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Discrete-Time Steady-State Minimum-Variance Prediction and Filtering

5.1 Introduction

This chapter presents the minimum-variance filtering results simplified for the case when the model parameters are time-invariant and the noise processes are stationary. The filtering objective remains the same, namely, the task is to estimate a signal in such as way to minimise the filter error covariance.

A somewhat naïve approach is to apply the standard filter recursions using the time-invariant problem parameters. Although this approach is valid, it involves recalculating the Riccati difference equation solution and filter gain at each time-step, which is computationally expensive. A lower implementation cost can be realised by recognising that the Riccati difference equation solution asymptotically approaches the solution of an algebraic Riccati equation. In this case, the algebraic Riccati equation solution and hence the filter gain can be calculated before running the filter.

The steady-state discrete-time Kalman filtering literature is vast and some of the more accessible accounts [1] – [14] are canvassed here. The filtering problem and the application of the standard time-varying filter recursions are described in Section 2. An important criterion for checking whether the states can be uniquely reconstructed from the measurements is observability. For example, sometimes states may be internal or sensor measurements might not be available, which can result in the system having hidden modes. Section 3 describes two common tests for observability, namely, checking that an observability matrix or an observability gramian are of full rank. The subject of Riccati equation monotonicity and convergence has been studied extensively by Chan [4], De Souza [5], [6], Bitmead [7], [8], Wimmer [9] and Wonham [10], which is discussed in Section 4. Chan, *et al* [4] also showed that if the underlying system is stable and observable then the minimum-variance filter is stable. Section 6 describes a discrete-time version of the Kalman-Yakubovich-Popov Lemma, which states for time-invariant systems that solving a Riccati equation is equivalent to spectral factorisation. In this case, the Wiener and Kalman filters are the same.

"Science is nothing but trained and organized common sense differing from the latter only as a veteran may differ from a raw recruit: and its methods differ from those of common sense only as far as the guardsman's cut and thrust differ from the manner in which a savage wields his club." *Thomas Henry Huxley*

5.2 Time-Invariant Filtering Problem

5.2.1 The Time-Invariant Signal Model

A discrete-time time-invariant system (or plant) $\mathcal{G} : \mathbb{R}^m \to \mathbb{R}^p$ is assumed to have the state-space representation

$$x_{k+1} = Ax_k + Bw_k ,\tag{1}$$

$$y_k = Cx_k + Dw_k ,\tag{2}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times p}$, w_k is a stationary process with $E\{w_k\} = 0$ and $E\{w_j w_k^T\} = Q\delta_{jk}$. For convenience, the simplification $D = 0$ is initially assumed within the developments. A nonzero feedthrough matrix, D, can be accommodated as described in Chapter 4. Observations z_k of the system output y_k are again modelled as

$$z_k = y_k + v_k ,\tag{3}$$

where v_k is a stationary measurement noise sequence over an interval $k \in [1, N]$, with $E\{v_k\} = 0$, $E\{w_j v_k^T\} = 0$, $E\{v_j v_k^T\} = R\delta_{jk}$. An objective is to design a filter $\mathcal{H}$ that operates on the above measurements and produces an estimate, $\hat{y}_{k/k} = C_k \hat{x}_{k/k}$, of y_k so that the covariance, $E\{\tilde{y}_{k/k} \tilde{y}_{k/k}^T\}$, of the filter error, $\tilde{y}_{k/k} = y_k - \hat{y}_{k/k}$, is minimised.

5.2.2 Application of the Time-Varying Filter Recursions

A naïve but entirely valid approach to state estimation is to apply the standard minimum-variance filter recursions of Section 4 for the problem (1) – (3). The predicted and corrected state estimates are given by

$$\hat{x}_{k+1/k} = (A - K_k C_k)\hat{x}_{k/k-1} + K_k z_k ,\tag{4}$$

$$\hat{x}_{k/k} = (I - L_k C_k)\hat{x}_{k/k-1} + L_k z_k ,\tag{5}$$

where $L_k = P_{k/k-1}C^T(CP_{k/k-1}C + R)^{-1}$ is the filter gain, $K_k = AP_{k/k-1}C^T(CP_{k/k-1}C + R)^{-1}$ is the predictor gain, in which $P_{k/k-1} = E\{\tilde{x}_{k/k-1}\tilde{x}_{k/k-1}^T\}$ is obtained from the Riccati difference equation

$$P_{k+1} = AP_k A^T - AP_k C^T(CP_k C^T + R)^{-1}CP_k A^T + BQB^T .\tag{6}$$

As before, the above Riccati equation is iterated forward at each time k from an initial condition P_0. A necessary condition for determining whether the states within (1) can be uniquely estimated is observability which is discussed below.

"We can understand almost anything, but we can't understand how we understand." *Albert Einstein*

5.3 Observability

5.3.1 The Discrete-time Observability Matrix

Observability is a fundamental concept in system theory. If a system is unobservable then it will not be possible to recover the states uniquely from the measurements. The pair (A, C) within the discrete-time system (1) – (2) is defined to be completely observable if the initial states, x_0, can be uniquely determined from the known inputs w_k and outputs y_k over an interval $k \in [0, N]$. A test for observability is to check whether an observability matrix is of full rank. The discrete-time observability matrix, which is defined in the lemma below, is the same the continuous-time version. The proof is analogous to the presentation in Chapter 3.

Lemma 1 [1], [2]: *The discrete-time system (1) – (2) is completely observable if the observability matrix*

$$O_N = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix}, N \geq n - 1, \tag{7}$$

is of rank n .

Proof: *Since the input w_k is assumed to be known, it suffices to consider the unforced system*

$$x_{k+1} = Ax_k , \tag{8}$$

$$y_k = Cx_k . \tag{9}$$

It follows from (8) – (9) that

$$y_0 = Cx_0$$

$$y_1 = Cx_1 = CAx_0$$

$$y_2 = Cx_2 = CA^2x_0 \tag{10}$$

$$\vdots$$

$$y_N = Cx_N = CA^Nx_0 ,$$

"What happens depends on our way of observing it or the fact that we observe it." *Werner Heisenberg*

which can be written as

$$
y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = C \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x_0 .
\tag{11}
$$

From the Cayley-Hamilton Theorem, A^k, for $k \geq n$, can be expressed as a linear combination of A^0, A^1, ..., A^{n-1} . Thus, with $N \geq n - 1$, equation (11) uniquely determines x_0 if O_N has full rank n. □

Thus, if O_N is of full rank then its inverse exists and so x_0 can be uniquely recovered as $x_0 = O_N^{-1} y$. Observability is a property of the deterministic model equations (8) – (9). Conversely, if the observability matrix is not rank n then the system (1) – (2) is termed unobservable and the unobservable states are called unobservable modes.

5.3.2 Discrete-time Observability Gramians

Alternative tests for observability arise by checking the rank of one of the observability gramians that are described below.

Lemma 2: The pair (A, C) is completely observable if the observability gramian

$$
W_N = O_N^T O_N = \sum_{k=0}^{N} (A^T)^k C^T C A^k , N \geq n\text{-}1
\tag{12}
$$

is of full rank.

Proof: *It follows from (8) – (9) that*

$$
y^T y = x_0^T \begin{bmatrix} I & A^T & (A^T)^2 & \cdots & (A^T)^N \end{bmatrix} C^T C \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x_0 .
\tag{13}
$$

From the Cayley-Hamilton Theorem, A^k, for $k \geq n$, can be expressed as a linear combination of A^0, A^1, ..., A^{n-1} . Thus, with $N = n - 1$,

$$
y^T y = x_0^T O_N^T O_N x_0 = x_0^T W_N x_0 = x_0^T \left(\sum_{k=0}^{n-1} (A^T)^k C^T C A^k \right) x_0
\tag{14}
$$

is unique provided that W_N is of full rank. □

"You affect the world by what you browse." *Tim Berners-Lee*

It is shown below that an equivalent observability gramian can be found from the solution of a Lyapunov equation.

Lemma 3: *Suppose that the system (8) – (9) is stable, that is, $|\lambda_i(A)| < 1$, $i = 1$ to n, then the pair (A, C) is completely observable if the nonnegative symmetric solution of the Lyapunov equation*

$$W = A^T W A + C^T C .$$

(15)

is of full rank.

Proof: *Pre-multiplying $C^T C = W - A^T W A$ by $(A^T)^k$, post-multiplying by A^k and summing from $k = 0$ to N results in*

$$\sum_{k=0}^{N}(A^T)^k C^T C A^k = \sum_{k=0}^{N}(A^T)^k W A^k - \sum_{k=0}^{N}(A^T)^{k+1} W A^{k+1}$$

$$= W_N - (A^T)^{k+1} W_N A^{k+1} .$$

(16)

Since $\lim_{k \to \infty}(A^T)^{k+1} W_N A^{k+1} = 0$, by inspection of (16), $W = \lim_{k \to \infty} W_N$ is a solution of the Lyapunov equation (15). Observability follows from Lemma 2. ☐

It is noted below that observability is equivalent to asymptotic stability.

Lemma 4 [3]: *Under the conditions of Lemma 3, $x_0 \in \ell_2$ implies $y \in \ell_2$.*

Proof: *It follows from (16) that $\sum_{k=0}^{N}(A^T)^k C^T C A^k \le W_N$ and therefore*

$$\|y\|_2^2 = \sum_{k=0}^{N} y_k^T y_k = x_0^T \left(\sum_{k=0}^{N}(A^T)^k C^T C A^k \right) x_0 \le x_0^T W_N x_0 ,$$

from which the claim follows. ☐

Another criterion that is encountered in the context of filtering and smoothing is detectability. A linear time-invariant system is said to be detectable when all its modes and in particular its unobservable modes are stable. An observable system is also detectable.

Example 1. (i) Consider a stable second-order system with $A = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

The observability matrix from (7) and the observability gramian from (12) are $O_1 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.1 & 0.6 \end{bmatrix}$ and $W_1 = O_1^T O_1 = \begin{bmatrix} 1.01 & 1.06 \\ 1.06 & 1.36 \end{bmatrix}$, respectively. It can easily be verified that the

---

"It is a good morning exercise for a research scientist to discard a pet hypothesis every day before breakfast." *Konrad Zacharias Lorenz*

solution of the Lyapunov equation (15) is $W = \begin{bmatrix} 1.01 & 1.06 \\ 1.06 & 1.44 \end{bmatrix} = W_4$ to three significant figures.

Since rank(O_1) = rank(W_1) = rank(W_4) = 2, the pair (A, C) is observable.

(ii) Now suppose that measurements of the first state are not available, that is, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

Since $O_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0.4 \end{bmatrix}$ and $W_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1.16 \end{bmatrix}$ are of rank 1, the pair (A, C) is unobservable. This

system is detectable because the unobservable mode is stable.

5.4 Riccati Equation Properties

5.4.1 Monotonicity

It will be shown below that the solution $P_{k+1/k}$ of the Riccati difference equation (6) monotonically approaches a steady-state asymptote, in which case the gain is also time-invariant and can be precalculated. Establishing monotonicity requires the following result. It is well known that the difference between the solutions of two Riccati equations also obeys a Riccati equation, see Theorem 4.3 of [4], (2.12) of [5], Lemma 3.1 of [6], (4.2) of [7], Lemma 10.1 of [8], (2.11) of [9] and (2.4) of [10].

Theorem 1: Riccati Equation Comparison Theorem [4] – [10]: *Suppose for a $t \geq 0$ and for all $k \geq 0$ the two Riccati difference equations*

$$P_{t+k} = AP_{t+k-1}A^T - AP_{t+k-1}C^T(CP_{t+k-1}C^T + R)^{-1}CP_{t+k-1}A^T + BQB^T ,\tag{17}$$

$$P_{t+k+1} = AP_{t+k}A^T - AP_{t+k}C^T(CP_{t+k}C^T + R)^{-1}CP_{t+k}A^T + BQB^T ,\tag{18}$$

have solutions $P_{t+k} \geq 0$ and $P_{t+k-1} \geq 0$, respectively. Then $\overline{P}_{t+k} = P_{t+k} - P_{t+k+1}$ satisfies

$$\overline{P}_{t+k+1} = \overline{A}_{t+k+1}\overline{P}_{t+k}\overline{A}_{t+k}^T - \overline{A}_{t+k+1}\overline{P}_{t+k}C^T(C\overline{P}_{t+k}C^T + \overline{R}_k)^{-1}C\overline{P}_{t+k}\overline{A}_{t+k+1}^T ,\tag{19}$$

where $\overline{A}_{t+k+1} = A - AP_{t+k-1}C^T(CP_{t+k+1}C^T + \overline{R}_{t+k+1})^{-1}C_{t+k}$ and $\overline{R}_{t+k} = CP_{t+k}C^T + R.$

The above result can be verified by substituting $\overline{A}_{t+k+1}$ and $\overline{R}_{t+k+1}$ into (19). The above theorem is used below to establish Riccati difference equation monotonicity.

Theorem 2 [6], [9], [10], [11]: *Under the conditions of Theorem 1, suppose that the solution of the Riccati difference equation (19) has a solution $\overline{P}_{t+k} \geq 0$ for a $t \geq 0$ and $k = 0$. Then $P_{t+k} \geq P_{t+k+1}$ for all $k \geq 0$.*

"We follow abstract assumptions to see where they lead, and then decide whether the detailed differences from the real world matter." *Clinton Richard Dawkins*

Proof: *The assumption $\overline{P}_{t+k} \geq 0$ is the initial condition for an induction argument. For the induction step, it follows from $C\overline{P}_{t+k}C^T(C\overline{P}_{t+k}C^T + \overline{R}_k)^{-1} \leq I$ that $\overline{P}_{t+k} \leq \overline{P}_{t+k}C^T(C\overline{P}_{t+k}C^T + \overline{R}_k)^{-1}C\overline{P}_{t+k}$, which together with Theorem 1 implies $\overline{P}_{t+k} \geq 0$.* †□

The above theorem serves to establish conditions under which a Riccati difference equation solution monotonically approaches its steady state solution. This requires a Riccati equation convergence result which is presented below.

5.4.2 Convergence
When the model parameters and second-order noise statistics are constant then the predictor gain is also time-invariant andpre-calculated as

$$K = APC^T(CPC^T + R)^{-1}, \tag{20}$$

where P is the symmetric positive definite solution of the algebraic Riccati equation

$$P = APA^T - APC^T(CPC^T + R)^{-1}CPA^T + BQB^T \tag{21}$$

$$= (A - KC)P(A - KC)^T + BQB^T + KRK^T. \tag{22}$$

A real symmetric nonnegative definite solution of the Algebraic Riccati equation (21) is said to be a *strong solution* if the eigenvalues of $(A - KC)$ lie inside or on the unit circle [4], [5]. If there are no eigenvalues on the unit circle then the strong solution is termed the *stabilising solution*. The following lemma by Chan, Goodwin and Sin [4] sets out conditions for the existence of[7] solutions for the algebraic Riccati equation (21).

Lemma 5 [4]: *Provided that the pair (A, C) is detectable, then*

i) the strong solution of the algebraic Riccati equation (21) exists and is unique;

ii) if A has no modes on the unit circle then the strong solution coincides with the stabilising solution.

A detailed proof is presented in [4]. If the linear time-invariant system (1) – (2) is stable and completely observable and the solution P_k of the Riccati difference equation (6) is suitably initialised, then in the limit as k approaches infinity, P_k will asymptotically converge to the solution of the algebraic Riccati equation. This convergence property is formally restated below.

Lemma 6 [4]: *Subject to:*

i) the pair (A, C) is observable;

ii) $|\lambda_i(A)| \leq 1$, $i = 1$ to n;

iii) $(P_0 - P) \geq 0$;

"We know very little, and yet it is astonishing that we know so much, and still more astonishing that so little knowledge can give us so much power." *Bertrand Arthur William Russell*

then the solution of the Riccati difference equation (6) satisfies

$$\lim_{k \to \infty} P_k = P \ . \tag{23}$$

A proof appears in [4]. This important property is used in [6], which is in turn cited within [7] and [8]. Similar results are reported in [5], [13] and [14]. Convergence can occur exponentially fast which is demonstrated by the following numerical example.

Example 2. Consider an output estimation problem where $A = 0.9$ and $B = C = Q = R = 1$. The solution to the algebraic Riccati equation (21) is $P = 1.4839$. Some calculated solutions of the Riccati difference equation (6) initialised with $P_0 = 10P$ are shown in Table 1. The data in the table demonstrate that the Riccati difference equation solution converges to the algebraic Riccati equation solution, which illustrates the Lemma.

k	P_k	$P_{k-1} - P_k$
1	1.7588	13.0801
2	1.5164	0.2425
5	1.4840	$4.7955*10^{-4}$
10	1.4839	$1.8698*10^{-8}$

Table. 1. Solutions of (21) for Example 2.

5.5 The Steady-State Minimum-Variance Filter

5.5.1 State Estimation

The formulation of the steady-state Kalman filter (which is also known as the limiting Kalman filter) follows by allowing k to approach infinity and using the result of Lemma That is, the filter employs fixed gains that are calculated using the solution of the algebraic Riccati equation (21) instead of the Riccati difference equation (6). The filtered state is calculated as

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + L(z_k - C\hat{x}_{k/k-1})$$
$$= (I - LC)\hat{x}_{k/k-1} + Lz_k \ , \tag{24}$$

where $L = PC^T(CPC^T + R)^{-1}$ is the time-invariant filter gain, in which P is the solution of the algebraic Riccati equation (21). The predicted state is given by

$$\hat{x}_{k+1/k} = A\hat{x}_{k/k}$$
$$= (A - KC)\hat{x}_{k/k-1} + Kz_k \ , \tag{25}$$

where the time-invariant predictor gain, K, is calculated from (20).

"Great is the power of steady misrepresentation - but the history of science shows how, fortunately, this power does not endure long". *Charles Robert Darwin*

5.5.2 Asymptotic Stability

The asymptotic stability of the filter (24) – (25) is asserted in two ways. First, recall from Lemma 4 (ii) that if $|\lambda_i(A)| < 1$, $i = 1$ to n, and the pair (A, C) is completely observable, then $|\lambda_i(A - KC)| < 1$, $i = 1$ to n. That is, since the eigenvalues of the filter's state matrix are within the unit circle, the filter is asymptotically stable. Second, according to the Lyapunov stability theory [1], the unforced system (8) is asymptotically stable if there exists a scalar continuous function $V(x)$, satisfying the following.

    (i) $V(x) > 0$ for $x \neq 0$.

    (ii) $V(x_{k+1}) - V(x_k) \leq 0$ for $x_k \neq 0$.

    (iii) $V(0) = 0$.

    (iv) $V(x) \rightarrow \infty$ as $\|x\|_2 \rightarrow \infty$.

Consider the function $V(x_k) = x_k^T P x_k$ where P is a real positive definite symmetric matrix. Observe that $V(x_{k+1}) - V(x_k) = x_{k+1}^T P x_{k+1} - x_k^T P x_k = x_k^T (A^T P A - P) x_k \leq 0$. Therefore, the above stability requirements are satisfied if for a real symmetric positive definite Q, there exists a real symmetric positive definite P solution to the Lyapunov equation

$$A P A^T - P = - Q . \tag{26}$$

By inspection, the design algebraic Riccati equation (22) is of the form (26) and so the filter is said to be stable in the sense of Lyapunov.

5.5.3 Output Estimation

For output estimation problems, the filter gain, L, is calculated differently. The output estimate is given by

$$\hat{y}_{k/k} = C\hat{x}_{k/k}$$

$$= C\hat{x}_{k/k-1} + L(z_k - C\hat{x}_{k/k-1}) \tag{27}$$

$$= (C - LC)\hat{x}_{k/k-1} + Lz_k ,$$

where the filter gain is now obtained by $L = CPC^T (CPC^T + R)^{-1}$. The output estimation filter (24) – (25) can be written compactly as

$$\begin{bmatrix} \hat{x}_{k+1/k} \\ \hat{y}_{k/k} \end{bmatrix} = \begin{bmatrix} (A - KC) & K \\ (C - LC) & L \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ z_k \end{bmatrix} , \tag{28}$$

from which its transfer function is

$$H_{OE}(z) = (C - LC)(zI - A + KC)^{-1} K + L . \tag{29}$$

"The scientists of today think deeply instead of clearly. One must be sane to think clearly, but one can think deeply and be quite insane." *Nikola Tesla*

5.6 Equivalence of the Wiener and Kalman Filters

As in continuous-time, solving a discrete-time algebraic Riccati equation is equivalent to spectral factorisation and the corresponding Kalman-Yakubovich-Popov Lemma (or Positive Real Lemma) is set out below. A proof of this Lemma makes use of the following identity

$$P - APA^T = (zI - A)P(z^{-1}I - A^T) + AP(z^{-1}I - A^T) + (zI - A)PA^T . \tag{30}$$

Lemma 7. Consider the spectral density matrix

$$\Delta\Delta^H(z) = \begin{bmatrix} C(zI-A)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} (z^{-1}I-A^T)^{-1}C^T \\ I \end{bmatrix} . \tag{31}$$

Then the following statements are equivalent.

(i) $\Delta\Delta^H(e^{j\omega}) \geq 0$, *for all* $\omega \in (-\pi, \pi)$.

(ii) $\begin{bmatrix} BQB^T - P + APA^T & APC^T \\ CPA^T & CPC^T + R \end{bmatrix} \geq 0 .$

(iii) There exists a nonnegative solution P of the algebraic Riccati equation (21).

Proof: *Following the approach of [12], to establish equivalence between (i) and (iii), use (21) within (30) to obtain*

$$BQB^T - APC^T(CPC^T + R)CPA^T = (zI - A)P(z^{-1}I - A^T) + AP(z^{-1}I - A^T) + (zI - A)PA^T . \tag{32}$$

Premultiplying and postmultiplying (32) by $C(zI - A)^{-1}$ *and* $(z^{-1}I - A^T)^{-1}C^T$, *respectively, results in*

$$C(zI - A)^{-1}(BQB^T - APC^T\Omega CPA^T)(z^{-1}I - A^T)C^T = CPC^T + C(zI - A)^{-1}APC^T + CPA^T(z^{-1}I - A^T)^{-1}C^T ,$$

where $\Omega = CPC^T + R$. *Hence,*

$$\Delta\Delta^H(z) = GQG^H(z) + R$$

$$= C(zI - A)^{-1}BQB^T(z^{-1}I - A^T)^{-1}C^T + R$$

$$= C(zI - A)^{-1}APC^T\Omega CPA^T(z^{-1}I - A^T)^{-1}C^T + C(zI - A)^{-1}APC^T + CPA^T(z^{-1}I - A^T)^{-1}C^T + \Omega$$

$$= \left(C(zI - A)^{-1}K + I\right)\Omega\left(K^T(z^{-1}I - A^T)^{-1}C^T + I\right) \tag{33}$$

$$\geq 0 .$$

The Schur complement formula can be used to verify the equivalence of (ii) and (iii).   □

"Any intelligent fool can make things bigger and more complex... It takes a touch of genius - and a lot of courage to move in the opposite direction." *Albert Einstein*

In Chapter 2, it is shown that the transfer function matrix of the optimal Wiener solution for output estimation is given by

$$H_{OE}(z) = I - R\{\Delta^{-H}\}_+ \Delta^{-1}(z) , \qquad (34)$$

where $\{ \}_+$ denotes the causal part. This filter produces estimates $\hat{y}_{k/k}$ from measurements z_k. By inspection of (33) it follows that the spectral factor is

$$\Delta(z) = C(zI - A)^{-1}K\Omega^{1/2} + \Omega^{1/2} . \qquad (35)$$

The Wiener output estimator (34) involves $\Delta^{-1}(z)$ which can be found using (35) and a special case of the matrix inversion lemma, namely, $[I + C(zI - A)^{-1}K]^{-1} = I - C(zI - A + KC)^{-1}K$. Thus, the spectral factor inverse is

$$\Delta^{-1}(z) = \Omega^{-1/2} - \Omega^{-1/2}C(zI - A + KC)^{-1}K . \qquad (36)$$

It can be seen from (36) that $\{\Delta^{-H}\}_+ = \Omega^{-1/2}$. Recognising that $I - R\Omega^{-1} = (CPC^T + R)(CPC^T + R)^{-1} - R(CPC^T + R)^{-1} = CPC^T(CPC^T + R)^{-1} = L$, the Wiener filter (34) can be written equivalently

$$
\begin{aligned}
H_{OE}(z) &= I - R\Omega^{-1}\Delta^{-1}(z) \\
&= I - R\Omega^{-1} + R\Omega^{-1}C(zI - A + KC)^{-1}K \\
&= L + (C - LC)(zI - A + KC)^{-1}K ,
\end{aligned}
\qquad (37)
$$

which is identical to the transfer function matrix of the Kalman filter for output estimation (29). In Chapter 2, it is shown that the transfer function matrix of the input estimator (or equaliser) for proper, stable, minimum-phase plants is

$$H_{IE}(z) = G^{-1}(z)(I - R\{\Delta^{-H}\}_+\Delta^{-1}(z)) . \qquad (38)$$

Substituting (35) into (38) gives

$$H_{IE}(z) = G^{-1}(z)H_{OE}(z) . \qquad (39)$$

The above Wiener equaliser transfer function matrices require common poles and zeros to be cancelled. Although the solution (39) is not minimum-order (since some pole-zero cancellations can be made), its structure is instructive. In particular, an estimate of w_k can be obtained by operating the plant inverse on $\hat{y}_{k/k}$, provided the inverse exists. It follows immediately from $L = CPC^T(CPC^T + R)^{-1}$ that

$$\lim_{R \to 0} L = I . \qquad (40)$$

By inspection of (34) and (40), it follows that

$$\lim_{R \to 0} \sup_{\omega \in \{-\pi, \pi\}} \left| H_{OE}(e^{j\omega}) \right| = I . \qquad (41)$$

Thus, under conditions of diminishing measurement noise, the output estimator will be devoid of dynamics and its maximum magnitude will approach the identity matrix.

"It is not the possession of truth, but the success which attends the seeking after it, that enriches the seeker and brings happiness to him." *Max Karl Ernst Ludwig Planck*

Therefore, for proper, stable, minimum-phase plants, the equaliser asymptotically approaches the plant inverse as the measurement noise becomes negligible, that is,

$$\lim_{R \to 0} H_{IE}(z) = G^{-1}(z) \,. \tag{42}$$

Time-invariant output and input estimation are demonstrated below.

```
w=sqrt(Q)*randn(N,1);                  % process noise
x=[0;0];                               % initial state
for k = 1:N
    y(k) = C*x + D*w(k);               % plant output
    x = A*x + B*w(k);
end
v=sqrt(R)*randn(1,N);                  % measurement noise
z = y + v;                            % measurement
omega=C*P*(C') + D*Q*(D') + R;
K = (A*P*(C')+B*Q*(D'))*inv(omega);    % predictor gain
L = Q*(D')*inv(omega);                 % equaliser gain
x=[0;0];                               % initial state
for k = 1:N
    w_estimate(k) = - L*C*x + L*z(k);  % equaliser output
    x = (A - K*C)*x + K*z(k);          % predicted state
end
```

Figure 1. Fragment of *Matlab®* script for Example 3.

Example 3. Consider a time-invariant input estimation problem in which the plant is given by

$$G(z) = (z + 0.9)^2(z + 0.1)^{-2}$$

$$= (z^2 + 1.8z + 0.81)(z^2 + 0.2z + 0.01)^{-1}$$

$$= (1.6z + 0.8)(z^2 + 0.2z + 0.01)^{-1} + 1,$$

together with $Q = 1$ and $R = 0.0001$. The controllable canonical form (see Chapter 1) yields the parameters $A = \begin{bmatrix} -0.2 & -0.1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1.6 & 1.8 \end{bmatrix}$ and $D = 1$. From Chapter 4, the corresponding algebraic Riccati equation is $P = APA^T - K\Omega K^T + BQB^T$, where $K = (APC^T + BQD^T)\Omega^{-1}$ and $\Omega = CPC^T + R + DQD^T$. The minimum-variance output estimator is calculated as

---

"There is no result in nature without a cause; understand the cause and you will have no need of the experiment." *Leonardo di ser Piero da Vinci*

$$\begin{bmatrix} \hat{x}_{k+1/k} \\ \hat{y}_{k/k} \end{bmatrix} = \begin{bmatrix} (A - KC) & K \\ (C - LC) & L \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ z_k \end{bmatrix},$$

where $L = (CPC^T + DQD^T)\Omega^{-1}$. The solution $P = \begin{bmatrix} 0.0026 & -0.0026 \\ -0.0026 & 0.0026 \end{bmatrix}$ for the algebraic Riccati

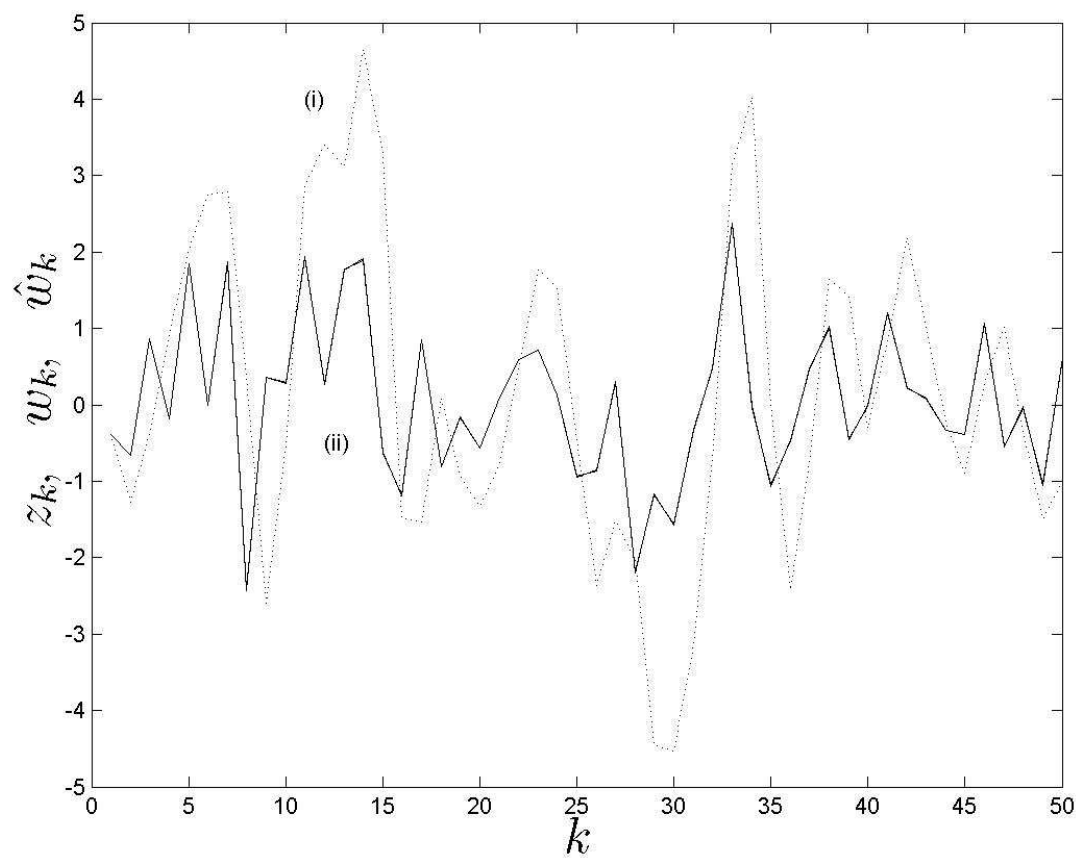equation was found using the Hamiltonian solver within *Matlab®*.

Figure 2. Sample trajectories for Example 5: (i) measurement sequence (dotted line); (ii) actual and estimated process noise sequences (superimposed solid lines).

The resulting transfer function of the output estimator is

$$H_{OE}(z) = (z + 0.9)^2(z + 0.9)^{-2},$$

which illustrates the low-measurement noise asymptote (41). The minimum-variance input estimator is calculated as

$$\begin{bmatrix} \hat{x}_{k+1/k} \\ \hat{w}_{k/k} \end{bmatrix} = \begin{bmatrix} (A - KC) & K \\ -LC & L \end{bmatrix} \begin{bmatrix} \hat{x}_{k/k-1} \\ z_k \end{bmatrix},$$

where $L = QD^T\Omega^{-1}$. The input estimator transfer function is

---

"Your theory is crazy, but its not crazy enough to be true." *Niels Henrik David Bohr*

$$H_{IE}(z) = (z + 0.1)^2 (z + 0.9)^{-2},$$

which corresponds to the inverse of the plant and illustrates the asymptote (42). A simulation was generated based on the fragment of *Matlab®* script shown in Fig. 1 and some sample trajectories are provided in Fig. 2. It can be seen from the figure that the actual and estimated process noise sequences are superimposed, which demonstrates that an equaliser can be successful when the plant is invertible and the measurement noise is sufficiently low. In general, when measurement noise is not insignificant, the asymptotes (41) – (42) will not apply, as the minimum-variance equaliser solution will involve a trade-off between inverting the plant and filtering the noise.

	ASSUMPTIONS	MAIN RESULTS
Signals and system	$E\{w_k\} = E\{v_k\} = 0.$ $E\{w_k w_k^T\} = Q$ and $E\{v_k v_k^T\} = R$ are known. A, B and C are known.	$x_{k+1} = Ax_k + Bw_k$ $y_k = Cx_k$ $z_k = y_k + v_k$
Filtered state and output estimate		$\hat{x}_{k+1/k} = (A - KC)\hat{x}_{k/k-1} + Kz_k$ $\hat{y}_{1,k/k} = (C - LC)\hat{x}_{k/k-1} + Lz_k$
Predictor gain, filter gain and algebraic Riccati equation	$Q > 0$, $R > 0$ and $CPC^T + R_k > 0$. The pair (A, C) is observable.	$K = APC^T(CPC^T + R)^{-1}$ $L = CPC^T(CPC^T + R)^{-1}$ $P = APA^T - K(CPC^T + R)K^T + BQB^T$

Table 2. Main results for time-invariant output estimation.

5.7 Conclusion

In the linear time-invariant case, it is assumed that the signal model and observations can be described by $x_{k+1} = Ax_k + Bw_k$, $y_k = Cx_k$, and $z_k = y_k + v_k$, respectively, where the matrices A, B, C, Q and R are constant. The Kalman filter for this problem is listed in Table 2. If the pair (A, C) is completely observable, the solution of the corresponding Riccati difference equation monotonically converges to the unique solution of the algebraic Riccati equation that appears in the table.

The implementation cost is lower than for time-varying problems because the gains can be calculated before running the filter. If $|\lambda_i(A)| < 1$, $i = 1$ to n, and the pair (A, C) is completely

---

"Clear thinking requires courage rather than intelligence." *Thomas Stephen Szasz*

observable, then $|\lambda_i(A - KC)| < 1$, that is, the steady-state filter is asymptotically stable. The output estimator has the transfer function

$$H_{OE}(z) = C(I - LC)(zI - A + KC)^{-1}K + CL .$$

Since the task of solving an algebraic Riccati equation is equivalent to spectral factorisation, the transfer functions of the minimum-mean-square error and steady-state minimum-variance solutions are the same.

5.8 Problems

Problem 1. Calculate the observability matrices and comment on the observability of the following pairs.

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 4 \end{bmatrix}$. (ii) $A = \begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 4 \end{bmatrix}$.

Problem 2. Generalise the proof of Lemma 1 (which addresses the unforced system $x_{k+1} = Ax_k$ and $y_k = Cx_k$) for the system $x_{k+1} = Ax_k + Bw_k$ and $y_k = Cx_k + Dw_k$.

Problem 3. Consider the two Riccati difference equations

$$P_{t+k} = AP_{t+k-1}A^T - AP_{t+k-1}C^T(CP_{t+k-1}C^T + R)^{-1}CP_{t+k-1}A^T + BQB^T$$

$$P_{t+k+1} = AP_{t+k}A^T - AP_{t+k}C^T(CP_{t+k}C^T + R)^{-1}CP_{t+k}A^T + BQB^T .$$

Show that a Riccati difference equation for $\overline{P}_{t+k} = P_{t+k-1} - P_{t+k}$ is given by

$$\overline{P}_{t+k+1} = \overline{A}_k\overline{P}_{t+k}\overline{A}_k^T - \overline{A}_k\overline{P}_{t+k}C^T(C\overline{P}_{t+k}C^T + \overline{R}_k)^{-1}C\overline{P}_{t+k}\overline{A}_k^T$$

where $\overline{A}_{t+k} = A_{t+k} - A_{t+k}P_{t+k}C^T(CP_{t+k}C^T + \overline{R}_{t+k})^{-1}C_{t+k}$ and $\overline{R}_{t+k} = CP_{t+k}C^T + R$.

Problem 4. Suppose that measurements are generated by the single-input-single-output system $\underline{x}_{k+1} = ax_k + w_k$, $z_k = x_k + v_k$, where $a \in \mathbb{R}$, $E\{v_k\} = 0$, $E\{w_jw_k^T\} = (1-a^2)\delta_{jk}$, $E\{v_jv_k^T\} = \delta_{jk}$, $E\{w_jv_k^T\} = 0$.

(a) Find the predicted error variance.
(b) Find the predictor gain.

(c) Verify that the one-step-ahead minimum-variance predictor is realised by

$$\hat{x}_{k+1/k} = \frac{a}{1 + \sqrt{1-a^2}}\hat{x}_{k/k-1} + \frac{a\sqrt{1-a^2}}{1 + \sqrt{1-a^2}}z_k .$$

(d) Find the filter gain.
(e) Write down the realisation of the minimum-variance filter.

"Thoughts, like fleas, jump from man to man. But they don't bite everybody." *Baron Stanislaw Jerzy Lec*

Problem 5. Assuming that a system G has the realisation $x_{k+1} = A_k x_k + B_k w_k$, $y_k = C_k x_k + D_k w_k$, expand $\Delta\Delta^H(z) = GQG(z) + R$ to obtain $\Delta(z)$ and the optimal output estimation filter.

5.9 Glossary

In addition to the terms listed in Section 2.6, the notation has been used herein.

A, B, C, D	A linear time-invariant system is assumed to have the realisation $\underline{x}_{k+1} = Ax_k + Bw_k$ and $y_k = Cx_k + Dw_k$ in which A, B, C, D are constant state space matrices of appropriate dimension.
Q, R	Time-invariant covariance matrices of stationary stochastic signals w_k and v_k, respectively.
O	Observability matrix.
W	Observability gramian.
P	Steady-state error covariance matrix.
K	Time-invariant predictor gain matrix.
L	Time-invariant filter gain matrix.
$\Delta(z)$	Spectral factor.
$H_{OE}(z)$	Transfer function matrix of output estimator.
$H_{IE}(z)$	Transfer function matrix of input estimator.

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"Nothing in life is to be feared, it is only to be understood. Now is the time to understand more, so that we may fear less." *Marie Sklodowska Curie*

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"Man is but a reed, the most feeble thing in nature, but he is a thinking reed." *Blaise Pascal*
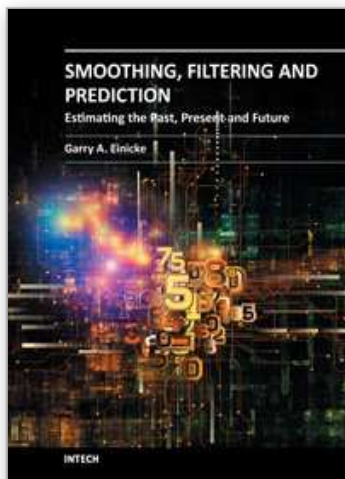
Smoothing, Filtering and Prediction - Estimating The Past, Present and Future

Edited by

This book describes the classical smoothing, filtering and prediction techniques together with some more recently developed embellishments for improving performance within applications. It aims to present the subject in an accessible way, so that it can serve as a practical guide for undergraduates and newcomers to the field. The material is organised as a ten-lecture course. The foundations are laid in Chapters 1 and 2, which explain minimum-mean-square-error solution construction and asymptotic behaviour. Chapters 3 and 4 introduce continuous-time and discrete-time minimum-variance filtering. Generalisations for missing data, deterministic inputs, correlated noises, direct feedthrough terms, output estimation and equalisation are described. Chapter 5 simplifies the minimum-variance filtering results for steady-state problems. Observability, Riccati equation solution convergence, asymptotic stability and Wiener filter equivalence are discussed. Chapters 6 and 7 cover the subject of continuous-time and discrete-time smoothing. The main fixed-lag, fixed-point and fixed-interval smoother results are derived. It is shown that the minimum-variance fixed-interval smoother attains the best performance. Chapter 8 attends to parameter estimation. As the above-mentioned approaches all rely on knowledge of the underlying model parameters, maximum-likelihood techniques within expectation-maximisation algorithms for joint state and parameter estimation are described. Chapter 9 is concerned with robust techniques that accommodate uncertainties within problem specifications. An extra term within Riccati equations enables designers to trade-off average error and peak error performance. Chapter 10 rounds off the course by applying the afore-mentioned linear techniques to nonlinear estimation problems. It is demonstrated that step-wise linearisations can be used within predictors, filters and smoothers, albeit by forsaking optimal performance guarantees.

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