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Waveguides, Resonant Cavities, Optical Fibers and Their Quantum Counterparts

Victor Barsan

Department of Theoretical Physics, National Institute of Physics and Nuclear Engineering,
Bucharest, Magurele
Romania

1. Introduction

In this chapter, we shall expose several analogies between oscillatory phenomena in mecahanics and optics. The main subject will be the analogy between propagation of electromagnetic waves in dielectrics and of electrons in various time-independent potentials. The basis of this analogy is the fact that both wave equations for electromagnetic monoenergetic waves (i.e. with well-defined frequency), obtained directly from the Maxwell equations, and the time-independent Schrodinger equation are Helmholtz equations; when specific restrictions - like behaviour at infinity and boundary conditions - are imposed, they generate similar eigenvalues problems, with similar solutions. The benefit of such analogies is twofold. First, it could help a researcher, specialized in a specific field, to better understand a new one. For instance, they might efficiently explain the fiber-optics properties to people already familiar with quantum mechanics. Also, even if such researchers work frequently with the quantum mechanical wave function, the electromagnetic modal field may provide an interesting vizualization of quantum probability density field [1]. Second, it provides the opportunity of cross-fertilization between (for instance) electromagnetism and optoelectronics, through the development of ballistic electron optics in two dimensional (2D) electron systems (2DESs); transferring concepts, models of devices and experiments from one field to another stimulate the progress in both domains.

Even if in modern times the analogies are not credited as most creative approaches in physics, in the early days of developement of science the perception was quite different. "Men's labour ... should be turned to the investigation and observation of the resemblances and analogies of things... for these it is which detect the unity of nature, and lay the fundation for the constitution of the sciences.", considers Francis Bacon, quoted by [1]. Some two centuries later, Goethe was looking for the "ultimate fact" - the Urphänomenon - specific to every scientific discipline, from botanics to optics [2], and, in this investigation, attributed to analogies a central role. However, if analogies cannot be considered anymore as central for the scientific investigation, thay could still be pedagogically useful and also inspiring for active scientific research.

Let us describe now shortly the structure of this chapter. The next two sections are devoted to a general description of the two fields to be hereafter investigated: metallic and dielectric waveguides (Section 2) and 2DESs (Section 3). The importance of these topics for the development of optical fibers, integrated optics, optoelectronics, transport

phenomena in mesoscopic and nanoscopic systems, is explained. In Section 4, some very general considerations about the physical basis of analogies between mechanical (classical or quantum) and electromagnetic phenomena, are outlined. Starting from the main experimental laws of electromagnetism, the Maxwell's equations are introduced in Section 5. In the next one, the propagation of electromagnetic waves in metalic and dielectric structures is studied, and the transverse solutions for the electric and magnetic field are obtained. These results are applied to metalic waveguides and cavities in Section 7. The optical fibers are described in Section 8, and the behaviour of fields, including the modes in circular fibers, are presented. Although the analogy between wave guide- and quantum mechanical- problems is treated in a huge number of references, the subject is rarely discussed in full detail. This is why, in Section 9, the analogy between the three-layer slab optical waveguide and the quantum rectangular well is mirrored and analyzed with utmost attention. The last part of the chapter is devoted to transport phenomena in 2DESs and their electromagnetic counterpart. In Section 10, the theoretical description of ballistic electrons is sketched, and, in Section 11, the transverse modes in electronic waveguides are desctibed. A rigorous form of the effective mass approach for electrons in semiconductors is presented in Section 12, and a quantitative analogy between the electronic wave function and the electric or magnetic field is established. Section 13 is devoted to optics experiments made with ballistic electrons. Final coments and conclusions are exposed in Section 14.

2. Metallic and dielectric waveguides; optical fibers

Propagation of electromagnetic waves through metallic or dielectric structures, having dimensions of the order of their wavelength, is a subject of great interest for applied physics. The only practical way of generating and transmitting radio waves on a well-defined trajectory involves such metallic structures [3]. For much shorter wavelengths, i.e. for infrared radiation and light, the propagation through dielectric waveguides has produced, with the creation of optical fibers, a huge revolution in telecommunications. The main inventor of the optical fiber, C. Kao, received the 2009 Nobel Prize in Physics (together with W. S. Boyle and G. E. Smith). As one of the laurees remarks, "it is not often that the awards is given for work in applied science". [4]

The creation of optical fibers has its origins in the efforts of improving the capabilities of the existing (at the level of early '60s) communication infrastructure, with a focus on the use of microwave transmission systems. The development of lasers (the first laser was produced in May 16, 1960, by Theodore Maiman) made clear that the coherent light can be an information carrier with 5-6 orders of magnitude more performant than the microwaves, as one can easily see just comparing the frequencies of the two radiations. In a seminal paper, Kao and Hockham [5] recognized that the key issue in producing "a successful fiber waveguide depends... on the availability of suitable low-loss dielectric material", in fact - of a glass with very small $(< 10^{-6})$ concentration of impurities, particularly of transition elements (Fe, Cu, Mn). Besides telecommunication applications, an appropriate bundle of optical fibers can transfer an image - as scientists sudying the insect eye realized, also in the early '60s. [6]

Another domain of great interest which came to being with the development of dielectric waveguides and with the progress of thin-film technology is the integrated optics. In the early '70s, thin films dielectric waveguides have been used as the basic element of all the components of an optical circuit, including lasers, modulators, detectors, prisms, lenses, polarizers and couplers [7]. The transmission of light between two optical components

became a problem of interconnecting of two waveguides. So, the traditional optical circuit, composed of separate devices, carefully arranged on a rigid support, and protected against mechanical, thermal or atmospheric perturbations, has been replaced with a common substrate where all the thin-film optical components are deposited [7].

3. 2DESs and ballistic electrons

Electronic transport in conducting solids is generally diffusive. Its flow follows the gradient in the electrochemical potential, constricted by the physical or electrostatic edges of the specimen or device. So, the mean free path of electrons is very short compared to the dimension of the specimen. [8] One of the macroscopic consequences of this behaviour is the fact that the conductance of a rectangular 2D conductor is directly proportional to its width (W) and inversely proportional to its length (L). Does this ohmic behaviour remain correct for arbitrary small dimensions of the conductor? It is quite natural to expect that, if the mean free path of electrons is comparable to W or L - conditions which define the ballistic regime of electrons - the situation should change. Although the first experiments with ballistic electrons in metals have been done by Sharvin and co-workers in the mid '60s [9] and Tsoi and co-workers in the mid '70s [10], the most suitable system for the study of ballistic electrons is the two-dimensional electron system (2DES) obtained in semiconductors, mainly in the $GaAs - Al_xGa_{1-x}As$ heterostructures, in early '80s. In such 2DESs, the mobility of electrons are very high, and the ballistic regime can be easily obtained. The discovery of quantum conductance is only one achievement of this domain of mesoscopic physics, which shows how deep is the non-ohmic behaviour of electrical conduction in mesoscopic systems. In the ballistic regime, the electrons can be described by a quite simple Schrodinger equation, and electron beams can be controlled via electric or magnetic fields. A new field of research, the classical ballistic electron optics in 2DESs, has emerged in this way. At low temperatures and low bias, the current is carried only by electrons at the Fermi level, so manipulating with such electrons is similar to doing optical experiments with a monochromatic source [11].

The propagation of ballistic electrons in mesoscopic conductors has many similarities with electromagnetic wave propagation in waveguides, and the ballistic electron optics opened a new domain of micro- or nano-electronics. The revealing of analogies between ballistic electrons and guided electromagnetic waves, or between optics and electric field manipulation of electron beams, are not only useful theoretical exercises, but also have a creative potential, stimulating the transfer of knowledge and of experimental techniques from one domain to another.

4. Mechanical and electrical oscillations

It is useful to begin the discussion of the analogies presented in this chapter with some very general considerations [12]. The most natural starting point is probably the comparison between the mechanical equation of motion of a mechanical oscillator having the mass m and the stiffness k:

$$m\frac{d^2x}{dt^2} + kx^2 = 0 \tag{1}$$

and the electrectromagnetical equation of motion of a LC circuit [12]:

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \tag{2}$$

which provides immediately an analogy between the mechanical energy:

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = \mathcal{E} \tag{3}$$

and the electromagnetic one:

$$\frac{1}{2}L\left(\frac{dq}{dt}\right)^2 + \frac{1}{2}\frac{q^2}{C} = \mathcal{E} \tag{4}$$

The analogy between these equations reveals a much deeper fact than a simple terminological dictionary of mechanical and electromagnetic terms: it shows the inertial properties of the magnetic field, fully expressed by Lenz's law. Actually, magnetic field inertia (defined by the inductance L) controls the rate of change of current for a given voltage in a circuit, in exactly the same way as the inertial mass controls the change of velocity for a given force. Magnetic inertial or inductive behaviour arises from the tendency of the magnetic flux threading a circuit to remain constant, and reaction to any change in its value generates a voltage and hence a current which flows to oppose the change of the flux. ([5], p.12) Even if, in the previous equations, the mechanical oscillator is a classical one, its deep connections with its quantum counterpart are wellknown ([13], vol.1, Ch. 12). Also, understanding of classical waves propagation was decissive for the formulation of quantum-wave theory [13], so the classical form of (1) and (3) is not an obstacle in the development of our arguments.

These basic remarks explain the similarities between the propagation of elastic and mechanical waves. The velocity of waves through a medium is determined by the inertial and elastical properties of the medium. They allow the storing of wave energy in the medium, and in the absence of energy dissipation, they also determine the impedance presented by the medium to the waves. In addition, when there is no loss mechanism, a plane wave solution will be obtained, but any resistive or loss term, will produce a decay with time or distance of the oscillatory solution.

Referring now to the electromagnetic waves, the magnetic inertia of the medium is provided by the inductive property of that medum, i.e. permeability μ, allowing storage of magnetic energy, and the elasticity or capacitive property - by the permittivity ϵ, allowing storage of the potential or electric field energy. ([12], p.199)

5. Maxwell's equations

The theory of electromagnetic phenomena can be described by four equations, two of them independent of time, and two - time-varying. The time-independent ones express the fact that the electric charge is the source of the electric field, but a "magnetic charge" does not exist:

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \tag{5}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0 \tag{6}$$

One time-varying equation expresses Faraday's (or Lenz's) law [12], relating the time variation of the magnetic induction, $\mu \mathbf{H} = \mathbf{B}$, with the space variation of $\mathbf{E}$:

$$\frac{\partial}{\partial t}(\mu \mathbf{H}) \text{ is connected with } \frac{\partial \mathbf{E}}{\partial z} \text{ (say)}$$

More exactly,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{7}$$

The other one expresses Ampere's law [12], relating that the time variation of $\epsilon \mathbf{E}$ defines the space variation of $\mathbf{H}$:

$$\frac{\partial}{\partial t} (\epsilon \mathbf{E}) \text{ is connected with } \frac{\partial \mathbf{H}}{\partial z} \text{ (say)}$$

More exactly,

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{8}$$

assuming that no free chages or electric current are present - a natural assumption for our approach, as we shall use Maxwell's equations only for studying the wave propagation. In this context, the only role played by (5) and (6) will be to demonstrate the transverse character of the vectors $\mathbf{E}, \mathbf{H}$.

6. Propagation of electromagnetic waves in waveguides and cavities

The propagation of electromagnetic waves in hollow metalic cylinders is an interesting subject, both for theoretical and practical reasons - e.g., for its applications in telecommunications. We shall consider that the metal is a perfect conductor; if the cylinder is infinite, we shall call this metallic structure *waveguide*; if it has end faces, we shall call it *cavity*. The transversal section of the cylinder is the same, along the cylinder axis. With a time dependence $\exp(-i\omega t)$, the Maxwell equations (5)-(8) for the fields inside the cylinder take the form [3]:

$$\nabla \times \mathbf{E} = i\omega \mathbf{B}, \ \ \nabla \cdot \mathbf{B} = 0, \ \ \nabla \times \mathbf{B} = -i\mu\epsilon\omega \mathbf{E}, \ \ \nabla \cdot \mathbf{E} = 0 \tag{9}$$

For a cylinder filled with a uniform non-dissipative medium having permittivity ϵ and permeability μ,

$$\left(\nabla^2 + \mu\epsilon\omega^2 \right) \begin{Bmatrix} \mathbf{E} \\ \mathbf{B} \end{Bmatrix} = 0 \tag{10}$$

The specific geometry suggests us to single out the spatial variation of the fields in the z direction and to assume

$$\begin{rcases} \mathbf{E}(x,y,z,t) \\ \mathbf{B}(x,y,z,t) \end{rcases} = \begin{cases} \mathbf{E}(x,y) \exp(\pm ikz - i\omega t) \\ \mathbf{B}(x,y) \exp(\pm ikz - i\omega t) \end{cases} \tag{11}$$

The wave equation is reduced to two variables:

$$\left[\nabla_t^2 + \left(\mu\epsilon\omega^2 - k^2 \right) \right] \begin{Bmatrix} \mathbf{E} \\ \mathbf{B} \end{Bmatrix} = 0 \tag{12}$$

where ∇_t^2 is the transverse part of the Laplacian operator:

$$\nabla_t^2 = \nabla - \frac{\partial^2}{\partial z^2} \tag{13}$$

It is convenient to separate the fields into components parallel to and transverse the oz axis:

$$\mathbf{E} = \mathbf{E}_z + \mathbf{E}_t, \text{ with } \mathbf{E}_z = \hat{\mathbf{z}} E_z, \ \ \mathbf{E}_t = (\hat{\mathbf{z}} \times \mathbf{E}) \times \hat{\mathbf{z}} \tag{14}$$

$\hat{z}$ is as usual, a unit vector in the $z-$direction. Similar definitions hold for the magnetic field **B**. The Maxwell equations can be expressed in terms of transverse and parallel fields as [3]:

$$\frac{\partial \mathbf{E}_t}{\partial z} + i\omega\hat{\mathbf{z}} \times \mathbf{B}_t = \boldsymbol{\nabla}_t E_z, \ \ \hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_t \times \mathbf{E}_t) = i\omega B_z \tag{15}$$

$$\frac{\partial \mathbf{B}_t}{\partial z} - i\mu\epsilon\omega\hat{\mathbf{z}} \times \mathbf{E}_t = \boldsymbol{\nabla}_t B_z, \ \ \hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_t \times \mathbf{B}_t) = -i\mu\epsilon\omega E_z \tag{16}$$

$$\boldsymbol{\nabla}_t \cdot \mathbf{E}_t = -\frac{\partial E_z}{\partial z}, \ \ \boldsymbol{\nabla}_t \cdot \mathbf{B}_t = -\frac{\partial B_z}{\partial z} \tag{17}$$

According to the first equations in (15) and (16), if E_z and B_z are known, the transverse components of **E** and **B** are determined, assuming the z dependence is given by (11). Considering that the propagation in the positive z direction (for the opposite one, k changes it sign) and that at least one E_z and B_z have non-zero values, the transverse fields are

$$\mathbf{E}_t = \frac{i}{\mu\epsilon\omega^2 - k^2} \left[k\boldsymbol{\nabla}_t E_z - \omega\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t B_z \right] \tag{18}$$

$$\mathbf{B}_t = \frac{i}{\mu\epsilon\omega^2 - k^2} \left[k\boldsymbol{\nabla}_t B_z + \omega\epsilon\omega\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t E_z \right] \tag{19}$$

Let us notice the existence of a special type of solution, called the *transverse electromagnetic* (TEM) wave, having only field components transverse to the direction of propagation [6]. From the second equation in (15) and the first in (16), results that $E_z = 0$ and $B_z = 0$ implies that $\mathbf{E}_t = \mathbf{E}_{ETM}$ satisfies

$$\boldsymbol{\nabla}_t \times \mathbf{E}_{ETM} = 0, \ \ \boldsymbol{\nabla}_t \cdot \mathbf{E}_{ETM} = 0 \tag{20}$$

So, $\mathbf{E}_{ETM}$ is a solution of an *electrostatic* problem in 2D. There are 4 consequences:

1. the axial wave number is given by the infinite-medium value,

$$k = k_0 = \omega\sqrt{\mu\epsilon} \tag{21}$$

as can be seen from (12).

2. the magnetic field, deduced from the first eq. in (16), is

$$\mathbf{B}_{ETM} = \pm\sqrt{\mu\epsilon}\hat{\mathbf{z}} \times \mathbf{E}_{ETM} \tag{22}$$

for waves propagating as $\exp(\pm ikz)$. The connection between $\mathbf{B}_{ETM}$ and $\mathbf{E}_{ETM}$ is just the same as for plane waves in an infinite medium.

3. the TEM mode cannot exist inside a single, hollow, cylindrical conductor of infinite conductivity. The surface is an equipotential; the electric field therefore vanishes inside. It is necessary to have two or more cylindrical surfaces to support the TEM mode. The familiar coaxial cable and the parallel-wire transmission line are structures for which this is the dominant mode.

4. the absence of a cutoff frequency (see below): the wave number (21) is real for all ω.

In fact, two types of field configuration occur in hollow cylinders. They are solutions of the eigenvalue problems given by the wave equation (12), solved with the following boundary conditions, to be fulfilled on the cylinder surface:

$$\mathbf{n} \times \mathbf{E} = 0, \ \ \mathbf{n} \cdot \mathbf{B} = 0 \tag{23}$$

where **n** is a normal unit at the surface S. From the first equation of (23):

$$\mathbf{n} \times \mathbf{E} = \mathbf{n} \times (-\mathbf{n}E_t + \hat{\mathbf{z}}E_z) = \mathbf{n} \times \hat{\mathbf{z}}E_z = 0$$

so:

$$E_z \mid_S = 0 \tag{24}$$

Also, from the second one:

$$\mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot (-\mathbf{n}B_t + \hat{\mathbf{z}}B_z) = -B_t = 0$$

With this value for B_t in the component of the first equation (16) parallel to **n**, we get:

$$\frac{\partial B_z}{\partial n} \mid_S = 0 \tag{25}$$

where $\partial/\partial n$ is the normal derivative at a point on the surface. Even if the wave equation for E_z and B_z is the same ((eq. (12)), the boundary conditions on E_z and B_z are different, so the eigenvalues for E_z and B_z will in general be different. The fields thus naturally divide themselves into two distinct categories:

Transverse magnetic (TM) waves:

$$B_z = 0 \text{ everywhere; boundary condition, } E_z \mid_S = 0 \tag{26}$$

Transverse electric (TE) waves:

$$E_z = 0 \text{ everywhere;} \quad \text{boundary condition, } \frac{\partial B_z}{\partial n} \mid_S = 0 \tag{27}$$

For a given frequency ω, only certain values of wave number k can occur (typical waveguide situation), or, for a given k, only certain ω values are allowed (typical resonant cavity situation).

The variuos TM and TE waves, plus the TEM waves if it can exist, constitute a complete set of fields to describe an arbitrary electromagnetic disturbance in a waveguide or cavity [3].

7. Waveguides

For the propagation of waves inside a hollow waveguide of uniform cross section, it is found from (18) and (19) that the transverse magnetic fields for both TM and TE waves are related by:

$$H_t = \pm \frac{1}{Z} \hat{\mathbf{z}} \times \mathbf{E}_t \tag{28}$$

where Z is called the *wave impedance* and is given by

$$Z = \begin{cases} \frac{k}{\epsilon\omega} = \frac{k}{k_0} \sqrt{\frac{\mu}{\epsilon}} & (TM) \\ \frac{\mu\omega}{k} = \frac{k_0}{k} \sqrt{\frac{\mu}{\epsilon}} & (TE) \end{cases} \tag{29}$$

and k_0 is given by (21). The $\pm$ sign in (28) goes with z dependence, $\exp{(\pm ikz)}$ [3]. The transverse fields are determined by the longitudinal fields, according to (18) and (19):

TM waves:

$$E_t = \pm \frac{ik}{\gamma^2} \boldsymbol{\nabla}_t \psi \tag{30}$$

TE waves:

$$H_t = \pm \frac{ik}{\gamma^2} \boldsymbol{\nabla}_t \psi \tag{31}$$

where $\psi \exp(\pm ikz)$ is E_z (H_z) for TM (TE) waves, and γ^2 is defined below. The scalar function ψ satisfies the 2D wave eq (12):

$$\left(\boldsymbol{\nabla}_t + \gamma^2\right) \psi = 0 \tag{32}$$

where

$$\gamma^2 = \mu\epsilon\omega^2 - k^2 \tag{33}$$

subject to the boundary condition,

$$\psi\,|_S = 0 \ \ \text{or} \ \ \frac{\partial\psi}{\partial n}\,|_S = 0 \tag{34}$$

for TM (TE) waves.

Equation (32) for ψ, together with boundary condition (34), specifies an eigenvalues problem. The similarity with non-relativistic quantum mechanics is evident.

7.1 Modes in a rectangular waveguide

Let us illustrate the previous general theory by considering the propagation of TE waves in a rectangular waveguide (the corners of the rectangle are situated in $(0,0), (a,0), (a,b), (0,b)$). In this case, is easy to obtain explicit solutions for the fields [3]. The wave equation for $\psi = H_z$ is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) \psi = 0 \tag{35}$$

with boundary conditions $\partial\psi/\partial n = 0$ at $x = 0, a$ and $y = 0, b$. The solution for ψ is easily find to be:

$$\psi_{mn}(x,y) = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \tag{36}$$

with γ givem by:

$$\gamma_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \tag{37}$$

with m, n - integers. Consequently, from (33),

$$k_{mn}^2 = \mu\epsilon\omega^2 - \gamma_{mn}^2 = \mu\epsilon\left(\omega^2 - \omega_{mn}^2\right), \ \ \omega_{mn}^2 = \frac{\gamma_{mn}^2}{\mu\epsilon} \tag{38}$$

As only for $\omega > \omega_{mn}$, k_{mn} is real, so the waves propagate without attenuation; ω_{mn} is called cutoff frequency. For a given ω, only certain values of k, namely k_{mn}, are allowed.

For TM waves, the equation for the field $\psi = E_z$ will be also (39), but the boundary condition will be different: $\psi = 0$ at $x = 0, a$ and $y = 0, b$. The solution will be:

$$\psi_{mn}(x,y) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{39}$$

with the same result for k_{mn}.

In a more general geometry, there will be a spectrum of eigenvalues γ_λ^2 and corresponding solutions ψ_λ, with λ taking discrete values (which can be integers or sets of integers, see for instance (37)). These different solutions are called the modes of the guide. For a given frequency ω, the wave number k is determined for each value of λ :

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2 \tag{40}$$

Defining a cutoff frequency ω_λ,

$$\omega_\lambda = \frac{\gamma_\lambda}{\sqrt{\mu\epsilon}}\sqrt{\omega^2 - \omega_\lambda^2} \tag{41}$$

then the wave number can be written:

$$k_\lambda = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_\lambda^2} \tag{42}$$

7.2 Modes in a resonant cavity

In a resonant cavity - i.e., a cylinder with metallic, perfect conductive ends perpendicular to the oz axis - the wave equation is identical, but the eigenvalue problem is somewhat different, due to the restrictions on k. Indeed, the formation of standing waves requires a $z-$dependence of the fields having the form

$$A \sin kz + B \cos kz \tag{43}$$

So, the wavenumber k is restricted to:

$$k = p\frac{\pi}{d}, \quad p = 0, 1, \dots \tag{44}$$

and the condition a(35) impose a quantization of ω :

$$\mu\epsilon\omega_{p\lambda}^2 = \left(p\frac{\pi}{d}\right)^2 + \gamma_\lambda^2 \tag{45}$$

So, the existence of quantized values of k implies the quantization of ω.

8. Electromagnetic wave propagation in optical fibers

Optical fibers belong to a subset (the most commercially significant one) of dielectric optical waveguides [6]. Although the first study in this subject was published in 1910 [14], the explosive increase of interest for optical fibers coincides with the technical production of low loss dielectrics, some six decenies later. In practice, they are highly clindrical flexible fibers made of nearly transparent dielectric material. These fibers - with a diameter comparable to a human hair - are composed of a central region, the *core* of radius a and reffractive index n_{co}, surrounded by the *cladding*, of refractive index $n_{cl} < n_{co}$, covered with a protective jacket [15]. In the core, n_{co} may be constant - in this case, one says that the refractive-index profile is a step profile (as also $n_{cl} = const.$), or may be graded, for instance:

$$n_{co}(r) = n_{co}(0)\left[1 - \Delta\left(\frac{r}{a}\right)^\alpha\right], \quad r < a \tag{46}$$

For $\alpha = 2$, the profile is called parabolic. One of the main parameters characterizing an optical fiber is the profile hight parameter Δ,

$$\Delta = \frac{1}{2}\left(1 - \frac{n_{cl}^2}{n_{co}^2}\right) \simeq 1 - \frac{n_{cl}}{n_{co}}, \quad n_{co} = \max n_{co}\left(r\right)|_{r \leq a} \tag{47}$$

Besides Δ, one usually also defines the fiber parameter V :

$$V = ka\sqrt{2\Delta} \tag{48}$$

Assimilating the propagating light with a geometric ray, it must be incident on the core-cladding interface at an angle smaller than the critical angle θ_c :

$$\theta_c = \arcsin \frac{n_{cl}}{n_{co}} \tag{49}$$

in order to be totally reflected at this interface, and therefore to remain inside the core. However, due to the wave character of light, it must satisfy a self-interference condition, in order to be trapped in the waveguide [6]. There are only a finite number of paths which satisfy this condition, and therefore a finite number of modes which propagate through the fiber. The fiber is multimode if $12.5\mu m < r < 100\mu m$ and $0.01 < \Delta < 0.03$, and single-mode if $2\mu m < r < 5\mu m$ and $0.003 < \Delta < 0.01$ [15]. By far the most popular fibers for long distance telecommunications applications allow only a single mode of each polarization to propagate [6].

8.1 Modes in circular fibers

We consider a fiber of uniform cross section with relative magnetic permeability = 1 and n varying only on transverse directions [3]. Assuming a $z-$ and $t-$ dependence $\exp\left(ik_z z - i\omega t\right)$, the Maxwell equations can be combined, to yield the Helmholtz wave equations for **H** and **E**:

$$\nabla^2 \mathbf{H} + \frac{n^2\omega^2}{c^2}\mathbf{H} = i\omega\epsilon_0\left(\nabla n^2\right) \times \mathbf{E} \tag{50}$$

$$\nabla^2 \mathbf{E} + \frac{n^2\omega^2}{c^2}\mathbf{E} = -\nabla\left[\frac{1}{n^2}\left(\nabla n^2\right)\cdot\mathbf{E}\right] \tag{51}$$

where we have written $\epsilon = n^2\epsilon_0$. Just as in Sect. 6, the transverse components of E and H can be expressed in terms of the longitudinal fields E_z, H_z, *i.e.*

$$E_t = \frac{i}{\gamma^2}\left[k_z\nabla_t E_z - \omega\mu_0\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t H_z\right] \tag{52}$$

and

$$H_t = \frac{i}{\gamma^2}\left[k_z\nabla_t H_z + \omega\epsilon_0 n^2\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t E_z\right] \tag{53}$$

where $\gamma^2 = n^2\omega^2/c^2 - k_z^2$ is the radial propagation constant, as for metallic waveguides. If we take the z component of the eqs (54), (55) and use (52) to eliminate the transverse

field components, assuming that $\partial n^2/\partial z = 0$, we find generalizations of the 2D scalar wave equation (32):

$$\boldsymbol{\nabla}_t^2 H_z + \gamma^2 H_z - \left(\frac{\omega}{\gamma c}\right)^2 \left(\boldsymbol{\nabla}_t n^2\right) \cdot \boldsymbol{\nabla}_t H_z = -\frac{\omega k_z \epsilon_0}{\gamma^2} \widehat{\mathbf{z}} \cdot \left[\boldsymbol{\nabla}_t n^2 \times \nabla_t E_z\right] \tag{54}$$

and

$$\boldsymbol{\nabla}_t^2 E_z + \gamma^2 E_z - \left(\frac{k_z^2}{\gamma n}\right)^2 \left(\boldsymbol{\nabla}_t n^2\right) \cdot \boldsymbol{\nabla}_t E_z = \frac{\omega k_z \mu_0}{\gamma^2 n^2} \widehat{\mathbf{z}} \cdot \left[\boldsymbol{\nabla}_t n^2 \times \nabla_t H_z\right] \tag{55}$$

In contrast to (32) for ideal metallic guides, the equations for E_z, H_z are coupled. In general, thee is no separation into purely TE and TM modes. The only simplification occurs in the case of a step-profile refractive index, where we can solve the equation (54) or (55) in each domain of constant refractive index, and match the two solutions, using appropriate boundary conditions. In this case, the radial part of the electric field (for the first mode) in the core is [6]:

$$R(r) = \frac{J_0\left(\sqrt{n_{co}^2 k_0^2 - \beta^2}\,(r/a)\right)}{J_0\left(\sqrt{n_{co}^2 k_0^2 - \beta^2}\right)}, \quad r < a \tag{56}$$

and in the cladding:

$$R(r) = \frac{K_0\left(\sqrt{n_{cl}^2 k_0^2 - \beta^2}\,(r/a)\right)}{K_0\left(\sqrt{n_{cl}^2 k_0^2 - \beta^2}\right)}, \quad r > a \tag{57}$$

These solutions are identical (using an appropriate "dictionary") with the solution of the Schrodinger equation for a particle moving in a potential with cylindrical symmetry, the radial part of the potential being a rectangular well of finite depth. However, this kind of analogies can be more easily developed for planar dielectric waveguides, namely for "step-index" dielectrics, consisting of a central slab of finite thickness and of higher refractive index (core), and two lateral, half-space medium of lower refractive index (cladding). Indeed, in such a situation, the quantum counterpart of the dielectric guide is much more extensively studied, in almost any textbook of quantum mechanics.

9. An optical-quantum analogy: the three-layer slab optical waveguide and the quantum rectangular well

We shall calculate in detail the TE modes of a three-layer slab optical waveguide, with a 1D structure, and the bound states of a particle in a rectangular well, and we shall find that these problems have identical solutions. Of course, the physical meaning of the parameters entering in each solution are different, but the mathematical structure of the solutions is identical.

9.1 The optical problem

We consider a three-layer slab optical waveguide, with a 1D structure [16]. The electromagnetic wave propagates along the x axis, and the slabs are: a semi-infinite medium of refractive index n_1, having as right border the yz plane; a slab of refractive index n_2, having as left border the plane yz and as left border a plane paralel to it, cutting the ox axis at $x_0 = W$;

and a semi-infinite medium of refractive index n_3, for the remaining space. The inner slab corresponds to the core, and the outer ones - to the cladding.

It is instructive to obtain the wave equation for the electric and magnetic field in this simple geometry, starting directly from the Maxwell equations (7), (8). Assuming that $\mu = \mu_0$ throughout the entire system and that the $t-$dependence is:

$$\mathbf{E} = \mathbf{E}\,(t = 0)\exp\left(-i\omega t\right),\quad \mathbf{H} = \mathbf{H}\,(t = 0)\exp\left(-i\omega t\right)$$

the equations for the field components are:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\mu_0\omega H_x \tag{58}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\mu_0\omega H_y \tag{59}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\mu_0\omega H_z \tag{60}$$

Also,

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = -i\epsilon\omega E_x \tag{61}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -i\epsilon\omega E_y \tag{62}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\epsilon\omega E_z \tag{63}$$

TE mode

We shall look for the TE mode. By definition, in this mode there is no electric field in longitudinal direction, $E_z = 0$, there is no space variation in the y direction, so $\partial/\partial y \to 0$, and the z-dependence is $\exp\left(-i\beta z\right)$, so $\partial/\partial z \to -i\beta$. The Maxwell equations (58)-(63) become:

$$H_x = \frac{\beta}{\mu_0\omega}E_y \tag{64}$$

$$-\beta E_x = \mu_0\omega H_y \tag{65}$$

$$H_z = -\frac{i}{\mu_0\omega}\frac{\partial E_y}{\partial x} \tag{66}$$

$$\beta H_y = -\epsilon\omega E_x \tag{67}$$

$$-i\beta H_x - \frac{\partial H_z}{\partial x} = -i\epsilon\omega E_y \tag{68}$$

$$\frac{\partial H_y}{\partial x} = 0 \tag{69}$$

From (69), $H_y = const$ and we can put $H_y = 0$, so from (65), (67), $E_x = 0$. So,

$$E_x = E_z = H_y = 0 \tag{70}$$

With (64), (66) in (68):

$$\frac{d^2 E_y}{dx^2} + \left(\epsilon\mu_0\omega^2 - \beta^2\right) E_y = 0 \tag{71}$$

Defining:

$$n_{eff} = \frac{\beta}{k_0} \rightarrow \beta = n_{eff}k_0, \ \ k_0 = \omega\sqrt{\epsilon_0\mu_0} \tag{72}$$

we have:

$$\frac{d^2 E_y}{dx^2} + k_0^2 \left(\epsilon_r - n_{eff}^2\right) E_y = 0 \tag{73}$$

With $k_0 = 2\pi/\lambda$, (73) becomes:

$$\frac{d^2 E_y}{dx^2} + \frac{4\pi^2}{\lambda^2} \left(\epsilon_r - n_{eff}^2\right) E_y = 0 \tag{74}$$

It is interesting to compare (74) with the Schrodinger equation for a particle of mass m moving in a potential V:

$$\frac{d^2 \psi}{dx^2} + \frac{4\pi^2}{h^2} \left(-2mV + 2m\mathcal{E}\right) \psi = 0 \tag{75}$$

For bound states, $\mathcal{E} = -|\mathcal{E}| < 0$. In (75), the energy is subject of quantization, similar to n_{eff}^2 in (73) - with appropriate boundary conditions, see below. So, the quantum-mechanical energy is proportional to ϵ_r, confirming the analogy stated in Sect.4. The opposite of the potential is proportional to the square of the refractive index - the so-called "upside-down correspondence" [1] between optical and mechanical propagation: a light wave tends to concentrate in the area with maximum refractive index, while a particle tends to propagate on the bottom of the potential. Also, the wavelength λ corresponds to the Planck constant h: when $\lambda \rightarrow 0$, the wave optics is replaced by geometrical optics, similarly with transition ftrom quantum to classical mechanics.

Let us discuss now the boundary conditions. In the absence of charges and current flow on surfaces, the boundary conditions for the electromagnetic fields are:

1. the tangential components of the electric field are continuous while crossing the border

2. the tangential components of the magnetic field are continuous

3. the normal components of the electric flux density $\mathbf{D} = \varepsilon\mathbf{E}$ are continuous

4. the normal components of the magnetic flux density $\mathbf{B} = \mu\mathbf{H}$ are continuous

The tangential electric field at the boundary is $E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}} = E_y\hat{\mathbf{y}}$ and the tangential magnetic field is $H_y\hat{\mathbf{y}} + H_z\hat{\mathbf{z}} = H_z\hat{\mathbf{z}}$. But $H_z = -\frac{i}{\mu_0\omega}\frac{dE_y}{dx}$, so the continuity of H_z is equivalent to the continuity of $\frac{dE_y}{dx}$. So, the conditions (1) and (2) impose the continuity of E_y and of its derivative, $\frac{dE_y}{dx}$. The normal component of the electric field E_x is identically zero, according to (70), so the condition (3) is automatically fulfilled. As $\mu = \mu_0$, condition (4) claims the continuity of the normal components of the magnetic flux, $H_x = \frac{\beta}{\mu_0\omega}E_y$, so this condition coincides with (1).

Consequently, in our case, the boundary conditions request the continuity of E_y and of its derivative dE_y/dx, at the slab boundaries. The equation (73), together with these boundary conditions, define a Sturm-Liouville problem, which determines the eigenvalues of n_{eff} or, equivalently, of β.

For the physics of optical fibers, the most interesting situation is that corresponding to an oscillatory solution inside the core and exponentially small ones outside the core (in the cladding):

$$E_y(x) = C_1 \exp(\gamma_1 x), \quad \gamma_1 = k_0 \sqrt{n_{eff}^2 - n_1^2} \tag{76}$$

$$= C_2 \sin(\gamma_2 x + \alpha), \quad \gamma_2 = k_0 \sqrt{n_2^2 - n_{eff}^2} \tag{77}$$

$$= C_3 \exp(-\gamma_3(x - W)), \quad \gamma_3 = k_0 \sqrt{n_{eff}^2 - n_3^2} \tag{78}$$

As we just have seen, the boundary conditions are equivalent to the continuity of $E_y(x)$ and of its derivative, $dE_y(x)/dx$:

$$\frac{dE_y(x)}{dx} = \gamma_1 C_1 \exp(\gamma_1 x), \quad \gamma_1 = k_0 \sqrt{n_{eff}^2 - n_1^2} \tag{79}$$

$$= \gamma_2 C_2 \cos(\gamma_2 x + \alpha) \tag{80}$$

$$= -\gamma_3 C_3 \exp(-\gamma_3(x - W)) \tag{81}$$

So, the continuity at $x = 0$ means:

$$C_1 = C_2 \sin \alpha \tag{82}$$

$$\gamma_1 C_1 = \gamma_2 C_2 \cos \alpha \tag{83}$$

Similarily, at $x = W$:

$$C_2 \sin(\gamma_2 W + \alpha) = C_3 \tag{84}$$

$$\gamma_2 C_2 \cos(\gamma_2 W + \alpha) = -\gamma_3 C_3 \tag{85}$$

Dividing (83) by (82), we get:

$$\frac{\gamma_1}{\gamma_2} = \cot \alpha, \tag{86}$$

$$\alpha = \operatorname{arccot} \frac{\gamma_1}{\gamma_2} + q_1 \pi, \quad q_1 = 0, 1, 2, \dots \tag{87}$$

Dividing (85) by (81), we get:

$$\frac{\gamma_3}{\gamma_2} = -\cot(\gamma_2 W + \alpha),$$

$$-\operatorname{arccot} \frac{\gamma_3}{\gamma_2} - \alpha + q_2 \pi = \gamma_2 W \tag{88}$$

Substitution of α from (87) into (88) gives:

$$-\operatorname{arccot} \frac{\gamma_3}{\gamma_2} - \operatorname{arccot} \frac{\gamma_1}{\gamma_2} + q\pi = \gamma_2 W \tag{89}$$

Written in the form:

$$-\arctan \frac{\gamma_2}{\gamma_3} - \arctan \frac{\gamma_2}{\gamma_1} + q\pi = \gamma_2 W \tag{90}$$

it coincides with eq. (20) Ch.III, vol.1, [13]. It is the energy eigenvalue equation for the Schrodinger equation of a particle of mass m, moving in the potential $V(x)$:

$$\left[\frac{d^2}{dx^2} + \frac{2m}{\hbar^2} \left(-V(x) + \mathcal{E} \right) \right] \psi(x) = 0 \tag{91}$$

where $V(x)$ is a piecewise-defined function ([13], III.1.6):

$$V(x) = \begin{cases} V_1, & x > a \\ V_2, & a > x > b \\ V_3, & b > x \end{cases} \tag{92}$$

$$V_2 < V_1 < V_3$$

It is useful to consider a particular situation, when $n_1 = n_3$ in the optical waveguide, respectively when $V_1 = V_3 = 0$, $V_2 < 0$, in the quantum mechanical problem. In (91), $-V(x) \geqslant 0$ is given, and we have to find the eigenvalues of the energy $\mathcal{E} < 0$. For the optical waveguide (71), $\epsilon \mu_0 \omega^2$ is given, and the eigenvalues of the quantity $-\beta^2$ (essentially, the propagation constant β) must be obtained. Let us note once again that the refractive index in the optical waveguide corresponds to the opposite of the potential, in the quantum mechanical problem.

Let us investigate in greater detail the consequences of the particular situation just mentioned, $n_1 = n_3$. With $q \to -q$, eq. (90) becomes:

$$\arctan \frac{\gamma_2}{\gamma_1} + \frac{q\pi}{2} = -\frac{\gamma_2 W}{2} \tag{93}$$

It gives, for q odd:

$$\frac{\gamma_1}{\gamma_2} = \tan \frac{\gamma_2 W}{2} \tag{94}$$

and for q even:

$$\frac{\gamma_2}{\gamma_1} = -\tan \frac{\gamma_2 W}{2} \tag{95}$$

Putting:

$$\frac{W}{2} = a, \ \gamma_1 a = a k_0 \sqrt{n_{eff}^2 - n_1^2} = \Gamma_1, \ \gamma_2 a = a k_0 \sqrt{n_2^2 - n_{eff}^2} = \Gamma_2 \tag{96}$$

we get, instead of (94), (95):

$$\frac{\Gamma_1}{\Gamma_2} = \tan \Gamma_2 \quad (q \ odd) \tag{97}$$

$$\frac{\Gamma_2}{\Gamma_1} = -\tan \Gamma_2 \quad (q \ even) \tag{98}$$

Defining K through the equation:

$$\Gamma_1^2 = K^2 - \Gamma_2^2 \tag{99}$$

the eigenvalue conditions (97), (98) take the form:

$$\frac{\sqrt{K^2 - \Gamma_2^2}}{\Gamma_2} = \tan \Gamma_2 \tag{100}$$

$$\frac{\Gamma_2}{\sqrt{K^2 - \Gamma_2^2}} = -\tan\Gamma_2 \tag{101}$$

So, the eigenvalue equation (90) splits into two simpler conditions (100), (101), carracterizing states with well defined parity, as we shall see further on (of course, the parity of q, mentioned just after (93), has nothing to do with the parity of states).

We shall analyze now the same problem, starting from the quantum mechanical side.

9.2 The quantum mechanical problem: the particle in a rectangular potential well

We discuss now the Schrodinger equation for a particle in a rectangular potential well ([17], v.1, pr.25), one of the simplest problems of quantum mechanics:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = \mathcal{E}\psi(x) \tag{102}$$

$$V(x) = \begin{cases} -U, & 0 < x < a \\ 0, & elsewere \end{cases} \tag{103}$$

Let be:

$$\mathcal{E} = -\frac{\hbar^2\varkappa^2}{2m}; \quad U = \frac{\hbar^2 k_0^2}{2m}; \quad k^2 = k_0^2 - \varkappa^2 \tag{104}$$

We are looking for bound states inside the well:

$$u_1(x) = A\exp(\varkappa x), \quad x < 0 \tag{105}$$

$$u_2(x) = B\sin(kx + \alpha), \quad 0 < x < a \tag{106}$$

$$u_3(x) = D\exp(\varkappa(a - x)), \quad a < x \tag{107}$$

The wavefunction and its derivatives:

$$u_1'(x) = \varkappa A\exp(\varkappa x), \quad x < 0 \tag{108}$$

$$u_2'(x) = kB\cos(kx + \alpha), \quad 0 < x < a \tag{109}$$

$$u_3'(x) = -\varkappa D\exp(\varkappa(a - x)), \quad a < x \tag{110}$$

must be continuous in $x = 0$:

$$A = B\sin\alpha \tag{111}$$

$$\varkappa A = kB\cos\alpha \tag{112}$$

and in $x = a$:

$$B\sin(ka + \alpha) = C \tag{113}$$

$$kB\cos(ka + \alpha) = -\varkappa C \tag{114}$$

Dividing (111), (112):

$$\frac{1}{\varkappa} = \frac{1}{k}\tan\alpha, \quad \alpha = \arctan\frac{k}{\varkappa} + n\pi \tag{115}$$

and (113), (114):

$$\frac{1}{k}\tan(ka + \alpha) = -\frac{1}{\varkappa}, \quad ka + \alpha = -\arctan\frac{k}{\varkappa} + n_1\pi \tag{116}$$

and substituting (115) in (116), we get:

$$\frac{k}{\varkappa} = \tan\left(-\frac{ka}{2} + \frac{n_2}{2}\pi\right) \tag{117}$$

For n_2 even:

$$\frac{k}{\varkappa} = -\tan\frac{ka}{2}, \qquad \arctan\frac{k}{\varkappa} = -\frac{ka}{2} \tag{118}$$

and for n_2 odd:

$$\frac{k}{\varkappa} = \tan\left(-\frac{ka}{2} + \frac{\pi}{2}\right), \qquad \arctan\frac{k}{\varkappa} = -\frac{ka}{2} + \frac{\pi}{2} \tag{119}$$

Putting

$$\xi = \frac{ka}{2}, \quad C = \frac{k_0 a}{2} \tag{120}$$

the conditions (118), (119) become respectively:

$$\tan\xi = -\frac{\xi}{\sqrt{C^2 - \xi^2}} \tag{121}$$

$$\cot\xi = \frac{\xi}{\sqrt{C^2 - \xi^2}} \tag{122}$$

Let us write now the wavefunction (106) using the expression (115) for α:

$$u_2(x) = B\sin\left(kx + \arctan\frac{k}{\varkappa} + n\pi\right), \quad 0 < x < a \tag{123}$$

With (118) and (119), the equation (123) splits in two equations:

$$u_2^*(x) = B\sin\left(kx - \frac{ka}{2} + n\pi\right) \tag{124}$$

$$u_2^{**}(x) = B\sin\left(kx - \frac{ka}{2} + \frac{\pi}{2} + n\pi\right) \tag{125}$$

We translate now the coordinate x, so that the origin of the new axis is placed in the center of the well:

$$x = y + \frac{a}{2} \tag{126}$$

and we get:

$$U_2^*(y) = u_2^*\left(y + \frac{a}{2}\right) = B\sin(ky + n\pi) = (-1)^n B\sin ky \tag{127}$$

$$U_2^{**}(y) = u_2^{**}\left(y + \frac{a}{2}\right) = B\sin\left(ky + \frac{\pi}{2} + n\pi\right) = (-1)^n B\cos ky \tag{128}$$

So, the wavefunctions corresponding to the eigenvalues obtained from (121), (122) have well defined parity.

Let us stress once more that the core of the optical-quantum analogy consists in eqs. (74), (75), which can be formulated as follows: the refractive index for the propagation of light plays a similar role to the potential, for the propagation of a quantum non-relativistic particle, and both the electric (or magnetic) field and the wave function are the solution of essentially

the same (Helmholtz) equation. So, if the dynamics of a particle, given by the Schrodinger equation, can be considered as the central aspect of quantum mechanics, the scattering of light by a medium with refractive index $n\left(\overrightarrow{r}\right)$ can be considered as the central aspect of optics, at least when the Maxwell equations can be reduced to a Helmholtz equation. Remembering Goethe's opinion, that the "Urphänomenon" of light science is the scattering of light on a "turbid" medium, one could remark that his theory of colours is not always as unrealistic as it was generally considered. [18]

10. Ballistic electrons in 2DESs

As already mentioned, the 2DES, formed at the interface of two semiconductors might play a central role in mesoscopic physics. The thin 2D conduction layer formed in the GaAs/AlGaAs heterojunction may reach a carrier concentration of 2×10^{12} cm^{-2} and can be depleted by applying a negative voltage to a metalic gate deposited on the surface [Datta]. The mobility can be as high as $10^6 cm^2/Vs$, two order of magnitude higher than in bulk semiconductors. The Fermi wavelength λ_M is about 35 nm, and the electron mean free path may be as long as $\lambda_m = 30\mu$m— the same order of magnitude as the liniar dimension of the sample; the ballistic regime of electrons is therefore easily reached.

At low temperature, the conduction in mesoscopic semiconductor is mainly due to electrons in the conduction band. Their dynamics can be described by an equation of the form:

$$\left[\mathcal{E}_c + \frac{(i\hbar\nabla + e\mathbf{A})^2}{2m} + U\left(\mathbf{r}\right)\right]\Psi\left(\mathbf{r}\right) = \mathcal{E}\Psi\left(\mathbf{r}\right) \tag{129}$$

where $\mathcal{E}_c$ referrs to the conduction band energy, $U\left(\mathbf{r}\right)$ is the potential energy due to space-charge etc., $\mathbf{A}$ is the vector potential and m is the effective mass. Any band discontinuity $\Delta\mathcal{E}_c$ at heterojunctions is incorporated by letting $\mathcal{E}_c$ be position-dependent [11].

In the case of a homogenous semiconductor, $U\left(\mathbf{r}\right) = 0$, assuming $\mathbf{A} = 0$ and $\mathcal{E}_c = const.$, the solution of (129) is given by plane waves, $\exp\left(i\mathbf{k}\cdot\mathbf{r}\right)$, and not by Bloch functions, $u_k\left(\mathbf{r}\right)\cdot\exp\left(i\mathbf{k}\cdot\mathbf{r}\right)$. So, the solutions of (129) are not true wavefunctions, but wavefunctions smoothed out over a mesoscopic distance, so any rapid variation at atomic scale is suppressed; eq. (129) is called single-band effective mass equation.

Let us consider a 2DES contained mainly in the xy plane. This means that, in the absence of any external potential, the electrons can move freely in the xy plane, but they are confined in the z-direction by some potential $V\left(z\right)$, so their wavefunctions will have the form:

$$\Psi\left(\mathbf{r}\right) = \phi_n\left(z\right)\exp\left(ik_x x\right)\exp\left(ik_y y\right) \tag{130}$$

The quantization on the $z-$direction, expressed by the functions $\phi_n\left(z\right)$, generate several subbands, with cut-off energy ε_n. At low temperature, only the first subband, corresponding to $n = 1$, is occupied, so, instead of (129), the electrons of the 2DES are described by:

$$\left[\mathcal{E}_s + \frac{(i\hbar\nabla + e\mathbf{A})^2}{2m} + U\left(\mathbf{r}\right)\right]\Psi\left(\mathbf{r}\right) = \mathcal{E}\Psi\left(\mathbf{r}\right) \quad (102) \tag{131}$$

where the subband energy is $\mathcal{E}_s = \mathcal{E}_c + \varepsilon_1$; so, the ‡−dimension enters in this equation only through ε_1, which depends on the confining potential $V\left(z\right)$. The eq. (131) correctly describes

the 2DESs formed in semiconductor heterostructures, but is inappropriate for metallic thin films, where the electron density is much higher, and even at nanoscopic scale, there are tens of occupied bands; so, the system is merely 3D. Consequently, the dimensionality of a system depends not only on its geometry, but also on its electron concentration. Let us remind that the conductive / dielectric properties of a sample depends on frequency of electromagnetic waves: so even basic classification of materials is not necessarely intrinsec, but it might depend of the value of some parameters.

11. Transverse modes (or magneto-electric subbands)

We shall discuss now the concept of transverse modes or subbands, which are analogous to the transverse modes of electromagnetic waveguides [11]. In narrow conductors, the different transverse modes are well separated in energy, and such conductors are often called electron waveguides.

We consider a rectangular conductor that is uniform in the $x-$direction and has some transverse confining potential $U(y)$. The motion of electrons in such a conductor is described by the effective mass eq (131):

$$\left[\mathcal{E}_s + \frac{(i\hbar \nabla + e\mathbf{A})^2}{2m} + U(y) \right] \Psi(x,y) = \mathcal{E}\Psi(x,y) \tag{132}$$

We assume a constant magnetic field B in the $z-$direction, perpendicular to the plane of the conductor, which can be represented by a vector potential defined by:

$$A_x = -By, \quad A_y = 0 \tag{133}$$

so that the effective-mass equationcan be rewritten as:

$$\left[\mathcal{E}_s + \frac{(p_x + eBy)^2}{2m} + \frac{p_y^2}{2m} + U(y) \right] \Psi(x,y) = \mathcal{E}\Psi(x,y) \tag{134}$$

Writing

$$\Psi(x,y) = \frac{1}{\sqrt{L}} \exp(ikx)\chi(y) \tag{135}$$

we get for the transverse function the equation:

$$\left[\mathcal{E}_s + \frac{(\hbar k + eBy)^2}{2m} + \frac{p_y^2}{2m} + U(y) \right] \chi(y) = \mathcal{E}\chi(y) \tag{136}$$

We are interested in the nature of the transverse eigenfunctions and eigenenergies for different combinations of the confining potential $U$and the magnetic field B. A parabolic potential

$$U(y) = \frac{1}{2}m\omega_0^2 y^2 \tag{137}$$

is often a good description of the actual potential in many electron wave guides.

Let us consider the case of confined electrons $(U \neq 0)$ in zero magnetic field $(B = 0)$. Eq. (134) becomes:

$$\left[\mathcal{E}_s + \frac{\hbar^2 k^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m \omega_0^2 y^2 \right] \chi(y) = \mathcal{E} \chi(y) \tag{138}$$

with solutions:

$$\chi_{n,k}(y) = u_n(q), \quad q = \sqrt{\frac{m \omega_0}{\hbar}} y \tag{139}$$

$$E(n,k) = E_s + \frac{(\hbar k)^2}{2m} + \left(n + \frac{1}{2} \right) \hbar \omega_0 \tag{140}$$

where:

$$u_n(q) = \exp \left(-\frac{1}{2} q^2 \right) H_n(q) \tag{141}$$

with H_n - Hermite polynomials. The velocity is obtained from the slope of the dispersion curve:

$$v(n,k) = \frac{1}{\hbar} \frac{\partial E(n,k)}{\partial k} = \frac{\hbar k}{m} \tag{142}$$

States with different index n are said to belong to different subbands; the situation is similar to that described in Sect. 8, where we have discussed the confinement due to the potential $V(z)$. However, the confinement in the $y-$direction is somewhat weaker, and several subbands are now normally occupied. The subbands are often referred to as *transverse modes*, in analogy with the modes of an electromagnetic waveguide [11].

12. Effective mass approximation revisited

A more attentive investigation of the effective-mass approximation for electrons in a semiconductor, introduced in a simplified form in Sect.10, will allow quantitative analogies between propagation ballistic electrons and guided electromagnetic waves past abrupt interfaces [19].

According to Morrow and Brownstein [20], out of the general class of Hamiltonians suggested by von Ross [21], only those of the form:

$$H \psi = -\frac{\hbar^2}{2} \left\{ m(\vec{r})^\alpha \nabla \cdot \left[m(\vec{r})^\beta \nabla \left(m(\vec{r})^\alpha \psi \right) \right] \right\} + V(\vec{r}) \psi = \mathcal{E} \psi \tag{143}$$

with the constraint

$$2\alpha + \beta = -1 \tag{144}$$

(where α and β have specific values for specific substances) can be used in the study of refraction of ballistic electrons at the interface between dissimilar semiconductors.

For the Hamiltonian (143), the boundary conditions for an electron wave at an interface are:

$$m^\alpha \psi = continous \tag{145}$$

and

$$m^{\alpha+\beta} \nabla \psi \cdot \hat{n} = continuous \tag{146}$$

with $\widehat{n}$ — the unit vector normal to the interface. Analogously, the boundary conditions for an electromagnetic waves at an interface between two dielectrics require the continuity of tangential components E_t, H_t across the interface. Based on these considerations, it is reasonable to look for analogies between

$$\Phi = m^\alpha \psi \tag{146}$$

and either E or H.

For bulk propagation in a homogenous medium, an exact analogy can be drawn between Φ and both E or H. In this case, eq. (143) reduces to a Helmholtz equation of the form:

$$\nabla^2 \Phi = -k^2 \Phi, \ \ k^2 = \frac{2m(E - V)}{\hbar^2} \tag{147}$$

The wave equation (148) is exactly analogous to the Helmholtz equation for an electromagnetic wave propagating in a homogenous dielectric of permittivity ϵ and permeability μ, with Φ replaced by E or H, and

$$k^2 = \omega^2 \mu \epsilon \tag{148}$$

So, an exact analogy can be drawn between Φ and both E or H. With these analogies, one can define a phase-refractive index for electron waves as:

$$n_{ph}^{EW} = m_r^{1/2} (E - V)_r^{1/2} \tag{149}$$

where $m_r = m/m_{ref}$ is the relative effective mass and

$$(\mathcal{E} - V)_r = \frac{\mathcal{E} - V}{\mathcal{E} - V_{ref}} \tag{150}$$

is the relative kinetic energy, where m_{ref}, V_{ref} are the effective mass and the potential energy in a reference region. This electron wave phase-refractive index is analogous to the phase-refracting index for electromagnetic waves,

$$n_{ph}^{EM} = \sqrt{\mu_r \epsilon_r} \tag{151}$$

With these results, phase-propagation effects, such as interference, can be analyzed using standard em results, where E (or H), n_{ph}^{EM} is replaced by Φ, n_{ph}^{EW}. These results are valid for all the Hamiltonians (143). For the electron wave amplitude, the index of refraction is defined as

$$n_{amp,l}^{EW} = m_{r,l}^{\beta+1/2} (\mathcal{E} - V)_{r,l}^{1/2} \tag{152}$$

These expressions are exactly the same as the anagous electromagnetic expressions for the reflection/refraction of an electromagnetic wave from an interface between two dielectrics [3].

The theory outlined in this section can be extended for 1D or 2D inhomogenous materials, but not for three dimensions [19].

Dragoman and Dragoman [22], [23], [24] obtained a quantum-mechanical - electromagnetic analogy, similar to [19], in the sense that the electronic wave function does not correspond to the fields, but to the vector potential:

$$m^\alpha \psi \rightarrow A \ \text{(TM wave) or} \ (\varepsilon/\mu) \ \text{(TE wave)} \tag{153}$$

13. Optics experiments with ballistic electrons

In 2DESs, there exists a unique oportunity to control ballistic carriers via electrostatic gates, which can act as refractive elements for the electron path, in complete analogy to refractive elements in geometrical optics. [8], [25], [26]. The refraction of a beam of ballistic electrons can be simply described, using elementary considerations. If in the "left" half-space, the potential has a constant value V, and in the "right" one, a different (but also constant) one, $V + \Delta V$, an incident electron arriving with an incident angle θ and kinetic energy

$$\mathcal{E} = \frac{\hbar^2 k^2}{2m^*} \tag{154}$$

emerges in the "right" half-plane under the angle θ', with the kinetic energy

$$\mathcal{E}' = \mathcal{E} + e\Delta V = \frac{\hbar^2 k'^2}{2m^*} \tag{155}$$

Translational invariance along the interface preserves the parallel component of electron momentum and thus

$$k \sin \theta = k' \sin \theta' \tag{156}$$

or

$$\frac{\sin \theta}{\sin \theta'} = \frac{k'}{k} = \sqrt{\frac{\mathcal{E}'}{\mathcal{E}}} \tag{157}$$

Considering that the energies $\mathcal{E}$, $\mathcal{E}'$ are Fermi energies, proportional (in a 2D system) to the electron densities n_{el} (not to be confused with refractive index!), the Snell's law takes the form:

$$\frac{\sin \theta}{\sin \theta'} = \sqrt{\frac{n'_{el}}{n_{el}}} \tag{158}$$

An electrostatic lens for ballistic electrons was set up in [8] and its focusing action was demonstrated in the GaAs / AlGaAs heterostructure. In this way, the close analogy between the propagation of ballistic electrons and geometrical optics has been put in evidence.

Another nice experiment used a refractive electronic prism to switch a beam of ballistic electrons between different collectors in the same 2DES [25]

The quantum character of ballistic electrons is clearly present, even they are regarded as beams of particles. Transmission and reflection of electrons on a sharp (compared to λ_F), rectangular barrier, induced via a surface gate at $T = 0.5K$, follow the laws obtained in quantum mechanics, for instance the following expression for the transmission coefficient:

$$T = \frac{1}{1 + 0.25 \left(\frac{k}{k_0} - \frac{k_0}{k} \right)^2 \sin^2 (ka)}$$

with k, k_0 - the wavevectors inside and outside the barrier, and a - its width (compare the previous formula with the results of III.I.7 of [13]). [27]

14. Conclusions

Several analogies between electromagnetic and quantum-mechanical phenomena have been analyzed. They rely upon the fact that both wave equation for electromagnetic and electric field with well defined frequency, obtained from the Maxwell equations, and the time-independent Schrodinger equation, have the same form - which is a Helmholtz equation.

However, the description of these analogies is by no way a simple dictionary between two formalisms. On the contrary, their physical basis has been discussed in detail, and they have been developed for very modern domains of physics - optical fibers, 2DESs, electron waveguides, electronic transport in mesoscopic and nanoscopic regime. So, the analogies examined in this chapter offers the opportunity of reviewing some very exciting, new and rapidly developing fields of physics, interesting from both the applicative and fundamental perspective. It has been stressed that the analogies are not simple curiosities, but they bear a significant cognitive potential, which can stimultate both scientific understanding and technological progress in fields like waveguides, optical fibers, nanoscopic transport.

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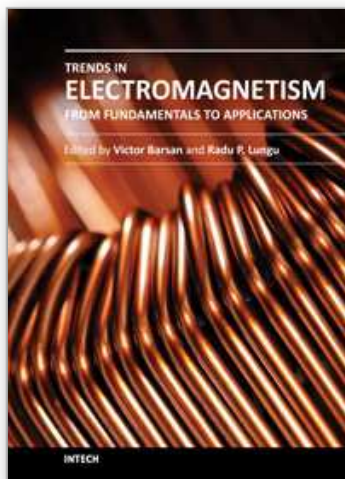
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