

We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

186,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com



The Lane-Emden-Fowler Equation and Its Generalizations – Lie Symmetry Analysis

Chaudry Masood Khalique

*International Institute for Symmetry Analysis and Mathematical Modelling,
Department of Mathematical Sciences, North-West University,
Mmabatho
Republic of South Africa*

1. Introduction

In the study of stellar structure the Lane-Emden equation (1; 2)

$$\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + y^r = 0, \quad (1)$$

where r is a constant, models the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. This equation was proposed by Lane (1) (see also (3)) and studied in detail by Emden (2). Fowler (4; 5) considered a generalization of Eq. (1), called Emden-Fowler equation (6), where the last term is replaced by $x^{\nu-1}y^r$.

The Lane-Emden equation (1) also models the equilibria of nonrotating fluids in which internal pressure balances self-gravity. When spherically symmetric solutions of Eq. (1) appeared in (7), they got the attention of astrophysicists. In the latter half of the twentieth century, some interesting applications of the isothermal solution (singular isothermal sphere) and its nonsingular modifications were used in the structures of collisionless systems such as globular clusters and early-type galaxies (8; 9).

The work of Emden (2) also got the attention of physicists outside the field of astrophysics who investigated the generalized polytropic forms of the Lane-Emden equation (1) for specific polytropic indices r . Some singular solutions for $r = 3$ were produced by Fowler (4; 5) and the Emden-Fowler equation in the literature was established, while the works of Thomas (10) and Fermi (11) resulted in the Thomas-Fermi equation, used in atomic theory. Both of these equations, even today, are being investigated by physicists and mathematicians. Other applications of Eq. (1) can be found in the works of Meerson et al (12), Gnutzmann and Ritschel (13), and Bahcall (14; 15).

Many methods, including numerical and perturbation, have been used to solve Eq. (1). The reader is referred to the works of Horedt (16; 17), Bender (18) and Lema (19; 20), Roxbough and Stocken (21), Adomian et al (22), Shawagfeh (23), Burt (24), Wazwaz (25) and Liao (26) for a sample. Exact solutions of Eq. (1) for $r = 0, 1$ and 5 have been obtained (see for example Chandrasekhar (7), Davis (27), Datta (28) and Wrubel (29)). Usually, for $r = 5$, only a one-parameter family of solutions is presented. A more general form of (1), in which the

coefficient of y' is considered an arbitrary function of x , was investigated for first integrals by Leach (30).

Many problems in mathematical physics and astrophysics can be formulated by the generalized Lane-Emden equation

$$\frac{d^2y}{dx^2} + \frac{n}{x} \frac{dy}{dx} + f(y) = 0, \quad (2)$$

where n is a real constant and $f(y)$ is an arbitrary function of y . For $n = 2$ the approximate analytical solutions to the Eq. (2) were studied by Wazwaz (25) and Dehghan and Shakeri (31).

Another form of $f(y)$ is given by

$$f(y) = (y^2 - C)^{3/2}. \quad (3)$$

Inserting (3) into Eq. (1) gives us the "white-dwarf" equation introduced by Chandrasekhar (7) in his study of the gravitational potential of degenerate white-dwarf stars. In fact, when $C = 0$ this equation reduces to Lane-Emden equation with index $r = 3$.

Another nonlinear form of $f(y)$ is the exponential function

$$f(y) = e^y. \quad (4)$$

Substituting (4) into Eq. (1) results in a model that describes isothermal gas spheres where the temperature remains constant.

Equation (1) with

$$f(y) = e^{-y}$$

gives a model that appears in the theory of thermionic currents when one seeks to determine the density and electric force of an electron gas in the neighbourhood of a hot body in thermal equilibrium was thoroughly investigated by Richardson (32).

Furthermore, the Eq. (1) appears in eight additional cases for the function $f(y)$. The interested reader is referred to Davis (27) for more detail.

The equation

$$\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + e^{\beta y} = 0, \quad (5)$$

where β is a constant, has also been studied by Emden (2). In a recent work (33) an approximate implicit solution has been obtained for Eq. (5) with $\beta = 1$.

Furthermore, more general Emden-type equations were considered in the works (34–38). See also the review paper by Wong (39), which contains more than 140 references on the topic.

The so-called generalized Lane-Emden equation of the first kind

$$x \frac{d^2y}{dx^2} + \alpha \frac{dy}{dx} + \beta x^\nu y^n = 0, \quad (6)$$

and generalized Lane-Emden equation of the second kind

$$x \frac{d^2y}{dx^2} + \alpha \frac{dy}{dx} + \beta x^\nu e^{ny} = 0, \quad (7)$$

where α , β , ν and n are constants, have been recently studied in (40; 41). In Goenner (41), the author uncovered symmetries of Eq. (6) to explain integrability of (6) for certain values of the parameters considered in Goenner and Havas (40). Recently, the integrability of the generalized Lane-Emden equations of the first and second kinds has been discussed in Muatjetjeja and Khalique (42).

In this chapter, firstly, a generalized Lane-Emden-Fowler type equation

$$x \frac{d^2 y}{dx^2} + n \frac{dy}{dx} + x^\nu f(y) = 0, \quad (8)$$

where n and ν are real constants and $f(y)$ is an arbitrary function of y will be studied. We perform the Lie and Noether symmetry analysis of this problem. It should be noted that Eq. (8) for the power function $F(y) = y^r$ is related to the Emden-Fowler equation $y'' + p(X)y' = 0$ by means of the transformation on the independent variable $X = x^{1-n}$, $n \neq 1$ and $X = \ln x$, $n = 1$.

Secondly, we consider a generalized coupled Lane-Emden system, which occurs in the modelling of several physical phenomena such as pattern formation, population evolution and chemical reactions. We perform Noether symmetry classification of this system and compute the Noether operators corresponding to the standard Lagrangian. In addition the first integrals for the Lane-Emden system will be constructed with respect to Noether operators.

2. Lie point symmetry classification of (8)

We start by determining the equivalence transformations of Eq. (8). We recall (43) that an equivalence transformation

$$\bar{x} = \bar{x}(x, y), \quad \bar{y} = \bar{y}(x, y)$$

is a nondegenerate change of variables such that the family of Eqs. (8) remains invariant, i.e., Eq. (8) becomes

$$\bar{x} \frac{d^2 \bar{y}}{d\bar{x}^2} + n \frac{d\bar{y}}{d\bar{x}} + \bar{x}^\nu \bar{f}(\bar{y}) = 0$$

with \bar{f} depending on \bar{y} . Equivalence transformations are essential for simplifying the determining equation and for obtaining disjoint classes.

For Eq. (8) the equivalence transformations are

$$\begin{aligned} \bar{x} &= e^{a_2} x, \\ \bar{y} &= e^{a_3} y + a_1, \\ \bar{f} &= e^{a_3 - (1+\nu)a_2} f, \end{aligned} \quad (9)$$

where a_1 , a_2 and a_3 are constants. For details of computations see (44).

If X , given by

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$$

is an admitted generator of a symmetry group of Eq. (8), then

$$X^{[2]} \left(x \frac{d^2 y}{dx^2} + n \frac{dy}{dx} + x^\nu f(y) \right) \Big|_{(8)} = 0, \quad (10)$$

where $X^{[2]}$ is the second prolongation of X , gives the determining equations for the symmetry. This gives rise to

$$\begin{aligned} \xi &= b(x), \\ \eta &= c(x)y + d(x), \end{aligned}$$

$$\begin{aligned} x^{\nu-1}(cy + d)f'(y) + \left[x^{\nu-1}(2b' - c) + (\nu - 1)x^{\nu-2}b \right] f(y) \\ + \left(\frac{n}{x}c' + c'' \right) y + \left(\frac{n}{x}d' + d'' \right) = 0. \end{aligned} \quad (11)$$

If f is an arbitrary function, the above system yields $\xi = 0$, $\eta = 0$, meaning that the principal Lie algebra of Eq. (8) is trivial.

The function f depends upon y only. Thus Eq. (11) only holds if its coefficients identically vanish or they are proportional to a function $\alpha = \alpha(x)$, i.e.,

$$\begin{aligned} c &= r\alpha, \quad d = q\alpha, \quad 2b' - c + (\nu - 1)x^{-1}b = p\alpha, \\ c''x^{-\nu+1} + nx^{-\nu}c' &= h\alpha, \quad d''x^{-\nu+1} + nx^{-\nu}d' = g\alpha, \end{aligned} \quad (12)$$

where r, q, p, h and g are constants. Thus Eq. (11) becomes

$$(ru + q)F'(u) + pF(u) + hu + g = 0, \quad (13)$$

which is our classifying relation. This relation is invariant under the equivalence transformations (9) if

$$\begin{aligned} \bar{r} &= r, \quad \bar{q} = (ra_1 + q)e^{-a_3}, \quad \bar{p} = p, \\ \bar{h} &= he^{(\nu+1)a_2}, \quad \bar{g} = e^{-a_3 + (\nu+1)a_2}(ha_1 + g). \end{aligned} \quad (14)$$

The relations in (14) are used to find the non-equivalent forms of f and this leads to the following eight cases.

Case 1. $n \neq (1 - \nu)/2$, $f(y)$ arbitrary but not of the form contained in Cases 3, 4, 5 and 6.

No Lie point symmetry exists in this case.

Case 2. $n = (1 - \nu)/2$, $f(y)$ arbitrary but not of the form contained in Cases 4, 5 and 6.

We obtain one Lie point symmetry

$$X = x^{(1-\nu)/2} \frac{\partial}{\partial x} \quad (15)$$

for the corresponding Eq. (8).

Case 3. $f(y)$ is linear in y .

This case is well known and the corresponding Eq. (8) has $sl(3, \mathbb{R})$ symmetry algebra. (See, for example, (45)).

Case 4. $f(y) = K - \delta y^2/2$, where $\delta = \pm 1$ and K is a constant.

Here we have six subcases:

4.1. $n = 2\nu + 3, K = 0$. The corresponding Eq. (8) admits a single Lie point symmetry

$$X_1 = x \frac{\partial}{\partial x} - (\nu + 1)y \frac{\partial}{\partial y}. \quad (16)$$

Note that this is subsumed in Case 5.1 below.

4.2. $n = 12\nu + 13, K = 0$. Here the corresponding Eq. (8) admits the same symmetry as in Case 4.1.

4.3. $n = (\nu + 4)/3, K = 0$. In this subcase the corresponding Eq. (8) admits a two-dimensional symmetry Lie algebra which is spanned by the operators (16) and

$$X_2 = x^{(2-\nu)/3} \frac{\partial}{\partial x} - \frac{\nu+1}{3} x^{-(\nu+1)/3} y \frac{\partial}{\partial y}.$$

Note that this is contained in Case 5.2 below.

4.4. $n = 7\nu + 8, K = 0$. The corresponding Eq. (8) admits the symmetry operator (16) and in addition, the symmetry operator

$$X_2 = x^{\nu+2} \frac{\partial}{\partial x} - \left[3(\nu+1)x^{\nu+1}y + \frac{24(\nu+1)^3}{\delta} \right] \frac{\partial}{\partial y}.$$

4.5. $n = (7\nu + 13)/6, K = 0$. In this subcase the corresponding Eq. (8) admits two Lie point symmetries, namely, the symmetry given by (16) and the symmetry

$$X_2 = x^{(5-\nu)/6} \frac{\partial}{\partial x} - \left[\frac{2}{3}(\nu+1)x^{-(\nu+1)/6}y - \frac{(\nu+1)^3}{9\delta}x^{-7(\nu+1)/6} \right] \frac{\partial}{\partial y}.$$

4.6. $n = (1 - \nu)/2, K = 0$. The corresponding Eq. (8) admits two Lie point symmetries and they are (15) and (16).

Case 5. $f(y) = -\delta_1/\sigma - y\delta_2/(\sigma + 1) + Ky^{-\sigma}$, where $\delta_1, \delta_2 = 0, \pm 1, \sigma \neq -1, 0$ and K is a constant.

Three subcases arise:

5.1. $n = \frac{\sigma - 2\nu - 1}{\sigma + 1}, \sigma \neq 3, \delta_1, \delta_2 = 0$. In this subcase we have one Lie point symmetry generator

$$X_1 = x \frac{\partial}{\partial x} + \frac{\nu+1}{\sigma+1} y \frac{\partial}{\partial y} \quad (17)$$

admitted by the corresponding Eq. (8).

5.2. $n = \frac{\sigma - \nu - 2}{\sigma - 1}$, $\sigma \neq 3$, $\delta_1, \delta_2 = 0$. Here the corresponding Eq. (8) admits a two-dimensional symmetry Lie algebra spanned by the operators (17) and

$$X_2 = x^{\frac{\sigma+\nu}{\sigma-1}} \frac{\partial}{\partial x} + \frac{\nu+1}{\sigma-1} x^{\frac{\nu+1}{\sigma-1}} y \frac{\partial}{\partial y}. \quad (18)$$

5.3. $n = \frac{1-\nu}{2}$, (This subcase corresponds to $\sigma = 3$), $\delta_1, \delta_2 = 0$. The corresponding Eq. (8) in this case admits three Lie point symmetry generators and these are given by (17), (18) with $\sigma = 3$ and (15).

Case 6. $f(y) = Ke^{-\delta_1 y} + \delta_2 y + \delta_3$, where $\delta_1 = \pm 1$, $\delta_2, \delta_3 = 0, \pm 1$ and K is a constant.

We have three subcases.

6.1. For all values of $n \neq 1$, $(1-\nu)/2$, $\delta_2, \delta_3 = 0$ one Lie point symmetry generator

$$X_1 = x \frac{\partial}{\partial x} + \frac{\nu+1}{\delta_1} \frac{\partial}{\partial y} \quad (19)$$

is admitted by the corresponding Eq. (8).

6.2. $n = 1$, $\delta_2, \delta_3 = 0$. In this subcase the corresponding Eq. (8) admits the Lie point symmetry (19) and in addition the Lie point symmetry

$$X_2 = x \ln x \frac{\partial}{\partial x} + \frac{1}{\delta_1} [2 + (\nu+1) \ln x] \frac{\partial}{\partial y}.$$

6.3. $n = (1-\nu)/2$, $\delta_2, \delta_3 = 0$. The corresponding Eq. (8) admits two Lie point symmetries. These symmetries are given by (15) and (19).

Case 7. $f(y) = -\delta_1 \ln y - \delta_2 y + K$, where $\delta_1, \delta_2 = 0, \pm 1$ and K is a constant.

This reduces to Case 2.

Case 8. $f(y) = -\delta_1 y \ln y + Ky + \delta_2$, where $\delta_1, \delta_2 = 0, \pm 1$ and K is a constant.

This also reduces to Case 2.

2.1 Integration of (8) for different f s

The main purpose for calculating symmetries is to use them to solve or reduce the order of differential equations. Here we use the symmetries calculated above to integrate Eq. (8) for three functions f . Other cases can be dealt in a similar manner. We recall that for any two-dimensional Lie algebra with symmetries G_1 and G_2 satisfying the Lie bracket relationship $[G_1, G_2] = \lambda G_1$, for some constant λ , the usual reduction of order is through the normal subgroup G_1 (46). We first consider Case 4.4. The corresponding Eq. (8) admits the two symmetries

$$X_1 = x \frac{\partial}{\partial x} - (\nu+1)y \frac{\partial}{\partial y}, \quad X_2 = x^{\nu+2} \frac{\partial}{\partial x} - \left[3(\nu+1)x^{\nu+1}y + \frac{24(\nu+1)^3}{\delta} \right] \frac{\partial}{\partial y}.$$

Since $[X_1, X_2] = (\nu + 1)X_2$, we may use X_2 to reduce the corresponding Eq. (8) to quadratures. The invariants of X_2 are found from

$$\frac{dx}{x^{\nu+2}} = \frac{dy}{- [3(\nu + 1)x^{\nu+1}y + 24(\nu + 1)^3/\delta]} = \frac{dy'}{- [3(\nu + 1)^2x^{\nu}y + (4\nu + 5)x^{\nu+1}y']}$$

and are

$$t = x^{3\nu+3}y + \frac{12}{\delta}(\nu + 1)^2x^{2\nu+2}, s = x^{4\nu+5}y' + 3(\nu + 1)x^{4\nu+4}y + \frac{24}{\delta}(\nu + 1)^3x^{3\nu+3}.$$

This leads to the first-order equation

$$\frac{ds}{dt} = \frac{\delta t^2}{2s}$$

which can be immediately integrated to give

$$s^2 = \frac{\delta}{3}t^3 + C_1,$$

where C_1 is an arbitrary constant of integration. Reverting to the x and y variables we obtain a first-order differential equation whose solution can be written as

$$y = x^{-3(\nu+1)}t - \frac{12}{\delta}(\nu + 1)^2x^{-(\nu+1)},$$

where t is given by

$$\int \frac{dt}{\pm \sqrt{C_1 + \delta t^3/3}} = -\frac{1}{\nu + 1}x^{-(\nu+1)} + C_2,$$

in which C_1 and C_2 are integration constants. Hence we have quadrature of Eq. (8) for given f .

We now consider Case 5.2. The two symmetries admitted by the corresponding Eq. (8) are

$$X_1 = x \frac{\partial}{\partial x} + \frac{\nu + 1}{\sigma + 1}y \frac{\partial}{\partial y}, \quad X_2 = x^{\frac{\sigma+\nu}{\sigma-1}} \frac{\partial}{\partial x} + \frac{\nu + 1}{\sigma - 1}x^{\frac{\nu+1}{\sigma-1}}y \frac{\partial}{\partial y}$$

with $[X_1, X_2] = \frac{(\nu + 1)}{\sigma - 1}X_2$. Following the above procedure we find that the solution of the corresponding Eq. (8) is

$$y = tx^{(\nu+1)/(\sigma-1)},$$

where t is defined by

$$\int \frac{dt}{\pm \sqrt{C_1 + 2Kt^{1-\sigma}/(1-\sigma)}} = \frac{1-\sigma}{1+\nu}x^{(1+\nu)/(1-\sigma)} + C_2,$$

in which C_1 and C_2 are arbitrary constants of integration.

Finally for Case 6.2 the corresponding Eq. (8) admits the two symmetries

$$X_1 = x \frac{\partial}{\partial x} + \frac{\nu+1}{\delta_1} \frac{\partial}{\partial y}, \quad X_2 = x \ln x \frac{\partial}{\partial x} + \frac{1}{\delta_1} [2 + (\nu+1) \ln x] \frac{\partial}{\partial y}$$

with $[X_1, X_2] = X_1$. In this case the solution of the corresponding Eq. (8) is

$$y = \frac{1}{\delta_1} \ln \left(\frac{x^{\nu+1}}{t} \right),$$

where t is given by

$$\int \frac{dt}{\pm \sqrt{2\delta_1 K t^3 + t^2[(\nu+1)^2 - 2\delta_1 C_1]}} = \ln x + C_2,$$

in which C_1 and C_2 are arbitrary integration constants.

3. Noether classification and integration of (8) for different f s

In this section we perform a Noether point symmetry classification of Eq. (8) with respect to the standard Lagrangian. We then obtain first integrals of the various cases, which admit Noether point symmetries and reduce the corresponding equations to quadratures.

It can easily be verified that the standard Lagrangian of Eq. (8) is

$$L = \frac{1}{2} x^n y'^2 - x^{n+\nu-1} \int f(y) dy. \quad (20)$$

The determining equation (see (47)) for the Noether point symmetries corresponding to L in (20) is

$$X^{[1]}(L) + LD(\xi) = D(B), \quad (21)$$

where X given by

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (22)$$

is the generator of Noether symmetry and $B(x, y)$ is the gauge term and D is the total differentiation operator defined by (48)

$$D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \dots \quad (23)$$

The solution of Eq. (21) results in

$$\xi = a(x),$$

$$\eta = \frac{1}{2} [a' - nx^{-1}a]y + b(x), \quad (24)$$

$$B = \frac{1}{4} x^n \left[a'' - n \left(\frac{a}{x} \right)' \right] y^2 + b' x^n y + c(x), \quad (25)$$

$$\begin{aligned}
& [-(n+\nu-1)x^{n+\nu-2}a - a'x^{n+\nu-1}] \int f(y)dy + [-\frac{1}{2}x^{n+\nu-1}a'y \\
& + \frac{1}{2}nx^{n+\nu-2}ay - x^{n+\nu-1}b]f(y) = \frac{1}{4}a'''x^ny^2 + \frac{1}{2}nx^{n-2}a'y^2 \\
& - \frac{1}{2}nx^{n-3}ay^2 - \frac{1}{4}n^2x^{n-1}\left(\frac{a}{x}\right)'y^2 + b''x^ny + b'nx^{n-1}y + c'(x). \quad (26)
\end{aligned}$$

The analysis of Eq. (26) leads to the following eight cases:

Case 1. $n \neq \frac{1-\nu}{2}$, $f(y)$ arbitrary but not of the form contained in cases 3, 4, 5 and 6.

We find that $\xi = 0$, $\eta = 0$, $B = \text{constant}$ and we conclude that there is no Noether point symmetry.

Noether point symmetries exist in the following cases.

Case 2. $n = \frac{1-\nu}{2}$, $f(y)$ arbitrary.

We obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = \text{constant}$. Therefore we have a single Noether symmetry generator $X = x^{\frac{1-\nu}{2}}\partial/\partial x$. For this case the integration is trivial even without a Noether symmetry. The Noetherian first integral (47) is

$$I = \frac{1}{2}x^{1-\nu}y'^2 + \int f(y)dy$$

from which, setting $I = C$, one gets quadrature.

Case 3. $f(y)$ is linear in y .

We have five Noether point symmetries associated with the standard Lagrangian for the corresponding differential equation (8) and $sl(3, \mathbb{R})$ symmetry algebra. This case is well-known, see, e.g., (45).

Case 4. $f = \alpha y^2 + \beta y + \gamma$, $\alpha \neq 0$

There are four subcases. They are as follows:

4.1. If $n = 2\nu + 3$, $\beta = 0$ and $\gamma = 0$, we obtain $\xi = x$, $\eta = -(\nu + 1)y$ and $B = \text{constant}$. This is contained in Case 5.1 below.

4.2. If $n = 2\nu + 3$, $\nu \neq -1$, $\beta^2 = 4\alpha\gamma$, we get $\xi = x$, $\eta = -(\nu + 1)(y + \beta/2\alpha)$ and $B = \frac{\beta\gamma}{6\alpha}x^{3\nu+3}$. We have

$$X = x\frac{\partial}{\partial x} - (\nu + 1)(y + \beta/2\alpha)\frac{\partial}{\partial y}.$$

In this case the Noetherian first integral (47) is

$$\begin{aligned}
I = & -\frac{1}{2}x^{2\nu+4}y'^2 - \frac{1}{3}\alpha x^{3\nu+3}y^3 - \frac{1}{2}\beta x^{3\nu+3}y^2 - \gamma x^{3\nu+3}y - (\nu + 1)x^{2\nu+3}yy' \\
& - (\nu + 1)\frac{\beta}{2\alpha}x^{2\nu+3}y' - \frac{\beta\gamma}{6\alpha}x^{3\nu+3}.
\end{aligned}$$

Thus the reduced equation is

$$\begin{aligned} \frac{1}{2}x^{2\nu+4}y'^2 + \frac{1}{3}\alpha x^{3\nu+3}y^3 + \frac{1}{2}\beta x^{3\nu+3}y^2 + \gamma x^{3\nu+3}y + (\nu+1)x^{2\nu+3}yy' \\ + (\nu+1)\frac{\beta}{2\alpha}x^{2\nu+3}y' + \frac{\beta\gamma}{6\alpha}x^{3\nu+3} = C, \end{aligned} \quad (27)$$

where C is an arbitrary constant. We now solve Eq. (27). For this purpose we use an invariant of X (see (49)) as the new dependent variable. This invariant is obtained by solving the Lagrange's system associated with X , viz.,

$$\frac{dx}{x} = \frac{dy}{-(\nu+1)(y + \beta/2\alpha)},$$

and is

$$u = x^{\nu+1}y + \frac{\beta}{2\alpha}x^{\nu+1}.$$

In terms of u Eq. (27) becomes

$$C = \frac{1}{2}(\nu+1)^2u^2 - \frac{1}{2}x^2u'^2 - \frac{1}{3}\alpha u^3,$$

which is a first-order variables separable ordinary differential equation. Separating the variables we obtain

$$\frac{du}{\pm\sqrt{(\nu+1)^2u^2 - (2/3)\alpha u^3 - 2C}} = \frac{dx}{x}.$$

Hence we have quadrature or double reduction of our Eq. (8) for the given f .

4.3. If $n = (\nu+4)/3$, $n \neq (1-\nu)/2, -1$, $\beta = 0$ and $\gamma = 0$, we find $\xi = x^{(2-\nu)/3}$, $\eta = -\frac{\nu+1}{3}x^{-(\nu+1)/3}y$ and $B = \frac{(\nu+1)^2}{18}y^2 + k$, k a constant. This is subsumed in Case 5.2 below.

4.4. If $n = (1-\nu)/2$, $n \neq (\nu+4)/3$, β and γ are arbitrary, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$. This reduces to Case 2.

Case 5. $f = \alpha y^r$, $\alpha \neq 0$, $r \neq 0, 1$.

Here we have two subcases.

5.1. If $n = \frac{r+2\nu+1}{r-1}$, we obtain $\xi = x$, $\eta = \frac{\nu+1}{1-r}y$ and $B = \text{constant}$. The solution of Eq. (8) for the above n and f is given by

$$y = ux^{\frac{\nu+1}{1-r}}, \quad (28)$$

where u satisfies

$$\int \frac{du}{\pm\sqrt{(\nu+1)^2(1-r)^{-2}u^2 - 2\alpha(1+r)^{-1}u^{1+r} - 2C_1}} = \ln x C_2, \quad (29)$$

in which, C_1 and C_2 are arbitrary constants of integration.

We note that when $r = 5$ and $\nu = 1$, we get $n = 2$. This gives us the Lane-Emden equation $y'' + (2/x)y' + y^5 = 0$. Its general solution is given by Eq. (29) and we recover the solution given in (50). Only a one-parameter family of solutions is known in the other literature, namely,

$y = [3a/(x^2 + 3a^2)]^{1/2}$, $a = \text{constant}$ (see, e.g., (27) or (51)). Here we have determined a two-parameter family of solutions. Another almost unknown exact solution of $y'' + (2/x)y' + y^5 = 0$, which is worth mentioning here, is given by

$$xy^2 = \left[1 + 3 \cot^2 \left(\frac{1}{2} \ln \frac{x}{c} \right) \right]^{-1}, \quad (30)$$

where c is an arbitrary constant.

5.2. If $n = \frac{r+\nu+2}{r+1}$, with $r \neq -1$, we have $\xi = x^{\frac{r-\nu}{r+1}}$, $\eta = -\left(\frac{\nu+1}{r+1}\right)x^{-\frac{\nu+1}{r+1}}y$ and $B = \frac{(\nu+1)^2}{2(r+1)^2}y^2 + k$, where k is a constant.

In this case the solution of the corresponding Eq. (8) is

$$y = ux^{-\frac{\nu+1}{r+1}}, \quad (31)$$

where u is given by

$$\int \frac{du}{\pm \sqrt{C_1 - 2\alpha(r+1)^{-1}u^{r+1}}} = \left(\frac{r+1}{\nu+1}\right)x^{\frac{\nu+1}{r+1}} + C_2, \quad (32)$$

in which, C_1 and C_2 are arbitrary constants.

5.3. If $n = \frac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = \text{constant}$. This reduces to Case 2.

Case 6. $f = \alpha \exp(\beta y) + \gamma y + \delta$, $\alpha \neq 0$, $\beta \neq 0$.

Here again we have two subcases.

6.1. If $n = \frac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

6.2. If $n = 1$, $\nu \neq -1$, $\gamma = 0$ and $\delta = 0$, we deduce that $\xi = x$, $\eta = -(\nu+1)/\beta$ and $B = k$, k a constant.

The solution of the corresponding Eq. (8) for this case is

$$y = \frac{\nu+1}{\beta} \ln \left(\frac{u}{x} \right), \quad (33)$$

where u is defined by

$$\int \frac{du}{\pm u \sqrt{1 - 2\alpha\beta(\nu+1)^{-2}u^{\nu+1} + 2C_1\beta^2(\nu+1)^{-2}}} = \ln x C_2, \quad (34)$$

in which, C_1 and C_2 are integration constants.

Case 7. $f = \alpha \ln y + \gamma y + \delta$, $\alpha \neq 0$.

If $n = \frac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

Case 8. $f = \alpha y \ln y + \gamma y + \delta$, $\alpha \neq 0$.

If $n = \frac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

4. Systems of Lane-Emden-Fowler equations

The modelling of several physical phenomena such as pattern formation, population evolution, chemical reactions, and so on (see, for example (52)), gives rise to the systems of Lane-Emden equations, and have attracted much attention in recent years. Several authors have proved existence and uniqueness results for the Lane-Emden systems (53; 54) and other related systems (see, for example (55–57) and references therein). Here we consider the following generalized coupled Lane-Emden system (58)

$$\frac{d^2u}{dt^2} + \frac{n}{t} \frac{du}{dt} + f(v) = 0, \quad (35)$$

$$\frac{d^2v}{dt^2} + \frac{n}{t} \frac{dv}{dt} + g(u) = 0, \quad (36)$$

where n is real constant and $f(v)$ and $g(u)$ are arbitrary functions of v and u , respectively. Note that system (35)-(36) is a natural extension of the well-known Lane-Emden equation. We will classify the Noether operators and construct first integrals for this coupled Lane-Emden system.

It can readily be verified that the natural Lagrangian of system (35)-(36) is

$$L = t^n \dot{u} \dot{v} - t^n \int f(v) dv - t^n \int g(u) du. \quad (37)$$

The determining equation (see (58)) for the Noether point symmetries corresponding to L in (37) is

$$X^{[1]}(L) + LD(\tau) = D(B), \quad (38)$$

where X is given by

$$X = \tau(t, u, v) \frac{\partial}{\partial t} + \xi(t, u, v) \frac{\partial}{\partial u} + \eta(t, u, v) \frac{\partial}{\partial v}, \quad (39)$$

with first extension (59)

$$X^{[1]} = X + (\dot{\xi} - \dot{u}\dot{\tau}) \frac{\partial}{\partial \dot{u}} + (\dot{\eta} - \dot{v}\dot{\tau}) \frac{\partial}{\partial \dot{v}}, \quad (40)$$

where $\dot{\tau}$, $\dot{\xi}$ and $\dot{\eta}$ denote total time derivatives of τ , ξ and η respectively. Proceeding as in Section 3, (see details of computations in (58)) we obtain the following seven cases:

Case 1. $n \neq 0$, $f(u)$ and $g(v)$ arbitrary but not of the form contained in cases 3, 4, 5 and 6.

We find that $\tau = 0$, $\xi = 0$, $\eta = 0$, $B = \text{constant}$ and we conclude that there is no Noether point symmetry.

Noether point symmetries exist in the following cases.

Case 2. $n = 0$, $f(u)$ and $g(v)$ arbitrary.

We obtain $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = \text{constant}$. Therefore we have a single Noether symmetry generator

$$X_1 = \frac{\partial}{\partial t} \quad (41)$$

with the Noetherian integral given by

$$I = \dot{u}\dot{v} + \int f(u)du + \int g(v)dv.$$

Case 3. $f(v)$ and $g(u)$ constants. We have eight Noether point symmetries associated with the standard Lagrangian for the corresponding system (35)-(36) and this case is well-known.

Case 4. $f = \alpha v + \beta$, $g = \gamma u + \lambda$, where α, β, γ and λ are constants, with $\alpha \neq 0$ and $\gamma \neq 0$.

There are three subcases, namely

4.1. For all values of $n \neq 0, 2$, we obtain $\tau = 0$, $\xi = a(t)$, $\eta = l(t)$ and $B = t^n \dot{l}u + t^n \dot{a}v - \lambda \int t^n a dt - \beta \int t^n l dt + C_1$, C_1 a constant. Therefore we obtain Noether point symmetry

$$X_1 = a(t) \frac{\partial}{\partial u} + l(t) \frac{\partial}{\partial v}, \quad (42)$$

where $a(t)$ and $l(t)$ satisfy the second-order coupled Lane-Emden system

$$\ddot{l} + \frac{n}{t} \dot{l} + \gamma a = 0, \quad \ddot{a} + \frac{n}{t} \dot{a} + \alpha l = 0. \quad (43)$$

The first integral in this case is given by

$$I_1 = t^n \dot{l}u + t^n \dot{a}v - \lambda \int t^n a dt - \beta \int t^n l dt - at^n \dot{v} - lt^n \dot{u}.$$

4.2. $n = 2$. In this subcase the Noether symmetries are X_1 given by the operator (42) and

$$X_2 = \frac{\partial}{\partial t} - ut^{-1} \frac{\partial}{\partial u} - vt^{-1} \frac{\partial}{\partial v}. \quad (44)$$

The value of B for the operator X_2 is given by $B = uv$.

The associated first integral for X_2 is given by

$$I_2 = uv + \frac{\alpha}{2} t^2 v^2 + \frac{\gamma}{2} t^2 u^2 + ut\dot{v} + vt\dot{u} + t^2 \dot{u}\dot{v}.$$

In this subcase, we note that the first integral corresponding to X_1 is subsumed in Case 4.1 above with $\beta, \lambda = 0$.

4.3. $n = 0$. Here the Noether operators are X_1 given by the operator (42) and

$$X_2 = \frac{\partial}{\partial t}, \quad \text{with } B = C_2, C_2 \text{ a constant.} \quad (45)$$

This reduces to Case 2.

We note also that the first integral associated with X_1 is contained in Case 4.1 above where $a(t)$ and $l(t)$ satisfy the coupled system

$$\ddot{l} + \gamma a = 0, \quad \ddot{a} + \alpha l = 0. \quad (46)$$

Case 5. $f = \alpha v^r$, $g = \beta u^m$, $m \neq -1$ and $r \neq -1$ where α, β are constants, with $\alpha \neq 0$ and $\beta \neq 0$.

There are three subcases, viz.,

5.1. If $n = \frac{2m+2r+mr+3}{rm-1}$, $rm \neq 1$, $m \neq -1$, $m \neq 1$ and $r \neq -1$, we obtain $\tau = t$, $\xi = -\frac{(1+n)}{m+1}u$, $\eta = -\frac{(1+n)}{r+1}v$ and $B = \text{constant}$.

Thus we obtain a single Noether point symmetry

$$X = t \frac{\partial}{\partial t} - \frac{(1+n)}{m+1} u \frac{\partial}{\partial u} - \frac{(1+n)}{r+1} v \frac{\partial}{\partial v} \quad (47)$$

with the associated first integral

$$I = \beta t^{n+1} \frac{u^{m+1}}{m+1} + \alpha t^{n+1} \frac{v^{r+1}}{r+1} + \frac{(n+1)}{m+1} t^n u \dot{v} + \frac{(n+1)}{r+1} t^n v \dot{u} + t^{n+1} \dot{u} \dot{v}.$$

We now consider the case when $m = -1$ and $r = -1$, in Case 5. Here we have two subcases

Case 5.2. $n = 0$, ($m = -1$, $r = -1$).

This case provides us with two Noether symmetries namely,

$$X_1 = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \text{ and } X_2 = \frac{\partial}{\partial t} \text{ with } B = 0 \text{ for both cases.} \quad (48)$$

We obtain the Noetherian first integrals corresponding to X_1 and X_2 as

$$I_1 = \dot{u}v - u\dot{v}, I_2 = \dot{u}\dot{v} + \ln u + \ln v,$$

respectively.

Case 5.3. $n = -1$ ($m = -1$, $r = -1$).

Here we obtain two Noether symmetry operators, viz.,

$$X_1 = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \text{ with } B = 0 \text{ and } X_2 = t \frac{\partial}{\partial t} + 2u \frac{\partial}{\partial u} \text{ with } B = -2 \ln t \quad (49)$$

and first integrals associated with X_1 and X_2 are given by

$$I_1 = \dot{u}vt^{-1} - u\dot{v}t^{-1}, I_2 = -2 \ln t + \ln u + \ln v - 2u\dot{v}t^{-1} + \dot{u}\dot{v},$$

respectively.

Case 6. $f = \alpha \exp(\beta v) + \lambda$, $g = \delta \exp(\gamma u) + \sigma$, $\alpha, \beta, \lambda, \gamma, \delta$, and σ are constants, with $\alpha \neq 0, \beta \neq 0, \delta \neq 0, \gamma \neq 0$.

There are two subcases. They are

6.1. If $n = 1$, $\lambda = 0$ and $\sigma = 0$, we obtain $\tau = t$, $\xi = -\frac{2}{\gamma}$, $\eta = -\frac{2}{\beta}$ and $B = C_3$, C_3 a constant.

Therefore we have a single Noether point symmetry

$$X_1 = t \frac{\partial}{\partial t} - \frac{2}{\gamma} \frac{\partial}{\partial u} - \frac{2}{\beta} \frac{\partial}{\partial v} \quad (50)$$

and this results in the first integral

$$I = t^2 \dot{u} \dot{v} + \frac{\alpha t^2}{\beta} \exp(\beta v) + \frac{\delta t^2}{\gamma} \exp(\gamma u) + \frac{2}{\gamma} t \dot{v} + \frac{2}{\beta} t \dot{u}.$$

6.2. If $n = 0$, $\lambda = 0$ and $\sigma = 0$, we deduce that $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = C_4$, C_4 a constant. The Noether operator is given by

$$X_1 = \frac{\partial}{\partial t}. \quad (51)$$

This reduces to Case 2.

Case 7. $f = \alpha \ln v + \beta$, $g = \gamma \ln u + \lambda$, where α, β, γ and λ are constants with $\alpha \neq 0$, $\gamma \neq 0$. If $n = 0$, we obtain $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = C_5$, C_5 a constant. This reduces to Case 2.

5. Concluding remarks

In this Chapter we gave a brief history of the Lane-Emden-Fowler equation and its applications in various fields. Several methods have been employed by scientists to solve the Lane-Emden-Fowler equation. Various generalizations of the Lane-Emden-Fowler equations were given which can be found in the literature. Also we gave the extension of the Lane-Emden equation to the System of Lane-Emden equations. We presented the complete Lie symmetry group classification of a generalized Lane-Emden-Fowler equation and performed the Lie and Noether symmetry analysis of this problem. It should be noted that Lie symmetry method is the most powerful tool to solve nonlinear differential equations. Finally, we classified a generalized coupled Lane-Emden system with respect to the standard first-order Lagrangian according to its Noether point symmetries and obtained first integrals for the corresponding Noether operators.

6. Acknowledgement

The author thanks Dr Ben Muatjetjeja for fruitful discussions.

7. References

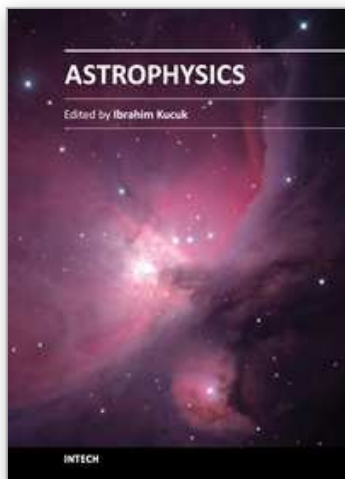
- [1] Lane JH, On the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment, *The American Journal of Science and Arts*, 2nd series 50 (1870), 57-74.
- [2] Emden R, *Gaskugeln, Anwendungen der mechanischen Warmen-theorie auf Kosmologie und meteorologische Probleme*, Leipzig, Teubner, 1907.
- [3] Thomson W, *Collected Papers*, Vol. 5, p. 266. Cambridge University Press, 1991.

- [4] Fowler RH, The form near infinity of real, continuous solutions of a certain differential equation of the second order, *Quart. J. Math. (Oxford)*, 45 (1914), 289-350.
- [5] Fowler RH, Further studies of Emden's and similar differential equations, *Quart. J. Math. (Oxford)*, 2 (1931), 259-288.
- [6] Mellin CM, Mahomed FM and Leach PGL, Solution of generalized Emden-Fowler equations with two symmetries, *Int. J. Nonlinear Mech.*, 29 (1994), 529-538.
- [7] Chandrasekhar S, *An Introduction to the Study of Stellar Structure*, Dover Publications Inc., New York, 1957.
- [8] Binney J and Tremaine S, *Galactic Dynamics*, Princeton University Press, Princeton, 1987.
- [9] Rix HW, de Zeeuw PT, Cretton N, van der Marel RP and Carollo, C. M. *ApJ*, 488 (1997), 702.
- [10] Thomas LH, *Proc. Cambridge Phil. Soc.*, 23, (1927), 542.
- [11] Fermi, E. *Rend. Accad. Naz. Lincei*, 6, (1927) 602.
- [12] Meerson E, Megged E and Tajima T, On the Quasi-hydrostatic Flowsof Radiatively Cooling Self-gravitating Gas Clouds, *Ap. J.*, 457 (1996), 321.
- [13] Gnutzmann S and Ritschel U, Analytic solution of Emden-Fowler equation and critical adsorption in spherical geometry, *Z. Phys. B: Condens Matter.*, 96 (1995), 391.
- [14] Bahcall NA, The Galaxy Distribution in the Cluster Abell 2199, *Ap. J.*, 186 (1973), 1179.
- [15] Bahcall NA, Core radii and central densities of 15 rich clusters of galaxies, *Ap. J.*, 198 (1975), 249.
- [16] Horedt GP, Seven-digit tables of Lane-Emden functions, *Astron. Astrophys.*, 126 (1986), 357-408.
- [17] Horedt GP, Approximate analytical solutions of the Lane-Emden equation in N-dimensional space, *Astron. Astrophys.*, 172 (1987), 359-367.
- [18] Bender CM, Milton KA, Pinsky SS and Simmons Jr LM, *et al.* A new perturbative approach to nonlinear problems, *J. Math. Phys.*, 30 (1989), 1447-1455.
- [19] Lima PM, Numerical methods and asymptotic error expansions for the Emden-Fowler equations, *J. Comput. Appl. Math.*, 70 (1996), 245-266.
- [20] Lima PM, Numerical solution of a singular boundary-value problem in non-newtonian fluid mechanics, *Appl. Num. Math.*, 30 (1999), 93-111.
- [21] Roxburgh IW and Stockman LM, Power series solutions of the polytrope equations, *Monthly Not. Roy. Astron. Soc.*, 303 (1999), 466-470.
- [22] Adomian G, Rach R and Shawagfen NT, On the analytic solution of the Lane-Emden equation, *Found. Phys. Lett.*, 8 (1995), 161-181.
- [23] Shawagfeh NT, Nonperturbative approximate solution for Lane-Emden equation, *J. Math. Phys.*, 34 (1993), 4364-4369.
- [24] Burt PB, Nonperturbative solution of nonlinear field equations, *Nuov. Cim.*, 100B (1987), 43-52.
- [25] Wazwaz AM, A new algorithm for solving differential equations of Lane-Emden type *Appl. Math. Comput.*, 118 (2001), 287-310.
- [26] Liao S, A new analytic algorithm of Lane-Emden type equations, *Appl. Math. Comput.*, 142 (2003), 1-16.
- [27] Davis HT, *Introduction to Nonlinear Differential and Integral Equations*, Dover Publications Inc., New York, 1962.
- [28] Datta BK, Analytic solution to the Lane-Emden equation, *Nuov. Cim.* 111B (1996), 1385-1388.

- [29] Wrubel MH, *Stellar Interiors. In Encyclopedia of Physics*, S. Flugge, Ed. Springer Verlag, Berlin, 1958, p 53.
- [30] Leach PGL, First integrals for the modified Emden equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$, *J. Math. Phys.*, 26 (1985), 2510-2514.
- [31] Dehghan M and Shakeri F, Approximate solution of a differential equation arising in astrophysics using the variational iteration method, *New Astronomy*, 13 (2008), 53-59.
- [32] Richardson OW, *The Emission of Electricity from Hot Bodies*. 2nd edition, London, 1921.
- [33] Momoniat E and Harley C, Approximate implicit solution of a Lane-Emden equation, *New Astronomy*, 11 (2006), 520-526.
- [34] Bozhkov Y and Martins ACG, Lie point symmetries of the Lane-Emden systems, *J. Math. Anal. Appl.*, 294 (2004), 334-344.
- [35] Bozhkov Y and Martins ACG, Lie point symmetries and exact solutions of quasilinear differential equations with critical exponents, *Nonlin. Anal.*, 57 (2004), 773-793.
- [36] Govinder KS and Leach PGL, Integrability analysis of the Emden-Fowler equation, *J. Nonlin. Math. Phys.*, 14 3 (2007), 435-453.
- [37] Kara AH and Mahomed FM, Equivalent Langrangians and solutions of some classes of nonlinear equations $\ddot{q} + p(t)\dot{q} + r(t)q = \mu\dot{q}^2q^{-1} + f(t)q^n$, *Int. J. Nonlin. Mech.*, 27 (1992), 919-927.
- [38] Kara AH and Mahomed FM, A note on the solutions of the Emden-Fowler equation, *Int. J. Nonlin. Mech.*, 28 (1993), 379-384.
- [39] Wong JSW, On the generalized Emden-Fowler equation, *SIAM Review*, 17 (1975), 339-360.
- [40] Goenner H and Havas P, Exact solutions of the generalized Lane-Emden equation, *J. Math. Phys.*, 41 (2000), 7029-7042.
- [41] Goenner H, Symmetry transformations for the generalized Lane-Emden equation, *Gen. Rel. Grav.*, 33 (2001), 833-841.
- [42] Muatjetjeja B and Khalique CM, Exact solutions of the generalized Lane-Emden equations of the first and second kind, *Pramana*, 77 (2011) 545-554.
- [43] Ibragimov NH, (ed) *CRC Handbook of Lie Group Analysis of Differential Equations*, Vols. 1, 2 and 3, CRC Press, Boca Raton, 1994-1996.
- [44] Khalique CM, Mahomed FM and Ntsime BP, Group classification of the generalized Emden-Fowler-type equation, *Nonlinear Analysis: Real World Applications*, 10 (2009) 3387-3395.
- [45] Lie S, *Differential Equations*, Chelsea, New York, (in German), 1967.
- [46] Olver PJ, *Applications of Lie groups to differential equations*, Springer-Verlag, New York, 1993.
- [47] Khalique CM and Ntsime BP, Exact solutions of the Lane-Emden-type equation, *New Astronomy*, 13 (2008) 476-480.
- [48] Ibragimov NH, *Elementary Lie Group Analysis and Ordinary Differential Equations*, Wiley, Chichester, 1999.
- [49] Adam AA and Mahomed FM, Integration of ordinary differential equations via nonlocal symmetries, *Nonlinear Dynamics*, 30 (2002), 267-275.
- [50] Khalique CM, Mahomed FM and Muatjetjeja B, Lagrangian formulation of a generalized Lane-Emden equation and double reduction, *J. Nonlinear Math. Phys.*, 15 (2008) 152-161.
- [51] Dresner L, *Similarity solutions of nonlinear partial differential equations*. Pitman Advanced Publishing Program, London, 1983.
- [52] Zou H, A priori estimates for a semilinear elliptic system without variational structure and their applications. *Mathematische Annalen*, 323 (2002) 713-735.

- [53] Serrin J and Zou H, Non-existence of positive solutions of the Lane-Emden system. *Differential Integral Equations*, 9 (1996) 635-653.
- [54] Serrin J and Zou H, Existence of positive solutions of Lane-Emden systems. *Atti del Sem. Mat. Fis. Univ. Modena*, 46 (Suppl) (1998) 369-380.
- [55] Qi Y, The existence of ground states to a weakly coupled elliptic system. *Nonlinear Analysis*, 48 (2002) 905-925.
- [56] Dalmaso R, Existence and uniqueness of solutions for a semilinear elliptic system. *International of Journal of Mathematics and Mathematical Sciences*, 10 (2005) 1507-1523.
- [57] Q. Dai and Tisdell CC, Non-degeneracy of positive solutions to homogeneous second order differential systems and its applications. *Acta Math. Sci.*, 29B (2009) 437-448.
- [58] Muatjetjeja B and Khalique CM, Lagrangian Approach to a Generalized Coupled Lane-Emden System: Symmetries and First Integrals, *Commun. Nonlin. Sci. Numer. Simulat.*, 15 (2010) 1166-1171.
- [59] Gorringe VM and Leach PGL, Lie point symmetries for systems of second order linear ordinary differential equations. *Queastiones Mathematicae*, 11 (1998) 95-117.

IntechOpen



Astrophysics

Edited by Prof. Ibrahim Kucuk

ISBN 978-953-51-0473-5

Hard cover, 398 pages

Publisher InTech

Published online 30, March, 2012

Published in print edition March, 2012

This book provides readers with a clear progress to theoretical and observational astrophysics. It is not surprising that astrophysics is continually growing because very sophisticated telescopes are being developed and they bring the universe closer and make it accessible. Astrophysics Book presents a unique opportunity for readers to demonstrate processes do occur in Nature. The unique feature of this book is to cover different aspects in astrophysics covering the topics: • Astronomy • Theoretical Astrophysics • Observational Astrophysics • Cosmology • The Solar System • Stars • Planets • Galaxies • Observation • Spectroscopy • Dark Matter • Neutron Stars • High Energy Astrophysics

How to reference

In order to correctly reference this scholarly work, feel free to copy and paste the following:

Chaudry Masood Khaliq (2012). The Lane-Emden-Fowler Equation and Its Generalizations - Lie Symmetry Analysis, Astrophysics, Prof. Ibrahim Kucuk (Ed.), ISBN: 978-953-51-0473-5, InTech, Available from: <http://www.intechopen.com/books/astrophysics/the-lane-emden-equation-and-its-generalizations>

INTech
open science | open minds

InTech Europe

University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
Phone: +385 (51) 770 447
Fax: +385 (51) 686 166
www.intechopen.com

InTech China

Unit 405, Office Block, Hotel Equatorial Shanghai
No.65, Yan An Road (West), Shanghai, 200040, China
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元
Phone: +86-21-62489820
Fax: +86-21-62489821

© 2012 The Author(s). Licensee IntechOpen. This is an open access article distributed under the terms of the [Creative Commons Attribution 3.0 License](https://creativecommons.org/licenses/by/3.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

IntechOpen

IntechOpen