Magnetohydrodynamic Rotating Flow of a Fourth Grade Fluid Between Two Parallel Infinite Plates

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1. Introduction

Mechanics of non-linear fluids present a special challenge to physicists, mathematician and engineers. The non-linearity can manifest itself in a variety of ways. Materials such as clay coatings and other suspensions, polymer melts, drilling muds, certain oils and greases, elastomers and many emulsions have been treated as non-Newtonian fluids. There is no single model which clearly exhibits all properties of non-Newtonian fluids and there has been much confusion over the classification of non-Newtonian fluids. However, non-Newtonian fluid may be classified as (1) fluids for which the shear stress depends only on the rate of shear; (2) fluids for which the relation between shear stress and shear rate depends on time; (3) the visco-elastoic fluids, which possess both elastic and viscous properties.

It is not possible to recommend a single constitutive equation which exhibits all properties of non-Newtonian fluids due to the great diversity in the physical structure of non-Newtonian fluids. For this reason, several non-Newtonian models or constitutive equations have been proposed and most of them are empirical or semi empirical. One of the simplest ways in which the visco-elastic fluids have been classified is the methodology given in [1,2]. They present constitutive relations for the stress tensor as a function of the symmetric part of the velocity gradient and its higher derivatives. Another class of models is the rate-type fluid models such as the Oldroyd model [3]. Although many constitutive equations have been suggested, many questions are still unsolved. Some of the continuum models do not give satisfactory results in accordance with the available experimental data. For this reason, in many practical applications, empirical or semi empirical equations have been used. A complete and thorough discussion of various models can be found in [4-7]. Various authors [8-12] investigated non-Newtonian fluids.

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The study of an electrically conducting fluid flows in channels under the action of a transversely applied magnetic field has important applications in many devices such as magnetohydrodynamic (MHD) pumps, aerodynamics heating, MHD power generators, accelerators, centrifugal separation of matter from fluid, flow meters, electrostatic precipitation, fluid droplets sprays, purification of crude oil, petroleum industries and polymer technology. Hartmann [13] first studied an incompressible viscous electrically conducting fluid under the action of a transverse magnetic field. Under different physical conditions it was considered by Sutton and Shermann [14], Hughes and Young [15], Cowling [16] and Pai [17]. Rajagopal and Na [18] studied the flow of a third grade fluid due to an oscillation of plate, Mollica and Rajagopal [19] examined secondary flows due to axial shearing of a third grade fluid between two eccentrically placed cylinders, Siddiqui and Kaloni [20] investigated plane flow of a third grade fluid. Rotating disk flows of conducting fluids have practical applications in many areas such as computer storage devices, lubrication, crystal growth processes, viscometry and rotating machinery. The effect of an external uniform magnetic field on the flow due to a rotating disk was studied [21-25], and eccentric rotation of disks was studied [26-29]. In many process of industries, the cooling of threads or sheets of some polymer materials is of great importance in the production line. Magneto convection plays an important role in various industrial applications including magnetic control of molten iron flow in the steel industry and liquid metal cooling in nuclear reactors. Palani and Abbas [30] investigated the combined effects of magnetohydrodynamic and radiation on free convection flow past an impulsively started isothermal vertical plate with Rosseland diffusion approximation, Farzaneh-Gord et al. [31] studied two-dimensional steady-state incompressible viscoelastic boundary layer magnetohydrodynamics flow and heat transfer over a stretching sheet in the presence of electric and magnetic fields. The highly non-linear momentum and heat transfer equations are solved analytically.

The MHD fluid flow as lubricant is of interest in industrial applications, because it prevents the unexpected variation of lubricant viscosity with temperature under certain extreme operating conditions. The MHD lubrication in an externally pressurized thrust bearing has been investigated both theoretically and experimentally by Maki et al. [32]. Hughes and Elco [33] and Kuzma et al. [34] have investigated the effects of a magnetic field in lubrication. These authors had neglected the inertial terms in the Navier–Stokes equations. Hamza [35] considered the squeezing flow between two discs in the presence of a magnetic field. The problem of squeezing flow between rotating discs has been studied by Hamza [36] and Bhattacharyya and Pal [37]. Considering two-dimensional unsteady MHD flow of a viscous fluid between two moving parallel plates, Sweet et al. [38] have shown that the flow is strongly influenced by the strength of the magnetic field and the density of the fluid. Abbas et al. [39] have investigated the unsteady MHD boundary layer flow and heat transfer in an incompressible rotating viscous fluid over a stretching continuous sheet. The resulting system of partial differential equations is solved numerically using Keller-box method. Turkylimazoglu [40] has analyzed the MHD time-dependent von Karman swirling electrically conducting viscous fluid flow having a temperature-dependent viscosity due to a rotating disk impulsively set into motion.

Hayat et al. [41] have considered the unsteady rotating MHD flow of an incompressible second grade fluid in a porous half space. The flow is induced by a suddenly moved plate in its own plane. Both the fluid and plate rotate in unison with the same angular velocity. Assuming the velocity field of the form $\mathbf{V} = [u(z,t), v(z,t), w(z,t)]$, analytical solutions are presented using Fourier sine transforms and it is shown that with an increase in MHD parameter the real and imaginary parts of velocity as well as the boundary layer thickness decreases.

The classical theories of continuum mechanics are inadequate to explicate the microscopic manifestations of microscopic events. The fluids with microstructure belonging to a class of fluid with non-symmetrical stress tensor referred to as polar fluids are called Micropolar fluids. Physically they represent fluids consisting of randomly oriented particles suspended in a viscous medium. Eringen [42] presented the earliest formulation of a general theory of fluid microcontinua taking into account the inertial characteristics of the substructure particles which are allowed to undergo rotation in 1964. This theory has been extended by Eringen [43] to take into account thermal effects. The theory of micropolar fluids and its extension thermomicropolar fluids [44] may form suitable non-Newtonian fluid models which can be used to explain the flow of colloidal fluids, polymeric suspensions, liquid crystals, animal blood, etc. Eldabe et al. [45] have discussed the problem of heat transfer to MHD flow of a micropolar fluid from a stretching sheet with suction and blowing through a porous medium. The numerical results indicate that the velocity and the angular velocity increase as the permeability parameter increases but they decrease as the magnetic field increases. On the other hand, the temperature decreases as the permeability parameter increases but it increases as the magnetic field increases.

The study of laminar boundary layer flow of non-Newtonian fluids over continuous moving surfaces is very important because of its practical importance in a number of engineering processes. For example, cooling of an infinite metallic plate in a cooling bath, the boundary layer along a liquid film in condensation processes, aerodynamic extrusion of plastic sheets and a polymer sheet or filament extruded continuously from a dye. Furthermore, it has several practical applications in the field of metallurgy and chemical engineering such as material manufactured by extrusion process and heat-treated materials traveling between a feed roll and a wind-up roll or on conveyor belt possess, the feature of a moving continuous surface. Also, glass blowing, continuous casting, and spinning of fibers involve the flow due to a stretching surface. Sarpakaya [46] studied the MHD flow of a non-Newtonian fluid, Char [47] studied the MHD flow of a viscoelastic fluid over a stretching sheet by considering the thermal diffusion in the energy equation. However, the effects of thermal radiation on the viscoelastic boundary layer flow and heat transfer can be quite significant at high operating temperatures. In view of this, Raptis [48], Raptis and Perdikis [49] and Raptis et al. [50] studied the viscoelastic flow and heat transfer over a flat plate with constant suction, thermal radiation and viscous dissipation. Recently, the effects of viscous dissipation, radiation, in presence of temperature dependent heat sources/sinks on heat transfer characteristics of a viscoelastic fluid is considered by Siddheshwar and Mahabaleswar [51]. Khan [52] extended the problem by including the effects of suction/injection, heat source/sink and radiation effects. Abel et al. [53] investigated the effects of viscous dissipation and non-uniform heat source on viscoelastic boundary layer flow over a linear stretching sheet. Abel and Nandeppanavar [54] studied the effect of non-uniform heat source/sink on MHD viscoelastic boundary layer flow, further Nandeppanavar et al. [55] studied the effects of elastic deformation and non-uniform heat source on viscoelastic boundary layer flow. Motivated by these studies, Mahantesh et al. [56] extended the results of researchers [53,54,55] for MHD viscoelastic boundary layer flow with combined effects of viscous dissipation, thermal radiation and non-uniform heat source which was ignored by [53,54,55]. Furthermore, they analyzed the effects of radiation, viscous dissipation, viscoelasticity, magnetic field on the heat transfer characteristics in the presence of non-uniform heat source with variable PST and PHF temperature boundary conditions. Kayvan Sadeghy et al. [57] have investigated theoretically the applicability of magnetic fields for controlling hydrodynamic separation in Jeffrey-Hamel flows of viscoelastic fluids. It is shown that for viscoelastic fluids, it is possible to delay flow separation in a diverging channel provided that the magnetic field is sufficiently strong. It is also shown that the effect of magnetic field on flow separation becomes more pronounced the higher the fluid's elasticity.

In the present paper we have modeled the unsteady flow equations of a fourth grade fluid bounded between two non-conducting rigid plates in a rotating frame of reference with imposed uniform transverse magnetic field. It is interesting to note that we are able to couple the equations arising for the velocity field. The steady rotating flow of the non-Newtonian fluid subject to a uniform transverse magnetic field is studied. The non-linear differential equations resulting from the balance of momentum and mass are solved numerically. The effects of exerted magnetic field, Ekman number and material parameter on the velocity distribution are presented graphically. The results for Newtonian and non-Newtonian fluids are compared.

2. Mathematical model of the problem

We introduce a Cartesian coordinate system with z-axis normal to the plane of the parallel plates. The plates are located at $z=0$ and $z=L$ and the plates and the fluid bounded between them are in a rigid body rotation with constant angular velocity Ω about the z-axis. The fluid is electrically conducting and assumed to be permeated by an imposed magnetic field B_0 perpendicular to the parallel plates. The disturbance in the fluid is produced by small amplitude non-torsional oscillations of the lower plate. For the present model we take the velocity field of the form.

$$\mathbf{V}=\left[u(z,t),\ v(z,t),\ 0\right], \tag{1}$$

where u and v are the x and y components of the velocity field. The Cauchy stress tensor for the fourth grade fluid can be obtained by the model introduced by Coleman and Noll [58]

$$\mathbf{T}=-p\mathbf{I}+\sum_{j=1}^{n}\mathbf{S}_j. \tag{2}$$

For the fourth grade fluid we have $n=4$ and the first four tensors S_j are given by

$$\mathbf{S}_1 = \mu\mathbf{A}_1, \tag{3}$$

$$\mathbf{S}_2 = \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{4}$$

$$\mathbf{S}_3 = \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \tag{5}$$

$$\begin{aligned}\mathbf{S}_4 = {} & \gamma_1\mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 \\ & + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) + \gamma_5(tr\mathbf{A}_2)\mathbf{A}_2 + \gamma_6(tr\mathbf{A}_2)\mathbf{A}_1^2 \\ & + \left\{\gamma_7(tr\mathbf{A}_3) + \gamma_8(tr\mathbf{A}_2\mathbf{A}_1)\right\}\mathbf{A}_1,\end{aligned} \tag{6}$$

where μ is the co-efficient of shear viscosity; and

$$\alpha_i \ (i=1,2),\ \beta_j \ (j=1,2,3),\ \gamma_k \ (k=1,2,...,8)$$

are material constants. The Rivlin- Ericken tensors $\mathbf{A}_n$ are defined by the recursion relation

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(grad\mathbf{V}) + (grad\mathbf{V})^T\mathbf{A}_{n-1},\ n>1, \tag{7}$$

$$\mathbf{A}_1 = (grad\mathbf{V}) + (grad\mathbf{V})^T, \tag{8}$$

where

$$\frac{d}{dt}(.) = \left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla\right)(.). \tag{9}$$

When $\gamma_k = 0\ (k=1,2,...,8)$, the fourth grade model reduces to third grade model, when $\beta_j = 0\ (j=1,2,3)$ and $\gamma_k = 0\ (k=1,2,...,8)$ then above model reduces to second grade model and if $\alpha_i = 0\ (i=1,2),\ \beta_j = 0\ (j=1,2,3),\ \gamma_k = 0\ (k=1,2,...,8)$ the flow model reduces to classical Navier-Stokes viscous fluid model.

The hydromagnetic flow is generated in the uniformly rotating fluid by small amplitude non-torsional oscillations of the plate located at $z=0$. With the Cartesian coordinate system O_{xyz} the unsteady motion of the incompressible fourth grade conducting fluid in the presence of magnetic field **B** is governed by the law of balance of linear momentum and balance of mass i.e.

$$\frac{d\mathbf{V}}{dt} + 2(\mathbf{\Omega}\times\mathbf{V}) + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = \frac{1}{\rho}\mathrm{div}\mathbf{T} + \frac{1}{\rho}(\mathbf{J}\times\mathbf{B}), \tag{10}$$

$$\mathrm{div}\,\mathbf{V} = 0, \tag{11}$$

where ρ is the density, $\mathbf{J}$ is the current density and $\mathbf{B}(=\mathbf{B}_0+\mathbf{b}$, $\mathbf{b}$ being the induced magnetic field) is the total magnetic field.

In the absence of displacement currents, the Maxwell equations and the generalized Ohm's law can be written as

$$\nabla\cdot\mathbf{B}=0,\ \nabla\times\mathbf{B}=\mu_m\mathbf{J},\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{12}$$

$$\mathbf{J}=\sigma(\mathbf{E}+\nabla\times\mathbf{B}), \tag{13}$$

where μ_m is the magnetic permeability, $\mathbf{E}$ is the electric field and σ is the electrical conductivity of the fluid.

The magnetic Reynolds number is assumed to be very small so that the induced magnetic field is negligible [14]. This assumption is reasonable for the flow of liquid metals, e.g. mercury or liquid sodium (which are electrically conducting under laboratory conditions). The electron–atom collision frequency is assumed to be relatively high so that the Hall effect can be included [14]. The Lorentz force per unit volume is given by

$$\mathbf{J}\times\mathbf{B}=-\sigma B_0^2\mathbf{V}. \tag{14}$$

For the velocity field defined in Eq. (1), the equation of continuity (11) is identically satisfied and Eq. (10) in component form can be written as

$$\begin{aligned}\frac{\partial u}{\partial t}-2\Omega v-\Omega^2 x=&-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 u}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 u}{\partial z^2\partial t^2}\\&+\frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{\begin{array}{l}2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)\\+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\end{array}\right\}\right]\\&+\frac{(\gamma_3+\gamma_5)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^2 u}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{\begin{array}{l}2\frac{\gamma_7}{\rho}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)\\+\frac{(\gamma_7+\gamma_8)}{\rho}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\end{array}\right\}\right]+\frac{\gamma_1}{\rho}\frac{\partial^5 u}{\partial z^2\partial t^3}-\frac{\sigma}{\rho}B_0^2 u,\end{aligned} \tag{15}$$

$$
\begin{aligned}
\frac{\partial v}{\partial t}+2\Omega u-\Omega^{2}y={}&-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\frac{\partial^{2}v}{\partial z^{2}}+\frac{\alpha_{1}}{\rho}\frac{\partial^{3}v}{\partial z^{2}\partial t}+\frac{\beta_{1}}{\rho}\frac{\partial^{4}v}{\partial z^{2}\partial t^{2}}\\
&+\frac{2\left(\beta_{2}+\beta_{3}\right)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^{2}+\left(\frac{\partial v}{\partial z}\right)^{2}\right\}\right]\\
&+\frac{\gamma_{2}}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{\begin{array}{l}2\dfrac{\partial}{\partial t}\left(\left(\dfrac{\partial u}{\partial z}\right)^{2}+\left(\dfrac{\partial v}{\partial z}\right)^{2}\right)\\+2\dfrac{\partial u}{\partial z}\dfrac{\partial^{2}u}{\partial z\partial t}+2\dfrac{\partial v}{\partial z}\dfrac{\partial^{2}v}{\partial z\partial t}\end{array}\right\}\right]\\
&+\frac{\left(\gamma_{3}+\gamma_{5}\right)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^{2}v}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^{2}+\left(\frac{\partial v}{\partial z}\right)^{2}\right\}\right]\\
&+\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{\begin{array}{l}2\dfrac{\gamma_{7}}{\rho}\dfrac{\partial}{\partial t}\left(\left(\dfrac{\partial u}{\partial z}\right)^{2}+\left(\dfrac{\partial v}{\partial z}\right)^{2}\right)\\+\dfrac{\left(\gamma_{7}+\gamma_{8}\right)}{\rho}\left(\dfrac{\partial u}{\partial z}\dfrac{\partial^{2}u}{\partial z\partial t}+\dfrac{\partial v}{\partial z}\dfrac{\partial^{2}v}{\partial z\partial t}\right)\end{array}\right\}\right]+\frac{\gamma_{1}}{\rho}\frac{\partial^{5}v}{\partial z^{2}\partial t^{3}}-\frac{\sigma}{\rho}B_{0}^{2}v,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
0={}&-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\left(2\alpha_{1}+\alpha_{2}\right)}{\rho}\frac{\partial}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^{2}+\left(\frac{\partial v}{\partial z}\right)^{2}\right\}+\frac{\beta_{1}}{\rho}\frac{\partial}{\partial z}\left\{\begin{array}{l}2\dfrac{\partial}{\partial t}\left(\left(\dfrac{\partial u}{\partial z}\right)^{2}+\left(\dfrac{\partial v}{\partial z}\right)^{2}\right)\\+2\dfrac{\partial u}{\partial z}\dfrac{\partial^{2}u}{\partial z\partial t}+2\dfrac{\partial v}{\partial z}\dfrac{\partial^{2}v}{\partial z\partial t}\end{array}\right\}\\
&+\frac{\beta_{2}}{\rho}\frac{\partial}{\partial z}\left\{2\frac{\partial u}{\partial z}\frac{\partial^{2}u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^{2}v}{\partial z\partial t}\right\}+\frac{\gamma_{1}}{\rho}\frac{\partial}{\partial z}\left\{\begin{array}{l}2\dfrac{\partial^{2}}{\partial t^{2}}\left(\left(\dfrac{\partial u}{\partial z}\right)^{2}+\left(\dfrac{\partial v}{\partial z}\right)^{2}\right)\\+\,2\dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial z}\dfrac{\partial^{2}u}{\partial z\partial t}+\dfrac{\partial v}{\partial z}\dfrac{\partial^{2}v}{\partial z\partial t}\right)\\+2\left(\dfrac{\partial u}{\partial z}\dfrac{\partial^{3}u}{\partial z\partial t^{2}}+\dfrac{\partial v}{\partial z}\dfrac{\partial^{3}v}{\partial z\partial t^{2}}\right)\end{array}\right\}\\
&+\frac{\gamma_{3}}{\rho}\frac{\partial}{\partial z}\left\{\begin{array}{l}\left(\dfrac{\partial^{2}u}{\partial z\partial t}\right)^{2}+\left(\dfrac{\partial^{2}v}{\partial z\partial t}\right)^{2}\\+4\left(\left(\dfrac{\partial u}{\partial z}\right)^{2}+\left(\dfrac{\partial v}{\partial z}\right)^{2}\right)^{2}\end{array}\right\}+\frac{(4\gamma_{4}+4\gamma_{5}+2\gamma_{6}}{\rho}\frac{\partial}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^{2}+\left(\frac{\partial v}{\partial z}\right)^{2}\right)^{2}.
\end{aligned}
\tag{17}
$$

Defining the modified pressure

$$\hat{p}=\frac{p}{\rho}-\frac{(2\alpha_1+\alpha_2)}{\rho}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}-\frac{\beta_1}{\rho}\left\{\begin{array}{l}2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)\\+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\end{array}\right\}$$
$$-\frac{\beta_2}{\rho}\left\{2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right\}-\frac{\gamma_1}{\rho}\left\{\begin{array}{l}2\frac{\partial^2}{\partial t^2}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)\\+2\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\\+2\left(\frac{\partial u}{\partial z}\frac{\partial^3 u}{\partial z\partial t^2}+\frac{\partial v}{\partial z}\frac{\partial^3 v}{\partial z\partial t^2}\right)\end{array}\right\} \tag{18}$$
$$-\frac{\gamma_3}{\rho}\left\{\begin{array}{l}\left(\frac{\partial^2 u}{\partial z\partial t}\right)^2+\left(\frac{\partial^2 v}{\partial z\partial t}\right)^2\\+4\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)^2\end{array}\right\}-\frac{(4\gamma_4+4\gamma_5+2\gamma_6)}{\rho}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)^2,$$

then Eqs. (15)-(17) become

$$\begin{aligned}\frac{\partial u}{\partial t}-2\Omega v-\Omega^2 x=&-\frac{\partial\hat{p}}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 u}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 u}{\partial z^2\partial t^2}\\&+\frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right\}\right]\\&+\frac{(\gamma_3+\gamma_5)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^2 u}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{2\frac{\gamma_7}{\rho}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+\frac{(\gamma_7+\gamma_8)}{\rho}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\right\}\right]\\&+\frac{\gamma_1}{\rho}\frac{\partial^5 u}{\partial z^2\partial t^3}-\frac{\sigma}{\rho}B_0^2 u,\end{aligned} \tag{19}$$

$$
\begin{aligned}
\frac{\partial v}{\partial t}+2\Omega u-\Omega^2 y=-\frac{\partial \hat{p}}{\partial y}+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 v}{\partial z^2\partial t^2}\\
+\frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\
+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right\}\right]\\
+\frac{(\gamma_3+\gamma_5)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^2 v}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\
+\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{2\frac{\gamma_7}{\rho}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+\frac{(\gamma_7+\gamma_8)}{\rho}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\right\}\right]\\
+\frac{\gamma_1}{\rho}\frac{\partial^5 v}{\partial z^2\partial t^3}-\frac{\sigma}{\rho}B_0^2 v,
\end{aligned}
\tag{20}
$$

$$
0=-\frac{\partial \hat{p}}{\partial z}. \tag{21}
$$

Since $r^2=x^2+y^2$, therefore $x=\frac{\partial}{\partial x}\left(\frac{1}{2}r^2\right)$ *and* $y=\frac{\partial}{\partial y}\left(\frac{1}{2}r^2\right)$. In view these substitutions we can write Eqs. (19)-(21) in the following manner:

$$
\begin{aligned}
\frac{\partial u}{\partial t}-2\Omega v=-\frac{\partial}{\partial x}\left(\hat{p}-\frac{1}{2}\Omega^2 r^2\right)+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 u}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 u}{\partial z^2\partial t^2}\\
+\frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\
+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right\}\right]\\
+\frac{(\gamma_3+\gamma_5)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^2 u}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\
+\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{2\frac{\gamma_7}{\rho}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+\frac{(\gamma_7+\gamma_8)}{\rho}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\right\}\right]\\
+\frac{\gamma_1}{\rho}\frac{\partial^5 u}{\partial z^2\partial t^3}-\frac{\sigma}{\rho}B_0^2 u,
\end{aligned}
\tag{22}
$$

$$\begin{aligned}\frac{\partial v}{\partial t}+2\Omega u=&-\frac{\partial}{\partial y}\left(\hat{p}-\frac{1}{2}\Omega^2r^2\right)+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 v}{\partial z^2\partial t^2}\\&+\frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right\}\right]\\&+\frac{(\gamma_3+\gamma_5)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^2 v}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{2\frac{\gamma_7}{\rho}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+\frac{(\gamma_7+\gamma_8)}{\rho}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\right\}\right]\\&+\frac{\gamma_1}{\rho}\frac{\partial^5 v}{\partial z^2\partial t^3}-\frac{\sigma}{\rho}B_0^2 v,\end{aligned}\tag{23}$$

$$0=-\frac{\partial}{\partial z}\left(\hat{p}-\frac{1}{2}\Omega^2r^2\right).\tag{24}$$

Redefining the modified pressure

$$\tilde{p}=\hat{p}-\frac{1}{2}\Omega^2r^2,\tag{25}$$

Eqs. (22)-(24) become

$$\begin{aligned}\frac{\partial u}{\partial t}-2\Omega v=&-\frac{\partial\tilde{p}}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 u}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 u}{\partial z^2\partial t^2}\\&+\frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right\}\right]\\&+\frac{(\gamma_3+\gamma_5)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^2 u}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{2\frac{\gamma_7}{\rho}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+\frac{(\gamma_7+\gamma_8)}{\rho}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\right\}\right]\\&+\frac{\gamma_1}{\rho}\frac{\partial^5 u}{\partial z^2\partial t^3}-\frac{\sigma}{\rho}B_0^2 u,\end{aligned}\tag{26}$$

$$\begin{aligned}\frac{\partial v}{\partial t}+2\Omega u=&-\frac{\partial \tilde{p}}{\partial y}+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 v}{\partial z^2\partial t^2}\\&+\frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right\}\right]\\&+\frac{(\gamma_3+\gamma_5)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^2 v}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{2\frac{\gamma_7}{\rho}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+\frac{(\gamma_7+\gamma_8)}{\rho}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\right\}\right]\\&+\frac{\gamma_1}{\rho}\frac{\partial^5 v}{\partial z^2\partial t^3}-\frac{\sigma}{\rho}B_0^2 v,\end{aligned}\tag{27}$$

$$0=-\frac{\partial \tilde{p}}{\partial z}.\tag{28}$$

Differentiating Eqs. (26) and (27) with respect to z and making use of Eq. (28), and then integrating with respect to z to obtain

$$\begin{aligned}\frac{\partial u}{\partial t}-2\Omega v=&\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 u}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 u}{\partial z^2\partial t^2}\\&+\frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right\}\right]\\&+\frac{(\gamma_3+\gamma_5)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^2 u}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}\right]\\&+\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left\{2\frac{\gamma_7}{\rho}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)+\frac{(\gamma_7+\gamma_8)}{\rho}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\right\}\right]\\&+\frac{\gamma_1}{\rho}\frac{\partial^5 u}{\partial z^2\partial t^3}-\frac{\sigma}{\rho}B_0^2 u+\lambda(t),\end{aligned}\tag{29}$$

$$
\begin{aligned}
\frac{\partial v}{\partial t}+2\Omega u=\frac{\mu}{\rho}\frac{\partial^2 v}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 v}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 v}{\partial z^2\partial t^2}& \\
+\ \frac{2\left(\beta_2+\beta_3\right)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\ \left(\frac{\partial v}{\partial z}\right)^2\right\}\right]& \\
+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{2\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\ \left(\frac{\partial v}{\partial z}\right)^2\right)+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right\}\right]& \\
+\frac{\left(\gamma_3+\gamma_5\right)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial^2 v}{\partial z\partial t}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\ \left(\frac{\partial v}{\partial z}\right)^2\right\}\right]& \qquad (30)\\
+\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\left\{2\frac{\gamma_7}{\rho}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial z}\right)^2+\ \left(\frac{\partial v}{\partial z}\right)^2\right)+\frac{\left(\gamma_7+\gamma_8\right)}{\rho}\left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial t}+\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z\partial t}\right)\right\}\right]& \\
+\frac{\gamma_1}{\rho}\frac{\partial^5 v}{\partial z^2\partial t^3}-\frac{\sigma}{\rho}B_0^2 v+\delta(t).&
\end{aligned}
$$

On multiplying Eq. (30) by *i* and then adding the resulting equation in Eq. (29) we get

$$
\begin{aligned}
\frac{\partial q}{\partial t}+2i\Omega q=\frac{\mu}{\rho}\frac{\partial^2 q}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 q}{\partial z^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 q}{\partial z^2\partial t^2}& \\
+\ \frac{4\left(\beta_2+\beta_3\right)}{\rho}\frac{\partial}{\partial z}\left[\left\{\left(\frac{\partial q}{\partial z}\right)^2\ \left(\frac{\partial \overline{q}}{\partial z}\right)\right\}\right]& \\
+\frac{\gamma_2}{\rho}\frac{\partial}{\partial z}\left[\left\{3\left(\frac{\partial q}{\partial z}\right)\frac{\partial}{\partial t}\left(\frac{\partial q}{\partial z}\frac{\partial \overline{q}}{\partial z}\right)\right\}\right]+\frac{\gamma_1}{\rho}\frac{\partial^5 q}{\partial z^3\partial t^2}& \\
+\frac{\left(\gamma_3+\gamma_5\right)}{\rho}\frac{\partial}{\partial z}\left[\left\{2\left(\frac{\partial^2 q}{\partial z\partial t}\right)\left(\frac{\partial q}{\partial z}\frac{\partial \overline{q}}{\partial z}\right)\right\}\right]& \qquad (31)\\
+\frac{\partial}{\partial z}\left[\begin{array}{l}\dfrac{2\gamma_7}{\rho}\left(\dfrac{\partial q}{\partial z}\right)\dfrac{\partial}{\partial t}\left(\dfrac{\partial q}{\partial z}\dfrac{\partial \overline{q}}{\partial z}\right)\\ +\dfrac{\left(\gamma_7+\gamma_8\right)}{2\rho}\left(\dfrac{\partial q}{\partial z}\right)\dfrac{\partial}{\partial t}\left(\dfrac{\partial q}{\partial z}\dfrac{\partial \overline{q}}{\partial z}\right)\end{array}\right]-\frac{\sigma}{\rho}B_0^2 q+\psi(t),&
\end{aligned}
$$

where

$$q = u + iv, \quad \bar{q} = u - iv, \quad \psi(t) = \lambda(t) + i\delta(t) \tag{32}$$

For steady state the Equations (29) and (30) reduce to

$$-2\Omega v = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} + \frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial u}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}\right] - \frac{\sigma}{\rho}B_0^2 u, \tag{33}$$

$$2\Omega u = \frac{\mu}{\rho}\frac{\partial^2 v}{\partial z^2} + \frac{2(\beta_2+\beta_3)}{\rho}\frac{\partial}{\partial z}\left[2\frac{\partial v}{\partial z}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}\right] - \frac{\sigma}{\rho}B_0^2 v, \tag{34}$$

Introducing the dimensionless variables

$$z^* = \frac{z}{L}, \; u^* = \frac{uL}{\nu}, \text{ and } v^* = \frac{vL}{\nu} \tag{35}$$

in above equations and simplifying the resulting equations and dropping '*' to obtain

$$-2vE^{-1} = \frac{\partial^2 u}{\partial z^2} + 4\beta\left[3\left(\frac{\partial^2 u}{\partial z^2}\right)\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial^2 u}{\partial z^2}\right)\left(\frac{\partial v}{\partial z}\right)^2 + 2\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial^2 v}{\partial z^2}\right)\right] - H^2 u \tag{36}$$

$$2E^{-1}u = \frac{\partial^2 v}{\partial z^2} + 4\beta\left[3\left(\frac{\partial^2 v}{\partial z^2}\right)\left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial^2 v}{\partial z^2}\right)\left(\frac{\partial u}{\partial z}\right)^2 + 2\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial^2 u}{\partial z^2}\right)\right] - H^2 v \tag{37}$$

where

$$E^{-1} = \frac{\Omega L^2}{\nu}, \; \beta = \frac{4(\beta_2+\beta_3)\nu}{\rho L^4}, \; H^2 = \frac{nL^2}{\nu}, \; n = \frac{\sigma B_0^2}{\rho} \tag{38}$$

and E is the Ekman number while H is the Hartmann number.

3. Numerical procedure

Consider a simplest boundary value problem

$$F(u'', u', u, z) = 0, \tag{39}$$

$$u(a) = A \text{ and } u(b) = B. \tag{40}$$

To solve the boundary value problem the derivatives u' and u'' involved in the problem are approximated by finite differences of appropriate order. If we employ second order central difference formulation, then we can write

$$u'(z) = \frac{u(z+h) - u(z-h)}{2h} + \mathrm{O}(h^2), \tag{41}$$

$$u''(z) = \frac{u(z+h) - 2u(z) + u(z-h)}{h^2} + O(h^2). \tag{42}$$

This converts the given boundary value problem into a linear system of equations involving values of the function u at a, $a+h$, $a+2h,\cdots$,b. For higher accuracy, one should choose h small. However, this increases the number of equations in the system which in turn increases the computational time.

Depending upon the size of this resulting system of linear equations, it can either be solved by exact methods or approximate methods.

In the present problem the governing differential equations (36) and (37) are highly non-linear which cannot be solved analytically. These equations are discretized using second order central finite difference approximations defined in Eqs. (41) and (42). The resulting system of algebraic equations is solved using successive under relaxation scheme. The difference equations are linearized employing a procedure known as lagging the coefficients [59]. The iterative procedure is repeated until convergence is obtained according to the following criterion

$$\max\left|u^{(n+1)} - u^{(n)}\right| < \varepsilon,$$

where superscript $'n'$ represents the number of iteration and $'\varepsilon'$ is the order of accuracy. In the present case ε is taken as 10^{-8}.

4. Numerical results and discussion

The steady velocity components u and v are plotted against independent variable z for different values of Ekman number E, Hartmann number H and material parameter β and results are compared for two types of fluids: the Newtonian fluid, for which $\beta_i = 0$ $(i = 1,2,3)$, and the non-Newtonian fluid, in which we choose $\beta = 1$. Fig. 1 shows the effect of Hartmann number H on the velocity components u and v. We fixed $E = 1$ and varied H= 0,1,3,5. It is observed that an increase in the Hartmann number reduces the velocity components u and v due to the effects of the magnetic force against the flow direction. Figs. 1a and 1b show that with an increase of Hartmann number H, the curvature of the velocity component u profile increases for both a Newtonian fluid and non-Newtonian fluid. Quite contrary, increasing Hartmann number H causes the velocity component v profile to become less parabolic, see Figs. 1c and 1d. It is also noted that decrease in u and v in the Newtonian fluid is larger as compared with non-Newtonian fluid. Furthermore, the boundary layer thickness is drastically decreased by increasing H. It means that the magnetic field provides some mechanism to control the boundary layer thickness.

The dependence of the velocity components u and v on the Ekman number is shown in Fig. 2. In Fig. 2 we fixed $H = 1$ and varied $E = 0.1$, 0.2, 0.3. It is observed that velocity component u increases with an increase in Ekman number E for the Newtonian fluid while it remains almost unaffected for non-Newtonian fluid (Figs. 2a, 2b). Moreover, a

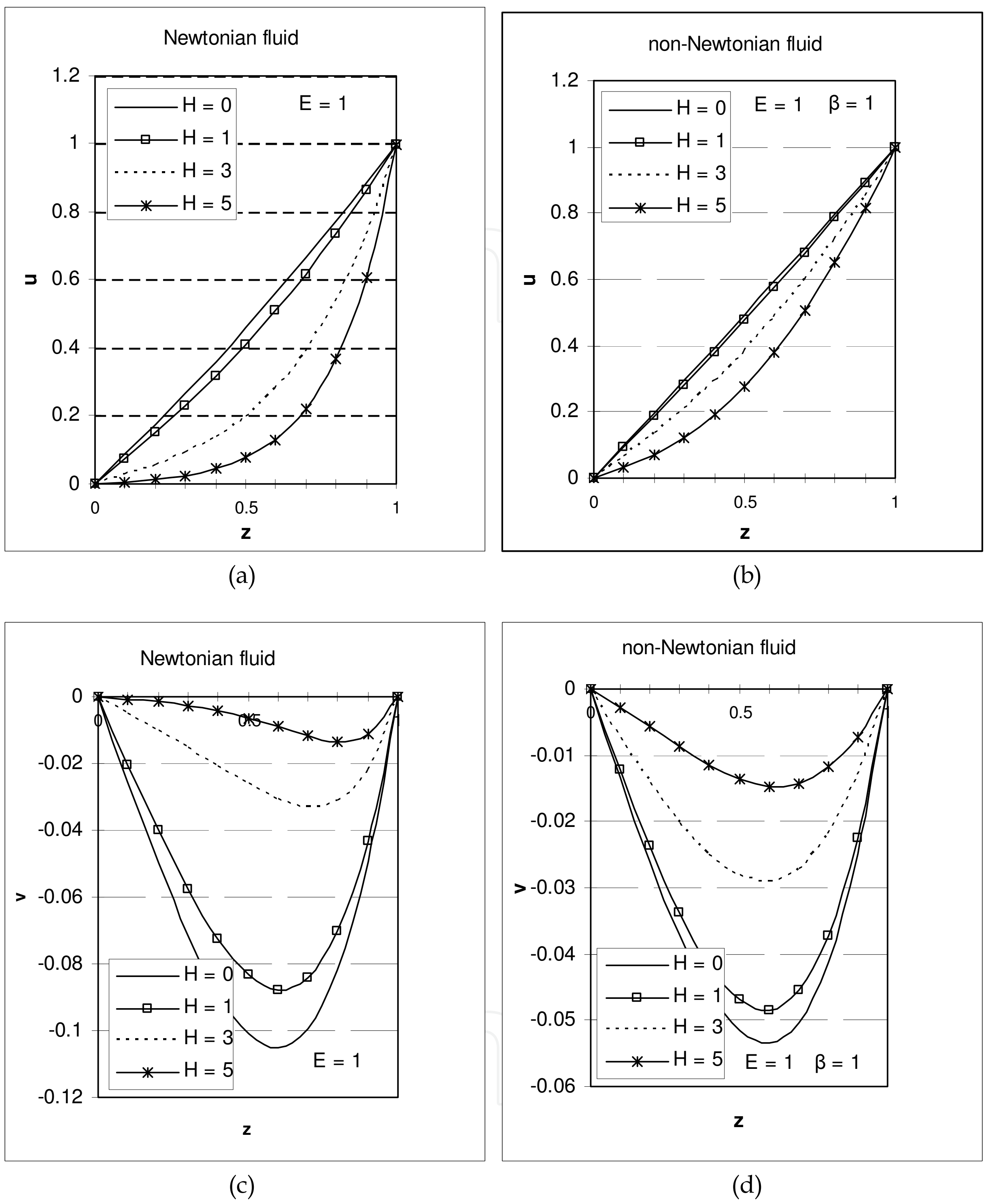


Fig. 1. Variation of velocity components u and v with z for $H = 0,1,3,5$; $E = 1, \beta = 0$ in (a), (c); $E = 1, \beta = 1$ in (b), (d).

backflow is observed near the boundary $z = 0$ for $E = 0.1$. On the contrary, the magnitude of velocity component v decreases with an increase in Ekman number E for the both types of the fluids (Figs. 2c, 2d). This velocity component has larger magnitude in Newtonian fluid as compared with non-Newtonian fluid.

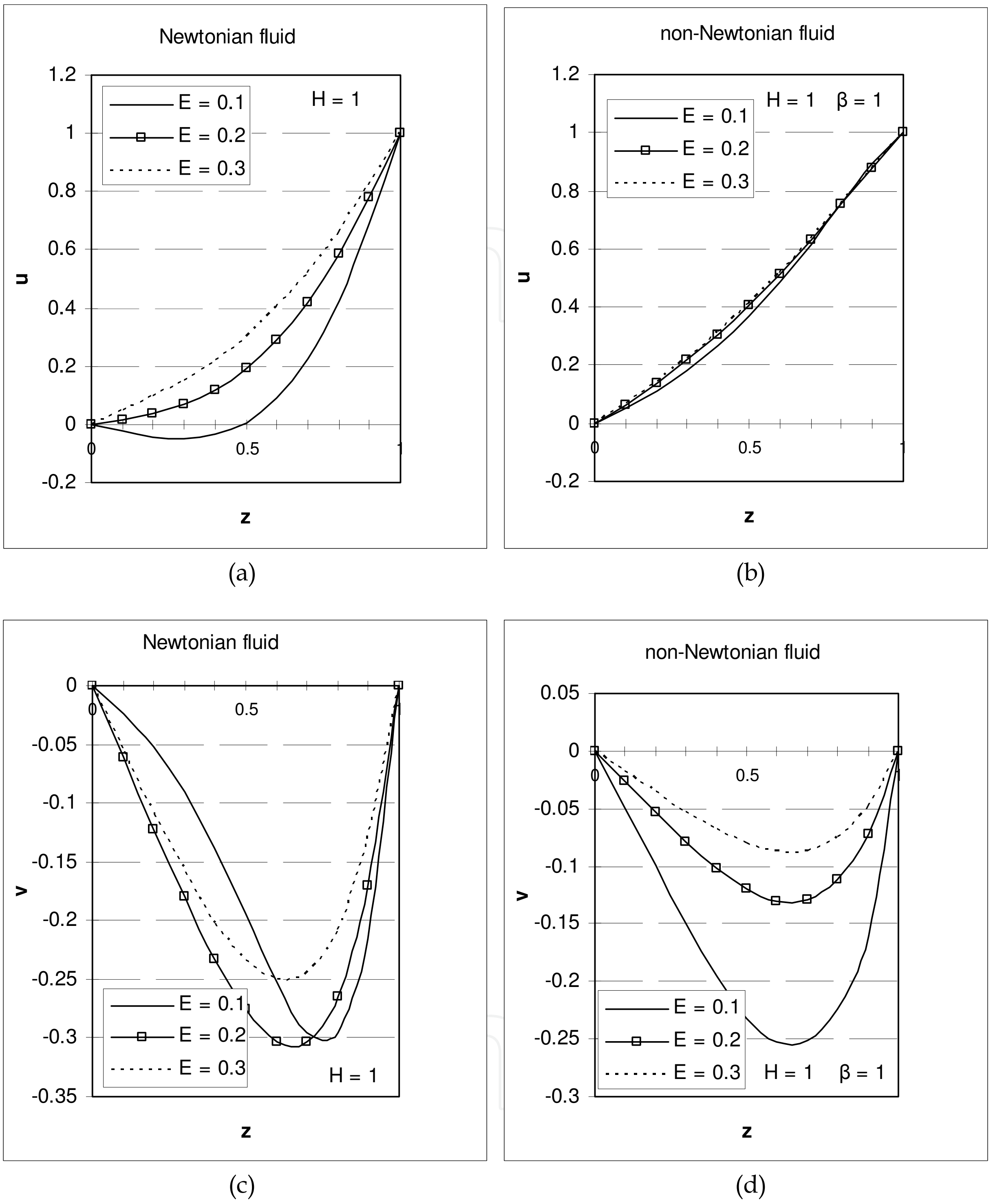


Fig. 2. Variation of velocity components u and v with z for $E = 0.1,\ 0.2,\ 0.3;\ H = 1, \beta = 0$ in (a), (c); $H = 1, \beta = 1$ in (b), (d).

Figs. 3,4 depict the variation of the velocity components u and v with z for various values of material parameter β fixing $E = 1$ and taking $H = 1$ in Figs. 3a and 3c, while $H = 5$ in 3b, 3d, and in Fig. 4. It is observed from Fig. 3b and 4a that when the material parameter β increases from $\beta = 1$ to a large value of 20, the velocity component u tend to approach the

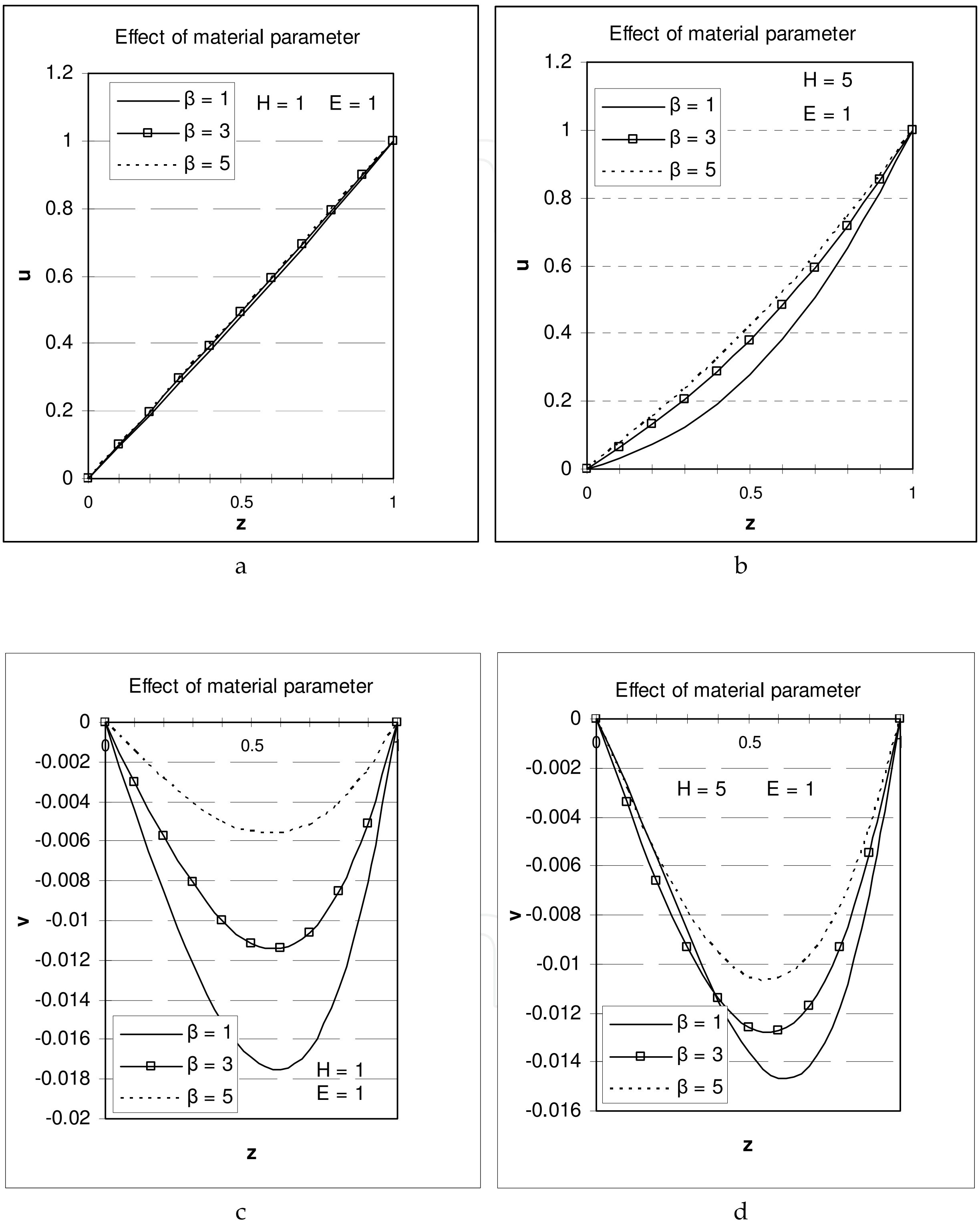
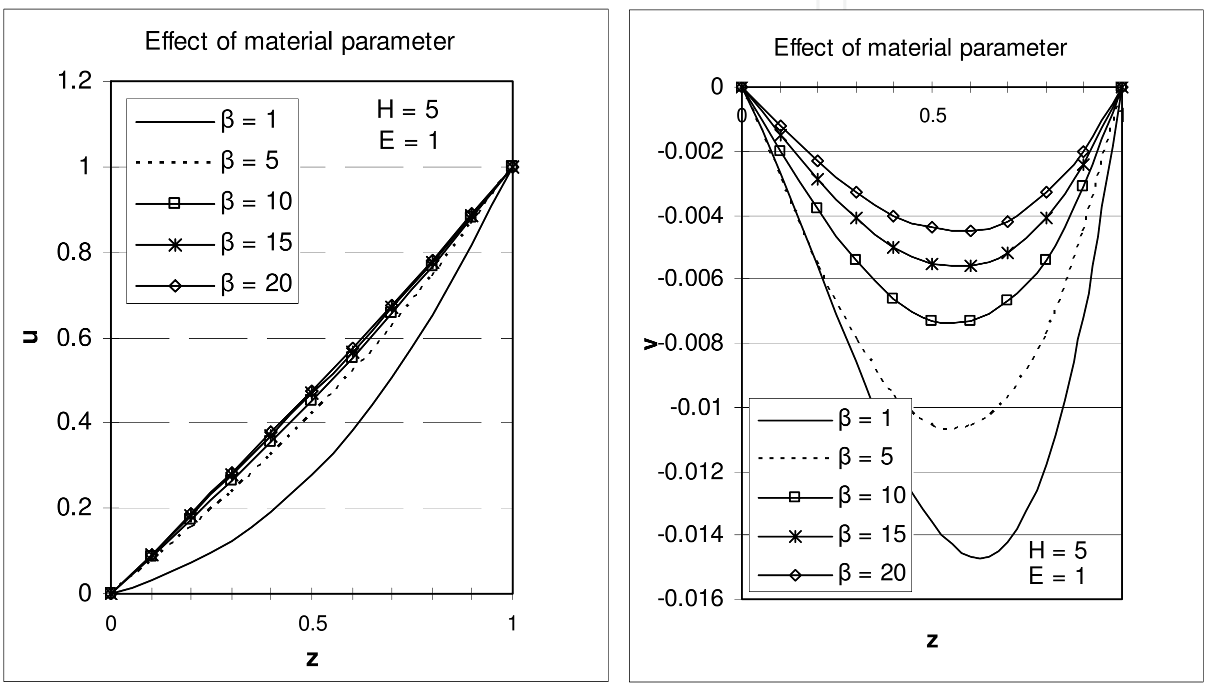


Fig. 3. Variation of velocity components u and v with z for $\beta = 1,\ 3,\ 5;$ fixing $E = 1,\ H = 5$.

linear distribution; thus, the shearing can unattenuately extend to the whole flow domain from the boundaries, corresponding to a shear-thickening phenomenon. A further increase of β will not effect this velocity component further. The magnitude of velocity component v decreases when β increases and the curvature of the velocity profile decreases with an increase in material parameter β (see Figs. 3c, 3d and 4b). It is also found that the flow behaviour depends strongly on the choice of the parameters, for example, for large H, u increases with an increase of material parameter β, whereas this velocity component is independent of β for small H. On the contrary, the magnitude of velocity component v decreases with β for both small and large values of H.



(a) (b)

Fig. 4. Variation of velocity components u and v with z for large values of $\beta = 1, 5,\ 10,\ 15, 20$; fixing $E = 1, H = 5$.

5. Conclusions

The unsteady rotating flow of a uniformly conducting incompressible fourth-grade fluid between two parallel infinite plates in the presence of a magnetic field is modeled. The steady rotating flow of the non-Newtonian fluid subject to a uniform transverse magnetic field is studied. The governing non-linear equations are solved numerically. The numerical results of the non-Newtonian fluid are compared with those of a Newtonian fluid. The major findings of the present works can be summarized as follows:

- The transverse magnetic field decelerates the fluid motion. When the strength of the magnetic field increases, the flow velocity decreases.
- It is observed that the boundary layer thickness decreases drastically by increasing H. It means that the magnetic field provides some mechanism to control the boundary layer thickness.

- It is noted that the flow behaviour depends strongly on the choice of the parameters.

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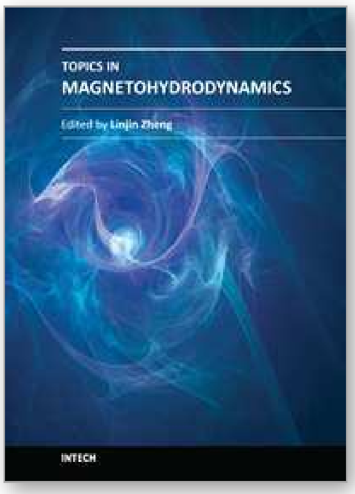

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