}{\sqrt{q^2 \Omega^2 v^2 + (U^2 + V^2) \{ (1 + U)^2 + V^2 \}}}. \tag{41}$$

From (40), the following can be obtained:

$$\eta(q_0, \omega_0) \Omega(\omega_0) V(\omega_0) = 0. \tag{42}$$

Obviously,  $\eta(q, \omega) > 0$ . Moreover, since  $0 < \omega_0 < \pi/h$ ,  $\Omega(\omega_0) > 0$  from (27). Then,

$$V(\omega_0) = 0 \tag{43}$$

is obtained. Thus,

$$\eta(q_0, \omega_0) = \frac{|1 + U(\omega_0)|}{|U(\omega_0)|} > \alpha \tag{44}$$

Inequality (44) corresponds to the stability condition which was determined for the time-invariant discrete-time system with a linear gain, i.e., the Nyquist stability condition for a discrete-time system.

Theorem-2 shows that the robust stability condition for a linear time-invariant system (the concept of interval set) can be applied to nonlinear discrete-time control systems, when (40) is satisfied. However, (32) is not always valid at the saddle point given in (40). In the following example, it can be shown that there are counter examples of Aizerman’s conjecture extended into the nonlinear discrete-time systems.

9. Numerical examples

In order to verify the theoretical result, two numerical examples for discrete control systems with saturation type nonlinearity are presented.

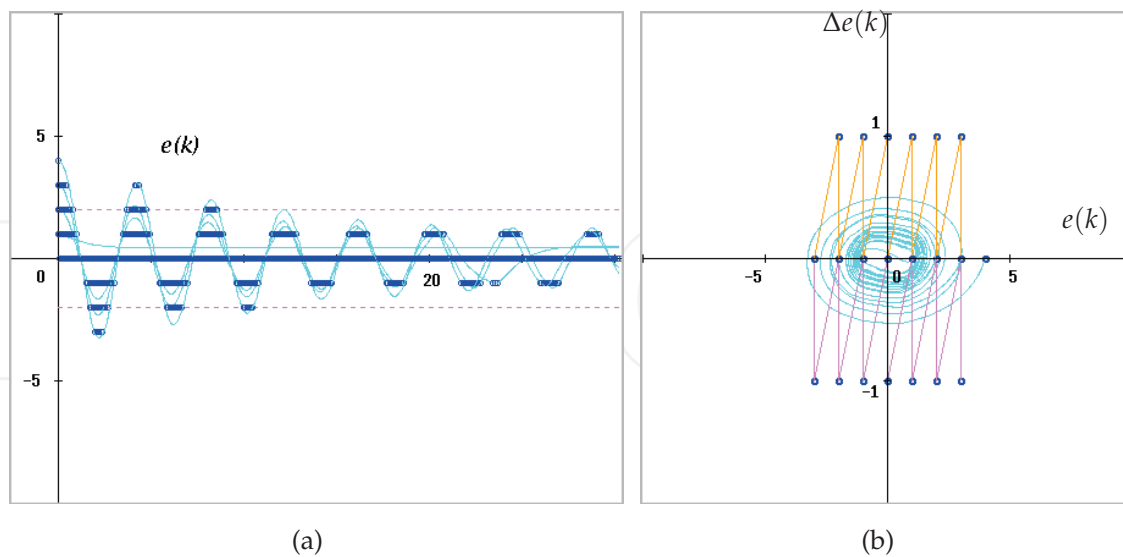


Fig. 7. Time responses of  $e(k)$  and  $e^+(k)$ , and phase traces  $(e(k), \Delta e(k))$  for Example-1 ( $r = 1.0, 2.0, 3.0, 4.0, 5.0$ ).

**[Example-1]** Consider the following controlled system:

$$G(s) = \frac{K_p(s+6)}{s(s+1)(s+2)}, \quad (45)$$

where  $K_p = 1.0$ . It is assumed that the discretized nonlinear characteristic (discretized sigmoid, i.e., arc tangent function (Okuyama et al., 2002b) is as shown in Fig. 6. Here, the resolution value is chosen as  $\gamma = 1.0$ . For C-language expression, it can be written as

$$\begin{aligned} e^+ &= \gamma * (\text{double})(\text{int})(e/\gamma), \\ v &= 0.4 * e^+ + 3.0 * \text{atan}(0.6 * e^+), \\ v^+ &= \gamma * (\text{double})(\text{int})(v/\gamma), \end{aligned}$$

where (int) and (double) denote the conversion into an integral number (a round-down discretization) and the reconversion into a double-precision real number, respectively.

When choosing the threshold  $\varepsilon = 2.0$ , the sectorial area of the stepwise (point-to-point) nonlinearity for  $\varepsilon \leq |e| < 35.0$  can be determined as  $[0.5, 1.5]$  drawn by dotted lines in the figure. In this example, the sampling period is chosen as  $h = 0.1$ . From (26) and (32), the max-min value can be calculated as follows:

$$\max_q \eta(q, \omega_0) = \eta(q_0, \omega_0) = 0.49,$$

when the nominal gain  $K = 1.0$ . Hence,  $\alpha < 0.49$  and the stable area is determined as  $[0.51, 1.49]$ . Obviously, this sector contains the area bounded by the dotted lines. Thus, the discrete control system is stable in a global sense.

The stability condition for linear gain  $K$  can be calculated as  $0 < K < 1.5$  when the sampling period is  $h = 0.1$ . In this example, Aizerman's conjecture for discrete-time system is satisfied. Figures 7 (a) and (b) show time responses  $e(k)$ ,  $e^+(k)$  and phase traces  $(e(k), \Delta e(k))$ ,

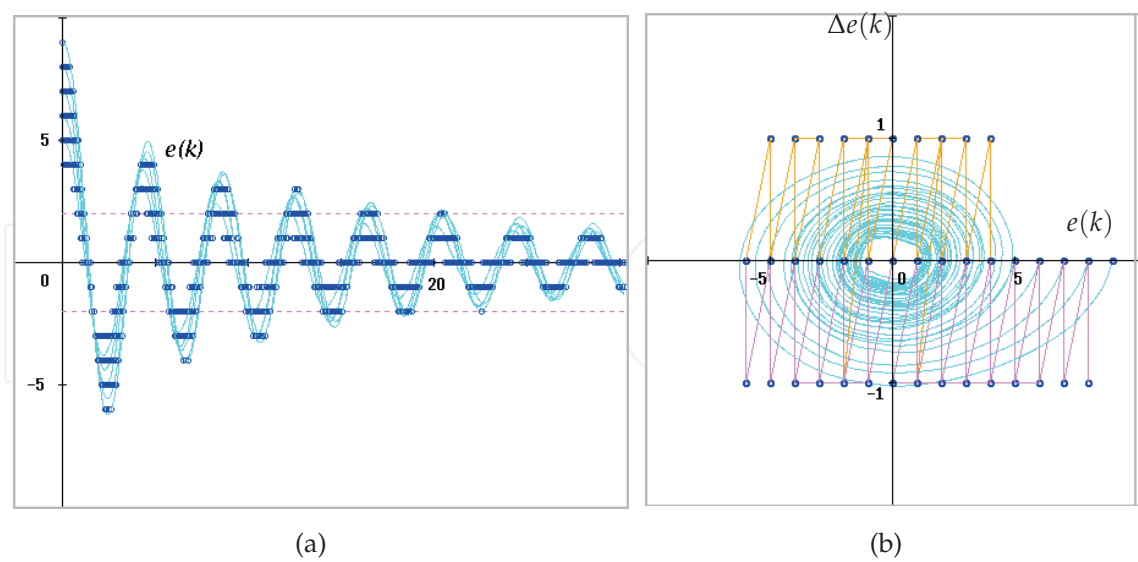


Fig. 8. Time responses of  $e(k)$ ,  $e^+(k)$ , and phase traces  $(e(k), \Delta e(k))$  for Example-1 ( $r = 5.0, 6.0, 7.0, 8.0, 9.0$ ).

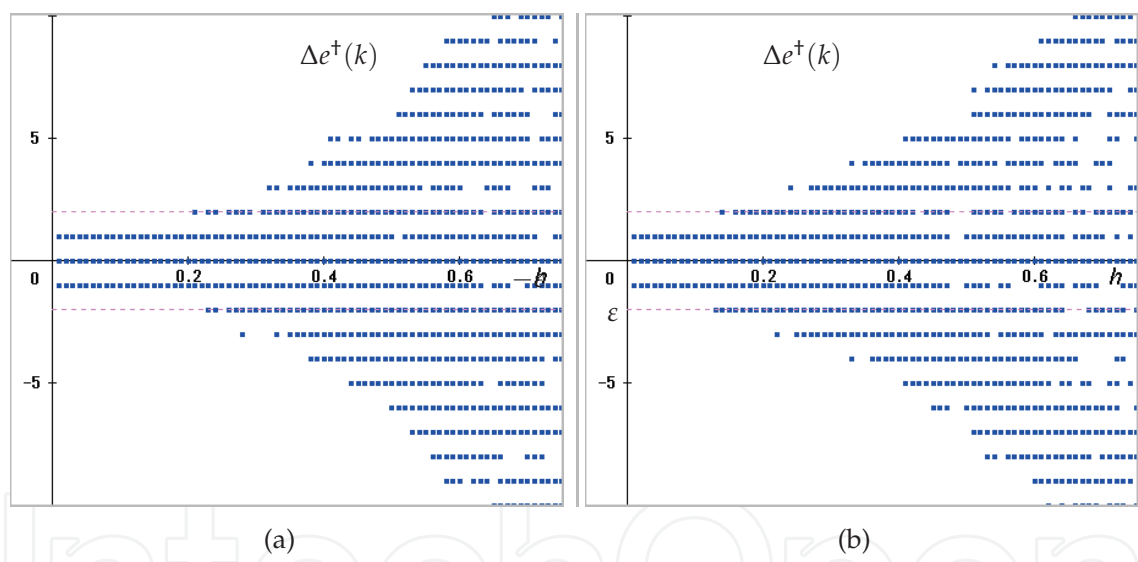


Fig. 9. Backward difference  $\Delta e^+(k)$  vs. sampling period  $h$  for Example-1 ( $r = 3.0$  and  $r = 9.0$ ).

$(e^+(k), \Delta e^+(k))$  of the discrete nonlinear control system when the reference inputs are  $r = 1.0, 2.0, 3.0$ . Figures 8 (a) and (b) show those responses when  $r = 5.0, 7.0, 9.0$ . Although the responses contain sustained oscillations, they do not exceed the threshold  $\varepsilon = 2.0$ . The input and the output of the nonlinearity lie in a parallelogram shown in Fig. 6. The robust stability in a global sense is guaranteed for all the reference inputs  $r$ . The above behavior can be estimated from the intersections of the highest gain of the sector and the stepwise nonlinear characteristic. Obviously, discrete-values  $(1.0, 2.0)$  and  $(-1.0, -2.0)$  lie in the outside of the stable sector.

Figures 9 (a) and (b) show the traces of backward difference  $\Delta e^+(k)$  when the sampling period  $h$  increases. As is obvious from the figure, the assumption of (10) is satisfied for  $h < 0.12$  when

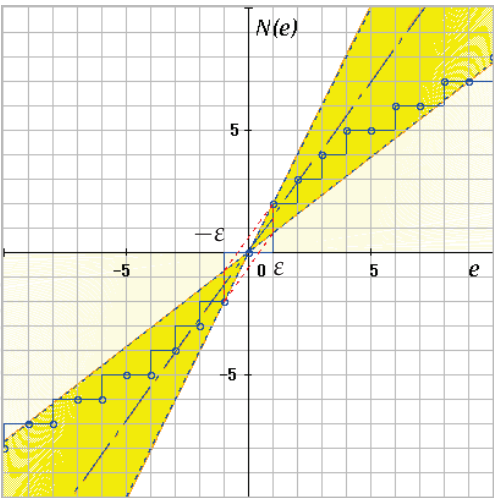


Fig. 10. Discretized nonlinear characteristic and stable sector for Example-2.

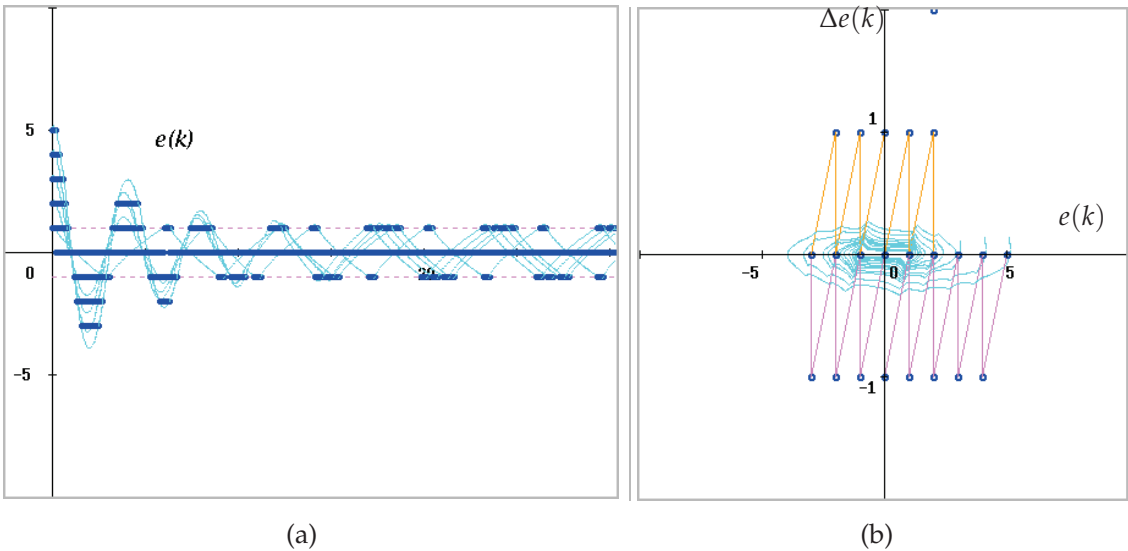


Fig. 11. Time responses of  $e(k)$  and  $e^\dagger(k)$  and phase traces  $(e(k), \Delta e(k))$  for Example-2 ( $r = 1.0, 2.0, 3.0, 4.0, 5.0$ ).  
 $r = 9.0$ , and for  $h < 2.0$  when  $r = 3.0$ . In either case, the assumption is satisfied in regard to  $h = 0.1$ .

[Example-2] Consider the following controlled system:

$$G(s) = \frac{K_p(-s + 8)(s + 4)}{s(s + 0.2)(s + 16)}, \tag{46}$$

where  $K_p = 1.0$ . Here, the same nonlinear characteristic is chosen as shown in Example-1. When the threshold  $\varepsilon = 1.0$  is specified, the sectorial area of the stepwise nonlinearity for  $\varepsilon \leq |e| < 10.0$  can be determined as  $[0.78, 2.0]$ . In this example, the sampling period is chosen as  $h = 0.04$ . The max-min value can be calculated as follows:

$$\max_q \eta(q, \omega_0) = \eta(q_0, \omega_0) = 0.45,$$

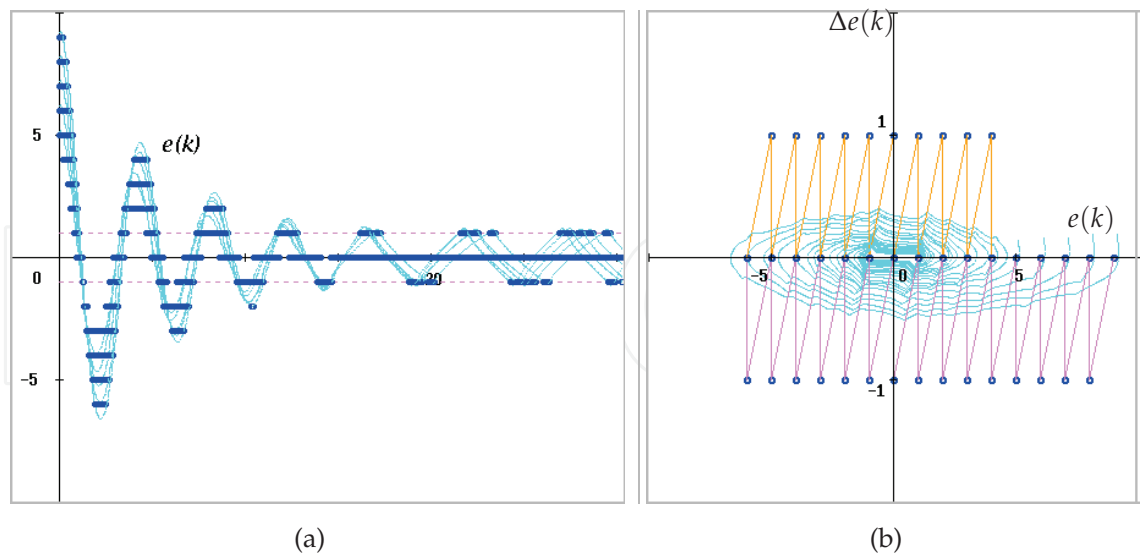


Fig. 12. Time responses of  $e(k)$  and  $e^{\dagger}(k)$  and phase traces  $(e(k), \Delta e(k))$  for Example-2 ( $r = 5.0, 6.0, 7.0, 8.0, 9.0$ ).

when the nominal gain  $K = 1.4$ . Hence,  $\alpha < 0.45$  and the stable area is determined as  $[0.77, 2.02]$ . This sector contains the above area. However, the stability region of control systems with linear gain  $K$  is given as  $0 < K < 6.3$  when the sampling period is  $h = 0.04$ . Obviously, the discrete nonlinear control system corresponds to a counter example of the Aizerman conjecture. Figures 11 and 12 show time responses  $e(k)$ ,  $e^{\dagger}(k)$  and phase traces  $(e(k), \Delta e(k))$ ,  $(e^{\dagger}(k), \Delta e^{\dagger}(k))$  of the discrete nonlinear control system, respectively. Although the nonlinear characteristic exists in the stable area for linear systems, a sustained oscillation is generated on account of a steep build-up characteristic in the lower side of the stable sector.

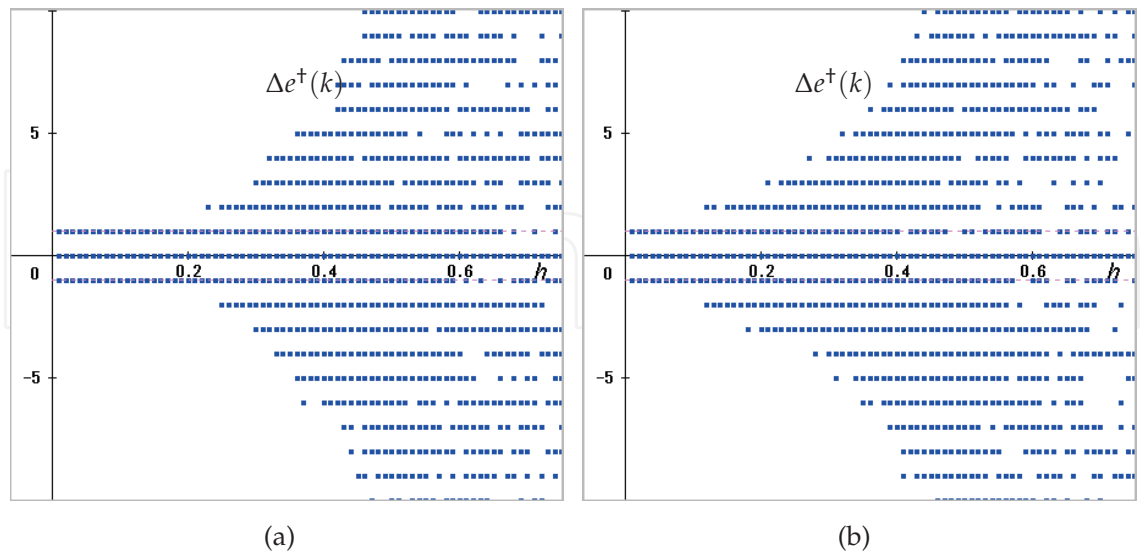


Fig. 13. Backward difference  $\Delta e^{\dagger}(k)$  vs. sampling period  $h$  for Example-2 ( $r = 3.0$  and  $r = 9.0$ ).



Figure 13 shows the traces of backward difference  $\Delta e^+(k)$  when the sampling period  $h$  increases. As is obvious from the figure, the assumption of (10) is satisfied for  $h < 0.11$  when  $r = 9.0$ , and for  $h < 2.2$  when  $r = 3.0$ . In either case, the assumption is satisfied in regard to  $h = 0.04$ .

## 10. Conclusions

This article analyzed the nonlinear phenomena and stability of discrete-time and discrete-value (discretized/quantized) control systems in a frequency domain. By partitioning the discretized nonlinear characteristic into two nonlinear sections and by defining a sectorial area over a specified threshold, the concept of the robust stability condition for nonlinear discrete-time systems was applied to the discrete nonlinear control systems. In consequence, the nonlinear phenomena of discrete control systems were clarified, and the robust stability of discrete nonlinear feedback systems was elucidated. The result described in this chapter will be useful in designing discrete (digital, event-driven, or packet transmission) control systems.

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A trend of investigation of Nonlinear Control Systems has been present over the last few decades. As a result the methods for its analysis and design have improved rapidly. This book includes nonlinear design topics such as Feedback Linearization, Lyapunov Based Control, Adaptive Control, Optimal Control and Robust Control. All chapters discuss different applications that are basically independent of each other. The book will provide the reader with information on modern control techniques and results which cover a very wide application area. Each chapter attempts to demonstrate how one would apply these techniques to real-world systems through both simulations and experimental settings.

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University Campus STeP Ri  
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51000 Rijeka, Croatia  
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Phone: +86-21-62489820  
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