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Approximate Solutions of the Dirac Equation for the Rosen-Morse Potential in the Presence of the Spin-Orbit and Pseudo-Orbit Centrifugal Terms

Kayode John Oyewumi
Theoretical Physics Section, Physics Department,
University of Ilorin, Ilorin
Nigeria

1. Introduction

In quantum mechanics, it is well known that the exact solutions play fundamental role, this is because, these solutions usually contain all the necessary information about the quantum mechanical model under investigation. In recent years, there has been a renewed interest in obtaining the solutions of the Dirac equations for some typical potentials under special cases of spin symmetry and pseudo-spin symmetry (Arima et al., 1969; Hecht and Adler, 1969).

The idea about spin symmetry and pseudo-spin symmetry with the nuclear shell model has been introduced in 1969 by Arima et al. (1969) & Hecht and Adler (1969). This idea has been widely used in explaining a number of phenomena in nuclear physics and related areas. Spin and pseudo-spin symmetric concepts have been used in the studies of certain aspects of deformed and exotic nuclei (Meng & Ring, 1996; Ginocchio, 1997; Ginocchio & Madland, 1998; Alberto et al., 2001; 2002; Lisboa et al., 2004a; 2004b; 2004c; Guo et al., 2005a; 2005b; Guo & Fang, 2006; Ginocchio, 2004; Ginocchio, 2005a; 2005b).

Spin symmetry (SS) is relevant to meson with one heavy quark, which is being used to explain the absence of quark spin orbit splitting (spin doublets) observed in heavy-light quark mesons (Page et al., 2001). On the other hand, pseudo-spin symmetry (PSS) concept has been successfully used to explain different phenomena in nuclear structure including deformation, superdeformation, identical bands, exotic nuclei and degeneracies of some shell model orbitals in nuclei (pseudo-spin doublets)(Arima, et al., 1969; Hecht & Adler, 1969; Meng & Ring, 1996; Ginocchio, 1997; Troltenier et al., 1994; Meng, et al., 1999; Stuchbery, 1999; 2002). Within this framework also, Ginocchio deduced that a Dirac Hamiltonian with scalar S(r) and vector $V(r)$ harmonic oscillator potentials when $V(r) = S(r)$ possesses a spin symmetry (SS) as well as a $U(3)$ symmetry, whereas a Dirac Hamiltonian for the case of $V(r) + S(r) = 0$ or $V(r) = -S(r)$ possesses a pseudo-spin symmetry and a pseudo-$U(3)$ symmetry (Ginocchio, 1997; 2004; 2005a; 2005b). As introduced in nuclear theory, the PSS refers to a quasi-degeneracy of the single-nucleon doublets which can be characterized with the non-relativistic quantum mechanics $(n, \ell, j = \ell + \frac{1}{2})$ and $(n-1, \ell+2, j = \ell + \frac{3}{2})$, where n, ℓ and j are the single-nucleon radial, orbital and total angular momentum quantum numbers for a single particle, respectively (Arima et al., 1969; Hecht & Adler, 1969; Ginocchio, 2004;

2005a; 2005b; Page et al., 2001). The total angular momentum is given as $j = \bar{\ell} + \bar{s}$, where $\bar{\ell} = \ell + 1$ is a pseudo- angular momentum and $\bar{s} = \frac{1}{2}$ is a pseudo-spin angular momentum.
Meng et al., (1998) deduced that in real nuclei, the PSS is only an approximation and the quality of approximation depends on the pseudo-centrifugal potential and pseudo-spin orbital potential. The orbital and pseudo-orbital angular momentum quantum numbers for SS ℓ and PSS $\bar{\ell}$ refer to the upper-and lower-spinor components (for instance, $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$, respectively.
Ginocchio (1997); (1999); (2004); (2005a); (2005b) and Meng et al., (1998) showed that SS occurs when the difference between the vector potential $V(r)$ and scalar potential $S(r)$ in the Dirac Hamiltonian is a constant (that is, $\Delta(r) = V(r) - S(r)$) and PSS occurs when the sum of two potential is a constant (that is, $\Sigma(r) = V(r) + S(r)$).
A large number of investigations have been carried out on the SS and PSS by solving the Dirac equation with various methods (Alberto et al., 2001; 2002; Lisboa et al., 2004a; 2004b; 2004c; Ginocchio, 2005a; 2005b; Xu et al., 2008; Guo et al., 2005a; 2005b; de Castro et al., 2006; Wei and Dong, 2009; Zhang, 2009; Zhang et al., 2009a; Setare & Nazari, 2009; Ginocchio, 1999; Soylu et al., 2007; 2008a; 2008b; Berkdermir, 2006; 2009; Berkdemir & Sever, 2009; Xu & Zhu, 2006; Jia et al., 2006; Zhang et al., 2009a; Zhang et al., 2008; Aydoğdu, 2009; Aydoğdu & Sever, 2009; Wei and Dong, 2008; Jia et al., 2009a; 2009b; Guo et al., 2007).
Some of these potentials are exactly solvable, these include: harmonic potential (Lisboa et al., 2004a; 2004b; 2004c; Ginocchio, 1999; 2005a; 2005b; Guo et al., 2005a; 2005b; de Castro et al., 2006; Akcay & Tezcan, 2009), Coulomb potential (Akcay, 2007; 2009), pseudoharmonic potential (Aydoğdu, 2009; Aydoğdu & Sever, 2009; Aydoğdu & Sever, 2010a), Mie-type potential (Aydoğdu, 2009; Aydoğdu & Sever, 2010b).
Also, for the $\bar{s}-$wave with zero pseudo-orbital angular momentum $\bar{\ell} = 0$ and spin-orbit quantum number $\kappa = 1$, exact analytical solutions have been obtained for some potentials with different methods, such as: Woods-Saxon potential (Aydoğdu, 2009; Guo & Sheng, 2005; Aydoğdu & Sever, 2010c), Eckart potential (Jia et al., 2006), Pöschl-Teller potential (Jia et al, 2009b), Rosen-Morse potential (Oyewumi & Akoshile, 2010), trigonometric Scarf potential (Wei et al., 2010).
However, exact analytical solution for any $\ell-$ states are possible only in a few instances. it is important to mention that most of these potentials can not be solved exactly for $\ell \neq 1 (\kappa \neq -1)$ or $\bar{\ell} \neq 0 (\kappa \neq 1)$ state, hence, a kind of approximation to the (pseudo or) - centrifugal term is necessary (Pekeris-type approximation) (Ikhdair, 2010; 2011; Ikhdair et al., 2011; Xu et al., 2008; Jia et al., 2009a; 2009b; Wei and Dong, 2009; Zhang et al., 2009b; Soylu et al., 2007; 2008a; 2008b; Zhang et al., 2008; Aydoğdu and Sever, 2010c; Aydoğdu and Sever, 2010d; Bayrak and Boztosun, 2007; Pekeris, 1934; Greene and Aldrich, 1976; Wei and Dong, 2010a; 2010b; 2010c).
With this kind of approximation to the (pseudo or) - centrifugal term, the SS and PSS problems have been solved using different methods to obtain the approximate solutions: AIM (Soylu et al., 2007; 2008a; 2008b; Aydoğdu & Sever, 2010c; Bayrak & Boztosun, 2007; Hamzavi et al., 2010c), Nikiforov- Uvarov method (Aydoğdu & Sever, 2010c; Hamzavi et al., 2010a; 2010b; Berkdemir, 2006; 2009; Berkdemir & Sever, 2009; Ikhdair, 2010; 2011; Ikhdair et al., 2011), functional analysis method (Xu et al., 2008; Wei & Dong, 2010d), SUSY and functional analysis (Jia et al., 2006; 2009a; 2009b; Wei & Dong, 2009; Zhang et al., 2009b; Setare & Nazari, 2009; Wei & Dong, 2010a; 2010b; 2010c). Therefore, by applying a Pekeris-type approximation to the (pseudo or) - centrifugal-like term, the relativistic bound state solutions can be obtained in the framework of the PSS and SS concepts.

In this study, the Rosen-Morse potential is considered, due to the important applications of in atomic, chemical and molecular Physics as well (Rosen & Morse, 1932). This potential is very useful in describing interatomic interaction of the linear molecules. The Rosen-Morse potential is given as

$$V(r) = -V_1 \text{sech}^2 \alpha r + V_2 \tanh \alpha r, \tag{1}$$

where V_1 and V_2 are the depth of the potential and α is the range of the potential, respectively. Thus, our aim is to employ the newly improved approximation scheme (or Pekeris-type approximation scheme) in order to obtain the PSS and SS solutions of the Dirac equations for the Rosen-Morse potential with the centrifugal term. This potential has been studied by various researchers in different applications (Rosen & Morse, 1932; Yi et al., 2004; Taşkin, 2009; Oyewumi & Akoshile and reference therein, 2010; Ikhdair, 2010; Ibrahim et al., 2011; Amani et al., 2011). In the light of this study, standard function analysis approach will be used (Yi et al, 2004; Taşkin, 2009).

In this chapter, Section 2 contains, the basic equations for the upper- and lower- component of the Dirac spinors. In Section 3, the approximate analytical solutions of the Dirac equation with the Rosen-Morse potential with arbitrary κ under pseudospin and spin symmetry conditions are obtained by means of the standard function analysis approach. Also, the solutions of some special cases are obtained. The bound state solutions of the relativistic equations (Klein-Gordon and Dirac) with the equally mixed Rosen-Morse potentials for any ℓ or κ are contained in Section 4. Section 5 contains contains the conclusions.

2. Basic Equations for the upper- and lower-components of the Dirac spinors

In the case of spherically symmetric potential, the Dirac equation for fermionic massive spin$-\frac{1}{2}$ particles interacting with the arbitrary scalar potential $S(r)$ and the time-component $V(r)$ of a four-vector potential can be expressed as (Greiner, 2000; Wei & Dong, 2009; 2010a; 2010b; 2010c; 2010d; Ikhdair, 2010; 2011; Oyewumi & Akoshile, 2010; Ikhdair et al., 2011):

$$\left[c\vec{\alpha}.\vec{P} + \beta[Mc^2 + S(\vec{r})] + V(\vec{r}) - E\right] \psi_{n\kappa}(\vec{r}) = 0, \tag{2}$$

where E is the relativistic energy of the system, M is the mass of a particle, $\vec{P} = -i\hbar\nabla$ is the momentum operator. $\vec{\alpha}$ and β are 4×4 Dirac matrices, given as

$$\vec{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \tag{3}$$

where $\mathbf{I}$ is the 2×2 identity matrix and $\sigma_i (i = 1, 2, 3)$ are the vector Pauli matrices.

Following the procedure stated in (Greiner, 2000; Wei & Dong 2009; 2010a; 2010b; 2010c; 2010d; Ikhdair, 2010; 2011; Ikhdair et al., 2011), the spinor wave functions can be written using the Pauli-Dirac representation as:

$$\psi_{n\kappa}(\vec{r}) = \frac{1}{r}\begin{bmatrix} F_{n\kappa}(r)\, Y^{\ell}_{jm}(\theta, \phi) \\ iG_{n\kappa}(r)\, Y^{\overline{\ell}}_{jm}(\theta, \phi) \end{bmatrix}; \; \kappa = \pm(j + \frac{1}{2}), \tag{4}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the radial wave functions of the upper and lower spinors components, respectively. $Y^{\ell}_{jm}(\theta, \phi)$ and $Y^{\overline{\ell}}_{jm}$ are the spherical harmonic functions coupled to the total angular momentum j and its projection m on the $z-$axis. The orbital and pseudo-orbital angular momentum quantum numbers for SS (ℓ) and PSS ($\bar{\ell}$) refer to the upper ($F_{n\kappa}(r)$) and lower ($G_{n\kappa}(r)$) spinor components, respectively, for which $\ell(\ell+1) = \kappa(\kappa+1)$ and $\bar{\ell}(\bar{\ell}+1) = \kappa(\kappa-1)$. For the relationship between the quantum number κ to the quantum numbers for SS (ℓ) and PSS ($\bar{\ell}$) (Ikhdair, 2010; 2011; Ikhdair et al., 2011; Jia et al., 2009a; 2009b; Xu et al., 2008; Wei & Dong, 2009; Ginocchio, 2004; Zhang et al., 2009b; Setare & Nazari, 2009). For comprehensive reviews, see Ginocchio (1997) and (2005b).

On substituting equation (4) into equation (2), the two-coupled second-order ordinary differential equations for the upper and lower components of the Dirac wave function are obtained as follows:

$$\left(\frac{d}{dr}+\frac{\kappa}{r}\right)F_{n\kappa}(r)=\left[Mc^2+E_{n\kappa}-\Delta(r)\right]G_{n\kappa}, \tag{5}$$

$$\left(\frac{d}{dr}-\frac{\kappa}{r}\right)G_{n\kappa}(r)=\left[Mc^2-E_{n\kappa}+\Sigma(r)\right]F_{n\kappa}. \tag{6}$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ from equations (5) and (6), the following two Schrödinger-like differential equations for the upper and lower radial spinors components are obtained, respectively as:

$$\begin{aligned}\left\{-\frac{d^2}{dr^2}+\frac{\kappa(\kappa+1)}{r^2}+\frac{1}{\hbar^2c^2}\left[Mc^2+E_{n\kappa}-\Delta(r)\right]\left[Mc^2-E_{n\kappa}+\Sigma(r)\right]\right\}F_{n\kappa}(r)\\ =\frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr}+\frac{\kappa}{r}\right)}{\left[Mc^2+E_{n\kappa}-\Delta(r)\right]}F_{n\kappa}(r),\end{aligned} \tag{7}$$

$$\begin{aligned}\left\{-\frac{d^2}{dr^2}+\frac{\kappa(\kappa-1)}{r^2}+\frac{1}{\hbar^2c^2}\left[Mc^2+E_{n\kappa}-\Delta(r)\right]\left[Mc^2-E_{n\kappa}+\Sigma(r)\right]\right\}G_{n\kappa}(r)\\ =-\frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr}-\frac{\kappa}{r}\right)}{\left[Mc^2-E_{n\kappa}+\Sigma(r)\right]}G_{n\kappa}(r),\end{aligned} \tag{8}$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$ are the difference and the sum of the potentials $V(r)$ and $S(r)$, respectively.

In the presence of the SS, that is, the difference potential $\Delta(r) = V(r) - S(r) = C_s =$ constant or $\frac{d\Delta(r)}{dr} = 0$, then, equation (7) reduces into

$$\begin{aligned}\left\{-\frac{d^2}{dr^2}+\frac{\kappa(\kappa+1)}{r^2}+\frac{1}{\hbar^2c^2}\left[Mc^2+E_{n\kappa}-C_s\right]\Sigma(r)\right\}F_{n\kappa}(r)\\ =\left[E_{n\kappa}^2-M^2c^4+C_s(Mc^2-E_{n\kappa})\right]F_{n\kappa}(r),\end{aligned} \tag{9}$$

where $\kappa(\kappa+1) = \ell(\ell+1)$, $\kappa = \begin{cases}\ell, & \text{for } \kappa<0\\ -(\ell+1), & \text{for } \kappa>0\end{cases}$. The SS energy eigenvalues depend on n and κ, for $\ell \neq 0$, the states with $j = \ell \pm \frac{1}{2}$ are degenerate. Then, the lower component

$G_{n\kappa}(r)$ of the Dirac spinor is obtained as

$$G_{n,\kappa}(r) = \frac{1}{Mc^2 + E_{n\kappa} - C_s}\left[\frac{d}{dr} + \frac{\kappa}{r}\right] F_{n\kappa}(r), \tag{10}$$

where $E_{n\kappa} + Mc^2 \neq 0$, only real positive energy state exist when $C_s = 0$ (Guo & Sheng, 2005; Ikhdair, 2010; Ikhdair et al., 2011).
Also, under the PSS condition, that is, the sum potential $\Sigma(r) = V(r) + S(r) = C_{ps}$ constant or $\frac{d\Sigma(r)}{dr} = 0$, then, equation (7) becomes

$$\begin{aligned}\left\{-\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2} - \frac{1}{\hbar^2 c^2}\left[Mc^2 - E_{n\kappa} + C_{ps}\right]\Delta(r)\right\} G_{n\kappa}(r) \\ = \left[E_{n\kappa}^2 - M^2c^4 + C_{ps}(Mc^2 - E_{n\kappa})\right] G_{n\kappa}(r),\end{aligned} \tag{11}$$

and the upper component $F_{n\kappa}(r)$ is obtained as

$$F_{n,\kappa}(r) = \frac{1}{Mc^2 - E_{n\kappa} + C_{ps}}\left[\frac{d}{dr} - \frac{\kappa}{r}\right] G_{n\kappa}(r), \tag{12}$$

where $E_{n\kappa} - Mc^2 \neq 0$, only real negative energy state exist when $C_{ps} = 0$. Also, κ is related to the pseudo-orbital angular quantum number $\bar{\ell}$ as $\kappa(\kappa-1) = \bar{\ell}(\bar{\ell}+1)$, $\kappa = \begin{cases} -\bar{\ell}, & \text{for } \kappa < 0 \\ (\bar{\ell}+1), & \text{for } \kappa > 0 \end{cases}$, which implies that $j = \bar{\ell} \pm \frac{1}{2}$ are degenerate for $\bar{\ell} \neq 0$ (Guo & Sheng, 2005; Ikhdair, 2010; Ikhdair et al., 2011). It is required that the upper and lower spinor components must satisfy the following boundary conditions $F_{n\kappa}(0) = G_{n\kappa}(0) = 0$ and $F_{n\kappa}(\infty) = G_{n\kappa}(\infty) = 0$ for bound state solutions.
Exact solutions of equations (9) and (11) with the Rosen-Morse potential (1) can be obtained only for the $s-$wave ($\kappa = 0, -1$) and ($\kappa = 0, 1$) due to the spin-orbit (or pseudo) centrifugal term $\frac{\kappa(\kappa+1)}{r^2}$ (or $\frac{\kappa(\kappa-1)}{r^2}$). Therefore, a newly improved approximation in dealing with the spin-orbit (or pseudo) centrifugal term to obtain the approximate solutions for the Rosen-Morse is adopted.
This type of approximation, (Pekeris-type) approximation can be traced back to Pekeris (1934), and for short-range potential, Greene & Aldrich (1976) proposed a good approximation to the centrifugal term ($1/r^2$). The idea about the use of approximation to centrifugal (or pseudo centrifugal) term has received much attention and considerable interest due to its wide range of applications (Wei & Dong, 2010a; 2010b; 2010c; 2010d; Aydoğdu & Sever, 2010; Zhang et al., 2009b; Jia et al., 2009a; 2009b; Lu, 2005; Ikhdair, 2010; Ikhdair et al., 2011). We adopt the centrifugal (or pseudo centrifugal) approximation introduced by Lu (2005) for values of κ that are not large and vibrations of the small amplitude about the minimum. This approximation to the centrifugal or (pseudo centrifugal) term near the minimum point $r = r_0$ introduced by Lu (2005) is given as follows:

$$\frac{1}{r^2} \approx \frac{1}{r_0^2}\left[c_0 + c_1\left(\frac{-e^{-2\alpha r}}{1+e^{-2\alpha r}}\right) + c_2\left(\frac{-e^{-2\alpha r}}{1+e^{-2\alpha r}}\right)^2\right], \tag{13}$$

where

$$
\begin{aligned}
C_0 &= 1 - \left(\frac{1+e^{-2\alpha r_0}}{2\alpha r_0}\right)^2 \left(\frac{8\alpha r_0}{1+e^{-2\alpha r_0}} - (3+2\alpha r_0)\right), \\
C_1 &= -2(e^{2\alpha r_0}+1)\left[3\left(\frac{1+e^{-2\alpha r_0}}{2\alpha r_0}\right) - (3+2\alpha r_0)\left(\frac{1+e^{-2\alpha r_0}}{2\alpha r_0}\right)\right], \\
C_2 &= (e^{2\alpha r_0}+1)^2\left(\frac{1+e^{-2\alpha r_0}}{2\alpha r_0}\right)^2\left[(3+2\alpha r_0) - \left(\frac{4\alpha r_0}{1+e^{-2\alpha r_0}}\right)\right],
\end{aligned} \tag{14}
$$

other higher terms are neglected.

3. Bound state solutions of the Dirac equation with the Rosen-Morse potential with arbitrary κ

3.1 Spin symmetry solutions of the Dirac equation with the Rosen-Morse potential with arbitrary κ

In equation (9), we adopt the choice of $\Sigma(r) = 2V(r) \to V(r)$ as earlier illustrated by Alhaidari et al. (2006), which enables us to reduce the resulting solutions into their non-relativistic limits under appropriate transformations, that is,

$$
\Sigma(r) = -4V_1\frac{e^{-2\alpha r}}{(1+e^{-2a\alpha r})^2} + V_2\frac{(1-e^{-2\alpha r})}{(1+e^{-2a\alpha r})}. \tag{15}
$$

Using the centrifugal term approximation in equation (13) and introducing a new variable of the form $z = e^{-2\alpha r}$ in equation (9), the following equation for the upper component spinor $F_{n\kappa}(r)$ is obtained as:

$$
\begin{aligned}
&z^2\frac{d^2}{dz^2}F_{n\kappa}(z) + z\frac{d}{dz}F_{n\kappa}(z) + \frac{1}{4\alpha^2}\left\{\frac{1}{\hbar^2c^2}\left[E_{n\kappa}^2 - M^2c^4 + C_s(Mc^2 - E_{n\kappa})\right]\right\}F_{n\kappa}(z) \\
&-\frac{\kappa(\kappa+1)}{4\alpha^2}\left\{\frac{1}{r_0^2}\left[C_0 + C_1\frac{z}{1-z} + C_2\frac{z^2}{(1-z)^2}\right] - \frac{4\widetilde{V}_1 z}{(1-z)^2} - \widetilde{V}_2 - \frac{2\widetilde{V}_2 z}{(1-z)}\right\}F_{n\kappa}(z),
\end{aligned} \tag{16}
$$

where

$$
\widetilde{V}_1 = \frac{V_1}{\hbar^2c^2}[Mc^2 + E_{n\kappa} - C_s] \text{ and } \widetilde{V}_2 = \frac{V_2}{\hbar^2c^2}[Mc^2 + E_{n\kappa} - C_s]. \tag{17}
$$

The upper component spinor $F_{n\kappa}(z)$ has to satisfy the boundary conditions, $F_{n\kappa}(z) = 0$ at $z \to 0$ ($r \to \infty$) and $F_{n\kappa}(z) = 1$ at $z \to 1$ ($r \to 0$). Then, the function $F_{nk}(z)$ can be written as

$$
F_{n\kappa}(z) = (1-z)^{1+q}z^{\beta}f_{n\kappa}(z), \tag{18}
$$

where

$$
q = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{\kappa(\kappa+1)C_2}{\alpha^2 r_0^2} + \frac{4\widetilde{V}_1}{\alpha^2}}\right] \tag{19}
$$

and

$$
-\beta^2 = \frac{1}{4\alpha^2}\left\{\frac{1}{\hbar^2c^2}\left[E_{n\kappa}^2 - M^2c^4 + C_s(Mc^2 - E_{n\kappa})\right] - \frac{\kappa(\kappa+1)}{r_0^2}C_0 - \widetilde{V}_2\right\}. \tag{20}
$$

On substituting equation (18) into equation (16) with equations (17), (19) and (20) , the second-order differential equation is obtained as

$$z(1-z)\frac{d^2}{dz^2}f_{n\kappa}(z)+[(2\beta+1)-(2q+2\beta+3)z]\frac{d}{dz}f_{n\kappa}(z) \\ -\left[(2\beta+1)(1+q)+\frac{\widetilde{V}_2+2\widetilde{V}_1}{2\alpha^2}+\frac{\kappa(\kappa+1)C_1}{4\alpha^2 r_0^2}\right]f_{n\kappa}(z), \tag{21}$$

whose solutions are the hypergeometric functions (Gradshteyn & Ryzhik, 2007), its general form can be expressed as

$$f_{n\kappa}(z)=A\,{}_2F_1(a,b;c;z)+Bz^{1-c}\,{}_2F_1(a-c+1,b-c+1;2-c;z), \tag{22}$$

in which the first term can be expressed as:

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)z^k}{\Gamma(c+k)k!}, \tag{23}$$

where

$$a=1+q+\beta-\gamma \\ b=1+q+\beta+\gamma \\ c=1+2\beta \\ \gamma=\sqrt{\beta^2-\frac{(\widetilde{V}_2+2\widetilde{V}_1)}{2\alpha^2}+q(1+q)-\frac{\kappa(\kappa+1)C_1}{4\alpha^2 r_0^2}}\,. \tag{24}$$

The hypergeometric function $f_{n\kappa}(z)$ can be reduced to polynomial of degree n, whenever either a or b equals to a negative integer $-n$. This implies that the hypergeometric function $f_{n\kappa}(z)$ given by equation (23) can only be finite everywhere unless

$$a=1+q+\beta-\gamma=-n;\ n=0,1,2,3,\dots\,. \tag{25}$$

Using equations (17), (19) and (20) in equation (25), an explicit expression for the energy eigenvalues of the Dirac equation with the Rosen-Morse potential under the spin symmetry condition is obtained as:

$$(Mc^2+E_{n\kappa}-C_s)(Mc^2-E_{n\kappa}+V_2)=-\frac{\kappa(\kappa+1)C_0}{r_0^2}\hbar^2c^2 \\ +4\alpha^2\hbar^2c^2\left[\frac{\frac{(C_2-C_1)}{4\alpha^2 r_0^2}\kappa(\kappa+1)-\frac{(Mc^2+E_{n\kappa}-C_s)V_2}{2\alpha^2\hbar^2c^2}}{2(n+q+1)}-\frac{(n+q+1)}{2}\right]^2. \tag{26}$$

It is observed that, the spin symmetric limit leads to quadratic energy eigenvalues. Hence, the solution of equation (26) consists of positive and negative energy eigenvalues for each n and κ. In 2005, Ginocchio has shown that there are only positive energy eigenvalues and no bound

negative energy eigenvalues exist in the spin limit. Therefore, in the spin limit, only positive energy eigenvalues are chosen for the spin symmetric limit.
Using equations (18) to (25), the radial upper component spinor can be obtained as

$$\begin{aligned} F_{n\kappa}(r) &= N_{n\kappa}(1+e^{-2\alpha r})^{1+q}(-e^{-2\alpha r})^{\beta}\ {}_2F_1(-n, n+2(\beta+q+1); 2\beta+1; -e^{-2\alpha r}) \\ &= N_{n\kappa}\frac{n!\Gamma(2\beta+1)}{\Gamma(n+\beta+1)}(1+e^{-2\alpha r})^{1+q}(-e^{-2\alpha r})^{\beta}P_n^{(2\beta,\,2q+1)}(1+2e^{-2\alpha r}), \end{aligned} \tag{27}$$

$N_{n\kappa}$ is the normalization constant which can be determined by the condition that $\int_0^{\infty} \mid F_{n\kappa}(r) \mid^2 dr = 1$.
By making use of the equation (23) and the following integral (see formula (7.512.12) in Gradshteyn & Ryzhik (2007)):

$$\int_0^1 (1-x)^{\mu-1}x^{\nu-1}\ {}_pF_q(a_1,..,a_p;\, b_1,..b_q;\, ax)dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\ {}_{p+1}F_{q+1}(\nu,\, a_1,..,a_p;\, \mu+\nu,\, b_1,..b_q; a)\ , \tag{28}$$

which is valid for $Re\mu > 0$, $Re\nu > 0$, $p \leq q+1$, if $p = q+1$, then $\mid q < 1 \mid$, this leads to

$$N_{n\kappa} = \left[\frac{\Gamma(2q+3)\Gamma(2\beta+1)}{2\alpha\Gamma(n)}\sum_{k=0}^{\infty}\frac{(-1)^k\,(n+2(1+\beta+q))_k\,\Gamma(n+k)}{k!(k+2\beta)!\Gamma\left(k+2(\beta+q+\frac{3}{2})\right)}A_{n\kappa}\right]^{-1/2}, \tag{29}$$

where $A_{n\kappa} = {}_3F_2(2\beta+k,\, -n,\, n+2(1+\beta+q);\, k+2(\beta+q+\frac{3}{2});\, 2\beta+1;\, 1)$ and $(x)_a = \frac{\Gamma(x+a)}{\Gamma(x)}$ (Pochhammer symbol). In order to find the lower component spinor, the recurrence relation of the hypergeometric function (Gradshteyn & Ryzhik, 2007)

$$\frac{d}{d\xi}\left[\,{}_2F_1\,(a,\, b,\, c; \xi)\right] = \left(\frac{ab}{c}\right)\frac{d}{d\xi}\,{}_2F_1\,(a+1,\, b+1,\, c+1; \xi)\ , \tag{30}$$

is used to evaluate equation (10) and this is obtained as

$$\begin{aligned} G_{n\kappa}(r) = \frac{N_{n\kappa}(1+e^{-2\alpha r})^{1+q}(-e^{-2\alpha r})^{\beta}}{[Mc^2+E_{n\kappa}-C_s]}&\left[-2\alpha\beta - \frac{2\alpha e^{-2\alpha r}}{1+e^{-2\alpha r}} + \frac{\kappa}{r}\right] \\ &\times {}_2F_1(-n, n+2(\beta+q+1); 2\beta+1; -e^{-2\alpha r}) + \\ \frac{N_{n\kappa}(1+e^{-2\alpha r})^{1+q}(-e^{-2\alpha r})^{\beta+1}}{[Mc^2+E_{n\kappa}-C_s]}&\left\{\frac{2\alpha n\,[n+2(\beta+q+1)]}{(2\beta+1)}\right\} \\ &\times {}_2F_1(-n+1, n+2(\beta+q+\tfrac{3}{2}); 2(\beta+1); -e^{-2\alpha r}). \end{aligned} \tag{31}$$

3.2 Pseudopin symmetry solutions of the Dirac equation with the Rosen-Morse potential with arbitrary κ

In the case of pseudospin symmetry, that is, the difference as in equation (11). $\frac{d\Sigma(r)}{dr} = 0$ or $\Sigma(r) = \text{Constant} = C_{ps}$, and taking into consideration the choice of $\Delta(r) = 2V(r) \rightarrow V(r)$ as earlier illustrated by Alhaidari et al. (2006). Then,

$$\Delta(r) = -4V_1 \frac{e^{-2\alpha r}}{(1+e^{-2a\alpha r})^2} + V_2 \frac{(1-e^{-2\alpha r})}{(1+e^{-2a\alpha r})}. \tag{32}$$

With the pseudo-centrifugal approximation in equation (13) and substituting $z = -e^{-2\alpha r}$, then, the following equation for the lower component spinor $G_{n\kappa}(r)$ is obtained as:

$$z^2 \frac{d^2}{dz^2} G_{n\kappa}(z) + z\frac{d}{dz} G_{n\kappa}(z) + \frac{1}{4\alpha^2}\left\{ \frac{1}{\hbar^2 c^2}\left[E_{n\kappa}^2 - M^2c^4 - C_{ps}(Mc^2 + E_{n\kappa}) \right] \right\} G_{n\kappa}(z)$$
$$-\frac{\kappa(\kappa-1)}{4\alpha^2}\left\{ \frac{1}{r_0^2}\left[C_0 + C_1 \frac{z}{1-z} + C_2 \frac{z^2}{(1-z)^2} \right] - \frac{4\widetilde{V}_3 z}{(1-z)^2} - \widetilde{V}_4 - \frac{2\widetilde{V}_4 z}{(1-z)} \right\} G_{n\kappa}(z), \tag{33}$$

where

$$\widetilde{V}_3 = \frac{V_1}{\hbar^2 c^2}[Mc^2 - E_{n\kappa} + C_{ps}] \text{ and } \widetilde{V}_4 = \frac{V_2}{\hbar^2 c^2}[Mc^2 - E_{n\kappa} + C_{ps}]. \tag{34}$$

With boundary conditions in the previous subsection, then, writing the function $G_{n\kappa}(z)$ as

$$G_{n\kappa}(z) = (1-z)^{1+\overline{q}} z^{\overline{\beta}} g_{n\kappa}(z), \tag{35}$$

where

$$\overline{q} = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{\kappa(\kappa-1)C_2}{\alpha^2 r_0^2} - \frac{4\widetilde{V}_3}{\alpha^2}} \right] \tag{36}$$

and

$$-\overline{\beta}^2 = \frac{1}{4\alpha^2}\left\{ \frac{1}{\hbar^2 c^2}\left[E_{n\kappa}^2 - M^2c^4 - C_{ps}(Mc^2 + E_{n\kappa}) \right] - \frac{\kappa(\kappa-1)}{r_0^2} C_0 + \widetilde{V}_4 \right\}. \tag{37}$$

On substituting equation (35) into equation (33) and using equations (34), (36) and (37), equation (33) becomes

$$z(1-z)\frac{d^2}{dz^2} g_{n\kappa}(z) + \left[(2\overline{\beta}+1) - (2\overline{q} + 2\overline{\beta} + 3)z \right] \frac{d}{dz} g_{n\kappa}(z)$$
$$-\left[(2\overline{\beta}+1)(1+\overline{q}) - \frac{\widetilde{V}_4 + 2\widetilde{V}_3}{2\alpha^2} + \frac{\kappa(\kappa-1)C_1}{4\alpha^2 r_0^2} \right] g_{n\kappa}(z), \tag{38}$$

whose solutions are the hypergeometric functions (Gradshteyn & Ryzhik, 2007), its general form can be expressed as

$$f_{n\kappa}(z) = A\, {}_2F_1(a,b;c;z) + Bz^{1-c}\, {}_2F_1(a-c+1, b-c+1; 2-c; z), \tag{39}$$

in which the first term can be expressed as:

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)z^k}{\Gamma(c+k)k!}, \tag{40}$$

where

$$a = 1 + \overline{q} + \overline{\beta} - \overline{\gamma}$$

$$b = 1 + \overline{q} + \overline{\beta} + \overline{\gamma}$$

$$c = 1 + 2\overline{\beta}$$

$$\overline{\gamma} = \sqrt{\overline{\beta}^2 + \frac{(\widetilde{V}_2 + 2\widetilde{V}_3)}{2\alpha^2} + \overline{q}(1+\overline{q}) - \frac{\kappa(\kappa-1)C_1}{4\alpha^2 r_0^2}}. \quad (41)$$

Also, in the similar fashion as obtained in the case of the spin symmetry condition, an explicit expression for the energy eigenvalues of the Dirac equation with the Rosen-Morse potential under the pseudospin symmetry is obtained as:

$$(Mc^2 - E_{n\kappa} + C_{ps})(Mc^2 + E_{n\kappa} - V_2) = -\frac{\kappa(\kappa-1)C_0}{r_0^2}\hbar^2 c^2$$

$$+4\alpha^2\hbar^2 c^2 \left[\frac{\frac{(C_2 - C_1)}{4\alpha^2 r_0^2}\kappa(\kappa-1) + \frac{(Mc^2 - E_{n\kappa} + C_{ps})V_2}{2\alpha^2\hbar^2 c^2}}{2(n+\overline{q}+1)} - \frac{(n+\overline{q}+1)}{2} \right]^2. \quad (42)$$

It is observed that, the pseudospin symmetric limit leads to quadratic energy eigenvalues. Therefore, the solution of equation (42) consists of positive and negative energy eigenvalues for each n and κ. Since, it has been shown that there are only negative energy eigenvalues and no bound positive energy eigenvalues exist in the pseudospin limit (Ginocchio, 2005). Therefore, in the pseudospin limit, only negative energy eigenvalues are chosen.
The radial lower component spinor can be obtained by considering equations (35)-(41) as

$$G_{n\kappa}(r) = \overline{N}_{n\kappa}(1+e^{-2\alpha r})^{1+\overline{q}}(-e^{-2\alpha r})^{\overline{\beta}}\ {}_2F_1(-n, n+2(\overline{\beta}+\overline{q}+1); 2\overline{\beta}+1; -e^{-2\alpha r})$$

$$= \overline{N}_{n\kappa}\frac{n!\Gamma(2\overline{\beta}+1)}{\Gamma(n+\overline{\beta}+1)}(1+e^{-2\alpha r})^{1+\overline{q}}(-e^{-2\alpha r})^{\beta} P_n^{(2\overline{\beta},\, 2\overline{q}+1)}(1+2e^{-2\alpha r}) \quad (43)$$

$\overline{N}_{n\kappa}$ is the normalization constant which can be determined by the condition that $\int_0^\infty |\ G_{n\kappa}(r)\ |^2\, dr = 1$ and by making use of the equations (23) and (28), we have

$$\overline{N}_{n\kappa} = \left[\frac{\Gamma(2\overline{q}+3)\Gamma(2\overline{\beta}+1)}{2\alpha\Gamma(n)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(n+2(1+\overline{\beta}+\overline{q})\right)_k \Gamma(n+k)}{k!(k+2\overline{\beta})!\Gamma\left(k+2(\overline{\beta}+\overline{q}+\frac{3}{2})\right)} \overline{A}_{n\kappa} \right]^{-1/2}, \quad (44)$$

where $\overline{A}_{n\kappa} = {}_3F_2(2\overline{\beta}+k,\ -n,\ n+2(1+\overline{\beta}+\overline{q});\ k+2(\overline{\beta}+\overline{q}+\frac{3}{2});\ 2\overline{\beta}+1;\ 1)$ and $(x)_a = \frac{\Gamma(x+a)}{\Gamma(x)}$ (Pochhammer symbol).

Similarly, by using equation (12) $F_{n\kappa}(r)$ can also be obtained as

$$F_{n\kappa}(r) = \frac{\overline{N}_{n\kappa}(1+e^{-2\alpha r})^{1+\overline{q}}(-e^{-2\alpha r})^{\overline{\beta}}}{\left[Mc^2 - E_{n\kappa} + C_{ps}\right]}\left[-2\alpha\overline{\beta} - \frac{2\alpha e^{-2\alpha r}}{1+e^{-2\alpha r}} - \frac{\kappa}{r}\right]$$
$$\times {}_2F_1(-n, n+2(\overline{\beta}+\overline{q}+1); 2\overline{\beta}+1; -e^{-2\alpha r}) +$$
$$\frac{\overline{N}_{n\kappa}(1+e^{-2\alpha r})^{1+\overline{q}}(-e^{-2\alpha r})^{\overline{\beta}+1}}{\left[Mc^2 - E_{n\kappa} + C_{ps}\right]}\left\{\frac{2\alpha n\left[n+2(\overline{\beta}+\overline{q}+1)\right]}{(2\overline{\beta}+1)}\right\}$$
$$\times {}_2F_1(-n+1, n+2(\overline{\beta}+\overline{q}+\tfrac{3}{2}); 2(\overline{\beta}+1); -e^{-2\alpha r}). \tag{45}$$

It is pertinent to note that, the negative energy solution for the pseudospin symmetry can be obtained directly from the positive energy solution of the spin symmetry using the parameter mapping (Berkdemir & Cheng, 2009; Ikhdair, 2010):

$F_{n\kappa}(r) \leftrightarrow G_{n\kappa}(r), V(r) \to -V(r),$ (or $V_1 \to -V_1$ and $V_2 \to -V_2$), $E_{n\kappa} \to -E_{n\kappa}$ and $C_s \to -C_{ps}$.

3.3 Remarks

In this work, solutions of some special cases are studied:

3.3.1 *s*-wave solutions:

Our results include any arbitrary κ values, therefore, there is need to investigate if our results will give similar results for *s*-wave for the spin symmetry when $\kappa = -1$ or $\ell = 0$ and for the pseudospin when $\kappa = 1$ or $\overline{\ell} = 0$.
For the SS, $\kappa = -1$ (or $\ell = 0$) in equation (26) gives

$$(Mc^2 + E_{n,-1} - C_s)(Mc^2 - E_{n,-1} + V_2) = 4\alpha^2\hbar^2c^2\left[\frac{\frac{(Mc^2+E_{n,-1}-C_s)V_2}{2\alpha^2\hbar^2c^2}}{2(n+q_1+1)} - \frac{(n+q_1+1)}{2}\right]^2, \tag{46}$$

where

$$q_1 = \frac{1}{2}\left[-1+\sqrt{1+\frac{4V_1[Mc^2+E_{n,-1}-C_s]}{\alpha^2\hbar^2c^2}}\right]. \tag{47}$$

For the PSS, $\kappa = 1$ (or $\overline{\ell} = 0$) in equation (42) gives

$$(Mc^2 - E_{n,1} + C_{ps})(Mc^2 + E_{n,1} - V_2) = 4\alpha^2\hbar^2c^2\left[\frac{\frac{(Mc^2-E_{n,1}+C_{ps})V_2}{2\alpha^2\hbar^2c^2}}{2(n+\overline{q}_1+1)} - \frac{(n+\overline{q}_1+1)}{2}\right]^2, \tag{48}$$

where

$$\overline{q}_1 = \frac{1}{2}\left[-1+\sqrt{1-\frac{4V_1[Mc^2-E_{n,1}+C_{ps}]}{\alpha^2\hbar^2c^2}}\right]. \tag{49}$$

The corresponding upper and lower component spinors for the SS and PSS can be obtained also. The above solutions are identical with the results obtained by Oyewumi & Akoshile (2010) and Ikhdair (2010).

3.3.2 Solutions for the standard Eckart potential:

By setting $V_1 = -V_1$ and $V_2 = -V_2$ in equation (1), we have the standard Eckart potential. The energy eigenvalues for the SS and the PSS are given, respectively as:

$$(Mc^2 + E_{n\kappa} - C_s)(Mc^2 - E_{n\kappa} - V_2) = -\frac{\kappa(\kappa+1)C_0}{r_0^2}\hbar^2c^2$$
$$+4\alpha^2\hbar^2c^2\left[\frac{\frac{(C_2-C_1)}{4\alpha^2r_0^2}\kappa(\kappa+1) + \frac{(Mc^2+E_{n\kappa}-C_s)V_2}{2\alpha^2\hbar^2c^2}}{2(n+q_2+1)} - \frac{(n+q_2+1)}{2}\right]^2 \tag{50}$$

and

$$(Mc^2 - E_{n\kappa} + C_{ps})(Mc^2 + E_{n\kappa} + V_2) = -\frac{\kappa(\kappa-1)C_0}{r_0^2}\hbar^2c^2$$
$$+4\alpha^2\hbar^2c^2\left[\frac{\frac{(C_2-C_1)}{4\alpha^2r_0^2}\kappa(\kappa-1) - \frac{(Mc^2-E_{n\kappa}+C_{ps})V_2}{2\alpha^2\hbar^2c^2}}{2(n+\overline{q}_2+1)} - \frac{(n+\overline{q}_2+1)}{2}\right]^2, \tag{51}$$

where q_2 and $\overline{q}_2$ are obtained, respectively as:

$$q_2 = \frac{1}{2}\left[-1+\sqrt{1+\frac{\kappa(\kappa+1)C_2}{\alpha^2r_0^2} - \frac{4V_1[Mc^2+E_{n\kappa}-C_s]}{\alpha^2\hbar^2c^2}}\right]$$
$$\overline{q}_2 = \frac{1}{2}\left[-1+\sqrt{1+\frac{\kappa(\kappa-1)C_2}{\alpha^2r_0^2} + \frac{4V_1[Mc^2-E_{n\kappa}+C_{ps}]}{\alpha^2\hbar^2c^2}}\right]. \tag{52}$$

The corresponding upper and lower component spinors for the SS and the PSS can easily be obtained from equations (27), (31), (43) and (45).

3.3.3 Solutions of the PT-Symmetric Rosen-Morse potential:

The choice of $V_2 = iV_2$ in equation (1) gives the PT-Symmetric Rosen-Morse potential (Jia et al., 2002; Yi et al., 2004; Taşkin, 2009; Oyewumi & Akoshile, 2010; Ikhdair, 2010):

$$V(r) = -V_1\text{sech}^2\alpha r + iV_2\tanh\alpha r. \tag{53}$$

For a given potential $V(r)$, if $V(-r) = V^*(r)$ (or $V(\eta - r) = V^*(r)$) exists, then, the potential $V(r)$ is said to be PT-Symmetric. Here, P denotes the parity operator (space reflection, $P: r \to -r$, or $r \to \eta - r$) and T denotes the time reversal operator ($T: i \to -i$).

For the case of the SS and the PSS solutions of this PT-Symmetric version of the Rosen-Morse potential, the energy eigenvalue equations are:

$$(Mc^2 + E_{n\kappa} - C_s)(Mc^2 - E_{n\kappa} + iV_2) = -\frac{\kappa(\kappa+1)C_0}{r_0^2}\hbar^2c^2$$
$$+4\alpha^2\hbar^2c^2\left[\frac{\frac{(C_2-C_1)}{4\alpha^2r_0^2}\kappa(\kappa+1) - i\frac{(Mc^2+E_{n\kappa}-C_s)V_2}{2\alpha^2\hbar^2c^2}}{2(n+q+1)} - \frac{(n+q+1)}{2}\right]^2 \tag{54}$$

and

$$(Mc^2 - E_{n\kappa} + C_{ps})(Mc^2 + E_{n\kappa} - iV_2) = -\frac{\kappa(\kappa-1)C_0}{r_0^2}\hbar^2 c^2$$
$$+4\alpha^2\hbar^2c^2\left[\frac{\frac{(C_2-C_1)}{4\alpha^2 r_0^2}\kappa(\kappa-1) + i\frac{(Mc^2-E_{n\kappa}+C_{ps})V_2}{2\alpha^2\hbar^2c^2}}{2(n+\bar{q}+1)} - \frac{(n+\bar{q}+1)}{2}\right]^2, \quad (55)$$

respectively. q and $\bar{q}$ have their usual values as in equations (19) and (36), the corresponding upper and lower component spinors for the SS and the PSS can be obtained directly from equations (27), (31), (43) and (45).

3.3.4 Solutions of the reflectionless-type potential:

If we choose $V_2 = 0$ and $V_1 = \frac{1}{2}\xi(\xi+1)$ in equation (1), then equation (1) becomes the reflectionless-type potential (Grosche & Steiner, 1995; 1998; Zhao et al., 2005):

$$V(r) = -\xi(\xi+1)\text{sech}^2\alpha r, \quad (56)$$

where ξ is an integer, that is, $\xi = 1, 2, 3, \ldots$.

For the SS solutions of the reflectionless-type potential, the energy eigenvalues, the upper and the lower component spinors are obtained, respectively as:

$$(Mc^2+E_{n\kappa}-C_s)(Mc^2-E_{n\kappa}) = -\frac{\kappa(\kappa+1)C_0}{r_0^2}\hbar^2c^2 + \alpha^2\hbar^2c^2\left[\frac{\frac{(C_2-C_1)}{4\alpha^2r_0^2}\kappa(\kappa+1)}{2(n+q_3+1)} - \frac{(n+q_3+1)}{2}\right]^2, \quad (57)$$

$$F_{n\kappa}(r) = N_{n\kappa}(1+e^{-2\alpha r})^{1+q_3}(-e^{-2\alpha r})^{\beta_3}\,{}_2F_1(-n, n+2(\beta_3+q_3+1); 2\beta_3+1; -e^{-2\alpha r})$$
$$= N_{n\kappa}\frac{n!\Gamma(2\beta_3+1)}{\Gamma(n+\beta_3+1)}(1+e^{-2\alpha r})^{1+q_3}(-e^{-2\alpha r})^{\beta_3}P_n^{(2\beta_3,\,2q_3+1)}(1+2e^{-2\alpha r}) \quad (58)$$

and

$$G_{n\kappa}(r) = \frac{N_{n\kappa}(1+e^{-2\alpha r})^{1+q_3}(-e^{-2\alpha r})^{\beta_3}}{[Mc^2+E_{n\kappa}-C_s]}\left[-2\alpha\beta_3 - \frac{2\alpha e^{-2\alpha r}}{1+e^{-2\alpha r}} + \frac{\kappa}{r}\right]$$
$$\times {}_2F_1(-n, n+2(\beta_3+q_3+1); 2\beta_3+1; -e^{-2\alpha r}) +$$
$$\frac{N_{n\kappa}(1+e^{-2\alpha r})^{1+q_3}(-e^{-2\alpha r})^{\beta_3+1}}{[Mc^2+E_{n\kappa}-C_s]}\left\{\frac{2\alpha n\,[n+2(\beta_3+q_3+1)]}{(2\beta_3+1)}\right\}$$
$$\times {}_2F_1(-n+1, n+2(\beta_3+q_3+\tfrac{3}{2}); 2(\beta_3+1); -e^{-2\alpha r}), \quad (59)$$

where

$$q_3 = \frac{1}{2}\left[-1+\sqrt{1+\frac{\kappa(\kappa+1)C_2}{\alpha^2r_0^2} + \frac{2\xi(\xi+1)[Mc^2+E_{n\kappa}-C_s]}{\alpha^2\hbar^2c^2}}\right] \quad (60)$$

and

$$\beta_3 = \sqrt{\frac{\kappa(\kappa+1)}{4\alpha^2 r_0^2} C_0 - \frac{1}{4\alpha^2\hbar^2 c^2}\left[E_{n\kappa}^2 - M^2c^4 + C_s(Mc^2 - E_{n\kappa})\right]}. \tag{61}$$

For the PSS solutions of the reflectionless-type potential, the energy eigenvalues, the upper and the lower component spinors are obtained, respectively as:

$$(Mc^2 - E_{n\kappa} + C_{ps})(Mc^2 + E_{n\kappa}) = -\frac{\kappa(\kappa-1)C_0}{r_0^2}\hbar^2c^2 + \alpha^2\hbar^2c^2\left[\frac{\frac{(C_2-C_1)}{\alpha^2 r_0^2}\kappa(\kappa-1)}{2(n+\overline{q}_3+1)} - \frac{(n+\overline{q}_3+1)}{2}\right]^2, \tag{62}$$

$$\begin{aligned} G_{n\kappa}(r) &= \overline{N}_{n\kappa}(1+e^{-2\alpha r})^{1+\overline{q}_3}(-e^{-2\alpha r})^{\overline{\beta}_3}\,{}_2F_1(-n, n+2(\overline{\beta}_3+\overline{q}_3+1); 2\overline{\beta}_3+1; -e^{-2\alpha r}) \\ &= \overline{N}_{n\kappa}\frac{n!\Gamma(2\overline{\beta}_3+1)}{\Gamma(n+\overline{\beta}_3+1)}(1+e^{-2\alpha r})^{1+\overline{q}_3}(-e^{-2\alpha r})^{\overline{\beta}_3} P_n^{(2\overline{\beta}_3,\, 2\overline{q}_3+1)}(1+2e^{-2\alpha r}) \end{aligned} \tag{63}$$

and

$$\begin{aligned} F_{n\kappa}(r) = {} & \frac{\overline{N}_{n\kappa}(1+e^{-2\alpha r})^{1+\overline{q}_3}(-e^{-2\alpha r})^{\overline{\beta}_3}}{\left[Mc^2 - E_{n\kappa} + C_{ps}\right]}\left[-2\alpha\overline{\beta}_3 - \frac{2\alpha e^{-2\alpha r}}{1+e^{-2\alpha r}} - \frac{\kappa}{r}\right] \\ & \times {}_2F_1(-n, n+2(\overline{\beta}_3+\overline{q}_3+1); 2\overline{\beta}_3+1; -e^{-2\alpha r}) + \\ & \frac{\overline{N}_{n\kappa}(1+e^{-2\alpha r})^{1+\overline{q}_3}(-e^{-2\alpha r})^{\overline{\beta}_3+1}}{\left[Mc^2 - E_{n\kappa} + C_{ps}\right]}\left\{\frac{2\alpha n\left[n+2(\overline{\beta}_3+\overline{q}_3+1)\right]}{(2\overline{\beta}_3+1)}\right\} \\ & \times {}_2F_1(-n+1, n+2(\overline{\beta}_3+\overline{q}_3+\tfrac{3}{2}); 2(\overline{\beta}_3+1); -e^{-2\alpha r}), \end{aligned} \tag{64}$$

where

$$\overline{q}_3 = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{\kappa(\kappa-1)C_2}{\alpha^2 r_0^2} - \frac{2\xi(\xi+1)[Mc^2 - E_{n\kappa} + C_{ps}]}{\alpha^2\hbar^2c^2}}\right] \tag{65}$$

and

$$\overline{\beta}_3 = \sqrt{\frac{\kappa(\kappa-1)}{4\alpha^2 r_0^2}C_0 - \frac{1}{4\alpha^2\hbar^2c^2}\left[E_{n\kappa}^2 - M^2c^4 - C_{ps}(Mc^2 + E_{n\kappa})\right]}. \tag{66}$$

3.3.5 Solutions of the non-relativistic limit

The approximate solutions of the Schrödinger equation for the Rosen-Morse potential including the centrifugal term can be obtained from our work. This can be done by equating $C_s = 0, S(r) = V(r) = \Sigma(r)$ in equations (26) and (27). By using the following appropriate transformations suggested by Ikhdair (2010):

$$\begin{gathered} \frac{(Mc^2 + E_{n\kappa})}{\hbar^2c^2} \to \frac{2\mu}{\hbar^2} \\ Mc^2 - E_{n\kappa} \to -E_{n\ell} \\ \kappa \to \ell, \end{gathered} \tag{67}$$

the non-relativistic limit of the energy equation and the associated wave functions, respectively become:

$$E_{n\ell} = V_2 + \frac{\ell(\ell+1)\hbar^2 C_0}{2\mu r_0^2} - \frac{\hbar^2 c^2}{2\mu}\left[\frac{(n+1)^2 + (2n+1)q_0 + \frac{\ell(\ell+1)C_1}{4\alpha^2 r_0^2} + \frac{\mu}{\alpha^2\hbar^2}(2V_1+V_2)}{(n+q_0+1)}\right]^2, \tag{68}$$

and

$$\begin{aligned} F_{n\ell}(r) &= N_{n\ell}(1+e^{-2\alpha r})^{1+q_0}(-e^{-2\alpha r})^{\beta_0}\,{}_2F_1(-n, n+2(\beta_0+q_0+1); 2\beta_0+1; -e^{-2\alpha r}) \\ &= N_{n\ell}\frac{n!\Gamma(2\beta_0+1)}{\Gamma(n+\beta_0+1)}(1+e^{-2\alpha r})^{1+q_0}(-e^{-2\alpha r})^{\beta_0} P_n^{(2\beta_0,\, 2q_0+1)}(1-2z)\,, \end{aligned} \tag{69}$$

where

$$q_0 = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{\ell(\ell+1)C_2}{\alpha^2 r_0^2} + \frac{8\mu V_1}{\alpha^2\hbar^2}}\right] \tag{70}$$

and

$$\beta_0 = \sqrt{\frac{\ell(\ell+1)}{4\alpha^2 r_0^2}C_0 + \frac{\mu V_2}{2\alpha^2\hbar^2} - \frac{\mu E_{n\ell}}{2\alpha^2\hbar^2}}. \tag{71}$$

By using the appropriate transformations suggested by Ikhdair (2010), the non-relativistic limit of energy equation and the associated wave functions of the Schrödinger equation for the Rosen-Morse potential are recovered completely. These results are identical with the results of Ikhdair (2010), Taşkin (2009)(note that Taşkin (2009) used $\hbar = \mu = 1$ in his calculations).

4. The relativistic bound state solutions of the Rosen-Morse potential with the centrifugal term

The Klein-Gordon and the Dirac equations describe relativistic particles with zero or integer and 1/2 integral spins, respectively (Landau & Lifshift 1999; Merzbacher, 1998; Greiner, 2000; Alhaidari et al., 2006; Dong, 2007). However, the exact solutions are only possible for a few simple systems such as the hydrogen atom, the harmonic oscillator, Kratzer potential and pseudoharmonic potential.

In the following specific examples, Soylu et al. (2008c) obtained the s-wave solutions of the Klein-Gordon equation with equal scalar and vector Rosen-Morse potential by using the asymptotic iteration method. Also, Yi et al. (2004) obtained the energy equation and the corresponding wave functions of the Klein-Gordon equation for the Rosen-Morse-type potential by using standard and functional method.

For the approximate solutions of the Schrödinger equation for the Rosen-Morse potential with the centrifugal term, that is $\ell \neq 0$ or $\kappa = 1$, with the standard function analysis method, Taşkin (2009) has used the newly improved Pekeris-type approximation introduced by Lu (2005). In addition, Ikot and Akpabio (2010) solved this same problem by using the Nikiforov-Uvarov method, they used the approximation scheme introduced by Jia et al. (2009a, 2009b) and Xu et al. (2010).

In the recent years, some researchers have used the Pekeris-type approximation scheme for the centrifugal term to solve the relativistic equations to obtain the ℓ or $\kappa-$ wave energy equations and the associated wave functions of some potentials. These include: Morse potential (Bayrak et al., 2010), hyperbolical potential (Wei & Liu, 2008), Manning-Rosen potential (Wei & Dong, 2010), Deng-Fan oscillator (Dong, 2011).
In the context of the standard function analysis approach, the approximate bound state solutions of the arbitrary ℓ-state Klein-Gordon and κ-state Dirac equations for the equally mixed Rosen-Morse potential will be obtained by introducing a newly improved approximation scheme to the centrifugal term.

4.1 Approximate bound state solutions of the Klein-Gordon equation for the Rosen-Morse potential for $\ell \neq 0$

The time-independent Klein-Gordon equation with the scalar $S(r)$ and vector $V(r)$ potentials is given as (Landau & Lifshift, 1999; Merzbacher, 1998; Greiner, 2000; Alhaidari et al., 2006):

$$\left\{-\hbar^2c^2\nabla^2+\left[Mc^2+S(r)\right]^2-[E-V(r)]^2\right\}\psi(r,\theta,\phi)=0, \tag{72}$$

where $M,\hbar$ and c are the rest mass of the spin-0 particle, Planck's constant and velocity of the light, respectively. For spherical symmetrical scalar and vector potentials, putting

$$\psi_{n,\ell,m}(r,\theta,\phi)=\frac{1}{r}U_{n,\ell}(r)Y_{\ell,m}(\theta,\phi), \tag{73}$$

where $Y_{\ell,m}(\theta,\phi)$ is the spherical harmonic function, we obtain the radial Klein-Gordon equation as

$$U''_{n,\ell}(r)+\frac{1}{\hbar^2c^2}\left\{E^2-M^2c^4-2\left[EV(r)+Mc^2S(r)\right]+\left[V^2(r)-S^2(r)\right]-\frac{\ell(\ell+1)\hbar^2c^2}{r^2}\right\}U_{n,\ell}(r)=0. \tag{74}$$

We are considering the case when the scalar and vector potentials are equal (that is, S(r)= V(r)), coupled with the resulting simplification in the solution of the relativistic problems as discussed by Alhaidari et al., 2006, we have

$$U''_{n,\ell}(r)+\frac{1}{\hbar^2c^2}\left\{E^2-M^2c^4-\left[E+Mc^2\right]V(r)-\frac{\ell(\ell+1)\hbar^2c^2}{r^2}\right\}U_{n,\ell}(r)=0. \tag{75}$$

This equation cannot be solved analytically for the Rosen-Morse potential with $\ell \neq 0$, unless, we introduce the approximation scheme (earlier discussed in this chapter) to the centrifugal term. With this approximation scheme, and the potential in (1) together with the transformation $z=-e^{-2\alpha r}$ in equation (75), we have

$$z^2U''_{n,\ell}(z)+zU'_{n,\ell}(z)$$

$$+\left[\frac{E^2-M^2c^4}{4\alpha^2\hbar^2c^2}-\frac{\widetilde{V}_5}{\alpha^2}\frac{z}{(1-z)^2}-\frac{\widetilde{V}_6}{4\alpha^2}\frac{(1+z)}{(1-z)}-\frac{\ell(\ell+1)C_0}{4\alpha^2r_0^2}-\frac{\ell(\ell+1)C_1}{4\alpha^2r_0^2}\frac{z}{(1-z)}-\frac{\ell(\ell+1)C_2}{4\alpha^2r_0^2}\frac{z^2}{(1-z)^2}\right]U_{n,\ell}(z)=0, \tag{76}$$

where

$$\widetilde{V}_5 = \frac{V_1}{\hbar^2 c^2}[E + Mc^2]$$
$$\widetilde{V}_6 = \frac{V_2}{\hbar^2 c^2}[E + Mc^2]. \tag{77}$$

In the similar manner, the energy equation of the arbitrary ℓ-state Klein-Gordon equation with equal scalar and vector potentials of the Rosen-Morse potential is obtained as follows:

$$(E_{n,\ell}^2 - M^2c^4) = (E_{n,\ell} + Mc^2)V_2 + \frac{\ell(\ell+1)C_0}{r_0^2}\hbar^2c^2$$
$$-4\alpha^2\hbar^2c^2\left[\frac{\frac{(C_2-C_1)}{4\alpha^2 r_0^2}\ell(\ell+1) - \frac{(E_{n,\ell}+Mc^2)V_2}{2\alpha^2\hbar^2c^2}}{2(n+\delta_1+1)} - \frac{(n+\delta_1+1)}{2}\right]^2, \tag{78}$$

where

$$\delta_1 = \frac{1}{2}\left[-1+\sqrt{1+\frac{\ell(\ell+1)C_2}{\alpha^2 r_0^2}+\frac{4(E_{n,\ell}+Mc^2)}{\alpha^2\hbar^2c^2}}\right]. \tag{79}$$

The associated wave function can be expressed as

$$U_{n,\ell}(r) = N_{n,\ell}(1+e^{-2\alpha r})^{1+\delta_1}(-e^{-2\alpha r})^{\xi_1}\,{}_2F_1(-n, n+2(\xi_1+\delta_1+1); 2\xi_1+1; -e^{-2\alpha r})$$
$$= N_{n,\ell}\frac{n!\Gamma(2\xi_1+1)}{\Gamma(n+\xi_1+1)}(1+e^{-2\alpha r})^{1+\delta_1}(-e^{-2\alpha r})^{\xi_1}P_n^{(2\xi_1,\,2\delta_1+1)}(1+2e^{-2\alpha r}) \tag{80}$$

where

$$\xi_1 = \sqrt{\frac{\ell(\ell+1)C_0}{4\alpha^2 r_0^2}+\frac{V_2(E_{n,\ell}+Mc^2)}{4\alpha^2\hbar^2c^2}-\frac{E^2-M^2c^4}{4\alpha^2\hbar^2c^2}} \tag{81}$$

and $N_{n,\ell}$ is the normalization constant which can easily be determined in the usual manner.

4.2 Approximate bound state solutions of the Dirac equation for the Rosen-Morse potential for any κ

In this subsection, we consider equations (2), (3) and (4), and on re-writing equations (5) and (6) for the case of equal scalar and vector, i. e. $V(r) = S(r)$, we have the following two coupled differential equations:

$$\left(\frac{d}{dr} - \frac{\kappa}{r}\right)F_{n\kappa}(r) = \left[Mc^2 + E_{n\kappa}\right]G_{n\kappa}, \tag{82}$$
$$\left(\frac{d}{dr} + \frac{\kappa}{r}\right)G_{n\kappa}(r) = \left[Mc^2 - E_{n\kappa}\right]F_{n\kappa}. \tag{83}$$

With the substitution of equation (82) into equation (83) and taking into consideration the suggestion of Alhaidari et al., (2006), a Schrödinger-like equation for the arbitrary spin-orbit coupling quantum number κ is obtained as

$$\frac{d^2F_{n\kappa}(r)}{dr^2} + \frac{1}{\hbar^2c^2}\left\{[E_{n\kappa}^2 - M^2c^4] - [Mc^2 + E_{n\kappa}]V(r) - \frac{\hbar^2c^2\kappa(\kappa-1)}{r^2}\right\}F_{n\kappa}(r) = 0. \tag{84}$$

Here, it is observed that equation (84) is identical with equation (75). Therefore, the energy equation of the Dirac equation with the equally mixed Rosen-Morse potential for arbitrary $\kappa-$state is obtained as

$$\begin{aligned}(E_{n\kappa}^2 - M^2c^4) &= (E_{n\kappa} + Mc^2)V_2 + \frac{\kappa(\kappa-1)C_0}{r_0^2}\hbar^2c^2 \\ &-4\alpha^2\hbar^2c^2\left[\frac{\frac{(C_2-C_1)}{4\alpha^2r_0^2}\kappa(\kappa-1) - \frac{(E_{n\kappa}+Mc^2)V_2}{2\alpha^2\hbar^2c^2}}{2(n+\delta_2+1)} - \frac{(n+\delta_2+1)}{2}\right]^2, \end{aligned} \tag{85}$$

where

$$\delta_2 = \frac{1}{2}\left[-1+\sqrt{1+\frac{\kappa(\kappa-1)C_2}{\alpha^2r_0^2}+\frac{4(E_{n\kappa}+Mc^2)V_1}{\alpha^2\hbar^2c^2}}\right]. \tag{86}$$

The associated upper component spinor $F_{n\kappa}(r)$ is obtained as

$$\begin{aligned}F_{n\kappa}(r) &= N_{n\kappa}(1+e^{-2\alpha r})^{1+\delta_2}(-e^{-2\alpha r})^{\xi_2}\,{}_2F_1(-n, n+2(\xi_2+\delta_2+1); 2\xi_2+1; -e^{-2\alpha r}) \\ &= N_{n\kappa}\frac{n!\Gamma(2\xi_2+1)}{\Gamma(n+\xi_2+1)}(1+e^{-2\alpha r})^{1+\delta_2}(-e^{-2\alpha r})^{\xi_2}P_n^{(2\xi_2,\, 2\delta_2+1)}(1+2e^{-2\alpha r}).\end{aligned} \tag{87}$$

On substituting equation (87) into equation (82) and by using the recurrence relation of the hypergeometric function in equation (30), the lower component spinor $G_{n\kappa}(r)$ can be obtained as

$$\begin{aligned}G_{n\kappa}(r) &= \frac{N_{n\kappa}(1+e^{-2\alpha r})^{1+\delta_2}(-e^{-2\alpha r})^{\xi_2}}{[E_{n\kappa}+Mc^2]}\left[-2\alpha\xi_2 - \frac{2\alpha e^{-2\alpha r}}{1+e^{-2\alpha r}} - \frac{\kappa}{r}\right] \\ &\times {}_2F_1(-n, n+2(\xi_2+\delta_2+1); 2\xi_2+1; -e^{-2\alpha r}) + \\ &\frac{N_{n\kappa}(1+e^{-2\alpha r})^{1+\delta_2}(-e^{-2\alpha r})^{\xi_2+1}}{[E_{n\kappa}+Mc^2]}\left\{\frac{2\alpha n\,[n+2(\xi_2+\delta_2+1)]}{(2\xi_2+1)}\right\} \\ &\times {}_2F_1(-n+1, n+2(\xi_2+\delta_2+\tfrac{3}{2}); 2(\xi_2+1); -e^{-2\alpha r}),\end{aligned} \tag{88}$$

where

$$\xi_2 = \sqrt{\frac{\kappa(\kappa-1)C_0}{4\alpha^2r_0^2}+\frac{V_2(E_{n\kappa}+Mc^2)}{4\alpha^2\hbar^2c^2}-\frac{E^2-M^2c^4}{4\alpha^2\hbar^2c^2}} \tag{89}$$

and $N_{n\kappa}$ is the normalization constant which can easily be determined in the usual manner. Substitution of $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ into equation (5) gives the bound state spinors of the Dirac equation with the equally mixed Rosen-Morse potential for the arbitrary spin-orbit coupling quantum number κ. In the similar manner, approximate solutions can be obtained when $S(r) = -V(r)$.

5. Conclusions

The approximate analytical solutions of the Dirac equation with the Rosen-Morse potential with arbitrary κ under the pseudospin and spin symmetry conditions have been studied, the standard function analysis approach has been adopted. The Pekeris-type approximation

scheme (a newly improved approximation scheme) has been used for the centrifugal (or pseudo centrifugal) term in order to solve for any values of κ.
Under the PSS and SS conditions, the energy equations, the upper- and the lower-component spinors for the Rosen-Morse potential for any κ have been obtained. The solutions of some special cases are also considered and the energy equations with their associated spinors for the PSS and SS are obtained, these include:
(i) the s-state solution,
(ii) the standard Eckart potential,
(iii) the PT-Symmetric Rosen-Morse potential,
(iv) the reflectionless-type potential,
(v) the non-relativistic limit.
Also, in the context of the standard function analysis approach, the approximate bound state solutions of the arbitrary ℓ-state Klein-Gordon and κ-state Dirac equations for the equally mixed Rosen-Morse potential are obtained by introducing a newly improved approximation scheme to the centrifugal term. The approximate analytical solutions with the Dirac-Rosen-Morse potential for any κ or ℓ have been obtained. The upper- and lower-component spinors have been expressed in terms of the hypergeometric functions (or Jacobi polynomials). The approximate analytical solutions obtained in this study are the same with other results available in the literature.

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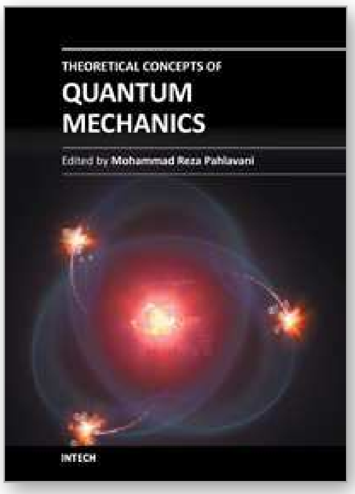

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中国上海市延安西路65号上海国际贵都大饭店办公楼405单元
Phone: +86-21-62489820
Fax: +86-21-62489821