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Hamiltonian Representation of Magnetohydrodynamics for Boundary Energy Controls

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1. Introduction

1.1 Brief summary of this chapter

This chapter shows that basic boundary control strategies for *magnetohydrodynamics* (MHD) can be derived from a formal system representation, called a *port-Hamiltonian system* (Van der Schaft and Maschke, 2002). The port-Hamiltonian formulation clarifies collocated input/output pairs used for stabilizing and assigning a global stable point. The controls called *passivity-based controls* (Arimoto, 1996; Ortega et al., 1998; Van der Schaft, 2000; Duindam et al., 2009) are simple and robust to disturbances. Moreover, port-Hamiltonian systems can be connected while keeping their consistency with respect to energy flows. Finally, we show that port-Hamiltonian systems can be used for boundary controls. In the future, this theory might be specialized, for instance, in order to control disruptions of Tokamak plasmas (Wesson, 2004; Pironti and Walker, 2005; Ariola and Pironti, 2008). This chapter emphasizes the versatility of control system representations.

1.2 Background and motivation

Control theory significantly progressed during the last two decades of the 20th century. *Linear control theory* (Zhou et al., 1996) was developed for systems whose states are limited to a neighborhood around stable points. The theory was extended to include particular classes of distributed parameter systems and nonlinear systems (Khalil, 2001; Isidori, 1995). However, dispite this progress, simpler and more intuitive methods like *PID controls* (Brogliato et al., 2006) are still in the mainstream of practical control designs. One reason for this trend is that advanced methods do not always remarkably produce significant improvements to the performance of controlled systems despite their theoretical complexity; rather, they are prone to modeling errors. The other reason is that simple methods are understandable and adjustable online, although the resulting performance is not exactly optimal.

On the other hand, actual controlled systems can be regarded as distributed parameter systems from a macroscopic viewpoint, e.g., as elastic continuums, and as discrete nonlinear

systems from a microscopic viewpoint, e.g., as molecular dynamics systems. Moreover, their stable points are not always unique and vary according to the environment. Multi-physics and multi-scaling models are becoming increasingly significant in science and engineering because of rapid advances in computational devices and micromachining technology. However, such complexities have tended to be ignored in system modeling of conventional control designs, because controllers have to be simple enough to be integrated with other mechanisms and be quickly adjustable. Moreover, numerical analyses using more detailed models can be executed off-line by trial and error and in circumstance where there are no physical size limitations on the computational devices. Hence, it would be desirable to have a new framework of simple control designs like PID controls, but for complex systems. The port-Hamiltonian system, which is introduced in this chapter, is one of the most promising frameworks for this purpose. This chapter addresses the issue of how to derive simple and versatile controls for partial differential equations (PDEs), especially, those of MHD, from considerations about the storage and dissipation of energy in port-Hamiltonian systems.

1.3 History of topic and relevant research

Port-Hamiltonian systems are a framework for passivity-based controls. *Passivity* (Van der Schaft, 2000) is a property by which the energy supplied from the outside of systems through input/output variables can be expressed as a function of the stored energy. The storage function is equivalent to a Hamiltonian in dynamical systems. The collocated input/output variable pairs, called *port variables*, are defined systematically in terms of port-Hamiltonian systems, and they are used as controls and for making observations. Passivity-based controls consist of *shaping Hamiltonians* and *damping assignments*. The Hamiltonians of these systems can be changed by "connecting" them to other port-Hamiltonian systems by means of the port variables. The Hamiltonian of controlled systems is equal to the sum of those of the original system and controllers. Thus, if we can design such a changed Hamiltonian beforehand, the connections give the Hamiltonian of the original system "shaping". Such connected port-Hamiltonian systems with a shaped Hamiltonian can be stabilized to the minimum of the storage function by adding dissipating elements to the port variables.

The energy preserving properties of port-Hamiltonian systems can be described in terms of a *Dirac structure* (Van der Schaft, 2000; Courant, 1990), which is the generalization of symplectic and Poisson structures (Arnold, 1989). Dirac structures enable us to model complex systems as port-Hamiltonian representations, e.g., distributed parameter systems with nonlinearity (Van der Schaft and Maschke, 2002), systems with higher order derivatives (Le Gorrec et al., 2005; Nishida, 2004), thermodynamical systems (Eberard et al., 2007), discretized distributed systems (Golo et al., 2004; Voss and Scherpen, 2011) and their coupled systems. This chapter mainly uses the port-Hamiltonian representation of PDEs for boundary controls based on passivity, i.e., the DPH system. The boundary integrability of DPH systems is derived from a *Stokes-Dirac structure* (Van der Schaft and Maschke, 2002), which is an extended Dirac structure in the sense of Stokes theorem. Because of this boundary integrability, the change in the internal energy of DPH systems is equal to the energy supplied through port variables defined on the boundary of the system domain. Hence, passivity-based controls for distributed parameter systems can be considered to be boundary energy controls.

1.4 Construction of this chapter

In Section 2, we derive the geometric formulation of MHD defined by using differential forms (Flanders, 1963; Morita, 2001). After that, we rewrite the model in terms of DPH systems. The modeling procedure is systematically determined by a given Hamiltonian. Next, we explain passivity-based controls that can be applied to the DPH system of MHD, and their energy flows by means of the bond graph (Karnopp et al., 2006). Finally, we show that the boundary power balance equation of the DPH system is the extended energy principle of MHD (Wesson, 2004) in the sense of dynamical systems and boundary controls.

In Section 3, we extend the DPH model of MHD to include non-Hamiltonian subsystems corresponding to external force terms in Euler-Lagrange equations. Actual controlled systems represented by MHD might be affected by model perturbations, e.g., disturbances or other controllers, or model improvements. Such variations cannot always be modeled in terms of Hamiltonian systems. Some systems of PDEs can be decomposed into a Hamiltonian subsystem, which we call an *exact subsystem*, and a non-Hamiltonian subsystem, which we call a *dual-exact subsystem* (Nishida et al., 2007a). Through this decomposition, a PDE system can be described as a coupled system consisting of a port-Hamiltonian subsystem determined by a *pseudo potential* and other subsystems representing, e.g., external forces, dissipations and distributed controls.

In Section 4, we derive a boundary observer for detecting symmetry breaking (Nishida et al., 2009) from the DPH system of conservation laws associated with MHD. For example, Hamiltonian systems can be regarded as the conservation law with a symmetry that is the invariance of energy with respect to the time evolution. If a symmetry is broken, the associated conservation law becomes invalid. Symmetry breaking can be detected by checking whether quantities are conserved with the boundary port variables of the DPH system. Furthermore, we present a basic strategy for detecting the topological transitions of the domain of DPH systems. The formulation using differential forms defined on Riemannian manifolds can describe systems affected by such transitions. We use a general decomposition of differential forms on Riemannian manifolds and of vector fields on three-dimensional Riemannian manifolds and derive the boundary controls for creating a desired topological energy flow from this decomposition.

The last section is devoted to a brief introduction of future work on this topic.

2. Port-Hamiltonian systems and passivity-based controls for MHD

2.1 Ideal magnetohydrodynamical equations

Magnetohydrodynamics (MHD) is a discipline involving modeling magnetically confined plasmas (Wesson, 2004; Pironti and Walker, 2005; Ariola and Pironti, 2008). The ideal MHD system is a coupled system consisting of a single fluid and an electromagnetic field with certain constitutive relations.

The fluid is described by the two equations in three dimensions. The first is the mass conservation law,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{1}$$

where $\boldsymbol{\rho}(t,x) \in \mathbb{R}$ is the local mass density at time $t \in \mathbb{R}$ at the spatial position $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, and $\boldsymbol{v}(t,x) \in \mathbb{R}^3$ is the fluid (Eulerian) velocity at t and x. The second is Newton's law applied to an infinitesimal plasma element with an electromagnetic coupling,

$$\boldsymbol{\rho}\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p + \boldsymbol{J}\times\boldsymbol{B}, \tag{2}$$

where $p(t,x) \in \mathbb{R}$ is the kinetic pressure in plasma, $\boldsymbol{J}(t,x) \in \mathbb{R}^3$ is the free current density, $\boldsymbol{B}(t,x) \in \mathbb{R}^3$ is the magnetic field induction, and the Lorentz force term $\boldsymbol{J}\times\boldsymbol{B}$ means the coupling.

The electromagnetic field satisfies the Maxwell's equations consisting of Ampere's law, Faraday's law, and Gauss's law for the magnetic induction field:

$$-\frac{\partial \boldsymbol{D}}{\partial t} = -\nabla\times\boldsymbol{H} + \boldsymbol{J}, \quad -\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times\boldsymbol{E}, \quad \nabla\cdot\boldsymbol{B} = 0, \tag{3}$$

where the time derivative of the electric field induction $\boldsymbol{D} \in \mathbb{R}^3$ is neglected in MHD.

The constitutive relations are given by

$$\boldsymbol{B} = \mu\boldsymbol{H}, \quad \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = \eta\boldsymbol{J}, \tag{4}$$

where μ is the magnetic permeability and η is the resistance coefficient that is assumed to be zero in an ideal MHD system.

2.2 Geometric formulation of MHD

The main framework of this chapter is the port-Hamiltonian system for PDEs called a *distributed port-Hamiltonian (DPH) system* (Van der Schaft and Maschke, 2002). DPH systems are expressed in terms of differential forms (Flanders, 1963; Morita, 2001). Moreover, a formulation using differential forms defined on Riemannian manifolds can describe the relation between the vector fields of systems and the topological properties of system domains (see *Section 4*). Thus, we shall rewrite the equations of MHD by using differential forms to derive the DPH representation of MHD.

Let Y be an n-dimensional smooth Riemannian manifold. Let Z be an n-dimensional smooth Riemannian submanifold of Y with a smooth boundary ∂Z. We assume that the time coordinate $t \in \mathbb{R}$ is split from the spatial coordinates $x = (x^1, \cdots, x^n) \in Z$ in the local chart of Z. We denote the space of differential k-forms on Z by $\Omega^k(Z)$ for $0 \le k \le n$. We denote the infinite-dimensional vector space of all smooth vector fields in Z by $\mathfrak{X}(Z)$. We identify the 1-from $\boldsymbol{v}$ with the vector field $\boldsymbol{v}^\sharp \in \mathfrak{X}(Z)$. The fluid equations (1) and (2) can be rewritten as follows:

$$\begin{cases} \dfrac{\partial \boldsymbol{\rho}}{\partial t} = -de_v, \quad \dfrac{\partial \boldsymbol{v}}{\partial t} = -de_\rho + g_1 + g_2, \\ e_v = \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{\rho}, \quad e_\rho = \dfrac{1}{2}\langle \boldsymbol{v}^\sharp, \boldsymbol{v}^\sharp\rangle + w(*\boldsymbol{\rho}), \\ g_1 = -(*\boldsymbol{\rho})^{-1}*(*d\boldsymbol{v}\wedge *e_v), \quad g_2 = (*\boldsymbol{\rho})^{-1}*(*\boldsymbol{J}\wedge *\boldsymbol{B}), \end{cases} \tag{5}$$

where $n = 3$, $\boldsymbol{\rho} \in \Omega^3(Z)$ is the mass density, $\boldsymbol{v} \in \Omega^1(Z)$ is the fluid velocity, $\boldsymbol{J} \in \Omega^2(Z)$ is the free current density, $\boldsymbol{B} \in \Omega^2(Z)$ is the magnetic field induction, $\langle \boldsymbol{v}^\sharp, \boldsymbol{v}^\sharp \rangle = \|\boldsymbol{v}^\sharp\|^2$ is the inner product with respect to $\boldsymbol{v}^\sharp$, and we have introduced the following operators:

- $d\colon \Omega^k(Z) \to \Omega^{k+1}(Z) \cdots$ The exterior differential operator d on Z is defined as

$$d\omega = \sum_{j=1}^{n} \frac{\partial f_{i_1 \cdots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{6}$$

  for $\omega = f_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(Z)$, where $i_1 \cdots i_k$ is the combination of k different integers selected from 1 to n, and $j \neq i_1 \neq \cdots \neq i_k$.

- $*\colon \Omega^k(Z) \to \Omega^{n-k}(Z) \cdots$ The Hodge star operator $*$ induced in terms of a Riemannian metric on Z is defined as

$$*\omega = \sum_{i_1 < \cdots < i_k} \operatorname{sgn}(I, J) f_{i_1 \cdots i_k}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}} \in \Omega^{n-k}(Z) \tag{7}$$

  for $\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(Z)$, where $j_1 < \cdots < j_{n-k}$ is the rearrangement of the complement of $i_1 < \cdots < i_k$ in the set $\{1, \cdots, n\}$ in ascending order, and $\operatorname{sgn}(I, J)$ is the sign of the permutation of $i_1, \cdots, i_k, j_1, \cdots, j_{n-k}$ generated by interchanging of the basic forms dx^i (if we interchange dx^i and dx^j in ω for arbitrary i and j, the sign of ω changes, i.e., it is alternating).

- $\boldsymbol{i}_{\boldsymbol{v}^\sharp}\colon \Omega^k(Z) \to \Omega^{k-1}(Z) \cdots$ The interior product $\boldsymbol{i}_{\boldsymbol{v}^\sharp}$ with respect to $\boldsymbol{v}^\sharp$ is defined as

$$\boldsymbol{i}_{\boldsymbol{v}^\sharp}\omega = \begin{cases} (-1)^{m-1} f_{i_1 \cdots i_k} g_{i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_{m-1}} \wedge dx^{i_{m+1}} \wedge \cdots \wedge dx^{i_k} & \text{if } j = i_m, \\ 0 & \text{if } j \neq i_m \end{cases} \tag{8}$$

  for $\boldsymbol{v}^\sharp = g_j(x)(\partial/\partial x^j)$ and $\omega = f_{i_1 \cdots i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

In (5), we used the formula $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = (1/2)\nabla(\boldsymbol{v} \cdot \boldsymbol{v}) + \operatorname{Curl} \boldsymbol{v} \times \boldsymbol{v}$, and the enthalpy $w(*\boldsymbol{\rho}) = (\partial/\partial *\boldsymbol{\rho})\,(*\boldsymbol{\rho}\, U(*\boldsymbol{\rho}))$ is related to the pressure $p(*\boldsymbol{\rho})$ by $(*\boldsymbol{\rho})^{-1} dp(*\boldsymbol{\rho}) = dw(*\boldsymbol{\rho})$, where $U(\boldsymbol{\rho})$ is the internal energy function of the fluid satisfying $p(*\boldsymbol{\rho}) = w(*\boldsymbol{\rho}) * \boldsymbol{\rho} - U(*\boldsymbol{\rho}) * \boldsymbol{\rho}$.

Next, Maxwell's equations are defined as follows:

$$-\frac{\partial \boldsymbol{D}}{\partial t} = -d\boldsymbol{H} + \boldsymbol{J}, \quad -\frac{\partial \boldsymbol{B}}{\partial t} = d\boldsymbol{E}, \quad d\boldsymbol{B} = 0, \quad d\boldsymbol{D} = \boldsymbol{\varrho}, \tag{9}$$

where $\boldsymbol{D} \in \Omega^2(Z)$ is the electric field induction, $\boldsymbol{H} \in \Omega^1(Z)$ is the magnetic field intensity, $\boldsymbol{E} \in \Omega^1(Z)$ is the electric field intensity, and $\boldsymbol{\varrho} \in \Omega^3(Z)$ is the free charge density.

The constitutive relations are written as follows:

$$\boldsymbol{B} = \mu * \boldsymbol{H}, \quad *(\boldsymbol{E} + \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{B}) = \eta \boldsymbol{J}. \tag{10}$$

2.3 Definition of port-Hamiltonian system

Let us recall the definition of DPH systems. The advantage of these systems will be explained from the viewpoint of passivity and boundary controls in later sections.

The inner product of k-forms can be defined on Z as

$$\langle \omega, \eta \rangle = \omega \wedge *\eta, \quad \langle \omega, \eta \rangle_Z = \int_Z \langle \omega, \eta \rangle \tag{11}$$

for $\omega, \eta \in \Omega^k(Z)$. Moreover, we can identify the 1-from v with the vector field $v^\sharp \in \mathfrak{X}(Z)$; therefore, (11) can be defined as the inner product of vector fields, as in (5). DPH systems are defined by Stokes-Dirac structures (Van der Schaft and Maschke, 2002; Courant, 1990) with respect to the inner product (11).

Definition 2.1. *Let*

$$\begin{cases} (f^p, f^q, f^b) \in \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z), \\ (e^p, e^q, e^b) \in \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z), \\ (f_d^p, f_d^q) \in \Omega^p(Z) \times \Omega^q(Z), \\ (e_d^p, e_d^q) \in \Omega^{n-p}(Z) \times \Omega^{n-q}(Z), \end{cases} \tag{12}$$

where all f^i and e^i for $i \in \{p, q, b\}$ and all f_d^i and e_d^i for $i \in \{p, q\}$ constitute the pairs with respect to the inner product $\langle \cdot, \cdot \rangle_Z$. The Stokes-Dirac structure is defined as follows:

$$\begin{bmatrix} f^p \\ f^q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix} - \begin{bmatrix} f_d^p \\ f_d^q \end{bmatrix}, \ \begin{bmatrix} e_d^p \\ e_d^q \end{bmatrix} = \begin{bmatrix} e^p \\ e^q \end{bmatrix}, \ \begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} e^p|_{\partial Z} \\ (-1)^p e^q|_{\partial Z} \end{bmatrix}, \tag{13}$$

where $r = pq + 1$, $p + q = n + 1$, $|_{\partial Z}$ is the restriction of differential forms to ∂Z, $df_d^p \neq 0$, and $df_d^q \neq 0$.

A DPH system is formed by substituting the following variables obtained from a Hamiltonian density in the above Stokes-Dirac structure.

Definition 2.2. *Let $\mathcal{H}(\alpha^p, \alpha^q) \in \Omega^n(Z)$ be a Hamiltonian density, where $\alpha^i \in \Omega^i(Z)$ for $i \in \{p, q\}$. A DPH system is defined by substituting*

$$f^p = -\frac{\partial \alpha^p}{\partial t}, \quad f^q = -\frac{\partial \alpha^q}{\partial t}, \quad e^p = \frac{\partial \mathcal{H}}{\partial \alpha^p}, \quad e^q = \frac{\partial \mathcal{H}}{\partial \alpha^q} \tag{14}$$

into (13), where $\partial / \partial \alpha^i$ means the variational derivative with respect to α^i. The variables f_d^p and f_d^q cannot be derived from any Hamiltonian.

DPH systems satisfy the following boundary integrable relation that comes from Stokes theorem (Flanders, 1963; Morita, 2001).

Proposition 2.1 (Van der Schaft and Maschke (2002))**.** *A DPH system satisfies the following power balance:*

$$\int_Z (e^p \wedge f^p + e^q \wedge f^q) + \int_Z \left(e_d^p \wedge f_d^p + e_d^q \wedge f_d^q \right) + \int_{\partial Z} e^b \wedge f^b = 0. \tag{15}$$

where each term $e^i \wedge f^i$ for $i \in \{p, q, b\}$ has the dimension of power.

In DPH systems, each f^i and e^i for $i \in \{p, q\}$ are called *port variables*, and f^b and e^b are called *boundary port variables* that are a pair of boundary inputs and outputs. We call

$e^b \wedge f^b$ a *boundary energy flow*. On the other hand, the terms $e_d^p \wedge f_d^p$ and $e_d^q \wedge f_d^q$ are non-boundary-integrable; therefore, we cannot detect changes in them from the boundary energy flows. We call $e_d^p \wedge f_d^p$ and $e_d^q \wedge f_d^q$ *distributed energy flows*.

2.4 Passivity and boundary integrability of energy flows

The advantages of DPH systems are grounded in the following stability.

Definition 2.3. *Consider a system with an input vector $u(t)$ and an output vector $y(t)$. The system is called* passive *if there exists a C^0 class non-negative function $V(x)$ such that $V(0) = 0$ and*

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} u^\top(s) y(s)\, ds \tag{16}$$

for all inputs $u(t)$ and an initial value $x(t_0)$, where $t_0 \leq t_1$ and $\top$ means the transpose of vectors.

$V(x)$ can be regarded as the internal energy of the systems, which is an extended Lyapunov function. The inequality in (16) means that the energy always decreases; therefore, the system is stable in the sense of Lyapunov. Controls using the relation (16) are called *passivity-based controls*. Standard control systems with pairs of inputs/outputs satisfying (16) are called *port-Hamiltonian systems*. In this case, $V(x)$ corresponds with the Hamiltonian of the system. Hence, in (15), all port variables e_j^i and f_j^i for $i \in \{p, q, b\}$ and $j \in \{f, e\}$ might be inputs and outputs for passivity-based controls.

The boundary port variables f_j^b and e_j^b in (15) can be used as passivity-based boundary controls (Van der Schaft, 2000; Duindam et al., 2009) (details are given in *Section 2.6*). In (15), the first integral means the time variation of Hamiltonian; i.e., it is calculated by taking the interior product between a possible variational vector field and the variational derivative of Hamiltonian: $\boldsymbol{i}_{\mathfrak{X}_i} d\mathcal{H}_i$ for $i \in \{f, e\}$, where $\mathfrak{X}_i = \sum_j (\partial \alpha_j / \partial t)(\partial / \partial \alpha_j)$ is the variational vector field and α_j is the variational variable. The power of the first integral can be transformed into that of the third integral by appealing to boundary integrability of Stokes theorem. The second integral means non-boundary-integrable energy flows. Hence, if the second integral is zero, we can detect the variation of energies distributed on system domains from the variation on the boundary. In this sense, the power balance (15) is the principle of passivity-based boundary controls.

2.5 Port-Hamiltonian representation of MHD

In this section, we derive the DPH representation of MHD from the geometric formulation presented in *Section 2.2*, which has been partially treated as Maxwell's equations and as an ideal fluid in (Van der Schaft and Maschke, 2002).

Let $n = 3$. The DPH representation can be systematically constructed in terms of the Hamiltonian densities of the fluid and the electromagnetic field

$$\mathcal{H}_f = \int_Z \frac{1}{2} \langle v^\sharp, v^\sharp \rangle \boldsymbol{\rho} + U(*\boldsymbol{\rho})\boldsymbol{\rho}, \quad \mathcal{H}_e = \int_Z \frac{1}{2} \left(\boldsymbol{E} \wedge \boldsymbol{D} + \boldsymbol{H} \wedge \boldsymbol{B} \right) \tag{17}$$

under constraints defined by the system equations (5), (9) and (10). Indeed, the DPH system of MHD can be constructed as

$$\begin{cases} \begin{bmatrix} -\boldsymbol{\rho}_t \\ -\boldsymbol{v}_t \end{bmatrix} = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_\rho \\ e_v \end{bmatrix} - \begin{bmatrix} 0 \\ g_1 + g_2 \end{bmatrix}, \quad \begin{bmatrix} f_f^b \\ e_f^b \end{bmatrix} = \begin{bmatrix} e_\rho|_{\partial Z} \\ -e_v|_{\partial Z} \end{bmatrix}, \\ \begin{bmatrix} -\boldsymbol{D}_t \\ -\boldsymbol{B}_t \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix} + \begin{bmatrix} \boldsymbol{J} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} f_e^b \\ e_e^b \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}|_{\partial Z} \\ \boldsymbol{H}|_{\partial Z} \end{bmatrix}, \end{cases} \tag{18}$$

where the subscript t means the partial derivative with respect to t, and we have defined

$$\begin{cases} e_v = \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{\rho}, \quad e_\rho = \dfrac{1}{2}\langle \boldsymbol{v}^\sharp, \boldsymbol{v}^\sharp \rangle + w(*\boldsymbol{\rho}), \\ g_1 = -(*\boldsymbol{\rho})^{-1} * (*d\boldsymbol{v} \wedge *e_v), \quad g_2 = (*\boldsymbol{\rho})^{-1} * (*\boldsymbol{J} \wedge *\boldsymbol{B}), \\ \boldsymbol{B} = \mu * \boldsymbol{H}, \quad *(\boldsymbol{E} + \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{B}) = \eta \boldsymbol{J}, \end{cases} \tag{19}$$

having set $p = 3$, $q = 1$, and $r = 3 \cdot 1 + 1$ for the fluid, and $p = 2$, $q = 2$, and $r = 2 \cdot 2 + 1$ for the electromagnetic field. The DPH system satisfies the following power balance equations:

$$\int_Z \left(-e_\rho \wedge \boldsymbol{\rho}_t - e_v \wedge \boldsymbol{v}_t\right) - \int_Z e_v \wedge g_2 - \int_{\partial Z} e_v \wedge e_\rho = 0, \tag{20}$$

$$\int_Z \left(-\boldsymbol{E} \wedge \boldsymbol{D}_t - \boldsymbol{H} \wedge \boldsymbol{B}_t\right) - \int_Z \boldsymbol{E} \wedge \boldsymbol{J} + \int_{\partial Z} \boldsymbol{H} \wedge \boldsymbol{E} = 0, \tag{21}$$

where $e_v \wedge g_1 = -(*\boldsymbol{\rho})^{-1} e_v \wedge *(*d\boldsymbol{v} \wedge *e_v) = -(*\boldsymbol{\rho})^{-1} * e_v \wedge *d\boldsymbol{v} \wedge *e_v = 0$ and (21) which corresponds to Poynting's theorem. Note that the definition of the boundary energy flow in (21) is invariant even if $\boldsymbol{D}_t$ is assumed to be zero, as is done in the standard theory of MHD. The first integrals of (20) and (21) correspond to the total change in energy of the system defined on Z, and the third integral is equal to the energy flowing across ∂Z.

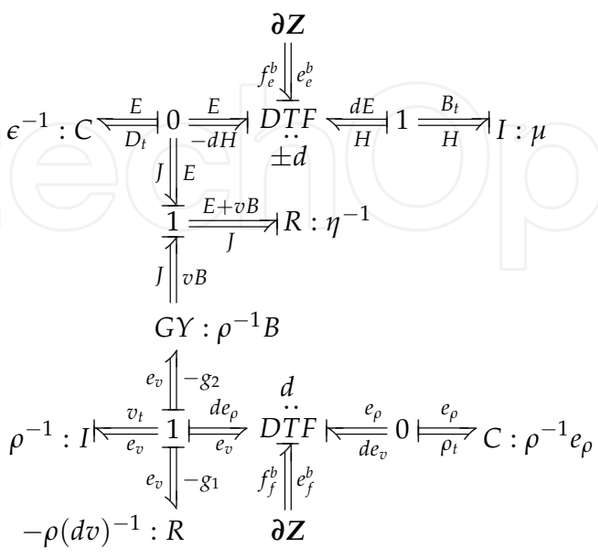
2.6 Passivity-based boundary controls

The basic strategy of passivity-based controls is to connect controllers through pairs of port variables, e.g., new port-Hamiltonian systems for changing the total Hamiltonians, or dissipative elements for stabilizing the system to the global minimum of the shaped Hamiltonian. The passivity-based boundary controls for DPH systems are applied to the boundary port variables f_j^b and e_j^b for $j \in \{f, e\}$. The product $e_j^b \wedge f_j^b$ has the dimension of power; therefore, f_j^b and e_j^b can be considered to be a generalized velocity and a generalized force in analogy to mechanical systems (the correspondence might be the inverse in some cases).

Applying the output f_j^b magnified by a negative gain to the input e_j^b means velocity feedback. This is one of most important passivity-based controls, i.e, damping assignment. Moreover, the boundary energy flow $e_j^b \wedge f_j^b$ balances the internal energy of DPH systems; therefore, the total energy of the controlled system decreases, and the system becomes stable in the sense of passivity (16).

On the other hand, the Hamiltonian of the original DPH system can be changed by connecting other DPH systems to the original. The connection by means of port variables is expressed by

bond graph theory (Karnopp et al., 2006), which is a generalized circuit theory for describing physical systems from the viewpoint of energy flows. For instance, the following diagram is the bound graph representation of the DPH system of MHD:

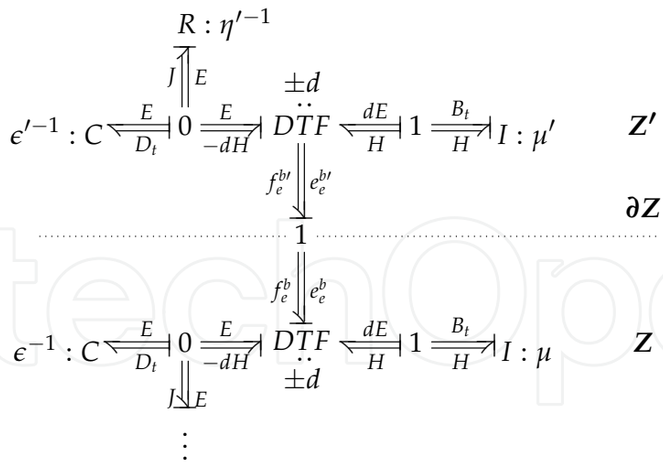


(22)

where ∂Z is the boundary of the systems, and we have defined the following bond graph elements:

- The arrow with the pair of variables e and f means the energy flow $e \wedge f$.
- The direction arrow indicates the sign of the energy flow.
- The causal stroke | at the edge of the arrows indicates the direction in which the effort signal is directed.
- The n pairs of variables e_i and f_i around the 0-junction satisfy $e_1 = e_2 = \cdots = e_n$ and $\sum_{i=1}^{n} s_i f_i = 0$, where $s_i = 1$ if the arrow is directed towards the junction and $s_i = -1$ otherwise.
- The n pairs of variables e_i and f_i around the 1-junction satisfy $f_1 = f_2 = \cdots = f_n$ and $\sum_{i=1}^{n} s_i e_i = 0$.
- The C element with a parameter K means the capacitor satisfies $e = K \int_{-\infty}^{t} f dt$.
- The I element with a parameter K means the inductor satisfies $f = K^{-1} \int_{-\infty}^{t} e dt$.
- The R element with a parameter K means the resister satisfies $e = Kf$.
- The GY element with a parameter M means the gyrator satisfies $e_2 = Mf_1$ and $e_1 = Mf_2$.
- The DTF element means the differential transformer that has a Stokes-Dirac structure. In the case with the symbol d, $e_2 = de_1$, $f_1 = df_2$, $f^b = e_1|_{\partial Z}$ and $e^b = -f_2|_{\partial Z}$. In the case with the symbol $\pm d$, $e_2 = de_1$, $f_1 = -df_2$, $f^b = e_1|_{\partial Z}$ and $e^b = f_2|_{\partial Z}$.

The Hamiltonian is shaped by connecting new systems to it through the pairs of boundary port variables (f_f^b, e_f^b) and (f_e^b, e_e^b). For example, we can connect an electromagnetic system as a controller on the boundary ∂Z of the upper part of (22) as follows:



(23)

where Z' is the domain of the new electromagnetic system and each system is connected through the common boundary $\partial Z = \partial Z'$. In this case, the original Hamiltonian $\mathcal{H}_f + \mathcal{H}_e$ is changed into the controlled Hamiltonian $\mathcal{H}_f + \mathcal{H}_e + \mathcal{H}'_e$, where $\mathcal{H}'_e$ is the Hamiltonian of the new electromagnetic system. Note that the Hamiltonians can only be shaped to control the energy flows of boundary port variables or energy levels of the original system, not to control the distributed states in the sense of boundary value problems.

The energy flow through the boundary $\partial Z = \partial Z'$ can be described as

$$\mathcal{H}_{\delta t} = \int_{\partial Z} e^b \wedge f^b - e^{b\prime} \wedge f^{b\prime}, \tag{24}$$

where $e^{b\prime}$ and $f^{b\prime}$ are the pair of the boundary port variables defined on $\partial Z'$. In general, when the port variable e^b is regarded as an input, the power balance (15) is changed into

$$\int_Z (e^p \wedge f^p + e^q \wedge f^q) + \int_Z \left\{ e^p_d \wedge (f^p_d + u^p_d) + e^q_d \wedge (f^q_d + u^q_d) \right\} + \int_{\partial Z} u^b \wedge f^b = 0, \tag{25}$$

where $e^b = u^b$ is the boundary control, and u^p_d and u^q_d are the distributed controls. If f^b is regarded as an input, then the boundary control is replaced by $f^b = u^b$.

Damping terms are assigned by connecting of resisters to the pair on the system domain; they are illustrated as R elements in the bond graph. If systems with dissipative elements are connected to the boundary of a controlled system, it corresponds to a boundary damping assignment that absorbs the energy of the original system through the boundary. For example, in (25), the controls

$$u^b = -K^b f^b, \quad u^p_d = -K^p_d \alpha^p, \quad u^q_d = -K^q_d \alpha^q \tag{26}$$

are equivalent to connecting an R element to the port variables, where K^b is the gain function defined on ∂Z, K^p_d and K^p_d are the gain functions defined on Z, and $f^i = -(\partial \alpha^i / \partial t)$. For eliminating distributed energy flows f^p_d and f^q_d that are exactly known, we can use the controls

$$u^p_d = -f^p_d, \quad u^q_d = -f^q_d, \tag{27}$$

where the inputs u_d^p and u_d^q distributed on Z. Moreover, in (23), $R\colon \eta'^{-1}$ distributed on Z' is considered as an element to create energy flowing across the boundary of the original MHD system.

A practical problem is whether the boundary port variables e_i^b and f_i^b can actually be used as inputs and outputs. In this section, we show all possible boundary port variables of MHD regardless of whether they are actually usable or not. The input/output pairs for the passivity-based boundary control of MHD are the boundary port variables

$$(e_f^b, f_f^b) = (-e_v|_{\partial Z}, e_\rho|_{\partial Z}), \quad (e_e^b, f_e^b) = (\boldsymbol{H}|_{\partial Z}, \boldsymbol{E}|_{\partial Z}). \tag{28}$$

(e_f^b, f_f^b) can be transformed as follows:

$$\begin{aligned}\int_{\partial Z} e_v \wedge e_\rho &= \int_{\partial Z} \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{\rho} \wedge \left(\frac{1}{2}\langle \boldsymbol{v}^\sharp, \boldsymbol{v}^\sharp\rangle + w(*\boldsymbol{\rho})\right) \\ &= \int_{\partial Z} \boldsymbol{i}_{\boldsymbol{v}^\sharp}\left(\frac{1}{2}\langle \boldsymbol{v}^\sharp, \boldsymbol{v}^\sharp\rangle\boldsymbol{\rho} + U(*\boldsymbol{\rho})\boldsymbol{\rho}\right) + \int_{\partial Z} \boldsymbol{i}_{\boldsymbol{v}^\sharp}(*p),\end{aligned} \tag{29}$$

where the first term corresponds to the boundary energy flow of convections and the second term means external work. Hence, the altered port variables are

$$(e_{f1}^b, f_{f1}^b) = (\mathcal{H}_f|_{\partial Z}, \boldsymbol{v}|_{\partial Z}), \quad (e_{f2}^b, f_{f2}^b) = (p|_{\partial Z}, \boldsymbol{v}|_{\partial Z}). \tag{30}$$

2.7 Port representation of balanced MHD

This section discusses the stability of the DPH systems of MHD (18) with (19) in a balanced state. If the change in the potential energy of MHD caused by physically admissible perturbation is positive, then the equilibrium of MHD is stable. This fact is called *the energy principle of MHD* (Wesson, 2004). We derive the basic equation of the energy principle from the DPH system.

If the 2-form $d\boldsymbol{v}$ is zero at a certain time $t = t_0$, it continues to be zero after t_0. Accordingly, (5) can be reduced as follows:

$$\frac{\partial \boldsymbol{\rho}}{\partial t} = 0, \quad \frac{\partial \boldsymbol{v}}{\partial t} = (*\boldsymbol{\rho})^{-1}\left\{-dp(*\boldsymbol{\rho}) + *(*\boldsymbol{J} \wedge *\boldsymbol{B})\right\} = 0. \tag{31}$$

Now, let us consider the variation in energy with respect to an infinitesimal variation in displacement:

$$\mathcal{W}_{\delta t} = \int_Z \delta\boldsymbol{x}\frac{\delta}{\delta t}\left\{-dp(*\boldsymbol{\rho}) + *(*\boldsymbol{J} \wedge *\boldsymbol{B})\right\}, \tag{32}$$

where the subscript δt means the variational derivative with respect to the time, and δ means an infinitesimal variation. From (9), we obtain

$$\frac{\delta \boldsymbol{J}}{\delta t} = d\frac{\delta \boldsymbol{H}}{\delta t}, \tag{33}$$

where we have assumed that $\boldsymbol{D}_t = 0$ and $\eta = 0$; therefore,

$$d\boldsymbol{D}_t = \boldsymbol{\varrho}_t = 0, \quad \boldsymbol{\varrho}_t = d\boldsymbol{J} = 0, \quad \boldsymbol{E} = -\boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{B}. \tag{34}$$

The DPH system of balanced MHD can be constructed as follows:

$$\begin{cases} \begin{bmatrix} -\boldsymbol{\rho}_t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \begin{bmatrix} w_{\delta t}(*\boldsymbol{\rho}) \\ \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{\rho} \end{bmatrix} - \begin{bmatrix} 0 \\ (*\boldsymbol{\rho})^{-1} dp_{\delta t} \end{bmatrix}, & \begin{bmatrix} f^b_{fs} \\ e^b_{fs} \end{bmatrix} = \begin{bmatrix} w_{\delta t}(*\boldsymbol{\rho})|_{\partial Z} \\ -\boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{\rho}|_{\partial Z} \end{bmatrix}, \\ \begin{bmatrix} 0 \\ -\boldsymbol{B}_t \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} -\boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{B} \\ \boldsymbol{H}_{\delta t} \end{bmatrix} + \begin{bmatrix} \boldsymbol{J}_{\delta t} \\ 0 \end{bmatrix}, & \begin{bmatrix} f^b_{es} \\ e^b_{es} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{B}|_{\partial Z} \\ \boldsymbol{H}_{\delta t}|_{\partial Z} \end{bmatrix}, \end{cases} \tag{35}$$

where $\delta \boldsymbol{x} = \boldsymbol{v}$. The DPH system (35) satisfies the power balance equations,

$$-\int_Z w_{\delta t}(*\boldsymbol{\rho}) \wedge \boldsymbol{\rho}_t + \int_Z \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{\rho} \wedge (*\boldsymbol{\rho})^{-1} dp_{\delta t} - \int_{\partial Z} \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{\rho} \wedge w_{\delta t}(*\boldsymbol{\rho}) = 0, \tag{36}$$

$$-\int_Z \boldsymbol{H}_{\delta t} \wedge \boldsymbol{B}_t + \int_Z \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{B} \wedge \boldsymbol{J}_{\delta t} - \int_{\partial Z} \boldsymbol{H}_{\delta t} \wedge \boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{B} = 0. \tag{37}$$

As a result, we obtain the boundary port variables

$$(f^b_{fs}, e^b_{fs}) = (w_{\delta t}(*\boldsymbol{\rho})|_{\partial Z}, -\boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{\rho}|_{\partial Z}), \quad (f^b_{es}, e^b_{es}) = (-\boldsymbol{i}_{\boldsymbol{v}^\sharp}\boldsymbol{B}|_{\partial Z}, \boldsymbol{H}_{\delta t}|_{\partial Z}) \tag{38}$$

from (35).

The energy principle is frequently used to analyze the stability of MHD. The DPH system of MHD generates the power balance equation (37) for an analysis. The boundary port variables of (35) correspond to those of the DPH system of dynamical MHD (18) except for the term depending on $\boldsymbol{v}$. Hence, (18) can be considered to be a generalized system following the energy principle of MHD. If active controls are used in MHD systems, e.g., in Tokamaks, the control side of the DPH system able to be used, e.g., as a boundary control for subdivided MHD systems.

3. Construction pseudo potentials for non-Hamiltonian subsystems

3.1 DPH systems of MHD with perturbations

Section 2 discussed the energy structure of the DPH system of MHD on the basis of its physical meaning. However, model perturbations caused by, for instance, disturbances, additional terms derived by using system identification methods for model refinements, or controllers designed by a control theory do not always have physical interpretations. In this section, we show a method of determining the energy structure of such perturbations. Precisely speaking, we decompose a given perturbation into a Hamiltonian subsystem and a non-Hamiltonian subsystem that can be regarded as an external force in terms of Euler-Lagrange equations (Nishida et al., 2007a).

In this section, we consider an n-dimensional smooth Riemannian manifold Y that is homeomorphic to an n-dimensional Euclidian space (i.e., topologically same, and one can be deformed into the other). Let Z be an n-dimensional smooth Riemannian submanifold of Y with a smooth boundary ∂Z. The DPH system (18) of MHD defined on a domain Z is

extended so as to have perturbations as follows:

$$\begin{cases} \begin{bmatrix} -\boldsymbol{\rho}_t \\ -\boldsymbol{v}_t \end{bmatrix} = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_\rho \\ e_v \end{bmatrix} - \begin{bmatrix} 0 \\ g_1 + g_2 \end{bmatrix} + \begin{bmatrix} \Delta_f^p \\ \Delta_f^q \end{bmatrix}, \quad \begin{bmatrix} f_f^b \\ e_f^b \end{bmatrix} = \begin{bmatrix} e_\rho|_{\partial Z} \\ -e_v|_{\partial Z} \end{bmatrix}, \\ \begin{bmatrix} -\boldsymbol{D}_t \\ -\boldsymbol{B}_t \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix} + \begin{bmatrix} \boldsymbol{J} \\ 0 \end{bmatrix} + \begin{bmatrix} \Delta_e^p \\ \Delta_e^q \end{bmatrix}, \quad \begin{bmatrix} f_e^b \\ e_e^b \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}|_{\partial Z} \\ \boldsymbol{H}|_{\partial Z} \end{bmatrix}, \end{cases} \tag{39}$$

where each Δ_j^i for $i \in \{p,q\}$ and $j \in \{f,e\}$ means a perturbation. Now, let us consider the subsystem of DPH systems, $\Delta_j^i(u_I^a)$, where $i \in \{p,q\}$, $j \in \{f,e\}$, u^a for $1 \le a \le l$ is the function defined by the local coordinates x^k of Y for $1 \le k \le n$, and we denote all possible derivatives up to the order r of u^a by u_I^a and denote the order by $0 \le |I| \le r$. For example, u_I^a for $r = 2$ means $\{u^a, u_t^a, u_y^a, u_z^a, u_{tt}^a, u_{ty}^a, u_{tz}^a, u_{yy}^a, u_{yz}^a, u_{zz}^a\}$ for $(x^1, x^2, x^3) = (t, y, z)$, and the subscript means the partial derivative.

3.2 Decomposition of model perturbations of DPH systems

Consider the DPH system (39) of MHD with perturbations. We assume that the DPH system includes up to second-order derivatives: $r = 2$. Accordingly, Δ_j^i can be uniquely decomposed into

$$\Delta_j^i = d\varphi_j^i + \gamma_j^i, \tag{40}$$

where φ_j^i is a *pseudo potential* derived from

$$\gamma_j^i\, du = \Delta\, du - d\tilde{\varphi}_j^i, \quad \varphi_j^i\, du = \Delta\, du - \gamma_j^i\, du, \tag{41}$$

the temporal variable $\tilde{\varphi}_j^i$ is calculated as

$$\tilde{\varphi}_j^i = h_v(\Delta_j^i\, du^a) = \int_0^1 u^a \cdot \Delta_j^i(x^k, \lambda u_I^a)\, d\lambda, \tag{42}$$

h_v is *the homotopy operator* for $\omega \in \Omega^k(Z)$ with respect to an equilibrium point u_{cI}^a, called a *homotopy center*, defined by

$$h_v(\omega) = \int_0^1 \boldsymbol{i}_{\bar{v}}\omega(x, \lambda\bar{u}_I)\, \lambda^{-1} d\lambda, \quad \bar{v} = \sum_{a,I}(u_I^a - u_{cI}^a)\frac{\partial}{\partial u_I^a}, \tag{43}$$

where $\bar{u}_I^a = u_{cI}^a + \lambda(u_I^a - u_{cI}^a)$, and usually $u_{cI}^a = 0$. In (40), we call $d\varphi_j^i$ an *exact system* and call γ_j^i a *dual exact system*, which corresponds to a distributed energy variable.

For example, let us consider $\Delta_j^i = 1 + w_t + w_{tt}$ for some i and j, where $u^1 = w$ and $x^0 = t$. The temporal variable

$$\tilde{\varphi}_j^i = w + \frac{1}{2}ww_t + \frac{1}{2}ww_{tt} \tag{44}$$

is derived from $h_v(\Delta^i_j\, dw)$. Hence,

$$d\tilde{\varphi}^i_j = (1 + w_{tt})\, dw, \quad \gamma^i_j = \Delta^i_j\, dw - d\tilde{\varphi}^i_j = w_t\, dw. \tag{45}$$

On the other hand, from the relation

$$\begin{aligned} \Delta^i_j\, dw &= (1 + w_t + w_{tt})\, dw \\ &= \left(1 + \frac{1}{2}w_t + w_{tt}\right) dw + \left(-\frac{1}{2}w - w_t\right) dw_t \end{aligned} \tag{46}$$

that is transformed in terms of an integration by parts, we obtain

$$\tilde{\varphi}^i_j = w - \frac{1}{2}w_t^2. \tag{47}$$

This result yields the same relation $d\tilde{\varphi}^i_j = (1 + w_{tt})\, dw$. Thus, the expression $\tilde{\varphi}^i_j$ has variations generated by an integration by parts; therefore, we should recalculate φ^i_j as in (41).

3.3 Necessary and sufficient condition of decomposition

We can check whether a given Δ^i_j is an exact system or a dual exact system from the self-adjointness of the differential operator $\mathcal{D}_{\Delta^i_j}$ defining Δ^i_j: $\mathcal{D}^*_{\Delta^i_j} = \mathcal{D}_{\Delta^i_j}$ (Olver, 1993, pp. 109, 307, 329 and 364). Here, *the Fréchet derivative* $\mathcal{D}_{\mathcal{F}}$ of a second-order subsystem $\mathcal{F}(u_I)$ is an $(l \times k)$-matrix with elements

$$(\mathcal{D}_{\mathcal{F}})_{ab}(h) = \left(\frac{\partial \mathcal{F}_a}{\partial u^b} + \sum_{i=0}^{n} \frac{\partial \mathcal{F}_a}{\partial u^b_{x^i}} \frac{\partial}{\partial x^i} + \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{\partial \mathcal{F}_a}{\partial u^b_{x^i x^j}} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}\right) h \tag{48}$$

and the adjoint operator $\mathcal{D}^*_{\mathcal{F}}$ of $\mathcal{D}_{\mathcal{F}}$ is a $(k \times l)$-matrix with elements

$$(\mathcal{D}^*_{\mathcal{F}})_{ba}(h) = \frac{\partial \mathcal{F}_a}{\partial u^b} h - \sum_{i=0}^{n} \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{F}_a}{\partial u^b_{x^i}} h\right) + \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial \mathcal{F}_a}{\partial u^b_{x^i x^j}} h\right) \tag{49}$$

for $a = 1, \cdots, k$ and $b = 1, \cdots, l$, where $h = h(u_I)$ is any function and we assume $k = l$.

For example, consider $\Delta^q_f = 1 + \nu \boldsymbol{v} + \boldsymbol{v}_t$ in (39), where $u^1 = w$, $w_t = \boldsymbol{v}$ and $x^0 = t$. Then, $\varphi^q_f = 1 + \boldsymbol{v}_t$ and $\gamma^q_f = \nu \boldsymbol{v}$, because $g = \nu \boldsymbol{v}$ is non-self-adjoint: $\mathcal{D}^*_g \neq \mathcal{D}_g$, and we have used (48) and (49) with $a = b = 1$, i.e.,

$$\mathcal{D}_g(h) = \frac{\partial g}{\partial u_{x^0}} \frac{\partial}{\partial x^0}(h) = \nu \frac{\partial h}{\partial t}, \quad \mathcal{D}^*_g(h) = -\frac{\partial}{\partial x^0}\left(\frac{\partial g}{\partial u_{x^0}} h\right) = -\nu \frac{\partial h}{\partial t}. \tag{50}$$

3.4 Elimination of decomposed perturbations

The uniqueness of the decomposition is determined by the topology of Y. That is, differential k-forms for $k \geq 1$ defined on such a domain can be always described as in (40). If a pseudo

potential can be defined for a perturbation, the perturbation can be included in the variables e^p or e^q of the Stokes-Dirac structure. Hence, such a perturbation can be detected in terms of the following boundary power balances:

$$\int_Z \left(-(e_\rho + \varphi_f^q) \wedge \boldsymbol{\rho}_t - (e_v + \varphi_f^p) \wedge \boldsymbol{v}_t\right) - \int_Z e_\rho \wedge \gamma_f^p - \int_Z e_v \wedge (g_2 + \gamma_f^q) - \int_{\partial Z} (e_v + \varphi_f^p) \wedge (e_\rho + \varphi_f^q) = 0, \tag{51}$$

$$\int_Z \left\{-(\boldsymbol{E} + \varphi_e^q) \wedge \boldsymbol{D}_t - (\boldsymbol{H} - \varphi_e^p) \wedge \boldsymbol{B}_t\right\} - \int_Z \boldsymbol{E} \wedge (\boldsymbol{J} + \gamma_e^p) - \int_Z \boldsymbol{H} \wedge \gamma_e^q + \int_{\partial Z} (\boldsymbol{H} - \varphi_e^p) \wedge (\boldsymbol{E} + \varphi_e^q) = 0. \tag{52}$$

Moreover, from these relations, we can see that the exact subsystem of perturbations can be controlled by boundary port variables. Indeed, we can construct the boundary controls in the fourth integrals of the power balance equations (51) and (52) as follows:

$$\int_{\partial Z} (e_v + \varphi_f^p + u_f^q) \wedge (e_\rho + \varphi_f^q + u_f^p), \tag{53}$$

$$\int_{\partial Z} (\boldsymbol{H} - \varphi_e^p + u_e^q) \wedge (\boldsymbol{E} + \varphi_e^q + u_e^p), \tag{54}$$

where u_j^i is the boundary input for compensating pseudo potentials such that

$$u_f^q = -\varphi_f^p, \quad u_f^p = -\varphi_f^q, \quad u_e^q = \varphi_e^p, \quad u_e^p = -\varphi_e^q. \tag{55}$$

On the other hand, the decomposed perturbations corresponding dual exact subsystems cannot be eliminated by boundary controls. Hence, we should introduce the distributed controls in the second and third integrals of the power balance equations (51) and (52) as follows:

$$-\int_Z e_\rho \wedge (\gamma_f^p + u_{df}^p) - \int_Z e_v \wedge (g_2 + \gamma_f^q + u_{df}^q), \tag{56}$$

$$-\int_Z \boldsymbol{E} \wedge (\boldsymbol{J} + \gamma_e^p + u_{de}^p) - \int_Z \boldsymbol{H} \wedge (\gamma_e^q + u_{de}^q), \tag{57}$$

where u_{dj}^i is the distributed input for eliminating dual exact subsystems such that

$$u_{df}^p = -\gamma_f^p, \quad u_{df}^q = -\gamma_f^q, \quad u_{de}^p = -\gamma_e^p, \quad u_{de}^q = -\gamma_e^q. \tag{58}$$

4. Boundary observer for detecting topological symmetry breaking

4.1 Symmetry and power balance equations

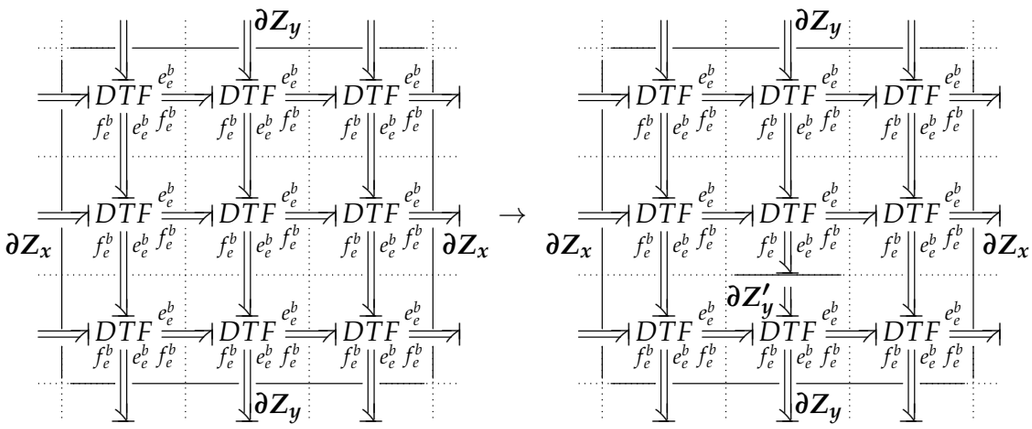
In this section, we first discuss the influence of topological variations in the system domains on the power balance equation of DPH systems. We can detect such changes by checking the boundary power balance of the original system; if there is an imbalance. According to Noether's theorem (Olver, 1993), conservation laws are associated with symmetries present in systems. That is, our purpose is to construct a boundary observer for detecting symmetry

breaking (Nishida et al., 2009). Finally, we derive a boundary control for creating desired energy flows from topological properties of manifolds.

We shall clarify the first problem by means of the following example. Consider a DPH system defined on a 2-dimensional domain Z. We assume that the energy flow of the system can be split along the x- and y-axis. Next, we divide the domain Z into subdomains, i.e., $Z = \bigcup_i Z^i$, where Z^i is the i-th subdomain of Z. We denote the common boundary between Z^i and Z^j by ∂Z^{ij}. The following power balance holds:

$$\mathcal{H}_{\delta t} = \int_{\partial Z^{ij}} \sum_{i,j} \left(e^{bi} \wedge f^{bi} - e^{bj} \wedge f^{bj} \right) = 0, \tag{59}$$

where e^{bi} and f^{bi} are the boundary port variables defined on ∂Z^i. The DPH system can be regarded as a connected structure of DPH systems defined on Z^i in terms of boundary port variables of ∂Z^{ij}. We shall simplify the shapes of Z and each Z^i to be squares as in the left diagram below:



(60)

Accordingly, we can split the original boundary ∂Z and denote the boundaries with respect to the x- and y-axis by ∂Z_x and ∂Z_y, respectively. Hence, the following power balance holds:

$$\mathcal{H}_{\delta t} = \mathcal{H}_{\delta t}|_{\partial Z_x} + \mathcal{H}_{\delta t}|_{\partial Z_y} = 0. \tag{61}$$

Now, let us assume that a structural change occurs in the inner part of Z on a segment along x-axis that we denote as $\partial Z'_y$ in the right diagram of (60). Such changes are caused by, for instance, energy dissipations, or energy transformations to other physical systems, and they can be illustrated as a new element connected to $\partial Z'_y$ in the bond graph. This means the energy preserving symmetry is broken along the x-axis. In this case, (61) should be revised to

$$\mathcal{H}'_{\delta t} = \mathcal{H}_{\delta t}|_{\partial Z_x} + \mathcal{H}_{\delta t}|_{\partial Z'_y} + \mathcal{H}_{\delta t}|_{\partial Z_y} = 0. \tag{62}$$

Hence, we can detect that the power on ∂Z_y: $\mathcal{H}_{\delta t}|_{\partial Z_y} = 0$ becomes imbalanced if the port variables in (61) are observable. In other words, this change can be regarded as a change in the topology of the system domain, i.e., a deformation from $Z \simeq \mathbb{R}^n$ to $Z \setminus \partial Z'_y \simeq \mathbb{R}^n \setminus \{0\}$,

where $\simeq$ means topological equivalence (i.e., homeomorphic), $\setminus$ means subtraction of sets, and $\{0\}$ is a point.

4.2 Topological decomposition of differential forms and vector fields

This section discusses the relation between the topology of the domain Z of DPH systems and the decomposable components of vector fields on Z. After this discussion, the symmetry breaking explained in the previous section will be extended to a change in energy flows of DPH systems defined on compact manifolds.

In *Section 2*, we assumed that the system domain Z is a subdomain of a manifold that is topologically the same as a Euclidian space. Actually, this assumption restricted the form of diffrential forms. In this case, differential k-forms for $k \geq 1$ can be decomposed into two types, i.e., an exact form and a dual exact form as in (40). That is, differential forms $\omega_e \in \Omega^k(Z)$ are called *exact forms* if there exists some $\eta \in \Omega^{k-1}(Z)$ such that $\omega_e = d\eta$, i.e., $d\omega_e = d(d\eta) = 0$ because of the nature of exterior differentiation. The forms $\omega_d \in \Omega^k(Z)$ such that $d\omega_d \neq 0$ are called *dual exact forms*. In general, there might also exist *harmonic forms* $\omega_h \in \Omega^k(Z)$ satisfying $\triangle\omega_h = 0$, where $\triangle = dd^\dagger + d^\dagger d$ is the Laplacian and $d^\dagger = (-1)^{n(k+1)+1} * d *$ is the adjoint operator of exterior differentiation. The components of differential forms depend on the topology of domains. All classifications of differential forms defined on a compact domain with a smooth boundary are given by *the Hodge decomposition theorem* (Morita, 2001); i.e., an arbitrary differential form on an oriented compact Riemannian manifold can be uniquely decomposed into an exact form, a dual exact form, and a harmonic form:

$$\omega = \omega_e + \omega_d + \omega_h \in \Omega^k(Z). \tag{63}$$

Moreover, a unique harmonic form on an oriented compact Riemannian manifold corresponds to a topological quantity of the manifolds called a *homology*. Precisely speaking, from *Hodge theorem*, *Poincaré duality thorem* and the duality between homology and (de Rham) cohomology, we obtain the isomorphism $H_k(Z, \partial Z) \cong \Omega_h^{n-k}(Z)$ (Morita, 2001; Gross and Kotiuga, 2004, pp. 102), where $H_k(Z)$ is the vector space with real coefficients of the k-th homology of Z, and $\Omega_h^k(Z)$ is the space of harmonic forms.

If $n = 3$, the homology of Z consists of the following vector spaces:

- $H_0(Z) \cdots$ The vector space is generated by such equivalence classes of points in Z as two points are equivalent if they can be connected by a path in Z. $\dim H_0(Z)$ is the number of components of Z. Note that $H_0(Z) \cong \mathbb{R}$ for a connected Z and the element of $H_0(Z)$ is a constant function.
- $H_1(Z) \cdots$ The vector space is generated by such equivalence classes of oriented loops in Z as two loops are equivalent if their difference is the boundary of an oriented surface in Z. The number of holes of closed surfaces is called a *genus*. $\dim H_1(Z)$ is the number of total genus of Z.
- $H_2(Z) \cdots$ The vector space is generated by such equivalence surfaces of points in Z as two surfaces are equivalent if their difference is the boundary of some oriented subregion of Z. $\dim H_2(Z)$ is the number of the difference between components of ∂Z and those of Z.
- $H_3(Z) \cdots \dim H_3(Z)$ is always 0.

On the other hand, the dual space of $H_k(Z)$ is $H_{n-k}(Z, \partial Z)$, where $H_k(Z, \partial Z)$ is called *the k-th relative homology of Z modulo* ∂Z. In $n = 3$, the relative homology of Z modulo ∂Z consists of the following vector spaces with real coefficients:

- $H_0(Z, \partial Z) \cdots \dim H_0(Z)$ is always 0.
- $H_1(Z, \partial Z) \cdots$ The vector space is generated by such equivalence classes of oriented paths whose endpoints lie on ∂Z as two such paths are equivalent if their difference (possibly paths on ∂Z) is the boundary of an oriented surface in Z.
- $H_2(Z, \partial Z) \cdots$ The vector space is generated by such equivalence classes of oriented surface whose boundaries lie on ∂Z as two such surfaces are equivalent if their difference (possibly portions of ∂Z) is the boundary of some oriented subregion of Z.
- $H_3(Z, \partial Z) \cdots$ The vector space has the oriented components of Z as a basis. Thus, $\dim H_3(Z, \partial Z)$ is the number of components of the subregions of Z whose boundaries lie on ∂Z. Note that $H_3(Z, \partial Z) \cong \mathbb{R}$ for a connected Z and the element of $H_3(Z, \partial Z)$ is a constant function.

Hence, $H_k(Z, \partial Z) \cong H_{3-k}(Z)$ for $0 \leq k \leq 3$.

As we mentioned before, the space of vector fields $\mathfrak{X}$ can be identified with that of 1-forms Ω^1 in the sense of a Riemannian metric. Thus, vector fields are affected by the decomposition of differential forms. Indeed, the space of vector fields on a compact domain Z in three-dimensional space with a smooth boundary can be decomposed as follows.

Theorem 4.1 (Cantarella et al. (2002))**.** *Consider vector fields $v^\sharp \in \mathfrak{X}(Z)$ on a compact domain Z with a smooth boundary ∂Z in three-dimensional space. Let W denote any smooth orientable surface in Z whose boundary ∂W lies on the boundary ∂Z: $W \subset Z$ and $\partial W \subset \partial Z$, and called it a* cross-sectional surface*. The space $\mathfrak{X}(Z)$ is the direct sum of five mutually orthogonal subspaces:*

$$\mathfrak{X}(Z) = \mathfrak{X}_K(Z) \oplus \mathfrak{X}_G(Z), \tag{64}$$

where $v \in \Omega^1(Z)$, $v^\sharp \in \mathfrak{X}(Z)$, $\varphi \in \Omega^0(Z)$,

$$\mathfrak{X}_K(Z) = \left\{ v^\sharp \in \mathfrak{X}(Z) \colon *d*v = 0,\ v^\sharp \cdot \boldsymbol{n}^\sharp = 0 \right\}, \quad \mathfrak{X}_G(Z) = \left\{ v^\sharp \in \mathfrak{X}(Z) \colon v = d\varphi \right\}, \tag{65}$$

which are called knots *and* gradients*, respectively, and $\boldsymbol{n}^\sharp$ means all unit vector fields normal to ∂Z. Furthermore,*

$$\mathfrak{X}_K(Z) = \mathfrak{X}_{FK}(Z) \oplus \mathfrak{X}_{HK}(Z), \quad \mathfrak{X}_G(Z) = \mathfrak{X}_{CG}(Z) \oplus \mathfrak{X}_{HG}(Z) \oplus \mathfrak{X}_{GG}(Z), \tag{66}$$

where

$$\mathfrak{X}_{FK}(Z) = \left\{ v^\sharp \in \mathfrak{X}(Z) \colon *d*v = 0,\ \langle v, \boldsymbol{n} \rangle_{\partial Z} = 0,\ \langle v, \boldsymbol{m} \rangle_W = 0 \right\}, \tag{67}$$

$$\mathfrak{X}_{HK}(Z) = \left\{ v^\sharp \in \mathfrak{X}(Z) \colon *d*v = 0,\ \langle v, \boldsymbol{n} \rangle_{\partial Z} = 0,\ dv = 0 \right\}, \tag{68}$$

$$\mathfrak{X}_{CG}(Z) = \left\{ v^\sharp \in \mathfrak{X}(Z) \colon v = d\varphi,\ *d*v = 0,\ \langle v, \boldsymbol{n} \rangle_{\partial Z} = 0 \right\}, \tag{69}$$

$$\mathfrak{X}_{HG}(Z) = \left\{ v^\sharp \in \mathfrak{X}(Z) \colon v = d\varphi,\ *d*v = 0,\ \varphi = C \right\}, \tag{70}$$

$$\mathfrak{X}_{GG}(Z) = \left\{ v^\sharp \in \mathfrak{X}(Z) \colon v = d\varphi,\ \varphi|_{\partial Z} = 0 \right\} \tag{71}$$

$$\dim H_1(Z) = \dim \mathfrak{X}_{HK}(Z), \quad \dim H_2(Z) = \dim \mathfrak{X}_{HG}(Z). \tag{72}$$

which are respectively called fluxless knots, harmonic knots, curly gradients, harmonic gradients *and* grounded gradients, *and* $\boldsymbol{m}^\sharp$ *means all unit vector fields normal to* W, *and* C *is a function on* ∂Z *that is locally constant.*

For example, consider a vector field defined on a three-dimensional disc. There is no $v^\sharp \in \mathfrak{X}_{HK}(Z)$ on the disc, because the genus is 0 and $\dim H_1(Z) = \dim \mathfrak{X}_{HK}(Z) = 0$. Thus, all rotation vector fields on the disc are $v^\sharp \in \mathfrak{X}_{FK}(Z)$ that is the rotating vector field whose axis is an inner point of the disc. $v^\sharp \in \mathfrak{X}_{CG}(Z)$ is a constant vector field flowing across the disc; therefore, it is divergence-free and zero flux through the one and only component of ∂Z. $v^\sharp \in \mathfrak{X}_{GG}(Z)$ is a radiational vector field flowing from an inner point of the disc, where the potential φ is constant on ∂Z. There is no $v^\sharp \in \mathfrak{X}_{HG}(Z)$ on the disc, because the numbers of components of ∂Z and Z are each 1, i.e., $\dim \mathfrak{X}_{HG}(Z) = 0$. However, a three-dimensional solid torus has a hole; therefore, $\dim \mathfrak{X}_{HK}(Z) = 1$, but $\dim \mathfrak{X}_{HG}(Z) = 0$. $v^\sharp \in \mathfrak{X}_{HK}(Z)$ is a circulative vector field flowing around the hole. Moreover, for a region between two concentric round spheres, $\dim \mathfrak{X}_{HG}(Z) = 1$. $v^\sharp \in \mathfrak{X}_{HK}(Z)$ is a radiational vector field flowing from a common center in the small sphere.

4.3 DPH systems with harmonic energy flows

In this section, we extend the DPH system of MHD to include the global energy flows originating from topological shapes of manifolds.

Let Z be a three-dimensional smooth Riemannian submanifold of Y with a smooth boundary ∂Z. The DPH system (18) of MHD defined on a domain Z is extended to have energy flows regarding harmonic knots and harmonic gradients that we call *harmonic energy flows* as follows:

$$\begin{cases} \begin{bmatrix} -\boldsymbol{\rho}_t \\ -\boldsymbol{v}_t \end{bmatrix} = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_\rho + e^p_{hf} \\ e_v + e^q_{hf} \end{bmatrix} - \begin{bmatrix} 0 \\ g_1 + g_2 \end{bmatrix} + \begin{bmatrix} f^p_{hf} \\ f^q_{hf} \end{bmatrix}, \quad \begin{bmatrix} f^b_f \\ e^b_f \end{bmatrix} = \begin{bmatrix} (e_\rho + e^p_{hf})|_{\partial Z} \\ -(e_v + e^q_{hf})|_{\partial Z} \end{bmatrix}, \\ \begin{bmatrix} -\boldsymbol{D}_t \\ -\boldsymbol{B}_t \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E} + e^p_{he} \\ \boldsymbol{H} + e^q_{he} \end{bmatrix} + \begin{bmatrix} \boldsymbol{J} \\ 0 \end{bmatrix} + \begin{bmatrix} f^p_{he} \\ f^q_{he} \end{bmatrix}, \quad \begin{bmatrix} f^b_e \\ e^b_e \end{bmatrix} = \begin{bmatrix} (\boldsymbol{E} + e^p_{he})|_{\partial Z} \\ (\boldsymbol{H} + e^q_{he})|_{\partial Z} \end{bmatrix}, \end{cases} \tag{73}$$

where we defined the following harmonic forms yielding harmonic energy flows:

$$\begin{cases} (f^p_{hf}, e^p_{hf}) \in \Omega^3(Z) \times \Omega^0(Z), \quad (f^q_{hf}, e^q_{hf}) \in \Omega^2(Z) \times \Omega^1(Z), \\ (f^p_{he}, e^p_{he}) \in \Omega^2(Z) \times \Omega^1(Z), \quad (f^q_{he}, e^q_{he}) \in \Omega^2(Z) \times \Omega^1(Z). \end{cases} \tag{74}$$

Note that $H_k(Z) \cong H_{n-k}(Z, \partial Z) \cong \Omega^k_h(Z)$, there is the dual from of ω_h with respect to $\langle\ ,\ \rangle_Z$, called a *Poincaré dual*: $\Omega^k_h(Z) \cong \Omega^{n-k}_h(Z)$, and f^p_{hf} and e^p_{hf} are constant functions. The system

(73) satisfies the power balances

$$\int_Z \left(-(e_\rho + e^p_{hf}) \wedge \boldsymbol{\rho}_t - (e_v + e^q_{hf}) \wedge \boldsymbol{v}_t\right) - \int_Z (e_\rho + e^p_{hf}) \wedge f^p_{hf} - \int_Z (e_v + e^q_{hf}) \wedge (g_2 + f^q_{hf}) - \int_{\partial Z} (e_v + e^q_{hf}) \wedge (e_\rho + e^p_{hf}) = 0, \tag{75}$$

$$\int_Z \left\{-(\boldsymbol{E} + e^p_{he}) \wedge \boldsymbol{D}_t - (\boldsymbol{H} + e^q_{he}) \wedge \boldsymbol{B}_t\right\} - \int_Z (\boldsymbol{E} + e^p_{he}) \wedge (\boldsymbol{J} + f^p_{he}) - \int_Z (\boldsymbol{H} + e^q_{he}) \wedge f^q_{he} + \int_{\partial Z} (\boldsymbol{H} + e^q_{he}) \wedge (\boldsymbol{E} + e^p_{he}) = 0. \tag{76}$$

4.4 Boundary detection and control of topological transitions

In fact, it is difficult to determine specific harmonic forms in (74). Hence, let us apply the classification of vector fields to the power balance equation for detecting topological transitions of systems and controlling energy flows.

Consider the cross-sectional surface W of Z such that $W \subset Z$ and $\partial W \subset \partial Z$. Let $\partial Z = \cup_i \partial Z^i$ be a set of subdivided domains of ∂Z or W in which each ∂Z^i is homeomorphic to Euclidian spaces (e.g., each component of ∂Z_x and ∂Z_y in (60)). In this setting, we can approximate port variables distributed on ∂Z^i, for instance, by using those on the boundary of each subdivided domain $\partial(\partial Z^i)$ if the subdivision is sufficiently fine. Let

$$(v^\sharp_1, v^\sharp_2, v^\sharp_3, v^\sharp_4, v^\sharp_5) \in \mathfrak{X}_{FK}(Z) \oplus \mathfrak{X}_{HK}(Z) \oplus \mathfrak{X}_{CG}(Z) \oplus \mathfrak{X}_{HG}(Z) \oplus \mathfrak{X}_{GG}(Z). \tag{77}$$

Then, we can rewrite (61) as follows:

$$\mathcal{H}_{\delta t} = \sum_i \left\{\mathcal{H}_{\delta t}(v^\sharp_1) + \mathcal{H}_{\delta t}(v^\sharp_2) + \mathcal{H}_{\delta t}(v^\sharp_3) + \mathcal{H}_{\delta t}(v^\sharp_4) + \mathcal{H}_{\delta t}(v^\sharp_5)\right\}\Big|_{\partial Z^i} = 0, \tag{78}$$

where $\mathcal{H}_{\delta t}(v^\sharp_r)$ means the split energy flow generated by $v^\sharp_r$ for $1 \leq r \leq 5$. If all boundary port variables are available as inputs and outputs, the balance of each decomposed energy flows can be confirmed from (78).

On the other hand, desired energy flows depending on the topology of system domain can be reinforced by servo feedback in terms of boundary port variables. If the cause of a change is a known structural perturbation and the boundary surrounds all energy flows generated by the perturbation, we can use the power balance defined on such appropriate boundaries to realize an energy flow control. Indeed, the control law is

$$\int_{\partial Z^j} \sum_{r=1}^{5} (e^b_j(v^\sharp_r) - u^q_{jr}) \wedge (f^b_j(v^\sharp_r) - u^p_{jr}), \tag{79}$$

$$u^i_{jr} = g^{ij}(e^b_j(v^\sharp_r) - \bar{e}^b_j(v^\sharp_r)) \wedge (f^b_j(v^\sharp_r) - \bar{f}^b_j(v^\sharp_r))\big|_{\partial Z^i}, \tag{80}$$

where e^{bi} is the boundary control input or output, f^{bi} is the boundary output or input, $\bar{e}^{bi}$ and $\bar{f}^{bi}$ are the desired energy flows, and g^{ij} is the feedback gain.

5. Conclusion

This chapter derived the boundary controls based on passivity for ideal magnetohydrodynamics (MHD) systems in terms of distributed port-Hamiltonian (DPH) representations. In *Section 2*, We first rewrote the geometric formulation of MHD as a DPH system. Next, we explained the passivity-based controls for the DPH system of MHD by using collocated input/output pairs, i.e., port variables for stabilizing and assigning a global stable point. The boundary power balance equation of the DPH system could be considered as an extended energy principle of MHD in the sense of dynamical systems and boundary controls. In *Section 3*, we considered the DPH model of MHD with model perturbations. The perturbation can be uniquely decomposed into a Hamiltonian subsystem, called an exact subsystem, and a non-Hamiltonian subsystem, called a dual-exact subsystem. We presented the method of creating a pseudo potential for an exact subsystem of the DPH model. In *Section 4*, we explained a symmetry breaking of conservation laws associated with the DPH system. The breaking can be detected by checking quantities with the boundary port variables of the DPH system. Finally, we showed that the boundary port variables can detect the topological change of the domain of DPH systems and can create desired topological energy flows.

These results open the way to active disturbance rejections or plasma shape controls. If an actual MHD system is not ideal or includes modeling errors, the power balance equations should be revised. In this case, the pseudo potential construction might be used for improving the model. The boundary control using the boundary port variables might be approximated by the discretization of port-Hamiltonian systems (Golo et al., 2004).

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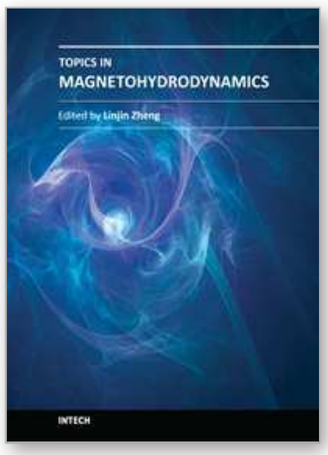
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