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Analysis of Modified Fifth Degree Chordal Rings

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1. Introduction

Implementation of new telecommunications services has always been associated with the need to ensure network efficiency required to implement these services. Network efficiency can be described by a number of parameters such as: network bandwidth, propagation time, quality, reliability and fault tolerance. More and better performance, and thus network efficiency is achieved mainly by using more and more advanced technical and technological solutions. There were milestones solutions such as the use of coaxial transmission cables, optical fibers, and various techniques of multiplication like TDM or WDM (Newton, 1996). Significant impact on the way to deliver services had wireless transmission, which has found widespread use in communication networks since the end of last century.

In addition to technical and technological solutions to improve network efficiency by using system solutions such as: protocols or topology (topology control) appropriate for the type of connection or service. Type of system solutions in the network is closely related to the technology involved in the network, so it can be said that technology determines the solutions. Examples may be the different topological approaches as a result or a consequence of the limiters for a specific technology; e. g.:

- Networks based on SONET/SDH have limitations as to the path length (number of nodes in the path) as result of synchronization signals distribution,
- In turn, networks based on WDM technology, where network nodes are OADM (ALU: Alcatel-Lucent, 2011) multiplexers there are restrictions as to the length of the path (in the literal sense) associated with the phenomenon of dispersion,
- Recent example is the WSN networks, which are increasingly common application in various areas of life, such as the implementation of communication solutions for Smart Grid (Al-Karaki, 2004), which dealt with the Authors of this chapter, in the case of WSN network must resolve a number of problems associated with reliable transmission over a large area using short-range devices.

From the above examples it follows that in order to provide high efficiency network technology solutions are not always sufficient and require additional system solutions, which should always go hand in hand with these technological ones. Therefore, proposed in this publication the solutions are always up to date.

2. Background

A critical issue in designing telecommunications systems is choosing the interconnection network topology as it has the biggest impact on efficiency, speed, and reliability of the entire system (Bhuyan, 1987). Nowadays, analysis of regular network structures is one of the most important issues in telecommunications and computer science.

These networks can be model by symmetric digraphs, i.e., a directed graph $\boldsymbol{G}$ with vertex set $\boldsymbol{V(G)}$ and edge set $\boldsymbol{E(G)}$, such that, if $[v_i, v_j]$ is in $\boldsymbol{E(G)}$, then $[v_j, v_i]$ is also in $\boldsymbol{E(G)}$. So any edge of digraph connecting vertices v_i and v_j can be replaced by two directed edges $[v_i, v_j]$ and $[v_j, v_i]$ (Narayanan et al., 2001).

It is obvious that the best service and reliability parameters one can obtain by forming complete networks (described by a complete graph), but only small networks can be built in this way. In (Kocis, 1992) a survey of known topologies has been presented. Among the analyzed topologies that would be used in designing the distributed structures, the authors of this publication have chosen rings as they are very simple and extensible. They are characterized by connectivity equal to 2 (damage of one edge or node ensures possibility of transmission), are not expensive (number of edges is equal to the number of nodes), are regular and symmetric, but possess poor transmission parameters.

Halfway between the complete graph and the ring is the chordal ring structure (Arden & Lee, 1981). The chordal ring is a ring with additional chords. It is defined by pair $(p,\boldsymbol{Q})$, where p is the number of nodes of the ring and $\boldsymbol{Q}$ is the set of chords. Each chord connects every pair of nodes of the ring that are at distance q_i in the ring.

The application of this type of structure is useful due to its simplicity, clear topology, resistance to damages, simplicity of routings, and good extension (Kocis, 1992).

The application of chordal rings in computer systems (Mans, 1999), TDM networks (communication between distributed switching modules) (Bujnowski, 2003), core optical networks (Freire & da Silva, 1999, 2001a, 2001b; Liestman et al., 1998; Narayanan & Opatrny, 1999; Narayanan et al., 2001), and optical access networks (Pedersen, 2005; Pedersen et al., 2004a, 2004b, 2005; Bujnowski et al., 2003) has been analyzed. The authors of this publication, in their earlier works on modeling of telecommunication and computer networks, present an analysis of chordal rings (Bujnowski et al. 2004a, 2004b, 2005).

In the beginning the general definition of chordal ring will be giving.

Definition 1. A chordal ring is a ring with additional edges called chords. A chordal ring is defined by the pair (p, Q), where p denotes the number of nodes of the ring and Q denotes the set of chord lengths $Q \subseteq \{1, 2, ..., \lfloor p/2 \rfloor\}$. Since it is a ring, every node is connected to exactly two other nodes (i.e. assume a numbering of the nodes $1,2,\ldots, p$ – then node i is connected to node $i-1$ and $i+1 \ (mod\ p)$. Node 0 is connected to p and 1). Each chord of length $q \in Q$ connects every two nodes of the ring that are at distance q. The chordal ring will be further denoted as $G(p; 1, q_1, ..., q_i)$, $q_1 < ... < q_i$. In general, the degree of chordal rings is $2i$, unless there is a chord of length $p/2$. In this case p should be even and rings' degree is $2i - 1$ (Gavoille, n.d).

In the papers (Bujnowski et al., 2008a, 2009b, 2010; Dubalski et al., 2007, 2008; Pedersen et al., 2009) the authors have previously analysed the transmission properties of third, fourth and sixth degree chordal rings and modified graphs of these types. These topologies are the subject of many publications of the researchers from Putra University (Farah et al, 2008, 2010a, 2010b; Azura et al., 2008, 2010; Farah et al. 2010, 2011).

In this publication the survey of the chordal rings consisting of fifth degree nodes (Fig. 1) will be presented. Until now this type of the regular structures is not widely examined, so authors decided to focus on it (Dubalski, 2010).

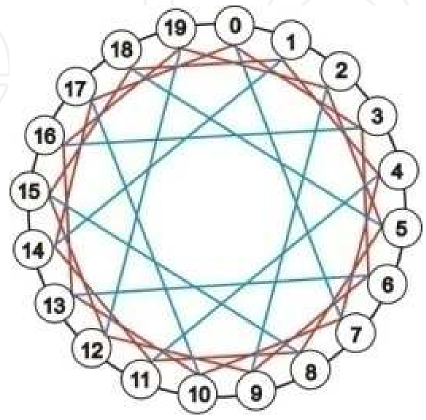

Fig. 1. An example of chordal ring fifth nodal degree

Average distance and diameter was chosen in order to provide a general and simple indication of transmission properties of the analyzed topologies. These follow standard definitions as summarized below. For more basic definitions of e.g. paths and path lengths, please refer to any basic graph theory book, such as (Distel, 2010).

Definition 2. The diameter *D(G)* is the largest path length among all of the shortest length of the paths between any pair of nodes. It is defined as follows:

$$D(G) = \max_{v_i v_j} \{d_{\min}(v_i, v_j)\} \tag{1}$$

where v_i means the number of the node, d_{min} minimal distance (number of edges) between *i-th* and *j-th* node.

Definition 3. The average path length d_{av} between all pairs of nodes is defined by the formula:

$$d_{av} = \frac{1}{p(p-1)} \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} d_{\min}(v_i, v_j) \tag{2}$$

where d_{min} (v_i, v_j) is the minimal number of edges between a source node v_i and every other chosen node v_j,, and p denotes the number of nodes.

A Reference Graph (a virtual example is shown in Fig. 2) can be determined, which presents a reference for all regular graphs of degree 5. It represents lower bounds for average distance and diameter for all these graphs, but since it is a "virtual graph" these bounds may not always be achievable.

The Reference Graph possesses parameters as follows:

1. The number of nodes p_{dr} in d-th layer is determined by formula:

$$p_{1r} = 5$$
$$p_{dr} = 20 \cdot 2^{2(d-2)} \ when \quad d > 1 \tag{3}$$

2. Total number of nodes $p_{D(G)r}$ versus graph diameter is described by expression:

$$p_{D(G)r} = \frac{5 \cdot 4^{D(G)r} - 2}{3} \tag{4}$$

3. Value of diameter versus total number of nodes can be calculated following formula:

$$D(G) = \left\lceil \log_4 \left(\frac{3p_r + 2}{5} \right) \right\rceil \tag{5}$$

4. Average path length d_{avr} in function of diameter is equal to:

$$d_{avr} = \frac{1 + \left(3 \cdot D(G)_r - 1\right) \cdot 4^{D(G)_r}}{3 \cdot \left(4^{D(G)_r} - 1\right)} \tag{6}$$

5. This graph is symmetrical, its all parameters are equal regardless from which node they are calculated.

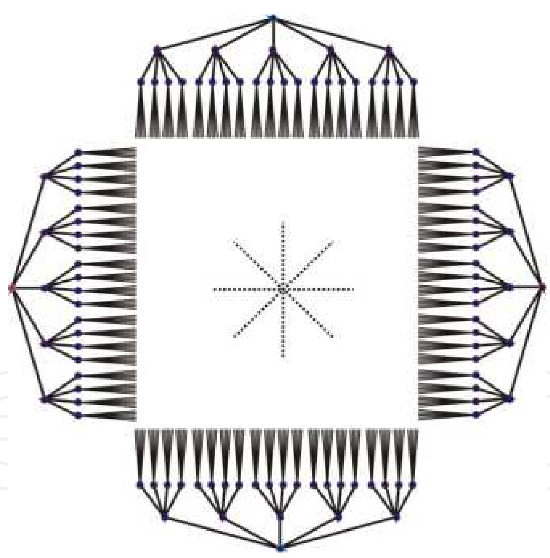

Fig. 2. General diagram of virtual infinite Reference Graph

Only one Reference Graph fifth nodal degree exists in reality, it is the complete graph consisting of 6 nodes.

Two other reference graphs, named as Ideal and Optimal graphs, are also useful for determining average distance and diameter of the chordal rings. They provide theoretical values, which in the following will be compared to values obtained in the real graphs. As for the reference graph mentioned above, the optimal and ideal graphs do not always exist.

In order to determine parameters of the theoretical calculated reference topologies of chordal rings two types of these structures were defined. The first one is called the ideal graph and the second one - optimal graph. In fact these graphs exist only in particular cases, but they are useful as reference models for evaluation expected parameters of tested graphs.

Definition 4. The ideal chordal ring with degree $D(G)$ is the regular graph with total number of nodes p_i given by the formula:

$$p_i = 1 + \sum_{d=1}^{D(G)-1} |p_d| + |p_{D(G)}| \tag{7}$$

where p_d means the number of nodes that belong to the *d-th* layer (the layer is the subset of nodes that are at a distance d from the source node), while $p_{D(G)}$ denotes the number of the remaining nodes which appear in the last layer. For ideal rings, for every n and $m < D(G)$ $p_n \cap p_m = \varnothing$. If for certain $D(G)$ the subset $p_{D(G)}$ of chordal ring reaches the maximal possible value, then such a ring is called the optimal ring (optimal graph).

For ideal chordal ring the average path length d_{avi} is expressed as:

$$d_{avi} = \frac{\sum_{d=1}^{d(G)-1} d|p_d| + D(G)|p_{d(G)}|}{p_i - 1} \tag{8}$$

whereas for the optimal graph the average path length d_{avo} is equal to:

$$d_{avo} = \frac{\sum_{d=1}^{d(G)} d\, p_d}{p_o - 1} \tag{9}$$

where d – layer number, p_d – number of nodes in *d-th* layer, p_o – number of nodes in optimal graph.

Optimal graphs were used to calculate the formulas describing parameters of each type of analyzed chordal ring, whereas ideal rings were served to compare calculated theoretically and obtained in reality parameters of analyzed structures.

The basic topology of fifth degree chordal rings in Fig. 3 is shown. The definition, short presentation and author's consideration concerned of this structure are given below.

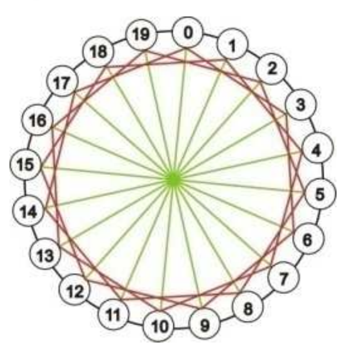

Fig. 3. Basic chordal ring fifth nodal degree CHR5(20; 3,10)

Definition 5. The basic chordal ring fifth nodal degree called CHR5 is an undirected graph, based on a cycle with additional connections (chords). It is denoted by CHR5(p; q_1,q_2) where p must be even and means number of nodes creating the ring, chord length $q_1 > p/2$ is odd, even too, chord length q_2 is equal to $p/2$. The values of p and q_1 must be prime each other (Bujnowski, 2011).

In order to calculate the diameters and average path lengths appearing in optimal graphs it is necessary to evaluate the maximal number of nodes appearing in each layer. In the table 1 the numbers of nodes in the first successive layers of virtual optimal ring are shown (d denotes the layer number, p_d - the number of nodes appearing in *d-th* layer).

d	1	2	3	4	5	6	7	8
p_{do}	5	12	20	28	36	44	52	60

Table 1. Maximal number of nodes in the layers

If $d > 1$ the power of these sets is described by formula:

$$p_{do} = 4(2d-1) \tag{10}$$

Using the formula given above, the total number of nodes p_o in the optimal graph with diameter $D(G)$ can be calculated ($D(G) > 1$):

$$p_o = 4D(G)^2 + 2\ . \tag{11}$$

The total number of nodes in optimal graphs versus its diameter is shown in table 2.

$d(G)$	1	2	3	4	5	6	7	8
p_{do}	6	18	38	66	102	146	198	258

Table 2. Total numbers of nodes forming optimal graphs versus diameter

The average path length in optimal graphs is given by formula:

$$d_{avo} = \frac{8D(G)^3 + 6D(G)^2 - 2D(G) + 3}{3\left(4D(G)^2 + 1\right)} \tag{12}$$

Only one optimal graph exists in reality. It is the complete graph which possesses 6 nodes, but the ideal chordal rings can be found. Whereas it founded two groups of ideal graphs consisting of p nodes, which can be described by formulas given below.

The graphs belonging to the first group are defined as follows:

$$p_i = 4D(G)^2 \quad \left(D(G) > 1\right) \tag{13}$$

so

$$D(G) = \frac{\sqrt{p_i}}{2} \tag{14}$$

In this case a chord length q_1 of ideal graphs is equal to:

$$\begin{gathered} q_1 = 2D(G) - 1 \quad or \quad q_1 = 2D(G) + 1 \\ q_1 = \sqrt{p_i} - 1 \quad or \quad q_1 = \sqrt{p_i} + 1 \end{gathered} \tag{15}$$

these both graphs are isomorphic each other.

The average path length can be express by formula:

$$d_{avi} = \frac{8D(G)^3 + 3D(G)^2 - 8D(G) + 3}{3\left(4D(G)^2 - 1\right)} \tag{16}$$

The graphs belonging to the second group are described as follows:

If $D(G) > 2$ then:

$$p_i = 4D(G)^2 - 4D(G) = 4D(G)\left[D(G) - 1\right] \tag{17}$$

So when the number of nodes is equal to p_i then

$$D(G) = \frac{1 + \sqrt{1 + p_i}}{2} \tag{18}$$

The lengths of chords used to construct ideal graphs can be calculated using formulas:

$$q_1 = 2D(G) - 1 \quad or \quad q_1 = \sqrt{1 + p_i} \tag{19}$$

When the number of nodes creating chordal ring is given by equation:

$$\begin{aligned} & p_i = 4(9i^2 + 9i + 2) \quad or \quad p_i = 6(6i^2 + 10i + 4) \quad where \quad i \in (1, 2, \cdots, n) \\ & then \quad q_1 = 2D(G) + 1 \quad or \quad q_1 = \sqrt{p_i + 1} + 2 \\ & if \quad p_i = 4(9i^2 + 9i + 2) \quad then \quad q_1 = \frac{4D(G)^2 - 2D(G) - 3}{3} \\ & if \quad p_i = 6(6i^2 + 10i + 4) \quad then \quad q_1 = \frac{4D(G)^2 - 6D(G) + 3}{3} \end{aligned} \tag{20}$$

The average path length of all these graphs is described by formula:

$$d_{avi} = \frac{8D(G)^3 - 6D(G)^2 - 8D(G) + 3}{3\left(4D(G)^2 + 4D(G) - 1\right)} \tag{21}$$

Unfortunately the parameters of CHR5 graphs are considerably different of Reference Graph parameters, what is shown in fig. 4 and 5 given above.

It follows from the difference of number of nodes appearing in successive layers and thus the difference of total number of nodes appearing in dependence of its diameter as well.

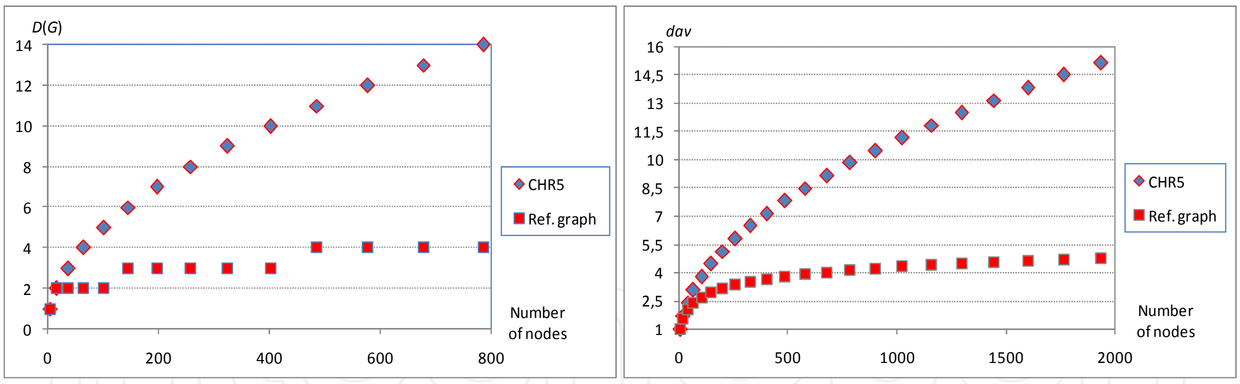

Fig. 4. Comparison of diameter and average path length of Reference Graphs and CHR5

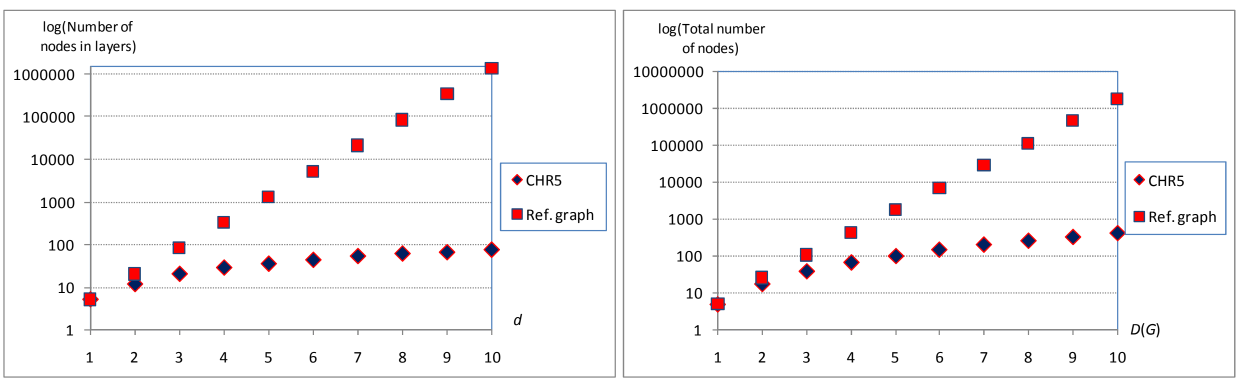

Fig. 5. Differences of number of nodes in successive layers and total number of nodes in Reference Graphs and CHR5

The aim of authors of this publication was to find structures possessing basic parameters which values would be closer to reference graph parameters.

3. Analysis of modified graphs fifth degree

The authors prepared two programs which were used to make it possible to examine the analysed graphs - “Program Graph Finder” and “Find the best distribution of nodes in the layers”. The first one - “Program Graph Finder” was used in the first stage of analysis for quite simple topologies, the second one “Find the best distribution of nodes in the layers” – for more complicated structures, when the number of variables describing the way of connections is greater than 4.

The real values of parameters of modified chordal rings were calculated using these programs and compared to those obtained in a theoretical way.

In the following sections, an analysis of 15 different regular structures based on chordal rings is presented. Each of the type of graphs is defined, examples are given, the distribution of nodes in different layers is analyzed, and the ideal and optimal graphs are compared to real graphs. Also, basing on the analysis of nodes in different layers, the average distance and diameter can be calculated as a function of the number of nodes.

The graphs are divided into 3 groups, each consisting of 5 types of graphs. The first group of graphs needs to have a number of nodes divisible by two, and the second group of graphs a number of nodes divisible by 4. The third group of graphs also has a number of nodes divisible by 4, but for these no mathematical expressions of node distribution (and thus the average distance and diameter) were found.

3.1 First group of chordal rings

As previously mentioned, for each type of graph we present:

- Definitions
- Descriptions
- Distribution of nodes
- Expressions for key parameters
- Comparisons of parameters for real and theoretical graphs.

Graph CHR5_a.

Definition 6. The modified fifth degree chordal rings called CHR5_a (Fig. 6) is denoted by CHR5_a(p; q_1,q_2), where p is even and means number of nodes; q_1, q_2 are chords. Chords q_1 and q_2 are odd and $< p/2$. Chord q_1 generates a Hamiltonian cycle. whereas q_2 is odd too and $< p/2$. Each even node i_{2k} is connected to five other nodes: i_{2k-1}, i_{2k+1}, $i_{2k-q1(\text{mod } p)}$, $i_{2k+q1(\text{mod } p)}$, $i_{2k+q2(\text{mod } p)}$, while odd node i_{2k+1} is connected to i_{2k}, i_{2k+2}, $i_{2k+1-q1(\text{mod } p)}$, $i_{2k+1+q1(\text{mod } p)}$ and $i_{2k+1-q2(\text{mod } p)}$ ($0 \le k < p/2$). The values of p and q_1 must be prime each other (this ensures that the Hamiltonian cycle is created). ☑

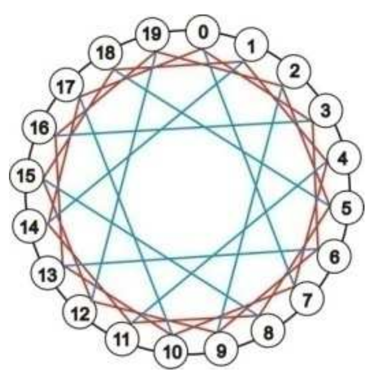

Fig. 6. Modified chordal ring CHR5_a(20; 3,7)

In table 3 the numbers of nodes in the layers of optimal rings as the function of node degree are shown.

d	1	2	3	4	5	6	7	8
p_{do}	5	16	33	58	89	128	173	226

Table 3. Maximal number of nodes in the successive layers

When d (layer number) is odd, then the power of these sets is described by the expression:

$$p_{do\ odd} = \frac{1}{2}(7d^2 + 3) \tag{22}$$

and when d is even:

$$p_{do\ even} = \frac{1}{2}(7d^2 + 4) \tag{23}$$

The general expression has the following form:

$$p_{do} = \frac{1}{2}(7d^2 + 4 - d(\mathrm{mod}\,2)) \tag{24}$$

The total number of nodes p_o forming an optimal graph which possesses diameter $D(G)$ is expressed as:

$$p_{o\ odd} = \frac{7}{12}\left(2D(G)^3 + 3D(G)^2 + 4D(G)\right) + \frac{3}{4}$$

$$p_{o\ even} = \frac{7}{12}\left(2D(G)^3 + 3D(G)^2 + 4D(G)\right) + 1$$

$$p_o = \frac{7}{12}\left(2D(G)^3 + 3D(G)^2 + 4D(G)\right) + 1 - \frac{D(G)(\mathrm{mod}\,2)}{4} \tag{25}$$

which confirms the results obtained by constructing the possible graphs, as shown in Table 4.

$D(G)$	1	2	3	4	5	6	7	8
p_o	6	22	55	113	202	330	503	729

Table 4. Diameters and total numbers of nodes in virtual, optimal graphs

The average path length in optimal graphs can be calculated as:

$$d_{avo} = \frac{3}{2}\frac{7d(G)^4 + 14d(G)^3 + 14d(G)^2 + 8d(G) - 2d(G)d(G)(\mathrm{mod}\,2) - d(G)\,\mathrm{mod}\,2}{14d(G)^3 + 21d(G)^2 + 28d(G) - 3d(G)\,\mathrm{mod}\,2} \tag{26}$$

Fig. 7 shows a comparison of diameters and average path lengths between theoretical and real graphs.

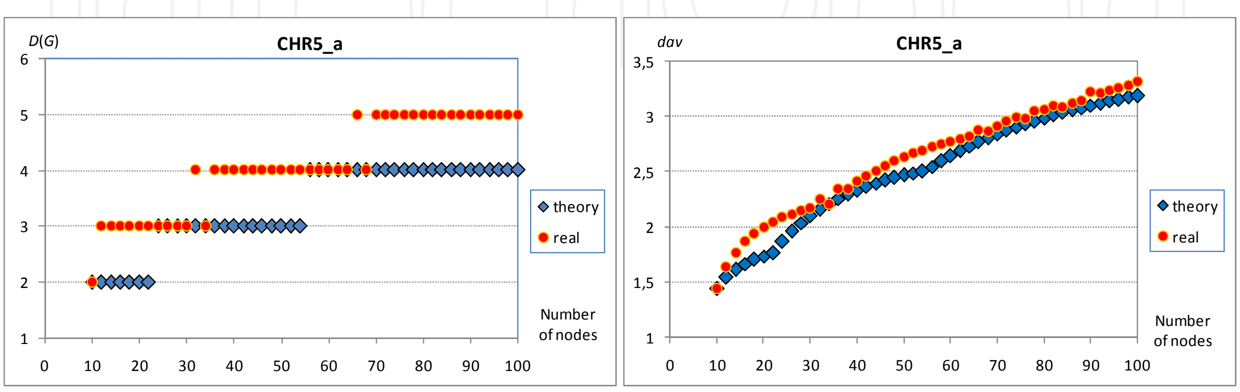

Fig. 7. Comparison of diameter and average path length of theoretical and real graphs CHR5_a

All graphs of this type are symmetrical, the values of their basic parameters do not depend on the number of source node.

Graph CHR5_b.

Definition 6. The modified fifth degree chordal ring called CHR5_b is denoted by CHR5_b(p; q_1,q_2,q_3) where p is even and means number of nodes; q_1, q_2, q_3 are chords, where chord q_1 and q_2 possess even lengths, whereas the length of q_3 is odd. The values of p and q_1, q_2, q_3 must be lower than $p/2$. Each even node i_{2k} is connected to five other nodes: i_{2k-1}, i_{2k+1}, $i_{2k-q1(\text{mod } p)}$, $i_{2k+q1(\text{mod } p)}$, $i_{2k+q3(\text{mod } p)}$, while odd node i_{2k+1} is connected to i_{2k}, i_{2k+2}, $i_{2k+1-q2(\text{mod } p)}$, $i_{2k+1+q2(\text{mod } p)}$ and $i_{2k+1-q3(\text{mod } p)}$ $(0 \leq k < p/2)$.

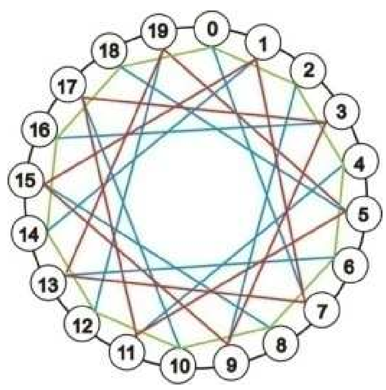

Fig. 8. Modified chordal ring CHR5_b(20; 2,6,7)

Fig. 8 shows an example of CHR5_b. In table 5, the numbers of nodes in the layers of an optimal graph is shown.

d	1	2	3	4	5	6	7	8
p_{do}	5	20	61	140	267	454	713	1056

Table 5. Maximal number of nodes in the layers

When d is bigger than 2, the maximal number of nodes which can appear in the successive layers is described by:

$$p_{do} = 2d^3 + 5d - 8 \tag{27}$$

The total number of nodes p_o in the optimal graph with diameter $D(G) > 1$ is given by:

$$p_o = \frac{1}{2}\left(D(G)^4 + 2D(G)^3 + 6D(G)^2 - 11D(G) + 18\right) \tag{28}$$

This was also confirmed by constructing the possible graphs. These results can be seen in Table 6.

$d(G)$	1	2	3	4	5	6	7	8
p_{do}	6	26	87	227	494	948	1661	2717

Table 6. Total numbers of nodes in optimal graphs versus diameter

The average path length in optimal graphs can be expressed as:

$$d_{avo} = \frac{12D(G)^5 + 30D(G)^4 + 70D(G)^3 - 45D(G)^2 - 97D(G) + 300}{15\left(D(G)^4 + 2D(G)^3 + 6D(G)^2 - 11D(G) + 16\right)} \qquad (29)$$

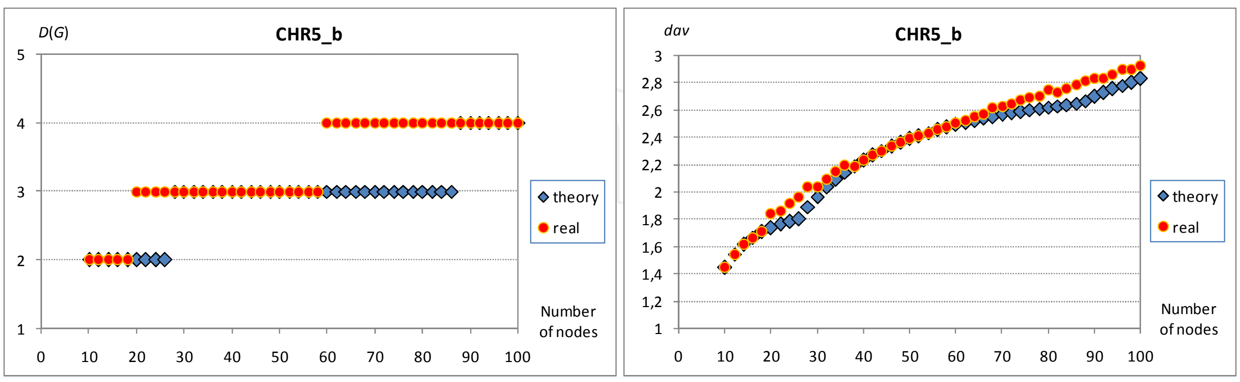

Fig. 9. Comparison of diameter and average path length of theoretical and real graphs CHR5_b

Fig. 9 shows a comparison of the diameter and average path length between theoretical and real graphs.

Not all these graphs are symmetric, but most of the graphs possessing parameters equal or close to ideal graphs are symmetric. In table 7 examples of the real and ideal chordal rings are presented.

Number of nodes	q_1	q_2	q_3	d_{av}
8	2	2	3	1.2857143
10	2	4	3	1.4444444
12	2	2	5	1.5454545
14	2	2	7	1.6153846
16	2	6	7	1.6666666
18	2	4	7	1.7058823
38	6	12	9	2.1891892
40	6	18	9	2.2307692
42	4	8	19	2.2682927
44	6	18	15	2.3023255
46	4	8	19	2.3333333
48	10	22	7	2.3617022
48	14	22	17	2.3617022
50	4	8	21	2.3877552
52	10	14	17	2.4117646
54	4	14	23	2.4339623
56	10	22	5	2.4545455
58	4	14	23	2.4736843

Table 7. Examples of ideal graphs CHR5_b

Graph CHR5_c

Definition 7. The modified fifth degree chordal ring called CHR5_c is denoted by CHR5_c(p; q_1,q_2,q_3), where p is even and means number of nodes; q_1, q_2, q_3 are chords, all chords possess odd lengths less then $p/2$. The values of p and q_1, q_2, q_3 must be prime each other. Each even node i_{2k} is connected to five other nodes: i_{2k-1}, i_{2k+1}, $i_{2k+q1(\mathrm{mod}\ p)}$, $i_{2k+q2(\mathrm{mod}\ p)}$, $i_{2k+q3(\mathrm{mod}\ p)}$, while odd node i_{2k+1} is connected to i_{2k}, i_{2k+2}, $i_{2k+1-q1(\mathrm{mod}\ p)}$, $i_{2k+1-q2(\mathrm{mod}\ p)}$ and $i_{2k+1-q3(\mathrm{mod}\ p)}$ where ($0 \leq k < p/2$). ☑

Fig. 10 shows an example of CHR5_c. In Table 8 the numbers of nodes in the layers of an optimal graph is shown.

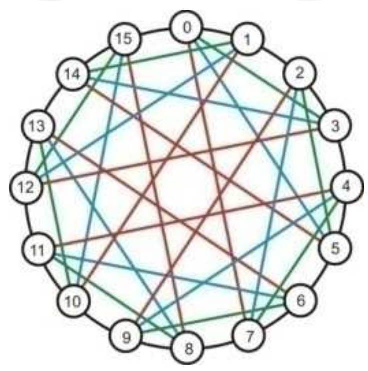

Fig. 10. Modified chordal ring CHR5_c(16; 3,5,,7)

d	1	2	3	4	5	6	7	8
p_{do}	5	20	50	110	200	340	550	850

Table 8. Number of nodes appearing in successive layers

In the case when d – number of layer is bigger than 2 the number of nodes in the layers can be described by the following expression:

$$p_{do} = 5\left(\frac{2}{3}d^3 - 5d^2 + \frac{67}{3}d - 30\right) \tag{30}$$

The total number of nodes p_o in the optimal graph with diameter $D(G) > 1$ is given by:

$$p_o = \frac{1}{6}\left(5D(G)^4 - 40D(G)^3 + 265D(G)^2 - 590D(G) + 516\right) \tag{31}$$

In table 9 the total number of nodes in virtual, optimal graphs, as described by the above expression, is shown.

$D(G)$	1	2	3	4	5	6	7	8
p_o	6	26	76	186	386	726	1276	2126

Table 9. Total numbers of nodes in optimal graphs

The average path length in optimal graphs can be calculated using the expression:

$$d_{avo} = \frac{8D(G)^5 - 55D(G)^4 + 310D(G)^3 - 305D(G)^2 - 678D(G) + 1260}{2\left(5D(G)^4 - 40D(G)^3 + 265D(G)^2 - 590D(G) + 510\right)} \tag{32}$$

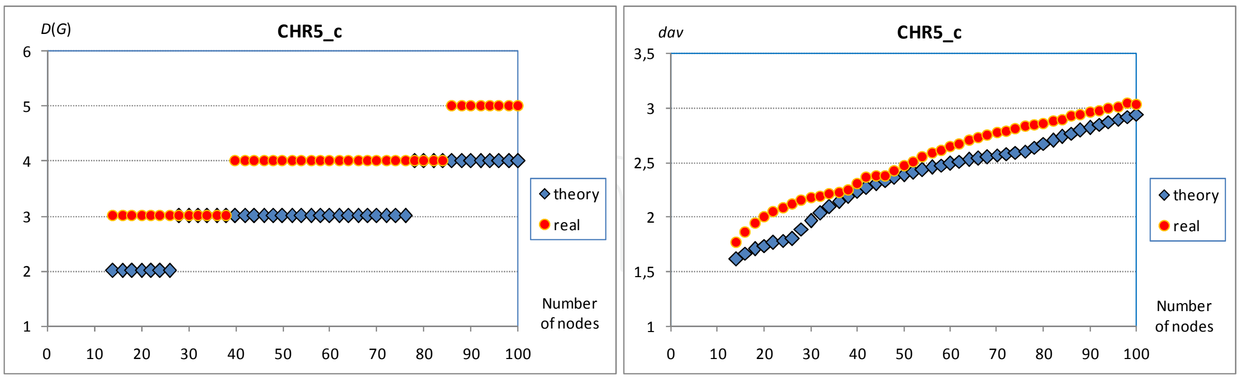

Fig. 11. Comparison of diameter and the average path length of theoretical and real CHR5_c graphs

Fig. 11 shows a comparison of diameter and the average path length between theoretical and real graphs. All graphs of this type are symmetrical, but they couldn't find any ideal graph.

Graph CHR5_d.

Definition 8. The modified fifth degree chordal ring called CHR5_d is denoted by CHR5_d(p; q_1,q_2,$p/2$), where p means the number of nodes and is positive and even; q_1, q_2 are chords which possess odd lengths less then $p/2$. The values of p and q_1, q_2 must be prime each other. Each even node i_{2k} is connected to five other nodes: i_{2k-1}, i_{2k+1}, $i_{2k+q1(\text{mod } p)}$, $i_{2k+q2(\text{mod } p)}$, $i_{2k+p/2(\text{mod } p)}$, while odd node i_{2k+1} is connected to i_{2k}, i_{2k+2}, $i_{2k+1-q1(\text{mod } p)}$, $i_{2k+1-q2(\text{mod } p)}$ and $i_{2k+1+p/2(\text{mod } p)}$ where $(0 \le k < p/2)$.

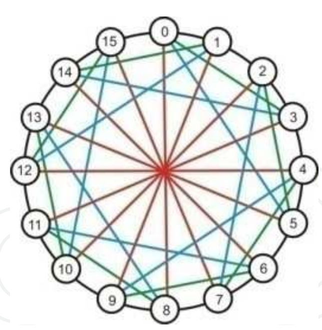

Fig. 12. Example of modified chordal ring CHR5_d(16; 3,5,,8)

Fig. 12 shows an example of CHR5_d. In table 10, the numbers of nodes in the layers of an optimal graph are shown. It should be noted that there are two different number of nodes in layers, depending on whether the total number of nodes is divisible by 4 or not.

d	1	2	3	4	5	6	7	8
$p_o = 0$ (mod 4)	5	16	36	66	106	156	216	286
$p_o \neq 0$ (mod 4)	5	18	39	72	113	166	227	300

Table 10. Maximal number of nodes in the successive layers

When the number of layer d is bigger than 1, the number of nodes in the layers can be described by the following expression:

$$\begin{aligned}&\textit{If } p=0 \,(\mathrm{mod}\,4)\textit{ then} \quad p_{do}=5d^2-5d+6\\&\textit{if } p\neq 0 \,(\mathrm{mod}\,4)\textit{ and layer number is even then} \quad p_{do}=5d^2-3d+4\\&\textit{if } p\neq 0 \,(\mathrm{mod}\,4)\textit{ and layer number is odd then} \quad p_{do}=5d^2-3d+3\end{aligned} \tag{33}$$

The total number of nodes p_o in the optimal graph depending on the diameter is given by:

$$\begin{aligned}p_o&=\frac{D(G)\left(4D(G)^2+13\right)}{3} \quad \textit{if } p_o=0 \text{ mod } 4\\p_o&=\frac{D(G)\left(10D(G)^2+6D(G)+17\right)}{6}+1 \quad \textit{if } p\neq 0 \text{ mod } 4 \textit{ and } D(G) \textit{ is even}\\p_o&=\frac{D(G)\left(10D(G)^2+6D(G)+17\right)}{6}+\frac{1}{2} \quad \textit{if } p\neq 0 \text{ mod } 4 \textit{ and } D(G) \textit{ is odd}\end{aligned} \tag{34}$$

In Table 11 the total number of nodes in virtual optimal graphs described by formula given above is shown.

$D(G)$	1	2	3	4	5	6	7	8
$p_o = 0$ (mod 4)	6	22	58	124	230	386	602	888
$p_o \neq 0$ (mod 4)	6	24	63	135	248	414	641	941

Table 11. Total numbers of nodes forming optimal graphs versus diameter

The average path length in optimal graphs can be calculated using the expressions:

$$\begin{aligned}&\textit{If } p_o=0 \text{ mod } 4\\&d_{avo}=\frac{D(G)\left(2D(G)^4+5D(G)^3+12D(G)^2+10D(G)+1\right)}{2\left(5D(G)^3+13D(G)-3\right)}\\&\textit{if } p_o\neq 0 \text{ mod } 4\\&d_{avo}=\frac{6D(G)\left(5D(G)^2-3D(G)+3+(D(G)+1)\,\mathrm{mod}\,2\right)}{D(G)\left(10D(G)^2+6D(G)+17\right)+3+3(D(G)+1)\,\mathrm{mod}\,2}\end{aligned} \tag{35}$$

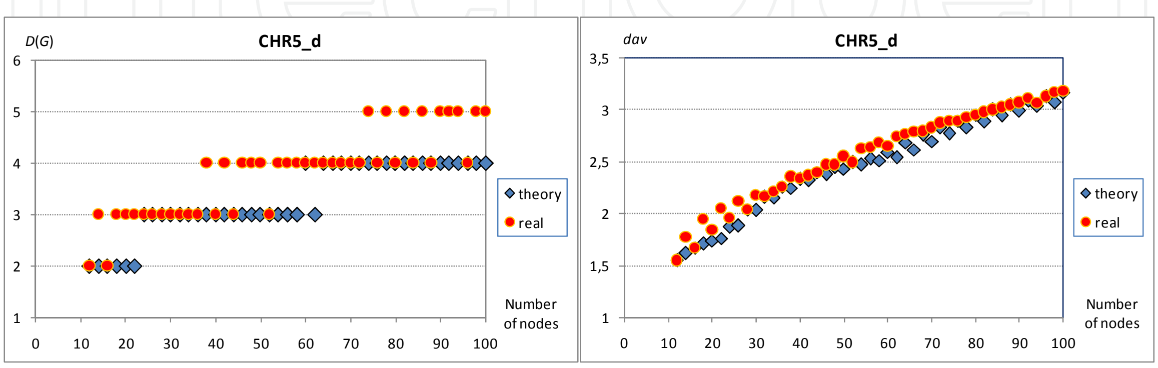

Fig. 13. Comparison of diameter and average path length of theoretical and real graphs CHR5_d

Fig. 13 shows a comparison of diameter and average path length between theoretical and real graphs.

All this type of chordal rings are symmetrical. Ideal graphs are only for the cases where the number of nodes is divisible by 4. Examples are given in Table 12.

Number of nodes	q_1	q_2	$p/2$	d_{av}
12	3	5	6	1,545455
16	3	5	8	1,666667
28	7	11	14	2,037037
32	5	13	16	2,16129
36	5	13	18	2,257143
40	7	17	20	2,333333
44	5	13	22	2,395349
52	5	17	26	2,490196
76	13	21	38	2,893333
80	7	25	40	2,949367
84	9	23	42	3,00000
88	7	27	44	3,045977
96	7	29	48	3,126316

Table 12. Examples of ideal graphs CHR5_d

Graph CHR5_e.

Definition 9. The modified fifth degree chordal ring called CHR5_e is denoted by CHR5_e(p; q_1,q_2,$\boldsymbol{p}/2$), where p means the number of nodes and is positive and even; q_1, q_2 are chords which possess even lengths less then $p/2$. The values of $p/2$ and q_1, q_2 must be prime each other. Each even node i_{2k} is connected to five other nodes: i_{2k-1}, i_{2k+1}, $i_{2k+q1(\mathrm{mod}\ p)}$, $i_{2k-q1(\mathrm{mod}\ p)}$, $i_{2k+p/2(\mathrm{mod}\ p)}$, while odd node i_{2k+1} is connected to i_{2k}, i_{2k+2}, $i_{2k+1+q2(\mathrm{mod}\ p)}$, $i_{2k+1-q2(\mathrm{mod}\ p)}$ and $i_{2k+1+\mathrm{p}/2(\mathrm{mod}\ p)}$ where ($0 \le k < p/2$).

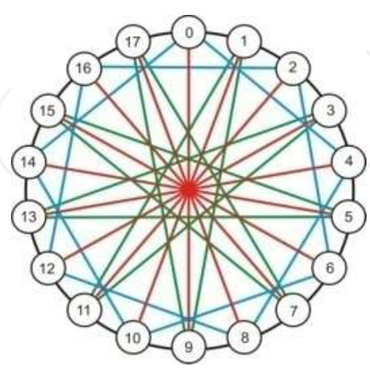

Fig. 14. Example modified chordal ring CHR5_e(18; 4,8,9)

Fig. 14 shows an example of CHR5_e. In Table 13 the number of nodes appearing in the successive layers of optimal graphs is shown. It should be observed that there are two different number of nodes in layers, depending on the total number of nodes are they divisible by 4 or not.

d	1	2	3	4	5	6	7	8
p_o= 0 (mod 4)	5	16	42	88	152	232	328	440
$p_o \neq$ 0 (mod 4)	5	18	48	96	160	240	336	448

Table 13. Maximal number of nodes in the successive layers

When the number of layers d – is bigger than 1 the number of nodes in the layers can be described by the following expressions:

$$\begin{aligned} &if\ \ p = 0\ (\mathrm{mod}\, 4)\ and\ d \rangle\ 2\ then \quad p_{do} = 8d(d-1) \\ &if\ \ p \neq 0\ (\mathrm{mod}\, 4)\ and\ d \rangle\ 3\ then \quad p_{do} = 8d(d-1) - 8 \end{aligned} \tag{36}$$

The total number of nodes p_o in the optimal graph depending on the diameter is given by:

$$\begin{aligned} p_o &= \frac{8D(G)^3 - 32d(G) + 72}{3} = \frac{8D(G)\left(D(G)^2 - 4\right)}{3} + 24 \quad if\ p_o = 0 \bmod 4\ and\ D(G) \rangle\ 2 \\ p_o &= \frac{8D(G)^3 - 8d(G) + 24}{3} = \frac{8D(G)\left(D(G)^2 - 1\right)}{3} + 8 \quad if\ p_o \neq 0 \bmod 4\ and\ D(G) \rangle\ 1 \end{aligned} \tag{37}$$

In Table 14 the total number of nodes in virtual optimal graphs described by the above expressions is shown.

$D(G)$	1	2	3	4	5	6	7	8
p_o = 0 (mod 4)	6	22	64	152	304	536	864	1304
$p_o \neq$ 0 (mod 4)	6	24	72	168	328	568	904	1352

Table 14. Total numbers of nodes forming optimal graphs versus diameter

The average path length in optimal graphs can be calculated using this expression:

$$\begin{aligned} &When\ p_o = 0 \bmod 4\ and\ D(G) \rangle\ 2 \\ &d_{avo} = \frac{6D(G)^4 + 4D(G)^3 - 18D(G)^2 - 16D(G) + 105}{8D(G)^3 - 32D(G) + 69} \\ &when\ p_o \neq 0 \bmod 4\ and\ D(G) \rangle\ 1 \\ &d_{avo} = \frac{6D(G)^4 + 4D(G)^3 - 6D(G)^2 - 4D(G) + 27}{8D(G)^3 - 8D(G) + 21} \end{aligned} \tag{38}$$

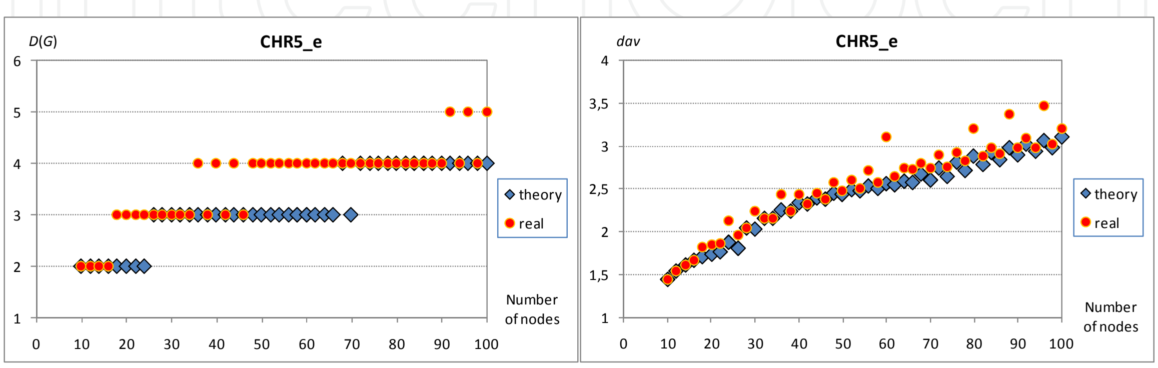

Fig. 15. Comparison of diameter and average path length of theoretical and real graphs CHR5_e

Fig. 15 shows diameter and average path lengths in theoretical and real graphs with up to 100 nodes.

Some but not all of these graphs are symmetric, and as illustrated in Fig. 15 the non symmetric graphs generally have parameters closer to those of ideal graphs. Only a few ideal graphs are found, of which some examples are shown in Table 15.

Number of nodes	q_1	q_2	$p/2$	d_{av}
10	2	4	5	1,444444
12	2	2	6	1,545455
14	2	4	7	1,615385
16	2	6	8	1,666667
28	6	10	14	2,037037
32	6	10	16	2,161290
34	4	10	17	2,151515
38	4	8	19	2,243243
42	4	8	21	2,317073

Table 15. Examples of ideal graphs CHR5_e

To sum up, in the first group of analyzed graphs the best parameters have CHR5_b graphs. They possess minimal diameter and average path length in comparison to the other analyzed chordal rings, and the parameters of the real graphs are close or equal to parameters of ideal graphs. Additionally, most of the best graphs are symmetric, what is also an advantage for the application in real networks.

Fig. 16 shows the comparisons of the real graphs in the first group.

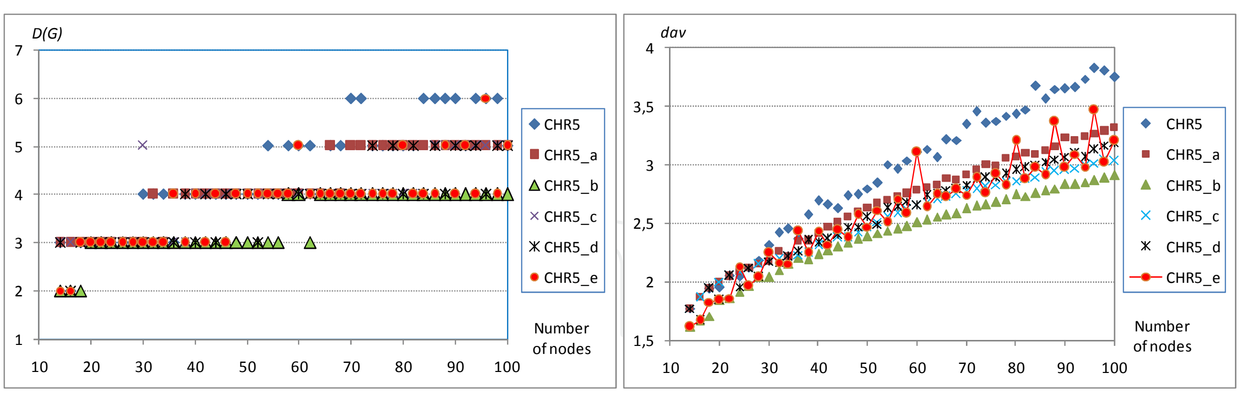

Fig. 16. Comparison of diameter and average path length of all real modified chordal rings belonging to the first group of analyzed graphs.

3.2 Second group of analyzed graphs

The chordal rings consisting of $4i$ nodes (i = 2, 3, 4, …,) belong to this group. These topologies are often more complicated, since they are less symmetric. Basing on patterns for ideal and optimal graphs it is possible to derive expressions for average distance and diameter for all of the different topologies in this group of graphs.

Graph CHR5_f.

Definition 10. The modified fifth degree chordal ring called CHR5_f is denoted by CHR5_f(p; q_1,q_2,q_3, $p/2$), where p means the number of nodes and is positive and divisible by 4; q_1, q_2, q_3, are chords which possess even lengths less then $p/2$. The values of $p/4$ and q_1, q_2, q_3 must be prime each other. Each even node i_{2k} is connected to five other nodes: i_{2k-1}, i_{2k+1}, $i_{2k+q1(\mathrm{mod}\ p)}$, $i_{2k-q1(\mathrm{mod}\ p)}$, $i_{2k+p/2(\mathrm{mod}\ p)}$, while odd node i_{2k+1} is connected to i_{2k}, i_{2k+2}, $i_{2k+1+q2(\mathrm{mod}\ p)}$, $i_{2k+1-q2(\mathrm{mod}\ p)}$ and $i_{2k+1+p/2(\mathrm{mod}\ p)}$ and node i_{2k-1} is connected to i_{2k}, i_{2k-2}, $i_{2k-1+q3(\mathrm{mod}\ p)}$, $i_{2k-1-q3(\mathrm{mod}\ p)}$ and $i_{2k-1+p/2(\mathrm{mod}\ p)}$ where $(0 \le k < p/2)$.

An example is shown in Fig. 17.

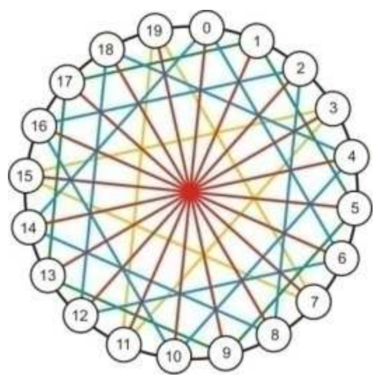

Fig. 17. Example of modified chordal ring CHR5_f(20; 4,6,,8,10)

This structure is more complicated since the number of nodes appearing in successive layers depends on whether the total number of nodes is divisible by 8 or not, and also on whether it seen from odd or even node number in the graph. This creates multiple cases, which also complicates deriving the basic parameters. Table 16 shows the experimentally obtained results, which are the basis for further analysis.

d	Number of nodes	1	2	3	4	5	6	7	8
$p_{do\ p=0\ (mod\ 8)}$	Even	5	16	44	112	248	488	888	1496
	Odd	5	16	46	116	262	536	984	1640
$p_{do\ p\neq 0\ (mod\ 8)}$	Even	5	16	48	136	312	616	1096	1784
	Odd	5	18	58	152	340	668	1172	1884

Table 16. Maximal number of nodes in the successive layers

The number of nodes in the layers can be described by the following formula:

$$\begin{aligned}
&\textit{When } p = 0\ (\mathrm{mod}\, 8) \textit{ and } d \rangle\ 5 \textit{ then}\\
&p_{do\ even} = 5\frac{1}{3}d^3 - 8d^2 - 173\frac{1}{3} + 664 \quad p_{do\ odd} = 5\frac{1}{3}d^3 - 8d^2 - 125\frac{1}{3} + 424\\
&\textit{when } p \neq 0\ (\mathrm{mod}\, 8) \textit{ and } d\ \rangle\ 4 \textit{ then}\\
&p_{do\ even} = 5\frac{1}{3}d^3 - 8d^2 - 93\frac{1}{3} + 312 \quad p_{do\ odd} = 5\frac{1}{3}d^3 - 8d^2 - 69\frac{1}{3} + 220
\end{aligned} \tag{39}$$

In Table 17 the total number of nodes in optimal graphs given by above expression is shown.

$d(G)$	Number of nodes	1	2	3	4	5	6	7	8
$p_{do\ p\,=\,0\ (mod\ 8)}$	even	6	22	66	178	426	914	1802	3298
	odd	6	22	68	184	446	982	1966	3606
$p_{do\ p\,\neq\,0\ (mod\ 8)}$	even	6	22	70	206	518	1134	2230	4014
	odd	6	24	82	234	574	1242	2414	4298

Table 17. Total numbers of nodes forming optimal graphs versus diameter

The total number of nodes p_o in the optimal graph as a function of the diameter is given by:

$$\begin{aligned}
&When\ p = 0\ (\mathrm{mod}\,8)\ and\ d \rangle 4\\
&p_{o\,even} = 1\frac{1}{3}D(G)^4 - 89\frac{1}{3}D(G)^2 + 576D(G) - 1054\\
&p_{o\,odd} = 1\frac{1}{3}D(G)^4 - 65\frac{1}{3}D(G)^2 + 360D(G) - 554\\
&when\ p \neq 0\ (\mathrm{mod}\,8)\ and\ d \rangle 3\\
&p_{o\,even} = 1\frac{1}{3}D(G)^4 - 49\frac{1}{3}D(G)^2 + 264D(G) - 402\\
&p_{o\,odd} = 1\frac{1}{3}D(G)^4 - 49\frac{1}{3}D(G)^2 + 264D(G) - 400
\end{aligned} \tag{40}$$

The average path length in optimal graphs can be calculated using expressions:

$$\begin{aligned}
&When\ p_o = 0\ \mathrm{mod}\ 8\ and\ D(G) \rangle 2\\
&d_{avo} = \frac{6D(G)^4 + 4D(G)^3 - 18D(G)^2 - 16D(G) + 105}{8D(G)^3 - 32D(G) + 69}\\
&when\ p_o \neq 0\ \mathrm{mod}\ 8\ and\ D(G) \rangle 1\\
&d_{avo} = \frac{6D(G)^4 + 4D(G)^3 - 6D(G)^2 - 4D(G) + 27}{8D(G)^3 - 8D(G) + 21}
\end{aligned} \tag{41}$$

Fig. 19 shows diameter and average path lengths in theoretical and real graphs with up to 100 nodes.

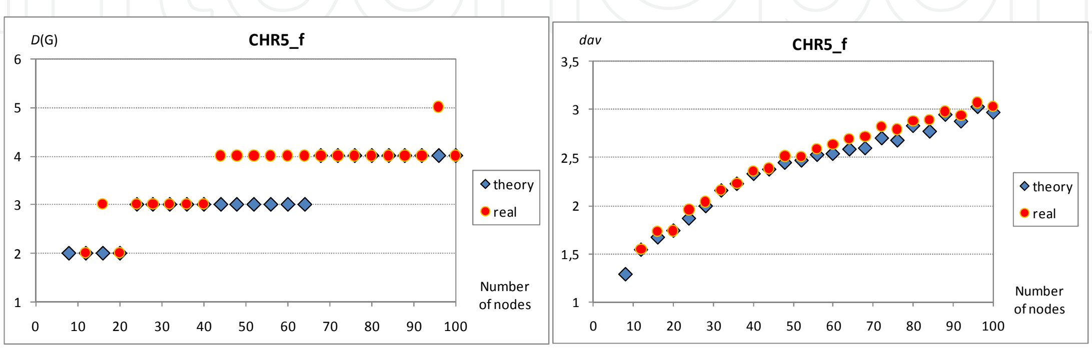

Fig. 18. Comparison of diameter and average path length of theoretical and real graphs CHR5_f

Only three ideal graphs found, which is presented in Table 18.

Number of nodes	q_1	q_2	q_3	$p/2$	d_{av}
12	2	4	4	6	1,545
20	6	4	4	10	1,737
32	6	12	12	16	2,161

Table 18. All founded ideal graphs CHR5_f with up to 100 nodes.

Graph CHR5_g.

Definition 11. The modified fifth degree chordal ring called CHR5_g is denoted by CHR5_g(p; q_1,q_2,q_3), where p means the number of nodes, is positive and divisible by 4; q_1 is chord has odd length; q_2, q_3 are chords which possess even lengths and less then $p/2$. The values of p and q_1 must be prime each other. Even node $i_{2k=0(\mathrm{mod}4)}$ is connected to five other nodes: i_{2k-1}, i_{2k+1}, $i_{2k+q1(\mathrm{mod}\ p)}$, $i_{2k-q1(\mathrm{mod}\ p)}$ and $i_{2k+q2(\mathrm{mod}\ p)}$, when $i_{2k=2(\mathrm{mod}4)}$ than this node is connected to i_{2k-1}, i_{2k+1}, $i_{2k+q1(\mathrm{mod}p)}$, $i_{2k-q1(\mathrm{mod}\ p)}$ and $i_{2k-q2(\mathrm{mod}p)}$; while odd node $i_{(2k+1)=1(\mathrm{mod}4)}$ is connected to i_{2k}, i_{2k+2}, $i_{2k+1+q1(\mathrm{mod}p)}$, $i_{2k+1-q1(\mathrm{mod}p)}$, $i_{2k+1+q3(\mathrm{mod}p)}$; and any node $i_{(2k+1)=3(\mathrm{mod}4)}$ is connected to i_{2k}, i_{2k+2}, $i_{2k+1+q1(\mathrm{mod}p)}$, $i_{2k+1-q1(\mathrm{mod}p)}$, $i_{2k+1-q3(\mathrm{mod}p)}$. ☑

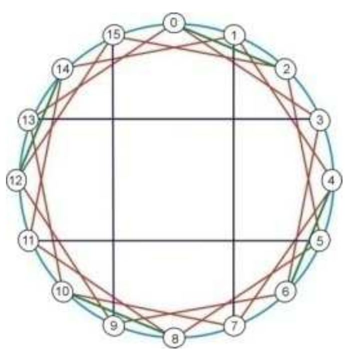

Fig. 19. Example of modified chordal ring CHR5_g(16; 3,2,6)

An example is shown in Fig. 20. The number of nodes in the layers of an optimal graph is given in table 19.

d	1	2	3	4	5	6	7	8
p_{do}	5	16	42	102	183	302	491	704

Table 19. Maximal number of nodes in the layers

In the case when the number of layer is bigger than 1, the number of nodes in the layers can be described by the following formula:

$$\begin{aligned} &When \quad d = 0\,(\mathrm{mod}\,3)\ then\ \ p_{do} = \frac{4}{3}d^3 + \frac{8}{3}d - 2 \\ &when \quad d = 1\,(\mathrm{mod}\,3)\ and\ d \rangle 2\ then\ \ p_{do} = \frac{4}{3}d^3 + \frac{2}{9}d^2 + \frac{29}{9}d + \frac{2}{9} \\ &when \quad d = 2\,(\mathrm{mod}\,3)\ and\ d \rangle 4\ then\ \ p_{do} = \frac{4}{3}d^3 - \frac{2}{9}d^2 + \frac{41}{9}d - \frac{8}{9} \end{aligned} \tag{42}$$

The total number of nodes p_o in the optimal graph can be calculated using the expressions:

$$\begin{aligned}
&\textit{When}\quad d = 0 \,(\mathrm{mod}\,3)\ \textit{then}\ \ p_o = \frac{1}{3}D(G)^4 + \frac{2}{3}D(G)^3 + 2D(G)^2 + \frac{2}{3}D(G) - 1 \\
&\textit{when}\quad d = 1 \,(\mathrm{mod}\,3)\ \textit{and}\ d \rangle\ 2\ \textit{then}\ \ p_o = \frac{1}{3}D(G)^4 + \frac{2}{3}D(G)^3 + \frac{20}{9}D(G)^2 + \frac{5}{9}D(G) + \frac{2}{9} \\
&\textit{when}\quad d = 2 \,(\mathrm{mod}\,3)\ \textit{and}\ d \rangle\ 2\ \textit{then}\ \ p_o = \frac{1}{3}D(G)^4 + \frac{2}{3}D(G)^3 + 2D(G)^2 + \frac{4}{3}D(G) + \frac{2}{3}
\end{aligned} \tag{43}$$

In Table 20 the total number of nodes in optimal graphs described by formula given above is shown.

$D(G)$	1	2	3	4	5	6	7	8
p_o	6	22	64	166	349	651	1142	1846

Table 20. Diameters and total numbers of nodes in optimal graphs

The average path length in the optimal graphs as a function of its diameter can be calculated using equations (44).

$$\begin{aligned}
&\textit{When}\ D(G) = 0\ (\mathrm{mod}\ 3) \\
&d_{avo} = \frac{64{,}8\left(\frac{D(G)}{3}\right)^5 + 54\left(\frac{D(G)}{3}\right)^4 + \frac{124}{3}\left(\frac{D(G)}{3}\right)^3 - 10\left(\frac{D(G)}{3}\right)^2 - \frac{47}{15}\frac{D(G)}{3} - 4}{\frac{1}{3}\left(\frac{D(G)}{3}\right)^4 + \frac{2}{3}\left(\frac{D(G)}{3}\right)^3 + 2\left(\frac{D(G)}{3}\right)^2 + \frac{2}{9}D(G) - 2} \\
&\textit{when}\ D(G) = 1\ (\mathrm{mod}\ 3) \\
&d_{avo} = \frac{64{,}8\left(\frac{D(G)-1}{3}\right)^5 - 54\left(\frac{D(G)-1}{3}\right)^4 + \frac{124}{3}\left(\frac{D(G)-1}{3}\right)^3 - 14\left(\frac{D(G)-1}{3}\right)^2}{\frac{1}{3}\left(\frac{D(G)-1}{3}\right)^4 + \frac{2}{3}\left(\frac{D(G)-1}{3}\right)^3 + \frac{20}{9}\left(\frac{D(G)-1}{3}\right)^2 + \frac{5}{27}(D(G)-1) - \frac{7}{9}} + \\
&+ \frac{-\frac{43}{15}\frac{D(G)-1}{3} - 4}{\frac{1}{3}\left(\frac{D(G)-1}{3}\right)^4 + \frac{2}{3}\left(\frac{D(G)-1}{3}\right)^3 + \frac{20}{9}\left(\frac{D(G)-1}{3}\right)^2 + \frac{5}{27}(D(G)-1) - \frac{7}{9}} \\
&\textit{when}\ D(G) = 2\ (\mathrm{mod}\ 3) \\
&d_{avo} = \frac{64{,}8\left(\frac{D(G)-2}{3}\right)^5 + 162\left(\frac{D(G)-2}{3}\right)^4 + \frac{574}{3}\left(\frac{D(G)-2}{3}\right)^3 + 117\left(\frac{D(G)-2}{3}\right)^2}{\frac{1}{3}\left(\frac{D(G)-2}{3}\right)^4 + \frac{2}{3}\left(\frac{D(G)-2}{3}\right)^3 + 2\left(\frac{D(G)-2}{3}\right)^2 + \frac{4}{9}(D(G)-2) - \frac{1}{3}} + \\
&+ \frac{-\frac{523}{15}\frac{D(G)-2}{3} + 1}{\frac{1}{3}\left(\frac{D(G)-2}{3}\right)^4 + \frac{2}{3}\left(\frac{D(G)-2}{3}\right)^3 + 2\left(\frac{D(G)-2}{3}\right)^2 + \frac{4}{9}(D(G)-2) - \frac{1}{3}}
\end{aligned} \tag{44}$$

Fig. 20 shows diameter and average path lengths in theoretical and real graphs with up to 100 nodes.

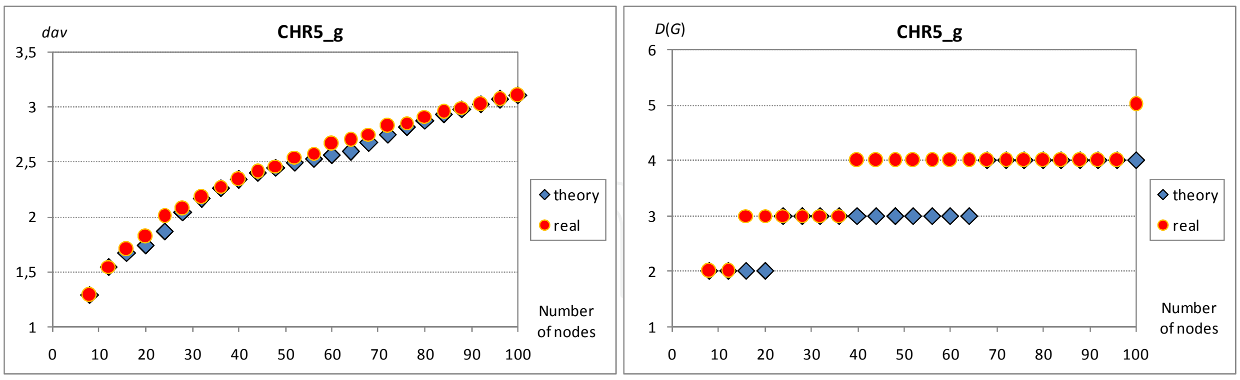

Fig. 20. Comparison of diameter and average path length of theoretical and real graphs CHR5_g

Graph CHR5_h.

Definition 12. The modified fifth degree chordal ring called CHR5_h is denoted by CHR5_h(p; q_1,q_2,q_3,q_4,q_5), where p means the number of nodes and is positive and divisible by 4; q_1, q_2, q_3, q_4 are chords which possess even lengths less then $p/2$, $q_5=p/2$. Even node i_{2k} is connected to five other nodes: i_{2k-1}, i_{2k+1}, $i_{2k+q5(\mathrm{mod}\ p)}$ and to $i_{2k+q1(\mathrm{mod}\ p)}$, $i_{2k-q1(\mathrm{mod}\ p)}$ when number of node is equal to 0 mod 4 or to $i_{2k+q2(\mathrm{mod}\ p)}$, $i_{2k-q2(\mathrm{mod}\ p)}$ when number of node is equal to 2 mod 4, while odd node i_{2k+1} is connected to i_{2k}, i_{2k+2}, $i_{2k+1+q5(\mathrm{mod}\ p)}$ and $i_{2k+1+q3(\mathrm{mod}\ p)}$, $i_{2k+1-q3(\mathrm{mod}\ p)}$ when number of node is equal to 1 mod 4 or to $i_{2k+1+q4\ (\mathrm{mod}\ p)}$, $i_{2k+1-q4\ (\mathrm{mod}\ p)}$ when number of node is equal to 3 mod 4.

An example is shown in Fig. 21, and the number of nodes in the layers of an optimal graph is shown in tables 21 and 22.

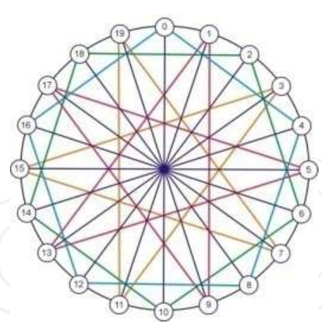

Fig. 21. Example of modified chordal ring CHR5_h(20; 4,4,8,8,10)

d	1	2	3	4	5	6	7	8
p_{do}	5	18	64	196	524	1244	2636	5068

Table 21. The number of nodes appearing in layers when p is not divided by 8

d	1	2	3	4	5	6	7	8
p_{do}	5	16	48	136	358	868	1908	3804

Table 22. The number of nodes appearing in layers when p is divided by 8

The number of nodes in the layers, as shown in Tables 21 and 22, can be described by the following expression:

$$\begin{aligned}
&\textit{When} \quad d > 5 \textit{ and } p \textit{ is not divisible by } 8 \textit{ then}\\
&p_{do} = \frac{8}{3}d^4 - \frac{16}{3}d^3 + 154\frac{2}{3}d^2 + 1133\frac{8}{3}d - 2292\\
&\textit{when} \quad d > 7 \textit{ and } p \textit{ is divisible by } 8 \textit{ then}\\
&p_{do} = \frac{8}{3}d^4 - \frac{16}{3}d^3 - 266\frac{2}{3}d^2 + 2285\frac{1}{3}d - 5604
\end{aligned} \tag{45}$$

In Table 23 the total number of nodes in optimal graphs described by expression given above is shown.

$D(G)$	1	2	3	4	5	6	7	8
p_o is not divisible by 8	6	24	88	284	808	2052	4688	9756
p_o is divisible by 8	6	22	70	206	564	1432	3340	7144

Table 23. Total numbers of nodes forming optimal graphs versus diameter

$$\begin{aligned}
&\textit{When} \quad D(G) > 4 \textit{ and } p \textit{ is not divisible by } 8 \textit{ then}\\
&p_o = \frac{8}{15}D(G)^5 - \frac{160}{3}D(G)^3 + 488d^2 + 1751\frac{1}{5}d + 2364\\
&\textit{when} \quad D(G) > 6 \textit{ and } p \textit{ is divisible by } 8 \textit{ then}\\
&p_o = \frac{8}{15}D(G)^5 - \frac{272}{3}D(G)^3 - 1008\frac{2}{3}D(G)^2 + 4505\frac{13}{15}D(G) + 7624
\end{aligned} \tag{46}$$

The average path length in optimal graphs:

$$\begin{aligned}
&\textit{When } D(G) > 4 \textit{ and } p \textit{ is not divisible by } 8\\
&d_{avo} = \frac{\frac{4}{9}D(G)^6 + \frac{4}{15}D(G)^5 - 40\frac{2}{9}D(G)^4 + 298\frac{2}{3}D(G)^3 - 618\frac{2}{9}D(G)^2 - 956\frac{14}{15}D(G)}{\frac{8}{15}D(G)^5 - \frac{160}{3}D(G)^3 + 488d^2 + 1751\frac{1}{5}d + 2363} +\\
&+\frac{3905}{\frac{8}{15}D(G)^5 - \frac{160}{3}D(G)^3 + 488d^2 + 1751\frac{1}{5}d + 2363}\\
&\textit{When } D(G) > 4 \textit{ and } p \textit{ is divisible by } 8\\
&d_{avo} = \frac{\frac{4}{9}D(G)^6 + \frac{4}{15}D(G)^5 - 68\frac{2}{3}D(G)^4 + 626\frac{2}{3}D(G)^3 + 1726\frac{2}{9}D(G)^2 - 2420\frac{14}{15}D(G)}{\frac{8}{15}D(G)^5 - \frac{272}{3}D(G)^3 - 1008\frac{2}{3}D(G)^2 + 4505\frac{13}{15}d + 7623} +\\
&+\frac{14695}{\frac{8}{15}D(G)^5 - \frac{272}{3}D(G)^3 - 1008\frac{2}{3}D(G)^2 + 4505\frac{13}{15}d + 7623}
\end{aligned} \tag{47}$$

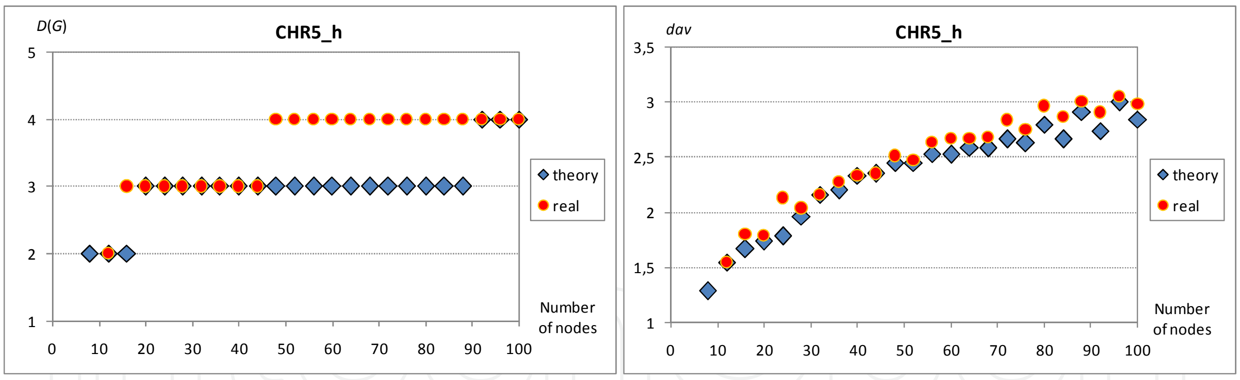

Fig. 22. Comparison of diameter and average path length of theoretical and real graphs CHR5_h

Fig. 22 shows diameter and average path lengths in theoretical and real graphs with up to 100 nodes.

There are only three chordal rings which possess basic parameters equal to parameters of theoretical graphs: These are: CHR5_h(12; 4,4,4,4,6), CHR5_h(40; 4,12,4,12,20) and CHR5_h(44; 4,8,12,16,22).

Graph CHR5_i.

Definition 13. The modified fifth degree chordal ring called CHR5_i is denoted by CHR5_i(p; q_1,q_2,q_3,q_4), where p means the number of nodes and is positive and divisible by 4; q_1, q_2, q_3, q_4 are chords which possess: q_1, odd length and q_2, q_3, q_4 - even lengths less then $p/2$. Each node is connected to five other nodes. Even node i_{2k} is connected to i_{2k-1}, i_{2k+1}, $i_{2k+q1(\mathrm{mod}\ p)}$, $i_{2k-q1(\mathrm{mod}\ p)}$ and $i_{2k+q2(\mathrm{mod}\ p)}$, while odd node $i_{(2k+1)=1(\mathrm{mod}4)}$ is connected to i_{2k}, i_{2k+2}, $i_{2k+1-q2(\mathrm{mod}\ p)}$, $i_{2k+1+q3(\mathrm{mod}\ p)}$, $i_{2k+1-q3(\mathrm{mod}\ p)}$ and odd node $i_{(2k+1)=3(\mathrm{mod}4)}$ is connected to i_{2k}, i_{2k+2}, $i_{2k+1-q2(\mathrm{mod}\ p)}$, $i_{2k+1+q4(\mathrm{mod}\ p)}$, $i_{2k+1-q4(\mathrm{mod}\ p)}$.

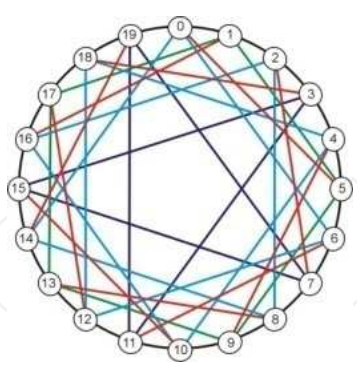

Fig. 23. Example of modified chordal ring CHR5_i(20; 5,6,4,8)

An example of a CHR5_i is shown in Fig. 23. The distribution of nodes in the layers depends on whether the graph is seen from odd or even node in Table 24 is given.

d	1	2	3	4	5	6	7	8	Node number
p_{do}	5	20	63	190	465	1010	2001	3594	even
	5	20	69	196	493	1094	2141	3790	odd

Table 24. Maximal number of nodes in the layers

The distribution of nodes in the layers can be described by the expression:

$$\begin{aligned}&When \quad d>4 \ \ and \ \ node \ \ number \ \ is \ \ even\\ &p_{do}=d^4+6d^2-192d+650\\ &when \quad d>4 \ \ and \ \ node \ \ number \ \ is \ \ odd\\ &p_{do}=d^4+6d^2-136d+398\end{aligned} \tag{48}$$

The total number of nodes in optimal graphs calculated depending on the source node number, given in Table 25, can be expressed as follows:

$$\begin{aligned}&When \quad D(G)>3 \text{ and node number is even}\\ &p_{o\ even}=\frac{D(G)^5}{5}+\frac{D(G)^4}{2}+\frac{7}{3}D(G)^3-93D(G)^2+554\frac{29}{30}D(G)-935\\ &when \quad D(G)>3 \text{ and node number is odd}\\ &p_{oodd}=\frac{D(G)^5}{5}+\frac{D(G)^4}{2}+\frac{7}{3}D(G)^3-65D(G)^2+330\frac{29}{30}D(G)-475\end{aligned} \tag{49}$$

$D(G)$	1	2	3	4	5	6	7	8	Node number
p_o	6	26	89	279	744	1754	3755	7349	even
	6	26	95	291	784	1878	4019	7809	odd

Table 25. Total numbers of nodes in the optimal graphs

The average path length in optimal graphs is equal to:

$$d_{avo}=\frac{d_{avoeven}+d_{avoodd}}{2}=$$
$$=\frac{\frac{D(G)^6}{6}+\frac{D(G)^5}{2}+1\frac{11}{12}D(G)^4-42\frac{1}{3}D(G)^3+132\frac{5}{12}D(G)^2+176\frac{1}{3}D(G)-764}{2\left(\frac{D(G)^5}{5}+\frac{D(G)^4}{2}+\frac{7}{3}D(G)^3-93D(G)^2+554\frac{29}{30}D(G)-936\right)}+$$
$$+\frac{\frac{D(G)^6}{6}+\frac{D(G)^5}{2}+1\frac{11}{12}D(G)^4-61\frac{1}{3}D(G)^3+230\frac{5}{12}D(G)^2+293D(G)-1646}{2\left(\frac{D(G)^5}{5}+\frac{D(G)^4}{2}+\frac{7}{3}D(G)^3-65D(G)^2+330\frac{29}{30}D(G)-476\right)} \tag{50}$$

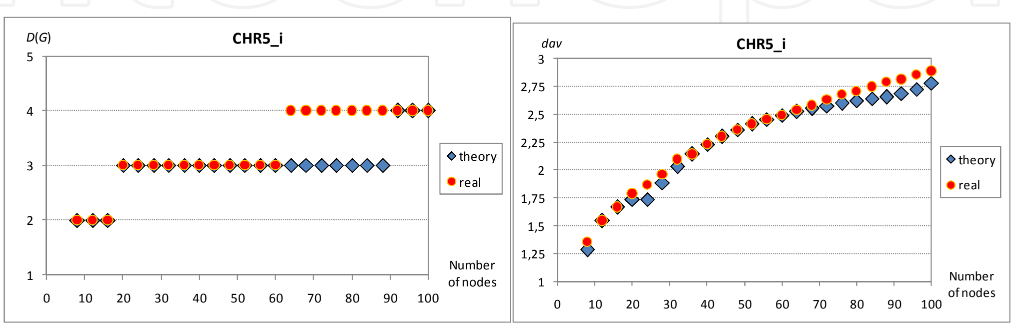

Fig. 24. Comparison of diameter and average path length of theoretical and real graphs CHR5_i

Fig. 24 shows diameter and average path lengths in theoretical and real graphs with up to 100 nodes.

For graphs with less than 72 nodes it is possible to find real graphs with parameters close to those of ideal graphs. However the difference becomes bigger for larger graphs. The differences seem to come from the different path lengths calculated from odd and even nodes. The ideal graphs found are shown in table 26.

Number of nodes	q_1	q_2	q_3	q_4	d_{av}
12	5	2	4	4	1,545
16	7	2	4	4	1,667
36	13	10	16	16	2,143
40	9	6	12	12	2,231
44	9	6	12	20	2,302
48	5	10	20	20	2,362
52	5	10	20	20	2,412

Table 26. Ideal graphs CHR5_i

Graph CHR5_j.

Definition 14. The modified fifth degree chordal ring called CHR5_j is denoted by CHR5_j(p; q_1,q_2,q_3,q_4), where p is the number of nodes. It must be positive and divisible by 4. Chords q_1 and q_2 have odd lengths; q_3 and q_4 possess even lengths, all chords lengths are less then $p/2$. Each node is connected to five other nodes. Even nodes i_{2k} are connected to i_{2k-1}, i_{2k+1} and to $i_{2k+q1(\mathrm{mod}\ p)}$, $i_{2k+q2(\mathrm{mod}\ p)}$ and $i_{2k+q3(\mathrm{mod}\ p)}$ when $2k = 0$ (mod4) or to $i_{2k-q3(\mathrm{mod}\ p)}$ when $2k = 2$ (mod4); while odd nodes $i_{(2k+1)}$ are connected to i_{2k}, i_{2k+2}, $i_{2k+1-q1(\mathrm{mod}\ p)}$, $i_{2k+1-q2(\mathrm{mod}\ p)}$ and to $i_{2k+1+q4(\mathrm{mod}\ p)}$ when $2k+1=1$ (mod4) or to $i_{2k+1-q4(\mathrm{mod}\ p)}$ when $2k+1 = 3$ (mod4).

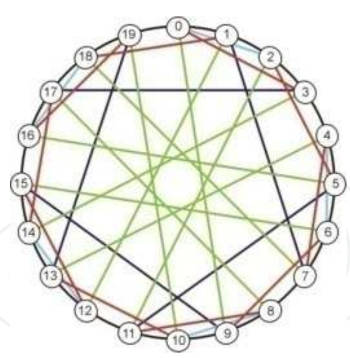

Fig. 25. Example of modified chordal ring CHR5_j(20; 3,9,2,6)

Based on going through all the real graphs, the distribution of nodes in the layers of CHR5_j is as follows (Table 27):

d	1	2	3	4	5	6	7	8
p_{do}	5	20	62	174	375	718	1303	2136

Table 27. Maximal number of nodes in the successive layers

The distribution of nodes in the layers of an optimal graph is described by expression:

$$\begin{aligned}
&\textit{When } d \rangle\, 2 \textit{ and } d = 0\ (\mathrm{mod}\, 3) \textit{ then}\\
&p_{do} = \frac{77}{162}\left(\frac{d}{3}\right)^4 + 27\frac{1}{2}\left(\frac{d}{3}\right)^2 - 4\\
&\textit{when } d \rangle\, 2 \textit{ and } d = 1\ (\mathrm{mod}\, 3) \textit{ then}\\
&p_{do} = \frac{77}{162}\left(\frac{d-1}{3}\right)^4 + 52\frac{1}{3}\left(\frac{d-1}{3}\right)^3 + 53\frac{1}{2}\left(\frac{d-1}{3}\right)^2 + 24\frac{2}{3}\left(\frac{d-1}{3}\right) + 5\\
&\textit{when } d \rangle\, 2 \textit{ and } d = 2\ (\mathrm{mod}\, 3) \textit{ then}\\
&p_{do} = \frac{77}{162}\left(\frac{d-2}{3}\right)^4 + 101\frac{2}{3}\left(\frac{d-2}{3}\right)^3 + 129\frac{1}{2}\left(\frac{d-2}{3}\right)^2 + 83\frac{1}{3}\left(\frac{d-2}{3}\right)^2 + 22
\end{aligned} \tag{51}$$

In Table 28 the total number of nodes in virtual, optimal graphs is shown as function of diameter.

$d(G)$	1	2	3	4	5	6	7	8
p_{do}	6	26	88	262	637	1355	2658	4794

Table 28. Total numbers of nodes in optimal graphs versus diameter

The total number of nodes in optimal graphs calculated as a function of its diameter can be expressed as follows:

$$\begin{aligned}
&\textit{When } D(G) \rangle\, 2 \textit{ and } D(G) = 0\ (\mathrm{mod}\, 3) \textit{ then}\\
&p_o = 23\frac{1}{10}\left(\frac{D(G)}{3}\right)^5 + 19\frac{1}{4}\left(\frac{D(G)}{3}\right)^4 + 31\frac{2}{3}\left(\frac{D(G)}{3}\right)^3 +\\
&+ 12\frac{3}{4}\left(\frac{D(G)}{3}\right)^2 + 2\frac{7}{30}\left(\frac{D(G)}{3}\right) - 1\\
&\textit{when } D(G) \rangle\, 2 \textit{ and } D(G) = 1\ (\mathrm{mod}\, 3) \textit{ then}\\
&p_o = 23\frac{1}{10}\left(\frac{D(G)-1}{3}\right)^5 + 57\frac{3}{4}\left(\frac{D(G)-1}{3}\right)^4 + 84\left(\frac{D(G)-1}{3}\right)^3 +\\
&+ 66\frac{1}{4}\left(\frac{D(G)-1}{3}\right)^2 + 26\frac{8}{9}\left(\frac{D(G)-1}{3}\right) + 4\\
&\textit{when } D(G) \rangle\, 2 \textit{ and } D(G) = 2\ (\mathrm{mod}\, 3) \textit{ then}\\
&p_o = 23\frac{1}{10}\left(\frac{D(G)-2}{3}\right)^5 + 96\frac{1}{4}\left(\frac{D(G)-2}{3}\right)^4 + 185\frac{2}{3}\left(\frac{D(G)-2}{3}\right)^3 +\\
&+ 195\frac{3}{4}\left(\frac{D(G)-2}{3}\right)^2 + 110\frac{7}{30}\left(\frac{D(G)-2}{3}\right) + 26
\end{aligned} \tag{52}$$

The average path length in optimal graphs is described by (53):

$$\begin{aligned}
&When\ \ D(G)\rangle\, 2\ and\ D(G)=0\ (\mathrm{mod}\,3)\ then\\
&\quad d_{avo}=57\frac{3}{4}\left(\frac{D(G)}{3}\right)^6+57\frac{3}{4}\left(\frac{D(G)}{3}\right)^5+77\frac{5}{12}\left(\frac{D(G)}{3}\right)^4+\\
&\quad +39\frac{7}{12}\left(\frac{D(G)}{3}\right)^3+4\frac{251}{300}\left(\frac{D(G)}{3}\right)^2-2\frac{101}{300}\left(\frac{D(G)}{3}\right)-4\\
&when\ \ D(G)\rangle\, 2\ and\ D(G)=1\ (\mathrm{mod}\,3)\ then\\
&\quad d_{avo}=57\frac{3}{4}\left(\frac{D(G)-1}{3}\right)^6-173\frac{1}{4}\left(\frac{D(G)-1}{3}\right)^5+272\frac{11}{12}\left(\frac{D(G)-1}{3}\right)^4+\\
&\quad -261\frac{3}{4}\left(\frac{D(G)-1}{3}\right)^3+146\frac{1}{3}\left(\frac{D(G)-1}{3}\right)^2-42\left(\frac{D(G)-1}{3}\right)+1\\
&when\ \ D(G)\rangle\, 2\ and\ D(G)=2\ (\mathrm{mod}\,3)\ then\\
&\quad d_{avo}=57\frac{3}{4}\left(\frac{D(G)}{3}\right)^6-57\frac{3}{4}\left(\frac{D(G)}{3}\right)^5+77\frac{5}{12}\left(\frac{D(G)}{3}\right)^4+\\
&\quad -42\frac{11}{12}\left(\frac{D(G)}{3}\right)^3+16\frac{5}{6}\left(\frac{D(G)}{3}\right)^2-2\frac{1}{3}\left(\frac{D(G)}{3}\right)-4
\end{aligned}\tag{53}$$

Fig. 27 shows diameter and average path lengths in theoretical and real graphs with up to 100 nodes.

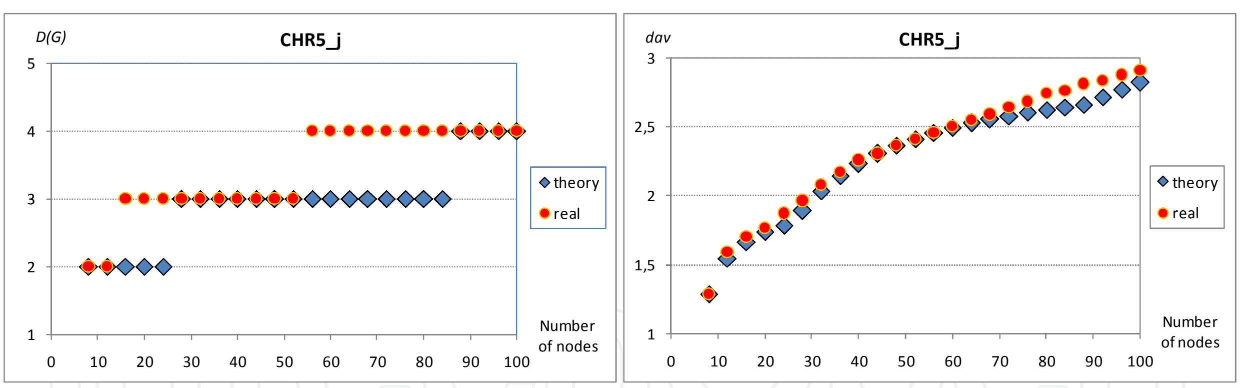

Fig. 26. Comparison of diameter and average path length of theoretical and real graphs CHR5_j

There are a few graphs having values of basic parameters equal to those of ideal graphs. For example: CHR5_j(44; 5,17,14,22), CHR5_j(48; 5,17,14,22), CHR5_j(52; 5,15,18,26), CHR5_j(56; 7,19,10,22).

Among all graphs belonging to the second group the best parameters (minimal diameter and minimal average path length given the number of nodes) were found in CHR5_i but other in minimal degree are slightly different it (especially from CHR5_j). In Fig. 27 the comparison of the second group of graphs is shown.

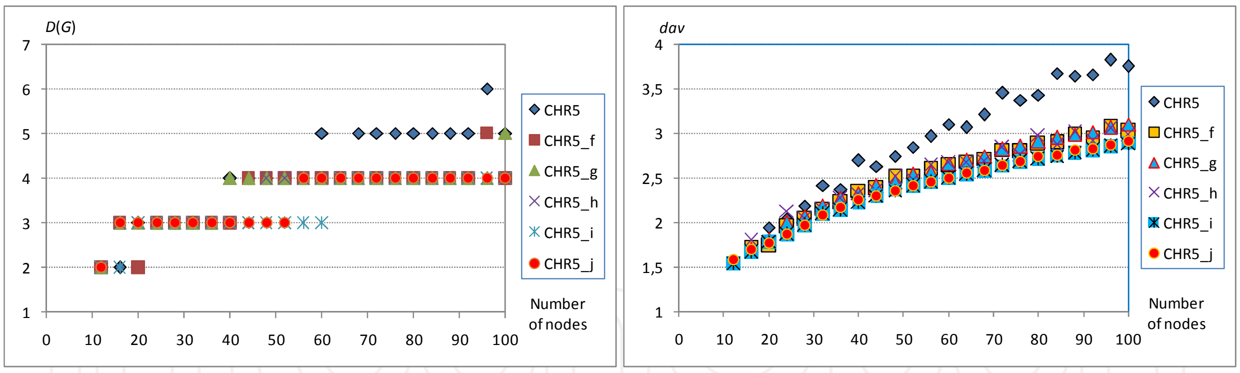

Fig. 27. Comparison of basic parameters of the chordal rings belonging to the second group of graphs

3.3 Analysed graphs – Third group

There are a number of other topologies, for which we have not found any nice expressions for the distribution of nodes in layers, and thus no expressions of the average distance and diameter could be derived. Due to the good basic parameters the topologies have been described, but further research is needed in order to provide more precise descriptions.

Graph CHR5_k.

Definition 15. The modified fifth degree chordal ring called CHR5_k is denoted by CHR5_k(p; q_1,q_2,q_3,q_4,q_5,q_6), where p is the number of nodes. It must be positive and divisible by 4. All chords have even lengths less than p/2. Each node is connected to five other nodes. Even nodes i_{2k} are connected to i_{2k-1}, i_{2k+1} and to $i_{2k+q1(\text{mod } p)}$, $i_{2k-q1(\text{mod } p)}$, $i_{2k+q5(\text{mod } p)}$ when 2k = 0 (mod4) or to $i_{2k+q2(\text{mod } p)}$, $i_{2k-q2(\text{mod } p)}$, $i_{2k-q5(\text{mod } p)}$ when 2k = 2 (mod4); while odd nodes $i_{(2k+1)}$ are connected to i_{2k}, i_{2k+2} and to $i_{2k+1+q3(\text{mod } p)}$, $i_{2k+1-q3(\text{mod } p)}$, $i_{2k+1+q6(\text{mod } p)}$ when 2k+1 = 1 (mod4) or to $i_{2k+1+q4(\text{mod } p)}$, $i_{2k+1-q4(\text{mod } p)}$, $i_{2k+1-q6(\text{mod } p)}$ when 2k+1 = 3 (mod4).

An example is shown in Fig. 28.

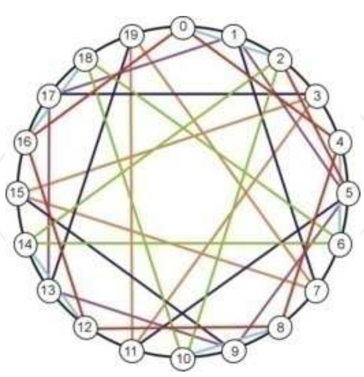

Fig. 28. Example of modified chordal ring CHR5_k(20; 4,8,4,8,2,6)

In Table 29 the distribution of nodes in layers is shown, based on observations of all graphs.

d	1	2	3	4	5	6	7	8
p_{do}	5	20	80	284	895	2520	6333	14334

Table 29. Maximal number of nodes in the successive layers

Using the results shown in Table 29, the counted total number of nodes in virtual optimal graphs is presented in table 30.

$D(G)$	1	2	3	4	5	6	7	8
p_o	6	26	106	390	1285	3805	10138	24472

Table 30. Total numbers of nodes forming optimal graphs versus diameter

In Fig. 29 comparison of diameter and average path length of theoretical and real graphs CHR5_k is shown.

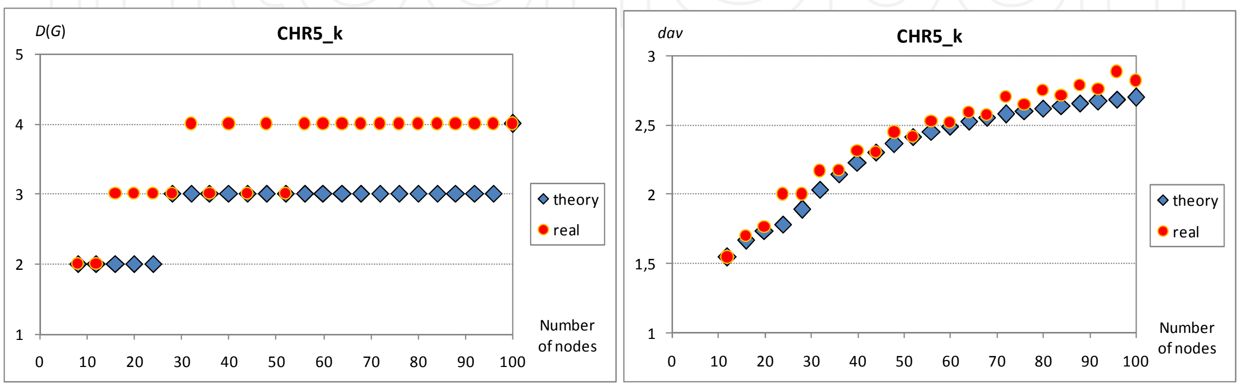

Fig. 29. Comparison of diameter and average path length of theoretical and real graphs CHR5_k

Only three graphs with average distance and diameter equal to those of ideal graphs can be found. These are CHR5_k(12; 4,4,4,4,2,6), CHR5_k(44; 4,8,12,16,18,22), CHR5_k(52; 4,8,12,16,22,26).

Graph CHR5_l.

Definition 16. The modified fifth degree chordal ring called CHR5_l is denoted by CHR5_l(p; q_1,q_2,q_3,q_4,q_5), where p means the number of nodes and is positive and divisible by 4. Chord q_1 has odd length, other chords have even length. The lengths of all chords are less than $p/2$. Each node is connected to five other nodes. Even nodes i_{2k} are connected to i_{2k-1}, i_{2k+1} and to $i_{2k+q1(\mathrm{mod}\ p)}$ and to $i_{2k+q2(\mathrm{mod}\ p)}$, $i_{2k+q3(\mathrm{mod}\ p)}$ when $2k$ = 0 (mod4) or to $i_{2k-q2(\mathrm{mod}\ p)}$, $i_{2k-q3(\mathrm{mod}\ p)}$ when $2k$ = 2 (mod4); while odd nodes $i_{(2k+1)}$ are connected to i_{2k}, i_{2k+2}, $i_{2k+1-q1(\mathrm{mod}\ p)}$, and to $i_{2k+1+q4(\mathrm{mod}\ p)}$, $i_{2k+1+q5(\mathrm{mod}\ p)}$ when $2k+1$ = 1 (mod4) or to $i_{2k+1-q4(\mathrm{mod}\ p)}$, $i_{2k+1-q5(\mathrm{mod}\ p)}$ when $2k+1$ = 3 (mod4).

An example is shown in Fig. 30.

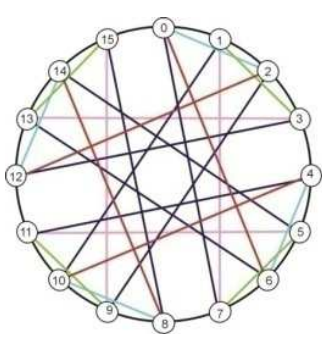

Fig. 30.Example of modified chordal ring CHR5_l(16; 7,6,2,2,6)

In Table 31 the distribution of nodes in the layers is shown.

d	1	2	3	4	5	6	7	8
p_{do}	5	20	71	228	555	1216	2442	4458

Table 31. Maximal number of nodes in the successive layers

Using the results shown in Table 31, the total number of nodes in virtual optimal graphs calculated in this way was presented in Table 32.

$D(G)$	1	2	3	4	5	6	7	8
p_o	6	26	97	325	880	2096	4538	8996

Table 32. Total numbers of nodes forming optimal graphs versus diameter

In Fig. 31 comparison of diameter and average path length of theoretical and real graphs CHR5_l is shown.

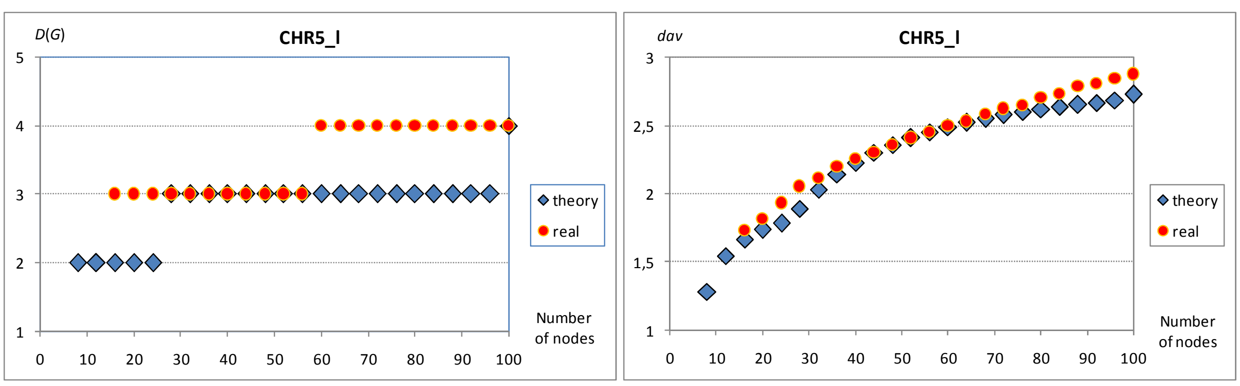

Fig. 31. Comparison of diameter and average path length of theoretical and real graphs CHR5_l

Examples of ideal graphs include: CHR5_l(16; 3,2,2,6,6), CHR5_l(44; 11,6,14,22,18), CHR5_l(48; 7,10,18,14,22), CHR5_l(52; 11,6,18,26,22), CHR5_l(56; 11,14,6,22,26).

Graph CHR5_m.

Definition 17. The modified fifth degree chordal ring called CHR5_m is denoted by CHR5_m(p; q_1,q_2,q_3,q_4), where p means the number of nodes and is positive and divisible by 4. All chords have even length less than $p/2$. Each node is connected to five other nodes. Even nodes i_{2k} are connected to i_{2k-1}, i_{2k+1}, $i_{2k+q1(\mathrm{mod}\ p)}$, $i_{2k-q1(\mathrm{mod}\ p)}$ and to $i_{2k+q2(\mathrm{mod}\ p)}$ when $2k = 0$ (mod4) or to $i_{2k-q2(\mathrm{mod}\ p)}$ when $2k = 2$ (mod4); while odd nodes $i_{(2k+1)}$ are connected to i_{2k}, i_{2k+2}, $i_{2k+1+q3(\mathrm{mod}\ p)}$, $i_{2k+1-q3(\mathrm{mod}\ p)}$, and to $i_{2k+1+q4(\mathrm{mod}\ p)}$ when $2k+1 = 1$ (mod4) or to $i_{2k+1-q4(\mathrm{mod}\ p)}$ when $2k+1 = 3$ (mod4).

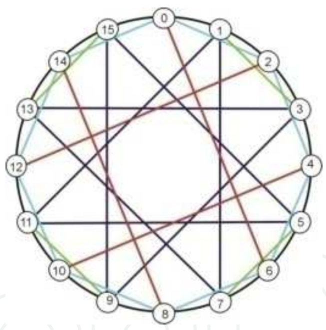

Fig. 32.Example of modified chordal ring CHR5_m(16; 2,6,6,2)

An example of CHR5_m is given in Fig. 32.

In Table 33 the distribution of nodes in layers is shown. The total number of nodes in optimal graphs, calculated based on these results, are shown in Table 34.

d	1	2	3	4	5	6	7	8
p_{do}	5	20	71	210	511	1064	1997	3440

Table 33. Maximal number of nodes in the layers

$D(G)$	1	2	3	4	5	6	7	8
p_o	6	26	97	307	818	1882	3879	7319

Table 34. Total numbers of nodes forming optimal graphs versus diameter

In Fig. 34 comparison of diameter and average path length of theoretical and real graphs CHR5_m is shown.

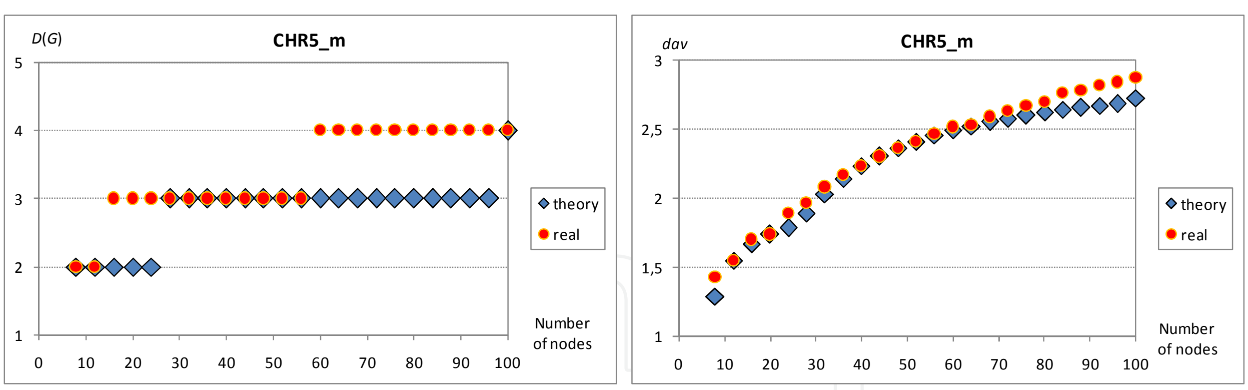

Fig. 33.Comparison of diameter and average path length of theoretical and real graphs CHR5_m

Examples of ideal chordal rings with up to 100 nodes include: CHR5_m(12; 2,2,6,6), CHR5_m(40; 6,14,10,18), CHR5_m(44; 6,14,10,18), CHR5_m(48; 10,22,6,14), CHR5_m(56; 18,26,14,6), CHR5_m(40; 6,14,10,18).

Graph CHR5_n.

Definition 18. The modified fifth degree chordal ring called CHR5_n is denoted by CHR5_n(p; q_1,q_2,q_3,q_4,q_5,q_6), where p means the number of nodes and is positive and divisible by 4. All chords have even length less than $p/2$. Each node is connected to five other nodes. Even nodes i_{2k} are connected to i_{2k-1}, i_{2k+1} and to $i_{2k+q1(\text{mod } p)}$, $i_{2k+q2(\text{mod } p)}$, $i_{2k+q3(\text{mod } p)}$ when $2k = 0$ (mod4) or to $i_{2k-q1(\text{mod } p)}$, $i_{2k-q2(\text{mod } p)}$, $i_{2k-q3(\text{mod } p)}$ when $2k = 2$ (mod4); while odd nodes $i_{(2k+1)}$ are connected to i_{2k}, i_{2k+2} and to $i_{2k+1+q4(\text{mod } p)}$, $i_{2k+1+q5(\text{mod } p)}$, $i_{2k+1+q6(\text{mod } p)}$ when $2k+1 = 1$ (mod4) or to $i_{2k+1-q4(\text{mod } p)}$, $i_{2k+1-q5(\text{mod } p)}$, $i_{2k+1-q6(\text{mod } p)}$ when $2k+1 = 3$ (mod4).

An example of a CHR5_n graph is shown in Fig. 34.

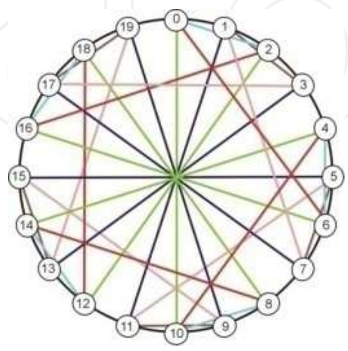

Fig. 34. Example of modified chordal ring CHR5_n(20; 2,6,10,2,6,10)

In Table 35 the distribution of nodes in layers is presented.

d	1	2	3	4	5	6	7	8
p_{do}	5	20	77	272	764	1916	4268	8696

Table 35. Maximal number of nodes in the layers

The total number of nodes in optimal graphs calculated based on the results given in Table 35 is shown in Table 36.

$D(G)$	1	2	3	4	5	6	7	8
p_o	6	26	103	375	1139	3055	7323	16019

Table 36. Total numbers of nodes in optimal graphs as a function of the diameter

In Fig. 36 shows the comparison of diameter and average path length of theoretical and real graphs CHR5_n.

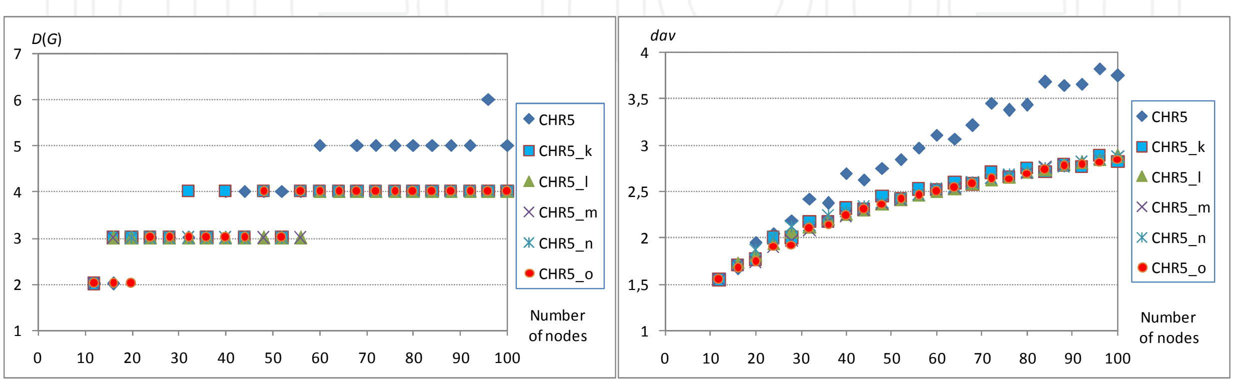

Fig. 35. Comparison of diameter and average path length of theoretical and real graphs CHR5_n

Going through all nodes with up to 100 nodes only one ideal graph was found, namely CHR5_n(52; 10,6,14,18,26,22) with 52 nodes.

Graph CHR5_o.

Definition 19. The modified fifth degree chordal ring called CHR5_o is denoted by CHR5_o(p; q_1,q_2,q_3,q_4,q_5) where p means the number of nodes and is positive and divisible by 4. All chords have even length less than (p/2+1). Each node is connected to five other nodes. Even nodes i_{2k} are connected to i_{2k-1}, i_{2k+1} and to $i_{2k+q1(\text{mod } p)}$, $i_{2k-q1(\text{mod } p)}$, $i_{2k+q2(\text{mod } p)}$ when $2k = 0$ (mod4) or to $i_{2k-q2(\text{mod } p)}$ when $2k = 2$ (mod4); while odd nodes $i_{(2k+1)}$ are connected to i_{2k}, i_{2k+2}, $i_{2k+1+q5(\text{mod } p)}$ and to $i_{2k+1+q3(\text{mod } p)}$, $i_{2k+1-q3(\text{mod } p)}$ when $2k+1 = 1$ (mod4) or to $i_{2k+1+q4(\text{mod } p)}$, $i_{2k+1-q4(\text{mod } p)}$ when $2k+1 = 3$ (mod4).

An example of a CHR5_o graph is shown in Fig. 36.

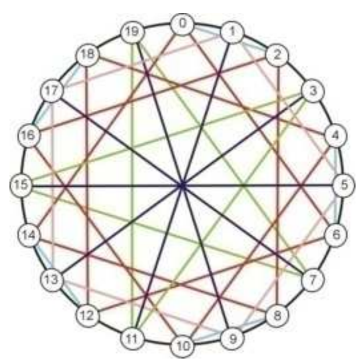

Fig. 36. Example of modified chordal ring CHR5_o(20; 6,2,4,8,10)

In Table 37 the distribution of nodes in layers is shown. Based on these numbers Table 38 is derived, showing the numbers of nodes in optimal graphs as a function of the diameter.

d	1	2	3	4	5	6	7	8	Node number
p_{do}	5	20	73	244	699	1726	3779	7498	even
	5	20	78	254	719	1778	3893	7696	odd

Table 37. Maximal number of nodes in the layers

$d(G)$	1	2	3	4	5	6	7	8	Node number
p_{do}	6	26	99	343	1042	2768	6547	14045	even
	6	26	104	358	1077	2855	6748	14444	odd

Table 38. Total numbers of nodes forming optimal graphs versus diameter

A number of ideal graphs can be found, such as CHR5_o(12; 6,2,2,4,4), CHR5_o(16; 2,6,6,4,4), CHR5_o(20; 10,10,6,4,4), CHR5_o(36; 10,18,14,4,8), CHR5_o(44; 6,14,10,20,4), CHR5_o(52; 6,10,14,20,24).

Fig. 37 shows the comparison of diameter and average path length of theoretical and real CHR5_o graphs.

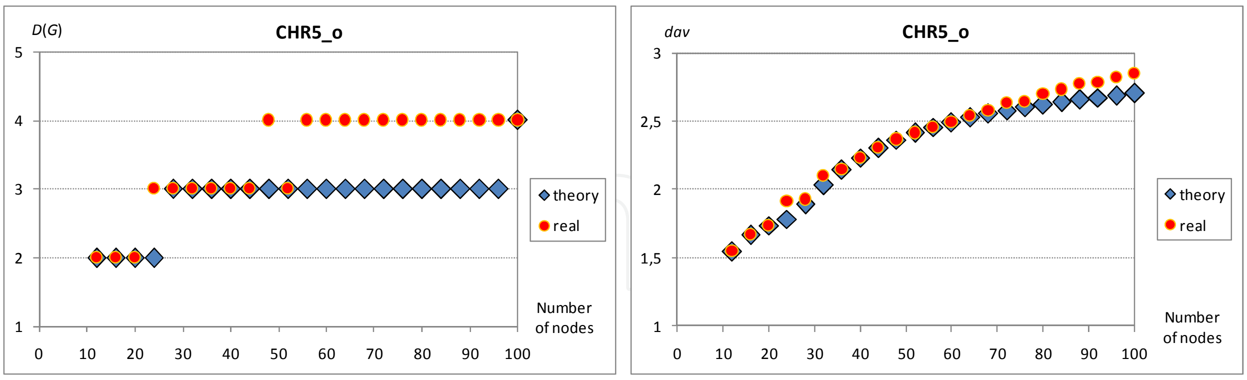

Fig. 37.Comparison of diameter and average path length of theoretical and real graphs CHR5_o

As for the previous group of graphs, the chordal rings of this group are compare with respect to the basic parameters. The comparison is given in Fig. 38.

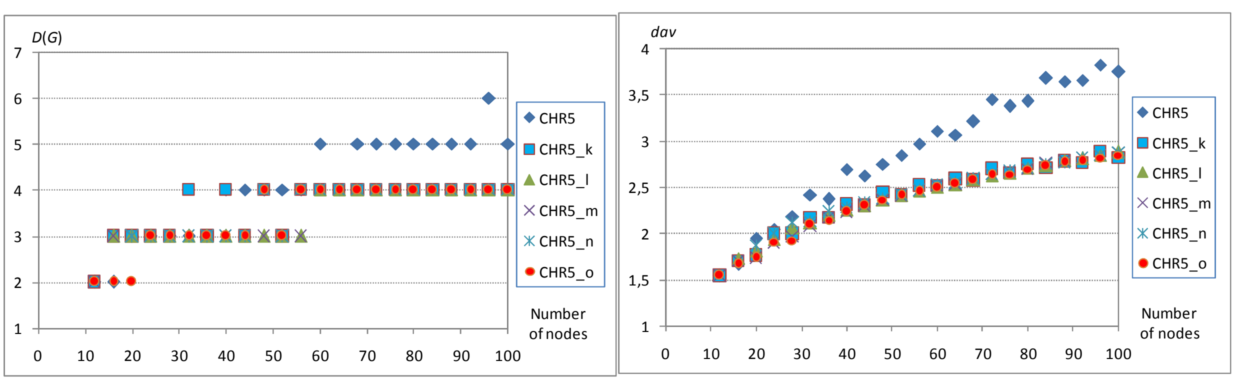

Fig. 38.Comparison of basic parameters of analyzing group chordal rings

From Fig. 38 it follows that all graphs belonging to third group of chordal rings have very similar properties when the number of nodes is smaller than 100.

4. Analysis of obtained results

Based on the obtained results for all 15 groups of graphs presented, the values of minimum diameter and average paths lengths can be compared. Despite the differences found between theoretical and real parameters the comparisons will be based on the theoretical values.

First, Fig. 39 presents the average path lengths as a function of the diameter in the graphs. This does not take into account the number of nodes in the graphs, which vary significantly between the different graphs as can be seen in Table 39 and Fig. 40. It can be seen that for a given number of nodes, CHR5_k has the smallest diameter.

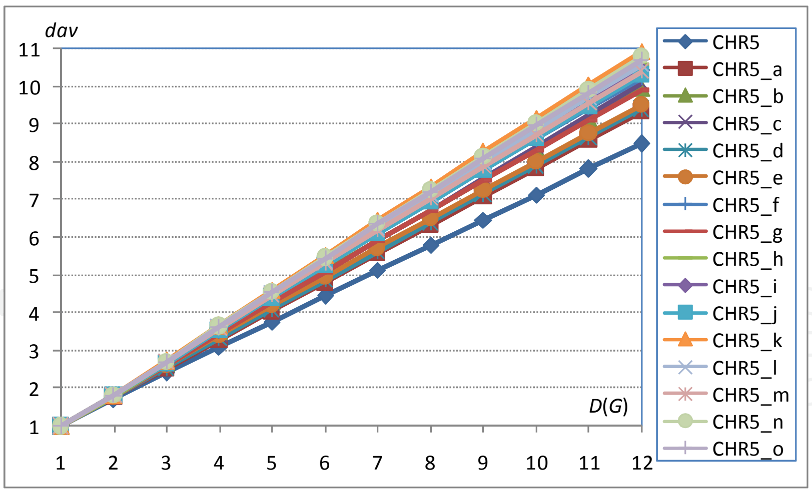

Fig. 39. Comparison of the calculated average paths length in the function of the diameter of graphs

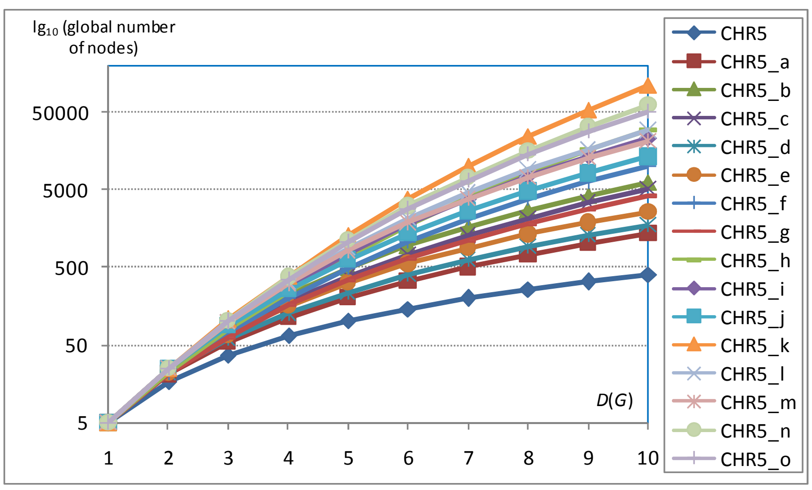

Fig. 40. Relationship between the total number of nodes and diameters

In the following section, a comparison of the average path length as a function of the number of nodes is presented. In order to compare graphs with different numbers of nodes, a "Normalized estimator of average path length" (E_{nav}) is introduced as follows:

$$E_{nav} = \frac{d_{avD(G)} \sum_{d=1}^{d(G)} p_{dr}}{d_{avrD(G)} \sum_{d=1}^{d(G)} p_d} \qquad (54)$$

Where $d_{avD(G)}$ means the average path length of a particular graph when its diameter is equal to $D(G)$, and $d_{avrD(G)}$ means the average path length of the reference graph, $\sum_{d=1}^{d(G)} p_{dr}$ is the total number of nodes in relation graph, $\sum_{d=1}^{d(G)} p_d$ - total number of nodes in the particular graph.

	D(G)									
Graph	1	2	3	4	5	6	7	8	9	10
CHR5	6	18	38	66	102	146	198	258	326	402
CHR5_a	6	22	55	113	202	330	503	729	1014	1366
CHR5_b	6	26	87	227	494	948	1661	2717	4212	6254
CHR5_c	6	26	76	186	386	726	1276	2126	3386	5186
CHR5_d	6	23	61	130	239	400	622	915	1288	1753
CHR5_e	6	23	68	160	316	552	884	1328	1900	2616
CHR5_f	6	23	72	201	491	1068	2103	3804	6411	10196
CHR5_g	6	22	64	166	349	651	1142	1846	2840	4228
CHR5_h	6	23	79	245	686	1742	4014	8450	16430	29842
CHR5_i	6	26	92	285	764	1816	3887	7579	13674	23158
CHR5_j	6	26	88	262	637	1355	2658	4794	8148	13240
CHR5_k	6	26	106	390	1285	3805	10138	24472	54108	110878
CHR5_l	6	26	97	325	880	2096	4538	8996	16706	29420
CHR5_m	6	26	97	307	818	1882	3879	7319	12876	21406
CHR5_n	6	26	103	375	1139	3055	7323	16019	32520	62092
CHR5_o	6	26	102	351	1060	2812	6648	14245	28120	51864

Table 39.Total number of nodes versus diameters.

The results are shown in Table 40 and Fig. 41.

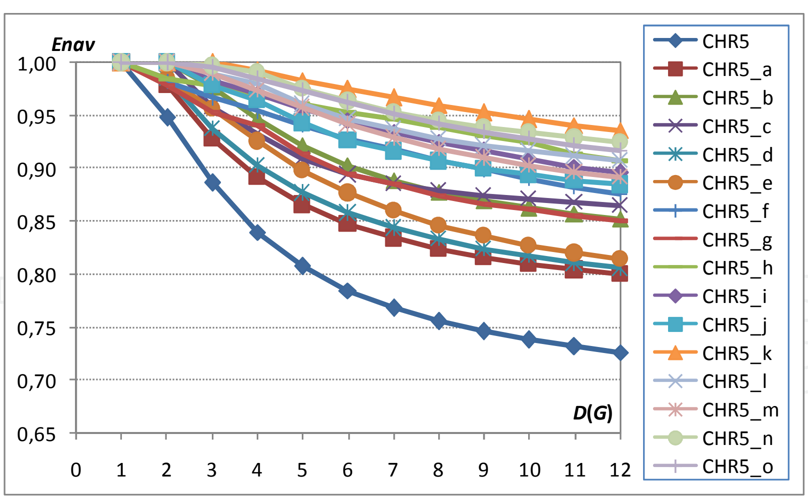

Fig. 41. Distribution of the normalized estimator of average path length versus graph diameter

From Fig. 41 it can be seen that he CHR5_k has the relatively shortest average path length.

Not surprisingly, the distributions of the number of nodes in the layers have a great impact on the two basic parameters. This can be seen also from expressions (7) and (9). The difference in the distribution for all graphs is shown in Fig. 42.

Graph	$D(G)$											
	1	2	3	4	5	6	7	8	9	10	11	12
CHR5	1,000	0,948	0,886	0,840	0,807	0,785	0,768	0,756	0,746	0,738	0,732	0,726
CHR5_a	1,000	0,979	0,928	0,892	0,866	0,848	0,834	0,824	0,816	0,810	0,804	0,800
CHR5_b	1,000	1,000	0,977	0,947	0,922	0,903	0,889	0,878	0,870	0,863	0,857	0,853
CHR5_c	1,000	1,000	0,958	0,932	0,909	0,894	0,885	0,879	0,875	0,871	0,868	0,865
CHR5_d	1,000	0,985	0,938	0,903	0,877	0,858	0,844	0,833	0,824	0,818	0,811	0,806
CHR5_e	1,000	0,985	0,957	0,926	0,898	0,877	0,860	0,847	0,836	0,828	0,820	0,814
CHR5_f	1,000	0,982	0,967	0,954	0,941	0,928	0,918	0,908	0,899	0,891	0,883	0,876
CHR5_g	1,000	0,979	0,953	0,940	0,914	0,895	0,885	0,874	0,866	0,861	0,855	0,851
CHR5_h	1,000	0,985	0,978	0,969	0,961	0,954	0,946	0,939	0,931	0,925	0,914	0,908
CHR5_i	1,000	1,000	0,984	0,971	0,957	0,945	0,935	0,925	0,916	0,909	0,901	0,896
CHR5_j	1,000	1,000	0,978	0,964	0,943	0,926	0,916	0,907	0,899	0,894	0,889	0,885
CHR5_k	1,000	1,000	1,000	0,992	0,983	0,975	0,967	0,959	0,953	0,946	0,941	0,936
CHR5_l	1,000	1,000	0,990	0,981	0,961	0,947	0,937	0,928	0,921	0,916	0,911	0,907
CHR5_m	1,000	1,000	0,990	0,974	0,957	0,941	0,929	0,919	0,910	0,903	0,897	0,892
CHR5_n	1,000	1,000	0,997	0,990	0,975	0,964	0,954	0,946	0,939	0,934	0,929	0,925
CHR5_o	1,000	1,000	0,995	0,985	0,973	0,962	0,952	0,943	0,935	0,928	0,922	0,917
	E_{nav}											

Table 40. Distribution of the normalized estimator of the average path length versus the graph diameter

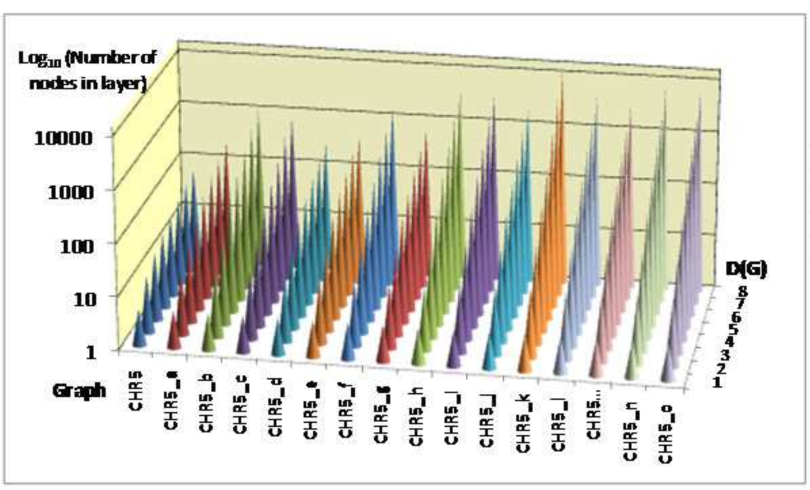

Fig. 42. Distribution of nodes numbers in the layers for all graphs

In the charts given below the comparison of the maximum number of nodes appearing in the layers of all analyzed graphs is presented.

On the basis of Fig. 42 and Fig. 43 it seems to be sufficient to analyze the distributions of numbers occurring in the first few layers to select a graph having the best basic parameters. This obviously reduces the time and effort for comparisons. The results again confirm that CHR5_k seems to be superior in terms of these basic parameters.

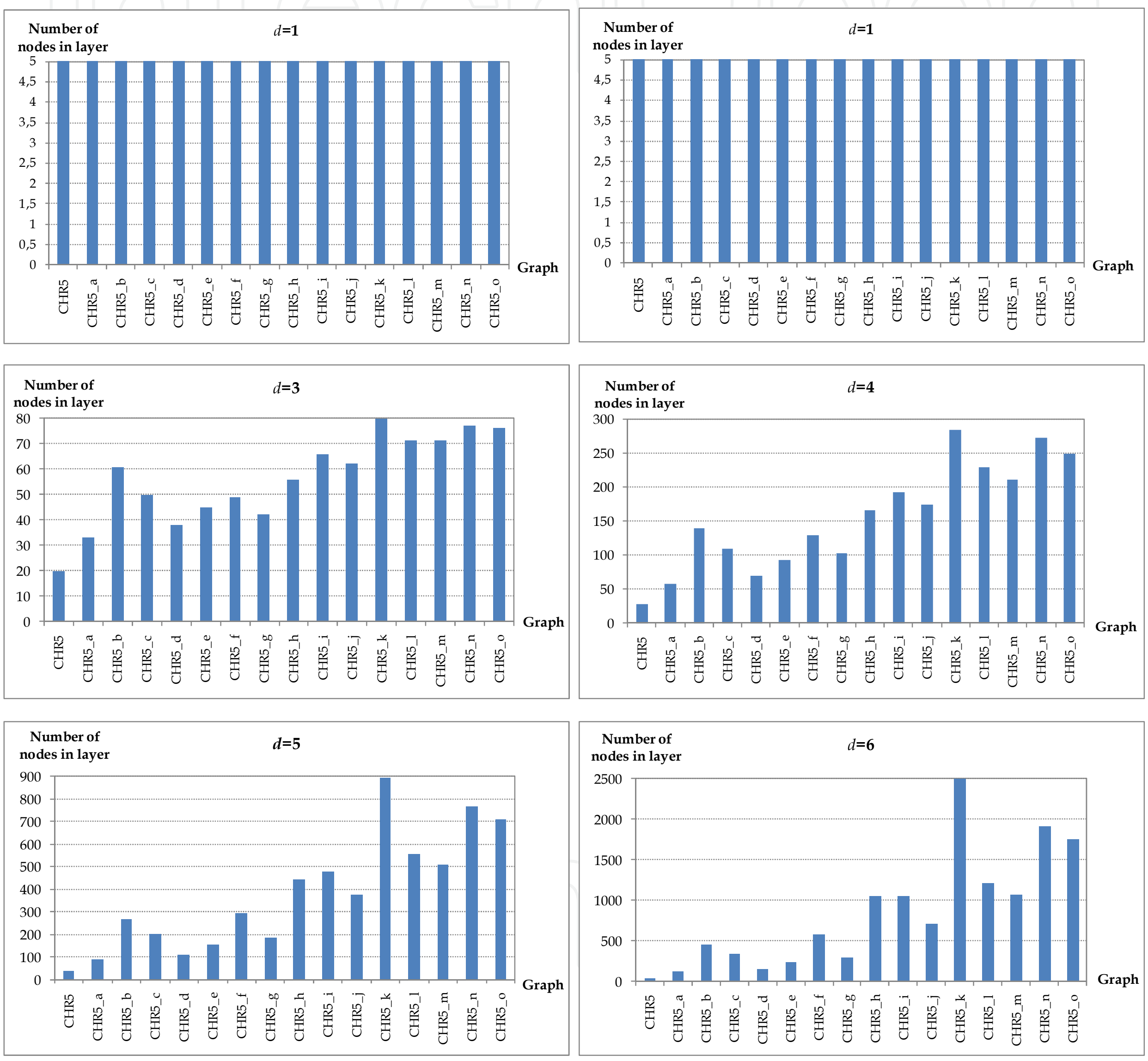

Fig. 43. Distribution of the number nodes in the first six layers

In order to make an objective assessment of the CHR5_k parameters, they were compared to the parameters of the Reference Graph as previously described. Table 41 and Fig. 44 show the distribution of nodes in different layers of these two graphs, as a function of their diameters.

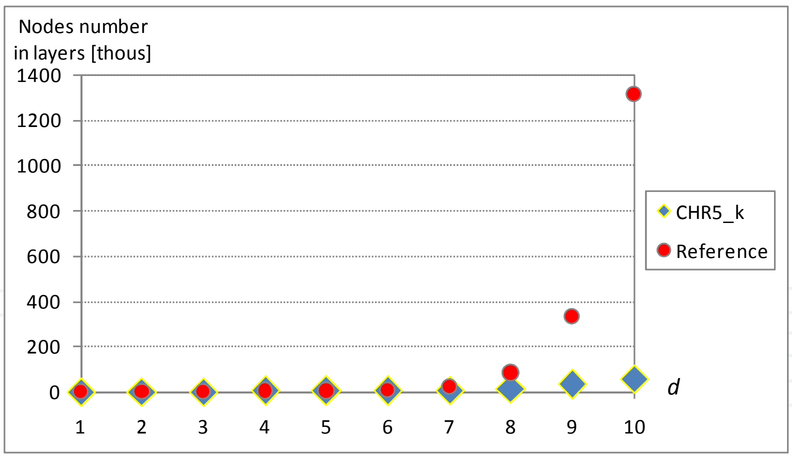

Fig. 44. Comparison of nodes number in successive layers

In Table 41 the total number of nodes in CHR5_k and reference graphs is compared for different diameters.

$D(G)$	1	2	3	4	5	6	7	8	9	10
CHR5_k	5	20	80	284	895	2520	6333	14334	29636	56770
Reference Graph	5	20	80	320	1280	5120	20480	81920	327680	1310720
	Total number of nodes									

Table 41. Total number of nodes in Reference Graph and CHR5_k

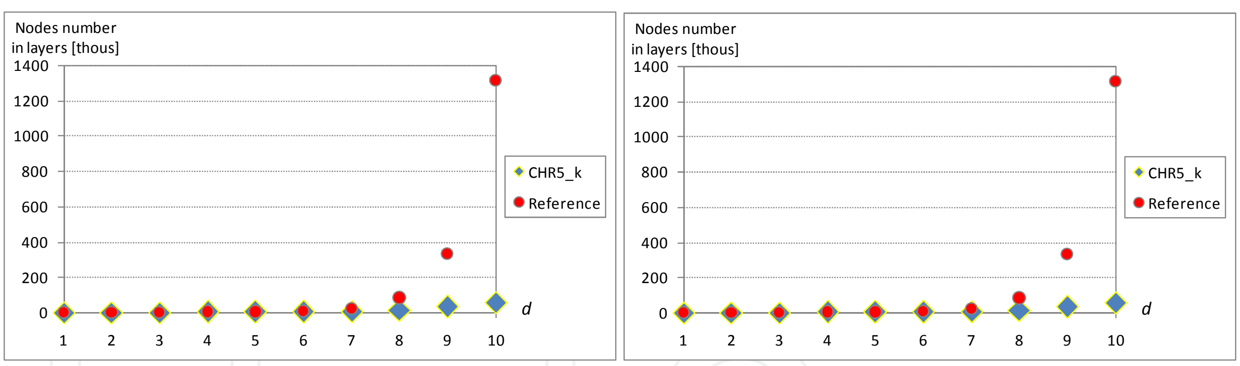

Fig. 45. Comparison the number of nodes in layers and total number of nodes versus layer number

In Fig. 45 a comparison of node numbers in successive layers and total number of nodes as a function of number layer in ideal CHR5_k and Reference Graph is shown.

Table 42 and Fig. 46 show a comparison of the average length as a function of the both graphs diameter, taking into account the total number of nodes corresponding to this diameter. As in the previous comparisons E_{nav} (54) is used.

$D(G)$	1	2	3	4	5	6	7	8	9	10
CHR5_k	1,0000	1,0000	1,0000	0,9920	0,9829	0,9747	0,9669	0,9595	0,9527	0,9465
	E_{nav}									

Table 42. The average path length as a function of the diameter

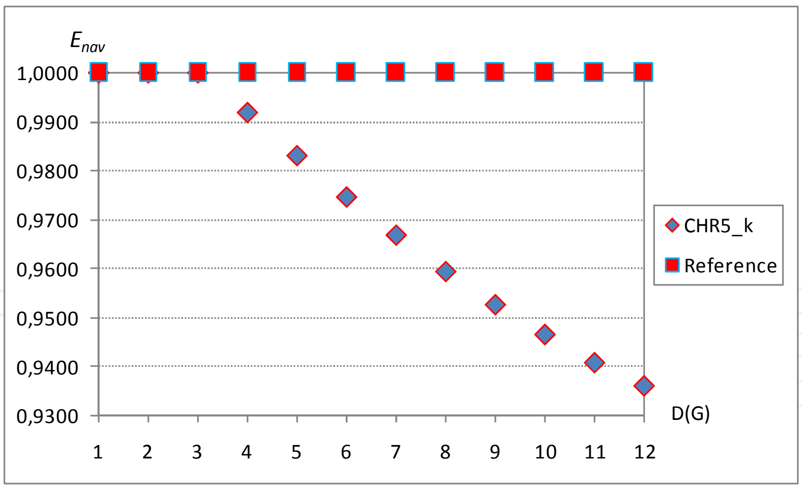

Fig. 46. Comparison of average path length as a function of the graphs diameter

5. Conclusions

In this publication the analysis of a different construction of 5th degree chordal rings was presented. The authors' main aim was to find structures which have the minimal diameter and minimal average of path in respect to number of nodes which create these graphs.

Presented considerations in the paper have rather theoretical nature, without a strict reference to practical applications. It is difficult to imagine a real regular WAN communication network that consists of thousands of nodes. However the interconnection structures connecting thousands of microprocessors, or sensors require the construction of such networks as the regular ones. The main objective function for regular interconnection structures is to minimize the network diameter or average path length. So, this is the main reason why such the structures were analyzed and studied in our paper.

In this regard the program was worked out which allows to calculate the analyzed parameters. It allowed describing virtual reference graphs namely optimal and ideal graphs. In this way we also found chordal rings which possess the smallest difference regards to average path length and diameter, which Reference Graphs have. They examined many types of structures and concluded that parameters of the real graphs are slightly different of the theoretically calculated graph parameter values. The obtained results became the basis for preparing the general formulas for determining these parameters without the need of simulation. As a side-result of the paper, we have shown that these reference graphs can be used for obtaining fairly good estimation of distance parameters in a simple manner. Additionally, they concluded that it is enough to inspect the maximal number of nodes which can appear in first few layers in aim to choose the best topology.

This publication presents the results of analysis of the modified chordal rings fifth degree. This analysis was carried out for 15 graphs divided into 3 groups. Each of the group included 5 types of graphs. Since the graphs were analyzed are regular graphs odd degree, hence all the graphs have to have an even number of nodes. The nodes number of all graphs belonging to the first group is divisible by two, the second and the third by four (it follows from used method of their construction).

For each group of graphs their analysis based on results obtained thanks to the application of testing programs constructed by authors. It made possible to carry out a distribution of maximal number of nodes in layers, to count total number of nodes in virtual, optimal graphs. Based on obtained results, for the first two groups of graphs they found strict mathematical expressions describing the distribution of nodes in layers, the total number of nodes, the average path length for optimal graphs, whilst for the third group such formulas were not found. Additionally the prepared programs allowed us to compare the basic parameters of real and theoretically constructed graphs.

Among the all analyzed graphs, the structures CHR5_f – CHR5_o have the most acceptable basic parameters. Those graphs however have a fundamental limitation: the network should consist of nodes with nodes number has to be divisible by 4. The parameters of these networks in more or less deviate from the parameters of the reference graph (graph ideal), and what's involved, they are usually asymmetrical (depending on the choice of the source node, obtained values differ). Also, computational process is rather complex, and takes long time.

From the point of view of application, according to the authors, the most appropriate structure of the regular network topology are graphs CHR5_b. These chordal rings are symmetrical, their parameters, if are not equal to the parameters of optimal graphs, they are very close to them; they are simple and easy to design and implement. The main limitation is the fact that the number of nodes, creating these networks, has to be even, but this follows from the assumption that the structure has to be regular, and the degree of nodes is five. Fig. 47 shows the comparison of the best two structures.

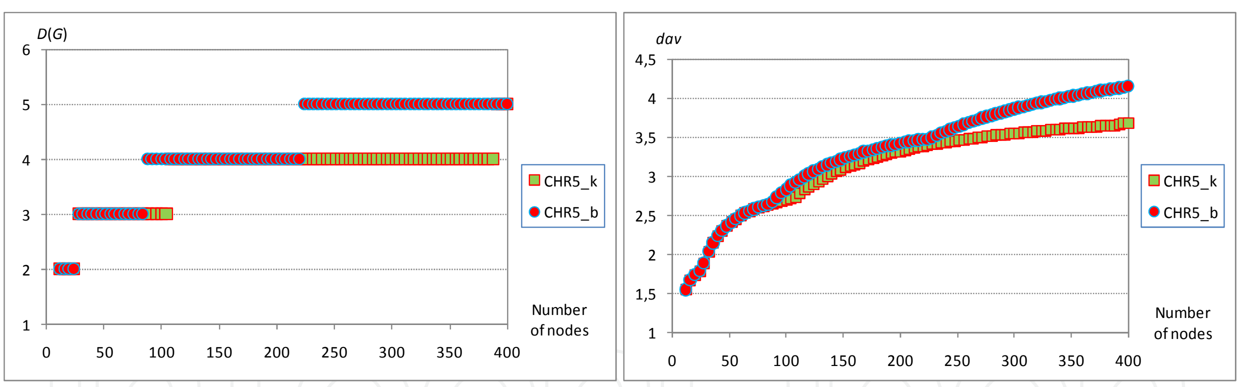

Fig. 47.Comparison of diameter and average length path both analyzing chordal rings versus the number of nodes

As the justification of this last conclusion the presented above diagrams can be used. In Fig. 47 we show the comparison of basic parameters of chordal rings CHR5_b and the best graph - CHR5_k. It can be observed that in up to 84 nodes, theoretically calculated parameters of both types of rings are identical, and up to 224 nodes - not much different from each another. Thus, taking into account the advantages of CHR5_b structure described before, it should be used to construct regular networks possessing not a huge number of nodes or in a large network consisting of a few identical regular structures.

Future work can focus on both theoretical and practical aspects. For the theoretical aspects, it would be a big help to find more precise and reachable bounds. This would make it possible to assess graph types, and to know how close to optimal they are. Moreover, a more thorough study of how precise distance estimates can be given using ideal and optimal graph would be interesting. Such a study could also cover other types of graphs. Another direction for further research would be to study new groups of graphs.

More practical aspects could deal with analyzing how well the good theoretical properties translate into good network properties. This could be done through simulation of different network configuration and traffic scenarios, and/or by studying how feasible the graphs are for assignment of physical, optical or logical links. In order to demonstrate that the topologies are useful in real-world settings, case studies would be a good place to start.

6. Acknowledgment

The authors would like to thank Associate Professor Jens Myrup Pedersen and Assistant Professor Tahir Riaz from Aalborg University, Department of Electronic Systems, and Professor Dr. Mohamed Othman from Universiti Putra Malaysia, Department of Communication Technology and Networks, for a fruitful collaboration with the work on analyzing chordal rings and other network topologies. Their inputs for this chapter have been particularly valuable.

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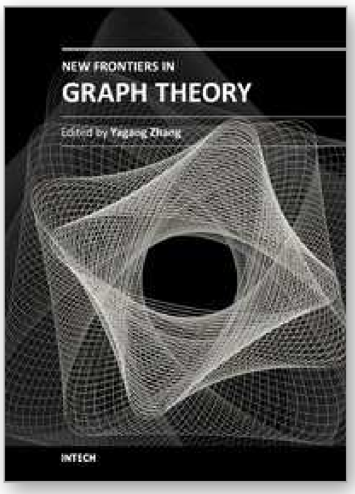

New Frontiers in Graph Theory
Edited by Dr. Yagang Zhang

ISBN 978-953-51-0115-4
Hard cover, 526 pages
Publisher InTech
Published online 02, March, 2012
Published in print edition March, 2012

Nowadays, graph theory is an important analysis tool in mathematics and computer science. Because of the inherent simplicity of graph theory, it can be used to model many different physical and abstract systems such as transportation and communication networks, models for business administration, political science, and psychology and so on. The purpose of this book is not only to present the latest state and development tendencies of graph theory, but to bring the reader far enough along the way to enable him to embark on the research problems of his own. Taking into account the large amount of knowledge about graph theory and practice presented in the book, it has two major parts: theoretical researches and applications. The book is also intended for both graduate and postgraduate students in fields such as mathematics, computer science, system sciences, biology, engineering, cybernetics, and social sciences, and as a reference for software professionals and practitioners.