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# Spontaneous Supersymmetry Breaking, Localization and Nicolai Mapping in Matrix Models

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## 1. Introduction

Supersymmetry (SUSY) is a symmetry between bosons and fermions. It leads to degeneracies of mass spectra between bosons and fermions. Although such degeneracies have not been observed yet, there is a possibility for SUSY being realized in nature as a spontaneously broken symmetry. From a theoretical viewpoint, SUSY provides a unified framework describing physics in high energy regime beyond the standard model (Sohnius, 1985). Spontaneous breaking of SUSY is one of the most interesting phenomena in quantum field theory. Since in general SUSY cannot be broken by radiative corrections at the perturbative level, its spontaneous breaking requires understanding of nonperturbative aspects of quantum field theory (Witten, 1981). In particular, recent developments in nonperturbative aspects of string theory heavily rely on the presence of SUSY. Thus, in order to deduce predictions to the real world from string theory, it is indispensable and definitely important to investigate a mechanism of spontaneous SUSY breaking in a nonperturbative framework of strings. Since one of the most promising approaches of nonperturbative formulations of string theory is provided by large- $N$  matrix models (Banks et al., 1997; Dijkgraaf et al., 1997; Ishibashi et al., 1997), it will be desirable to understand how SUSY can be spontaneously broken in the large- $N$  limit of simple matrix models as a first step. Analysis of SUSY breaking in simple matrix models would help us find a mechanism which is responsible for possible spontaneous SUSY breaking in nonperturbative string theory.

For this purpose, it is desirable to treat systems in which spontaneous SUSY breaking takes place in the path-integral formalism, because matrix models are usually defined by the path integrals, namely integrals over matrix variables. In particular, IIB matrix model defined in zero dimension can be formulated only by the path-integral formalism (Ishibashi et al., 1997). Motivated by this, we discuss in the next section the path-integral formalism for (discretized) SUSY quantum mechanics, which includes cases that SUSY is spontaneously broken. Analogously to the situation of ordinary spontaneous symmetry breaking, we introduce an external field to choose one of degenerate broken vacua to detect spontaneous SUSY breaking. The external field plays the same role as a magnetic field in the Ising model introduced to detect the spontaneous magnetization. For the supersymmetric system, we deform the boundary condition for fermions from the periodic boundary condition (PBC) to a twisted boundary condition (TBC) with twist  $\alpha$ , which can be regarded as such an external

field. If a supersymmetric system undergoes spontaneous SUSY breaking, the partition function with the PBC for all the fields,  $Z_{\text{PBC}}$ , which usually corresponds to the Witten index, is expected to vanish (Witten, 1982). Then, the expectation values of observables, which are normalized by  $Z_{\text{PBC}}$ , would be ill-defined or indefinite. By introducing the twist, the partition function is regularized and the expectation values become well-defined. It is an interesting aspect of our external field for SUSY breaking, which is not seen in spontaneous breaking of ordinary (bosonic) symmetry.

Notice that our argument can be applied to systems in less than one-dimension, for example discretized SUSY quantum mechanics with a finite number of discretized time steps. Spontaneous SUSY breaking is observed even in such simple systems with lower degrees of freedom. Also, we give some argument that an analog of the Mermin-Wagner-Coleman theorem (Coleman, 1973; Mermin & Wagner, 1966) does not hold for SUSY. Thus, cooperative phenomena are not essential to cause spontaneous SUSY breaking, which makes a difference from spontaneous breaking of the ordinary (bosonic) symmetry.

In this setup, we compute an order parameter of SUSY breaking such as the expectation value of an auxiliary field in the presence of the external field. If it remains nonvanishing after turning off the external field, it shows that SUSY is spontaneously broken because it implies that the effect of the infinitesimal external field we have introduced at the beginning remains. Here, it should be noticed that, if we are interested in the large- $N$  behavior of SUSY matrix models, we have to take the large- $N$  limit before turning off the external field, which is reminiscent of the thermodynamic limit of the Ising model taken before turning off the magnetic field in detecting the spontaneous  $Z_2$  breaking.

In view of this, it is quite important to calculate the partition function in the presence of the external field in the path integral for systems which spontaneously break SUSY. Especially it would be better to calculate it in matrix models at finite  $N$  in order to observe breaking/restoration of SUSY in the large- $N$  limit. We address this problem by utilizing two methods: localization and Nicolai mapping (Nicolai, 1979) in sections 3 and 4, respectively.

As for the localization, in section 3 we make change of integration variables in the path integral, which is always possible whether or not the SUSY is explicitly broken (the external field is on or off). It is investigated in detail how the integrand of the partition function with respect to the integral over the auxiliary field behaves as the auxiliary field approaches to zero. It plays a crucial role to understand the localization from the change of variables. For SUSY matrix models with  $Q$ -SUSY preserved, the path integral receives contributions only from the fixed points of  $Q$ -transformation, which are nothing but the critical points of superpotential, i.e. zeros of the first derivative of superpotential. However, in terms of eigenvalues of matrix variables, an interesting phenomenon arises. Localization attracts the eigenvalues to the critical points of superpotential, while the square of the Vandermonde determinant arising from the measure factor prevents the eigenvalues from collapsing. The dynamics of the eigenvalues is governed by balance of attractive force from the localization and repulsive force from the Vandermonde determinant. Without the external field, contribution to the partition function from each eigenvalue distributed around some critical point is derived for a general superpotential.

In the case that the external field is turned on, computation is still possible, but in section 4 we find that a method by the Nicolai mapping is more effective. Interestingly, the Nicolai mapping works for SUSY matrix models even in the presence of the external field which explicitly breaks SUSY. It enables us to calculate the partition function at least in the leading nontrivial order of an expansion with respect to the small external field for finite  $N$ . We can

take the large- $N$  limit of our result before turning off the external field and detect whether SUSY is spontaneously broken or not in the large- $N$  limit. For illustration, we obtain large- $N$  solutions for a SUSY matrix model with double-well potential.

Section 5 is devoted to summarize the results and discuss future directions.

This chapter is mainly based on the two papers (Kuroki & Sugino, 2010; 2011).

## 2. Preliminaries on SUSY quantum mechanics

As a preparation to discuss large- $N$  SUSY matrix models, in this section we present some preliminary results on SUSY quantum mechanics.

Let us start with a system defined by the Euclidean (Wick-rotated) action:

$$S^{\text{QM}} = \int_0^\beta dt \left[ \frac{1}{2} B^2 + iB (\dot{\phi} + W'(\phi)) + \bar{\psi} (\dot{\psi} + W''(\phi)\psi) \right], \quad (1)$$

where  $\phi$  is a real scalar field,  $\psi, \bar{\psi}$  are fermions, and  $B$  is an auxiliary field. The dot means the derivative with respect to the Euclidean time  $t \in [0, \beta]$ . For a while, all the fields are supposed to obey the PBC.  $W(\phi)$  is a real function of  $\phi$  called superpotential, and the prime ( $'$ ) represents the  $\phi$ -derivative.

$S^{\text{QM}}$  is invariant under one-dimensional  $\mathcal{N} = 2$  SUSY transformations generated by  $Q$  and  $\bar{Q}$ . They act on the fields as

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \quad (2)$$

and

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB + 2\dot{\phi}, \quad \bar{Q}B = 2i\dot{\psi}, \quad (3)$$

with satisfying the algebra

$$Q^2 = \bar{Q}^2 = 0, \quad \{Q, \bar{Q}\} = 2\partial_t. \quad (4)$$

Note that  $S^{\text{QM}}$  can be written as the  $Q$ - or  $Q\bar{Q}$ -exact form:

$$S^{\text{QM}} = Q \int dt \bar{\psi} \left\{ \frac{i}{2} B - (\dot{\phi} + W'(\phi)) \right\} \quad (5)$$

$$= Q\bar{Q} \int dt \left( \frac{1}{2} \bar{\psi}\psi + W(\phi) \right). \quad (6)$$

For demonstration, let us consider the case of the derivative of the superpotential

$$W'(\phi) = g(\phi^2 - \mu^2) \quad \text{with} \quad g, \mu^2 \in \mathbf{R}. \quad (7)$$

For  $\mu^2 < 0$ , the classical minimum is given by the static configuration  $\phi = 0$ , with its energy  $E_0 = \frac{1}{2}g^2\mu^4 > 0$  implying spontaneous SUSY breaking. Then,  $B = ig\mu^2 \neq 0$  from the equation of motion, leading to  $Q\bar{\psi}, \bar{Q}\psi \neq 0$ , which also means the SUSY breaking.

For  $\mu^2 > 0$ , the classical minima  $\phi = \pm\sqrt{\mu^2}$  are zero-energy configurations. It is known that the quantum tunneling (instantons) between the minima resolves the degeneracy giving positive energy to the ground state. SUSY is broken also in this case.

Next, let us consider quantum aspects of the SUSY breaking in this model. For later discussions on matrix models, it is desirable to observe SUSY breaking via the path-integral formalism, that is, by seeing the expectation value of some field. We take  $\langle B \rangle$  (or  $\langle B^n \rangle$ )

( $n = 1, 2, \dots$ ) as such an order parameter. Whichever  $\mu^2$  is positive or negative, the SUSY is broken, so the ground state energy  $E_0$  is positive. Then, for each of the energy levels  $E_n$  ( $0 < E_0 < E_1 < E_2 < \dots$ ), the SUSY algebra<sup>1</sup>

$$\{Q, \bar{Q}\} = 2E_n, \quad Q^2 = \bar{Q}^2 = 0 \quad (8)$$

leads to the SUSY multiplet formed by bosonic and fermionic states

$$|b_n\rangle = \frac{1}{\sqrt{2E_n}} \bar{Q}|f_n\rangle, \quad |f_n\rangle = \frac{1}{\sqrt{2E_n}} Q|b_n\rangle. \quad (9)$$

As a convention, we assume that  $|b_n\rangle$  and  $|f_n\rangle$  have the fermion number charges  $F = 0$  and  $1$ , respectively. Since the  $Q$ -transformation for  $B$  in (2) is expressed as  $[Q, B] = 0$  in the operator formalism, we can see that

$$\langle b_n|B|b_n\rangle = \langle f_n|B|f_n\rangle \quad (10)$$

holds for each  $n$ . Then, it turns out that the unnormalized expectation value of  $B$  vanishes<sup>2</sup>:

$$\begin{aligned} \langle B \rangle' &\equiv \int_{\text{PBC}} d(\text{fields}) B e^{-S^{\text{QM}}} = \text{Tr} [B(-1)^F e^{-\beta H}] \\ &= \sum_{n=0}^{\infty} (\langle b_n|B|b_n\rangle - \langle f_n|B|f_n\rangle) e^{-\beta E_n} = 0. \end{aligned} \quad (11)$$

This observation shows that, in order to judge SUSY breaking from the expectation value of  $B$ , we should choose either of the SUSY broken ground states ( $|b_0\rangle$  or  $|f_0\rangle$ ) and see the expectation value with respect to the chosen ground state. The situation is somewhat analogous to the case of spontaneous breaking of ordinary (bosonic) symmetry.

However, differently from the ordinary case, when SUSY is broken, the supersymmetric partition function vanishes:

$$Z_{\text{PBC}}^{\text{QM}} = \int_{\text{PBC}} d(\text{fields}) e^{-S^{\text{QM}}} = \text{Tr} [(-1)^F e^{-\beta H}] \quad (12)$$

$$= \sum_{n=0}^{\infty} (\langle b_n|b_n\rangle - \langle f_n|f_n\rangle) e^{-\beta E_n} = 0, \quad (13)$$

where the normalization  $\langle b_n|b_n\rangle = \langle f_n|f_n\rangle = 1$  was used. So, the expectation values normalized by  $Z_{\text{PBC}}^{\text{QM}}$  could be ill-defined (Kanamori et al., 2008a;b).

## 2.1 Twisted boundary condition

To detect spontaneous breaking of ordinary symmetry, some external field is introduced so that the ground state degeneracy is resolved to specify a single broken ground state. The external field is turned off after taking the thermodynamic limit, then we can judge whether spontaneous symmetry breaking takes place or not, seeing the value of the corresponding order parameter. (For example, to detect the spontaneous magnetization in the Ising model, the external field is a magnetic field, and the corresponding order parameter is the expectation value of the spin operator.)

<sup>1</sup> In the operator formalism,  $\bar{Q}$ ,  $\bar{\psi}$  are regarded as hermitian conjugate to  $Q$ ,  $\psi$ , respectively.

<sup>2</sup> Furthermore,  $\langle B^n \rangle' = 0$  ( $n = 1, 2, \dots$ ) can be shown.

We will do a similar thing also for the case of spontaneous SUSY breaking. For this purpose, let us change the boundary condition for the fermions to the TBC:

$$\psi(t + \beta) = e^{i\alpha} \psi(t), \quad \bar{\psi}(t + \beta) = e^{-i\alpha} \bar{\psi}(t), \quad (14)$$

then the twist  $\alpha$  can be regarded as an external field. Other fields remain intact. As seen shortly in section 2.1.1, the partition function with the TBC corresponds to the expression (12) with  $(-1)^F$  replaced by  $(-e^{-i\alpha})^F$ :

$$Z_\alpha^{\text{QM}} \equiv -e^{-i\alpha} \int_{\text{TBC}} d(\text{fields}) e^{-S^{\text{QM}}} = \text{Tr} \left[ (-e^{-i\alpha})^F e^{-\beta H} \right] \quad (15)$$

$$= \sum_{n=0}^{\infty} \left( \langle b_n | b_n \rangle - e^{-i\alpha} \langle f_n | f_n \rangle \right) e^{-\beta E_n} = \left( 1 - e^{-i\alpha} \right) \sum_{n=0}^{\infty} e^{-\beta E_n}. \quad (16)$$

Then, the normalized expectation value of  $B$  under the TBC becomes

$$\begin{aligned} \langle B \rangle_\alpha &\equiv \frac{1}{Z_\alpha^{\text{QM}}} \text{Tr} \left[ B (-e^{-i\alpha})^F e^{-\beta H} \right] \\ &= \frac{1}{Z_\alpha^{\text{QM}}} \sum_{n=0}^{\infty} \left( \langle b_n | B | b_n \rangle - e^{-i\alpha} \langle f_n | B | f_n \rangle \right) e^{-\beta E_n} \\ &= \frac{\sum_{n=0}^{\infty} \langle b_n | B | b_n \rangle e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = \frac{\sum_{n=0}^{\infty} \langle f_n | B | f_n \rangle e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}}. \end{aligned} \quad (17)$$

Note that the factors  $(1 - e^{-i\alpha})$  in the numerator and the denominator cancel each other, and thus  $\langle B \rangle_\alpha$  does not depend on  $\alpha$  even for finite  $\beta$ . As a result,  $\langle B \rangle_\alpha$  is equivalent to the expectation value taken over one of the ground states and its excitations  $\{|b_n\rangle\}$  (or  $\{|f_n\rangle\}$ ). The normalized expectation value of  $B$  under the PBC was of the indefinite form  $0/0$ , which is now regularized by introducing the parameter  $\alpha$ . The expression (17) is well-defined.

On the other hand, from the  $Q$ -transformation  $\psi = [Q, \phi]$ , we have

$$\langle b_n | \phi | b_n \rangle = \langle f_n | \phi | f_n \rangle + \frac{1}{\sqrt{2E_n}} \langle f_n | \psi | b_n \rangle. \quad (18)$$

The second term is a transition between bosonic and fermionic states via the fermionic operator  $\psi$ , which does not vanish in general. Thus, differently from  $\langle B \rangle_\alpha$ , the expectation value of  $\phi$  becomes

$$\begin{aligned} \langle \phi \rangle_\alpha &= \frac{1}{Z_\alpha^{\text{QM}}} \text{Tr} \left[ \phi (-e^{-i\alpha})^F e^{-\beta H} \right] \\ &= \frac{1}{Z_\alpha^{\text{QM}}} \sum_{n=0}^{\infty} \left( \langle b_n | \phi | b_n \rangle - e^{-i\alpha} \langle f_n | \phi | f_n \rangle \right) e^{-\beta E_n} \\ &= \frac{\sum_{n=0}^{\infty} \langle f_n | \phi | f_n \rangle e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} + \frac{1}{1 - e^{-i\alpha}} \frac{\sum_{n=0}^{\infty} \langle f_n | \psi | b_n \rangle \frac{1}{\sqrt{2E_n}} e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}}. \end{aligned} \quad (19)$$

When  $\langle f_n | \psi | b_n \rangle \neq 0$  for some  $n$ , the second term is  $\alpha$ -dependent and diverges as  $\alpha \rightarrow 0$ . The divergence comes from the transition between  $|b_n\rangle$  and  $|f_n\rangle$ . Since the two states are transformed to each other by the (broken) SUSY transformation, we can say that they should belong to the separate superselection sectors, in analogy to spontaneous breaking of ordinary (bosonic) symmetry. Thus, the divergence of  $\langle \phi \rangle_\alpha$  as  $\alpha \rightarrow 0$  implies that the superselection rule does not hold in the system.

### 2.1.1 Partition function with the twist $\alpha$

We here show that the partition function with the TBC for the fermions (14) can be expressed in the form (15).

Let  $\hat{b}, \hat{b}^\dagger$  be annihilation and creation operators of the fermions:

$$\hat{b}^2 = (\hat{b}^\dagger)^2 = 0, \quad \{\hat{b}, \hat{b}^\dagger\} = 1, \quad (20)$$

and they are represented on the Fock space  $\{|0\rangle, |1\rangle\}$  as

$$\hat{b}|0\rangle = 0, \quad \hat{b}^\dagger|0\rangle = |1\rangle. \quad (21)$$

We assume that  $|0\rangle, |1\rangle$  have the fermion numbers  $F = 0, 1$ , respectively.

The coherent states  $|\psi\rangle, \langle\bar{\psi}|$  satisfying

$$\hat{b}|\psi\rangle = \psi|\psi\rangle, \quad \langle\bar{\psi}|\hat{b}^\dagger = \langle\bar{\psi}|\bar{\psi} \quad (22)$$

( $\psi, \bar{\psi}$  are Grassmann numbers, and anticommute with  $\hat{b}, \hat{b}^\dagger$ .) are explicitly constructed as

$$|\psi\rangle = |0\rangle - \psi|1\rangle = e^{-\psi\hat{b}^\dagger}|0\rangle, \quad \langle\bar{\psi}| = \langle 0| - \langle 1|\bar{\psi} = \langle 0|e^{-\bar{\psi}\hat{b}}. \quad (23)$$

Also,

$$|0\rangle = \int d\psi \psi|\psi\rangle, \quad \langle 0| = \int d\bar{\psi} \langle\bar{\psi}|\bar{\psi}, \quad |1\rangle = -\int d\psi |\psi\rangle, \quad \langle 1| = \int d\bar{\psi} \langle\bar{\psi}|. \quad (24)$$

Thus, we can obtain

$$\begin{aligned} \text{Tr} \left[ (-e^{-i\alpha})^F e^{-\beta H} \right] &= \langle 0|e^{-\beta H}|0\rangle - e^{-i\alpha}\langle 1|e^{-\beta H}|1\rangle \\ &= \int d\bar{\psi}d\psi (e^{-i\alpha} + \psi\bar{\psi})\langle\bar{\psi}|e^{-\beta H}|\psi\rangle \\ &= e^{-i\alpha} \int d\bar{\psi}d\psi \exp(e^{i\alpha}\psi\bar{\psi}) \langle\bar{\psi}|e^{-\beta H}|\psi\rangle. \end{aligned} \quad (25)$$

Since the bosonic part of  $H$  is obvious, below we focus on the fermionic part  $H_F = \hat{b}^\dagger W'' \hat{b}$ . Dividing the interval  $\beta$  into  $M$  short segments of length  $\varepsilon$ :  $\beta = M\varepsilon$  in (25) and applying the relations

$$\langle\bar{\psi}|\psi\rangle = e^{\bar{\psi}\psi}, \quad 1 = \int d\bar{\psi}d\psi |\psi\rangle e^{\bar{\psi}\psi} \langle\bar{\psi}| \quad (26)$$

to each segment, we have the following expression:

$$\text{Tr} \left[ (-e^{-i\alpha})^F e^{-\beta H_F} \right] = -e^{-i\alpha} \int \left( \prod_{j=1}^M d\psi_j d\bar{\psi}_j \right) \exp \left[ -\varepsilon \sum_{j=1}^M \bar{\psi}_j \left( \frac{\psi_{j+1} - \psi_j}{\varepsilon} + W'' \psi_j \right) \right] \quad (27)$$

with

$$\psi_{M+1} = e^{i\alpha} \psi_1, \quad (28)$$

or

$$\text{Tr} \left[ (-e^{-i\alpha})^F e^{-\beta H_F} \right] = -e^{-i\alpha} \int \left( \prod_{j=1}^M d\psi_j d\bar{\psi}_j \right) \exp \left[ -\varepsilon \sum_{j=1}^M \left( -\frac{\bar{\psi}_j - \bar{\psi}_{j-1}}{\varepsilon} + \bar{\psi}_j W'' \right) \psi_j \right] \quad (29)$$

with

$$\bar{\psi}_0 = e^{i\alpha} \bar{\psi}_M. \quad (30)$$

Since (28) and (30) correspond to (14) in the continuum limit  $\varepsilon \rightarrow 0, M \rightarrow \infty$  with  $\beta = M\varepsilon$  fixed, we find that the formula (15) holds.

## 2.2 Discretized SUSY quantum mechanics

In this subsection, we consider a discretized system of (1), namely the Euclidean time is discretized as  $t = 1, \dots, T$ . The action is written as

$$S^{\text{dQM}} = Q \sum_{t=1}^T \bar{\psi}(t) \left\{ \frac{i}{2} B(t) - (\phi(t+1) - \phi(t) + W'(\phi(t))) \right\} \quad (31)$$

$$= \sum_{t=1}^T \left[ \frac{1}{2} B(t)^2 + iB(t) \{ \phi(t+1) - \phi(t) + W'(\phi(t)) \} + \bar{\psi}(t) \{ \psi(t+1) - \psi(t) + W''(\phi(t))\psi(t) \} \right], \quad (32)$$

where the  $Q$ -SUSY is of the same form as in (2). As is seen by the  $Q$ -exact form (31), the action is  $Q$ -invariant and the  $Q$ -SUSY is preserved upon the discretization (Catterall, 2003). On the other hand, the  $\bar{Q}$ -SUSY can not be preserved by the discretization in the case of  $T \geq 2$ .

When  $T$  is finite, the partition function or various correlators are expressed as a finite number of integrals with respect to field variables. So, at first sight, one might expect that spontaneous breaking of the SUSY could not take place, because of a small number of the degrees of freedom. In what follows, we will demonstrate that the expectation is not correct, and that the SUSY can be broken even in such a finite system.

Expressing as  $S_{\alpha}^{\text{dQM}}$  the action (32) under the TBC

$$\phi(T+1) = \phi(1), \quad \psi(T+1) = e^{i\alpha} \psi(1), \quad (33)$$

the partition function

$$Z_{\alpha}^{\text{dQM}} \equiv \left( \frac{-1}{2\pi} \right)^T \int \prod_{t=1}^T (dB(t) d\phi(t) d\psi(t) d\bar{\psi}(t)) e^{-S_{\alpha}^{\text{dQM}}} \quad (34)$$

is computed to be

$$Z_{\alpha}^{\text{dQM}} = (-1)^T (1 - e^{i\alpha}) C_T, \quad (35)$$

$$C_T \equiv \int \left( \prod_{t=1}^T \frac{d\phi(t)}{\sqrt{2\pi}} \right) \exp \left[ -\frac{1}{2} \sum_{t=1}^T (\phi(t+1) - \phi(t) + W'(\phi(t)))^2 \right]. \quad (36)$$

Here we used

$$\int \left( \prod_{t=1}^T \frac{d\phi(t)}{\sqrt{2\pi}} \right) \left[ \prod_{t=1}^T (-1 + W''(\phi(t))) - (-1)^T \right] \times \exp \left[ -\frac{1}{2} \sum_{t=1}^T (\phi(t+1) - \phi(t) + W'(\phi(t)))^2 \right] = 0 \quad (37)$$

for the superpotential (7), which is derived from the Nicolai mapping (Nicolai, 1979). (Note the factor  $\left[ \prod_{t=1}^T (-1 + W''(\phi(t))) - (-1)^T \right]$  is equal to the fermion determinant under the PBC.) Also,  $C_T$  is positive definite.

Similarly, for the normalized expectation value

$$\langle B(t) \rangle_\alpha \equiv \frac{1}{Z_\alpha^{\text{dQM}}} \left( \frac{-1}{2\pi} \right)^T \int \prod_{t=1}^T (dB(t) d\phi(t) d\psi(t) d\bar{\psi}(t)) B(t) e^{-S_\alpha^{\text{dQM}}}, \quad (38)$$

we use the Nicolai mapping to have

$$\begin{aligned} \langle B(t) \rangle_\alpha &= \frac{1}{Z_\alpha^{\text{dQM}}} (-1)^T (1 - e^{i\alpha}) \int \left( \prod_{t=1}^T \frac{d\phi(t)}{\sqrt{2\pi}} \right) (-i) (\phi(t+1) - \phi(t) + W'(\phi(t))) \\ &\quad \times \exp \left[ -\frac{1}{2} \sum_{t=1}^T (\phi(t+1) - \phi(t) + W'(\phi(t)))^2 \right] \\ &= \frac{1}{C_T} \int \left( \prod_{t=1}^T \frac{d\phi(t)}{\sqrt{2\pi}} \right) (-i) (\phi(t+1) - \phi(t) + W'(\phi(t))) \\ &\quad \times \exp \left[ -\frac{1}{2} \sum_{t=1}^T (\phi(t+1) - \phi(t) + W'(\phi(t)))^2 \right]. \end{aligned} \quad (39)$$

The factor  $(-1)^T (1 - e^{i\alpha})$  was canceled, and  $\langle B(t) \rangle_\alpha$  does not depend on  $\alpha$ , again. The result (39) is finite and well-defined. By using the Nicolai mapping, it is straightforward to generalize this result to the case of  $W'$  being a general polynomial

$$W'(\phi) = g_p \phi^p + g_{p-1} \phi^{p-1} + \dots + g_0. \quad (40)$$

We find that (39) holds and it is finite and well-defined for even  $p$ , and that  $\lim_{\alpha \rightarrow 0} \langle B(t) \rangle_\alpha = 0$  for odd  $p$ .

### 2.2.1 No analog of Mermin-Wagner-Coleman theorem for SUSY

As claimed in the Mermin-Wagner-Coleman theorem (Coleman, 1973; Mermin & Wagner, 1966), continuous bosonic symmetry cannot be spontaneously broken at the quantum level in the dimensions of two or lower. In dimensions  $D \leq 2$ , although the symmetry might be broken at the classical level, in computing quantum corrections to a classical (nonzero) value of a corresponding order parameter, one encounters infrared (IR) divergences from loops of a massless boson. It indicates that the conclusion of the symmetry breaking from the classical value is not reliable at the quantum level any more. It is a manifestation of the Mermin-Wagner-Coleman theorem.

Here, we consider whether an analog of the Mermin-Wagner-Coleman theorem for SUSY holds or not. Naively, since loops of a massless fermion ("would-be Nambu-Goldstone fermion") would be dangerous in the dimension one or lower, we might be tempted to expect that SUSY could not be spontaneously broken at the quantum level in the dimension of one or lower. However, this expectation is not correct. Because the twist  $\alpha$  in our setting can also be regarded as an IR cutoff for the massless fermion, the finiteness of (39) shows that  $\langle B(t) \rangle_\alpha$  is free from IR divergences and well-defined at the quantum level for less than one-dimension. (For one-dimensional case, (17) has no  $\alpha$ -dependence, thus no IR divergences.)

We can see it more explicitly in perturbative calculations. Let us consider the superpotential (7) with  $\mu^2 < 0$ , where the classical configuration  $\phi(t) = 0$  gives  $B(t) = ig\mu^2$ . If the theorem held, quantum corrections should modify this classical value to zero, and

there we should come across IR divergences owing to a massless fermion. Although we have obtained the finite result (39), the following perturbative analysis would clarify a role played by the massless fermion. We evaluate quantum corrections to the classical value of  $B(t)$  perturbatively. Under the mode expansions

$$\begin{aligned}\phi(t) &= \frac{1}{\sqrt{T}} \sum_{n=-(T-1)/2}^{(T-1)/2} \tilde{\phi}_n e^{i2\pi nt/T} \quad \text{with} \quad \tilde{\phi}_n^* = \tilde{\phi}_{-n}, \\ \psi(t) &= \frac{1}{\sqrt{T}} \sum_{n=-(T-1)/2}^{(T-1)/2} \tilde{\psi}_n e^{i(2\pi n + \alpha)t/T}, \\ \bar{\psi}(t) &= \frac{1}{\sqrt{T}} \sum_{n=-(T-1)/2}^{(T-1)/2} \tilde{\bar{\psi}}_n e^{-i(2\pi n + \alpha)t/T},\end{aligned}\tag{41}$$

free propagators are

$$\begin{aligned}\langle \tilde{\phi}_{-n} \tilde{\phi}_m \rangle_{\text{free}} &= \frac{\delta_{nm}}{4 \sin^2 \left( \frac{\pi n}{T} \right) + M^2}, \\ \langle \tilde{\psi}_n \tilde{\bar{\psi}}_m \rangle_{\text{free}} &= \frac{\delta_{nm}}{e^{i(2\pi n + \alpha)/T} - 1}\end{aligned}\tag{42}$$

with  $M^2 \equiv -2g^2\mu^2$ . Here we consider the case of odd  $T$  for simplicity of the mode expansion. Note that the boson is massive while the fermion is nearly massless regulated by  $\alpha$ . Also, there are three kinds of interactions in  $S_\alpha^{\text{dQM}}$  (after  $B$  is integrated out):

$$\begin{aligned}V_4 &= \sum_{t=1}^T \frac{1}{2} g^2 \phi(t)^4, \\ V_{3B} &= \sum_{t=1}^T g \phi(t)^2 (\phi(t+1) - \phi(t)), \\ V_{3F} &= \sum_{t=1}^T 2g \phi(t) \bar{\psi}(t) \psi(t).\end{aligned}\tag{43}$$

We perturbatively compute the second term of

$$\langle B(t) \rangle_\alpha = ig\mu^2 - i \langle g\phi(t)^2 + \phi(t+1) - \phi(t) \rangle_\alpha\tag{44}$$

up to the two-loop order, and directly see that the nearly massless fermion (“would-be Nambu-Goldstone fermion”) does not contribute and gives no IR singularity. It is easy to see that the tadpole  $\langle \phi(t+1) - \phi(t) \rangle_\alpha$  vanishes from the momentum conservation. For  $-i \langle g\phi(t)^2 \rangle_\alpha$ , the one-loop contribution comes from the diagram (1B) in Figure 1, which consists only of a boson line independent of  $\alpha$ . Also, the two-loop diagrams (2BBa), (2BBb), (2BBc) and (2BBd) do not contain fermion lines. The relevant diagrams for the IR divergence at the two-loop order are the last four (2FFa), (2FFb), (2BFa) and (2BFb), which are evaluated

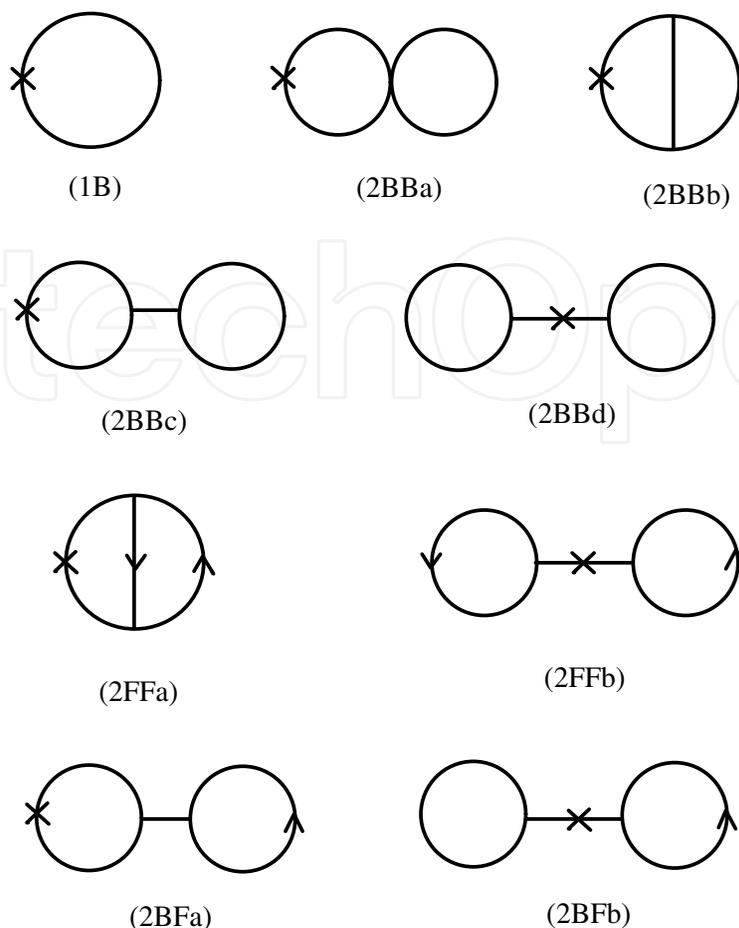


Fig. 1. One- and two-loop diagrams. The crosses represent the insertion of the operator  $-ig\phi(t)^2$ . The solid lines with (without) arrows mean the fermion (boson) propagators. (1B) is the one-loop diagram, and the other eight are the two-loop diagrams. The diagrams with the name “FF” (“BB”) are constructed by using the interaction vertices  $V_{3F}$  twice ( $V_4$  once or  $V_{3B}$  twice), and those with “BF” are by using each of  $V_{3B}$  and  $V_{3F}$  once.

as

$$\begin{aligned}
 (2FFa) &= i \frac{4g^3}{T^2} \sum_{m,k=-(T-1)/2}^{(T-1)/2} \left( \frac{1}{4 \sin^2 \left( \frac{\pi m}{T} \right) + M^2} \right)^2 \frac{1}{e^{i(2\pi k + \alpha)/T} - 1} \frac{1}{e^{i(2\pi(m+k) + \alpha)/T} - 1}, \\
 (2FFb) &= -i \frac{4g^3}{T^2} \frac{1}{M^4} \left( \sum_{m=-(T-1)/2}^{(T-1)/2} \frac{1}{e^{i(2\pi m + \alpha)/T} - 1} \right)^2, \\
 (2BFa) &= -i \frac{4g^3}{T^2} \frac{1}{M^2} \sum_{m=-(T-1)/2}^{(T-1)/2} \left( 1 - \frac{M^2}{4 \sin^2 \left( \frac{\pi m}{T} \right) + M^2} \right) \frac{1}{4 \sin^2 \left( \frac{\pi m}{T} \right) + M^2} \\
 &\quad \times \sum_{k=-(T-1)/2}^{(T-1)/2} \frac{1}{e^{i(2\pi k + \alpha)/T} - 1}, \\
 (2BFb) &= -i \frac{4g^3}{T^2} \frac{1}{M^4} \sum_{m=-(T-1)/2}^{(T-1)/2} \left( 1 - \frac{M^2}{4 \sin^2 \left( \frac{\pi m}{T} \right) + M^2} \right) \sum_{k=-(T-1)/2}^{(T-1)/2} \frac{1}{e^{i(2\pi k + \alpha)/T} - 1}.
 \end{aligned} \tag{45}$$

Each diagram is singular as  $\alpha \rightarrow 0$  due to the fermion zero-mode, however it is remarkable that the sum of them vanishes:

$$\begin{aligned} & (2FFa) + (2FFb) + (2BFa) + (2BFb) \\ &= -i \frac{4g^3}{T^2} \frac{1}{M^4} \sum_{m=1}^{T-1} \left[ 1 - \left( \frac{M^2}{4 \sin^2 \left( \frac{\pi m}{T} \right) + M^2} \right)^2 \right] F(m) \end{aligned} \quad (46)$$

with

$$\begin{aligned} F(m) &\equiv \sum_{k=1}^T \left( 1 + \frac{1}{e^{i(2\pi(m+k)+\alpha)/T} - 1} \right) \frac{1}{e^{i(2\pi k+\alpha)/T} - 1} \\ &= \sum_{k=1}^T \frac{1}{e^{i(2\pi k+\alpha)/T} - 1} \left[ 1 - \frac{e^{-i(2\pi k+\alpha)/T}}{1 - e^{i2\pi m/T}} - \frac{e^{-i(2\pi k+\alpha)/T}}{1 - e^{-i2\pi m/T}} \right] \\ &= \sum_{k=1}^T e^{-i(2\pi k+\alpha)/T} = 0. \end{aligned} \quad (47)$$

Thus, the two-loop contribution turns out to have no  $\alpha$ -dependence, and the quantum corrections come only from the boson loops which are IR finite, that is consistent with (39). Since the classical value  $ig\mu^2 = -i\frac{M^2}{2g}$  is regarded as  $\mathcal{O}(g^{-1})$ , and  $\ell$ -loop contributions are of the order  $\mathcal{O}(g^{2\ell-1})$ , the quantum corrections can not be comparable to the classical value in the perturbation theory. Thus, the conclusion of the SUSY breaking based on the classical value continues to be correct even at the quantum level.

### 3. Change of variables and localization in SUSY matrix models

As argued in the previous section, in order to discuss spontaneous SUSY breaking in the path-integral formalism of (discretized) SUSY quantum mechanics, we introduce an external field to twist the boundary condition of fermions in the Euclidean time direction and observe whether an order parameter of SUSY breaking remains nonzero after turning off the external field. This motivates us to calculate the partition function in the presence of the external field. In the following, we consider a matrix-model analog of (32)

$$\begin{aligned} S^M &= Q \sum_{t=1}^T N \operatorname{tr} \bar{\psi}(t) \left\{ \frac{i}{2} B(t) - (\phi(t+1) - \phi(t) + W'(\phi(t))) \right\} \\ &= \sum_{t=1}^T N \operatorname{tr} \left[ \frac{1}{2} B(t)^2 + iB(t) \{ \phi(t+1) - \phi(t) + W'(\phi(t)) \} \right. \\ &\quad \left. + \bar{\psi}(t) \{ \psi(t+1) - \psi(t) + QW'(\phi(t)) \} \right], \end{aligned} \quad (48)$$

where all variables are  $N \times N$  Hermitian matrices. Under the PBC, this action is manifestly invariant under  $Q$ -transformation defined in (2). When  $N = 1$ , it reduces to the discretized SUSY quantum mechanics in section 2.2. We will focus on the simplest case  $T = 1$  below. Under the twisted boundary condition (33) with  $T = 1$ , the action is

$$S_\alpha^M = N \operatorname{tr} \left[ \frac{1}{2} B^2 + iBW'(\phi) + \bar{\psi} \left( e^{i\alpha} - 1 \right) \psi + \bar{\psi} QW'(\phi) \right], \quad (49)$$

and the partition function is defined by

$$Z_\alpha^M \equiv (-1)^{N^2} \int d^{N^2} B d^{N^2} \phi \left( d^{N^2} \psi d^{N^2} \bar{\psi} \right) e^{-S_\alpha^M}, \quad (50)$$

where we fix the normalization of the measure as

$$\int d^{N^2} \phi e^{-N \text{tr}(\frac{1}{2} \phi^2)} = \int d^{N^2} B e^{-N \text{tr}(\frac{1}{2} B^2)} = 1, \quad (-1)^{N^2} \int \left( d^{N^2} \psi d^{N^2} \bar{\psi} \right) e^{-N \text{tr}(\bar{\psi} \psi)} = 1. \quad (51)$$

Explicitly, when  $W'(\phi)$  is a general polynomial (40), (49) becomes

$$S_\alpha^M = N \text{tr} \left[ \frac{1}{2} B^2 + i B W'(\phi) + \bar{\psi} \left( e^{i\alpha} - 1 \right) \psi + \sum_{k=1}^p g_k \sum_{\ell=0}^{k-1} \bar{\psi} \phi^\ell \psi \phi^{k-\ell-1} \right]. \quad (52)$$

Notice the ordering of the matrices in the last term. We see that the effect of the external field remains even after the reduction to zero dimension ( $T = 1$ ). When  $\alpha = 0$ ,  $S_{\alpha=0}^M$  is invariant under  $Q$  and  $\bar{Q}$ :

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \quad (53)$$

and

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0, \quad (54)$$

both of which become broken explicitly in  $S_\alpha^M$  by introducing the external field  $\alpha$ .

Now let us discuss localization of the integration in  $Z_\alpha^M$ . Some aspects are analogous to the discretized SUSY quantum mechanics with  $T \geq 2$  under the identification  $N^2 = T$  from the viewpoint of systems possessing multi-degrees of freedom, while there are also interesting new phenomena specific to matrix models<sup>3</sup>. We make a change of variables

$$\phi = \tilde{\phi} + \bar{\epsilon} \psi, \quad \bar{\psi} = \tilde{\bar{\psi}} - i \bar{\epsilon} B, \quad (55)$$

where in the second equation,  $\tilde{\bar{\psi}}$  satisfies

$$N \text{tr}(B \tilde{\bar{\psi}}) = 0, \quad (56)$$

namely,  $\tilde{\bar{\psi}}$  is orthogonal to  $B$  with respect to the inner product  $(A_1, A_2) \equiv N \text{tr}(A_1^\dagger A_2)$ . Let us take a basis of  $N \times N$  Hermitian matrices  $\{t^a\}$  ( $a = 1, \dots, N^2$ ) to be orthonormal with respect to the inner product:  $N \text{tr}(t^a t^b) = \delta_{ab}$ . More explicitly, we take

$$\bar{\epsilon} \equiv i \frac{\text{tr}(B \tilde{\bar{\psi}})}{\text{tr} B^2} = \frac{i}{\mathcal{N}_B^2} N \text{tr}(B \tilde{\bar{\psi}}) \quad (57)$$

with  $\mathcal{N}_B \equiv \|B\| = \sqrt{N \text{tr}(B^2)}$  the norm of the matrix  $B$ . Notice that for general  $N$   $\bar{\psi}$  is an  $N \times N$  matrix and that  $\bar{\epsilon}$  does not have enough degrees of freedom to parametrize the whole space of  $\bar{\psi}$ . In fact,  $\bar{\epsilon}$  is used to parametrize a single component of  $\bar{\psi}$  parallel to  $B$ .

If we write (50) as

$$Z_\alpha^M = \int d^{N^2} B \Xi_\alpha(B), \quad \Xi_\alpha(B) \equiv (-1)^{N^2} \int d^{N^2} \phi \left( d^{N^2} \psi d^{N^2} \bar{\psi} \right) e^{-S_\alpha^M}, \quad (58)$$

<sup>3</sup> Localization in the discretized SUSY quantum mechanics is discussed in appendix A in ref. (Kuroki & Sugino, 2011).

and consider the change of the variables in  $\Xi_\alpha(B)$ ,  $B$  may be regarded as an external variable. The measure  $d^{N^2} \bar{\psi}$  can be expressed by the measures associated with  $\tilde{\psi}$  and  $\bar{\epsilon}$  as

$$d^{N^2} \bar{\psi} = \frac{i}{\mathcal{N}_B} d\bar{\epsilon} d^{N^2-1} \tilde{\psi}, \tag{59}$$

where  $d^{N^2-1} \tilde{\psi}$  is explicitly given by introducing the constraint (56) as a delta-function:

$$\begin{aligned} d^{N^2-1} \tilde{\psi} &\equiv (-1)^{N^2-1} d^{N^2} \tilde{\psi} \delta \left( \frac{1}{\mathcal{N}_B} N \text{tr}(B\tilde{\psi}) \right) \\ &= (-1)^{N^2-1} \left( \prod_{a=1}^{N^2} d\tilde{\psi}^a \right) \frac{1}{\mathcal{N}_B} \sum_{a=1}^{N^2} B^a \tilde{\psi}^a. \end{aligned} \tag{60}$$

$\tilde{\psi}^a$  and  $B^a$  are coefficients in the expansion of  $\tilde{\psi}$  and  $B$  by the basis  $\{t^a\}$ :

$$\tilde{\psi} = \sum_{a=1}^{N^2} \tilde{\psi}^a t^a, \quad B = \sum_{a=1}^{N^2} B^a t^a. \tag{61}$$

Notice that the measure on the RHS of (59) depends on  $B$ . When  $B \neq 0$ , we can safely change the variables as in (55) and in terms of them the action becomes

$$S_\alpha^M = N \text{tr} \left[ \frac{1}{2} B^2 + i B W'(\tilde{\phi}) + \tilde{\psi} \left( (e^{i\alpha} - 1) \psi + Q W'(\tilde{\phi}) \right) - (e^{i\alpha} - 1) i \bar{\epsilon} B \psi \right] \tag{62}$$

with  $Q\tilde{\phi} = \psi$ .

**3.1  $\alpha = 0$  case**

Let us first consider the case of the PBC ( $\alpha = 0$ ).  $S_{\alpha=0}^M$  does not depend on  $\bar{\epsilon}$  as a consequence of its SUSY invariance, because (55) reads

$$\phi = \tilde{\phi} + \epsilon Q\tilde{\phi}, \quad \bar{\psi} = \tilde{\psi} + \bar{\epsilon} Q\tilde{\psi}. \tag{63}$$

Therefore, the contribution to the partition function from  $B \neq 0$

$$\tilde{Z}_{\alpha=0} = \int_{\|B\| \geq \epsilon} d^{N^2} B \Xi_{\alpha=0}(B) \quad (0 < \epsilon \ll 1) \tag{64}$$

vanishes due to the integration over  $\bar{\epsilon}$  according to (59). Namely, when  $\alpha = 0$ , the path integral of the partition function (50) is localized to  $B = 0$ .

For the contribution to the partition function from the vicinity of  $B = 0$

$$Z_{\alpha=0}^{(0)} = \int_{\|B\| < \epsilon} d^{N^2} B \Xi_{\alpha=0}(B), \tag{65}$$

when  $W'(\phi)$  is given by (40) of degree  $p \geq 2$ , rescaling as

$$\tilde{\phi} = \mathcal{N}_B^{-\frac{1}{p}} \phi', \quad \tilde{\psi} = \mathcal{N}_B^{\frac{p-1}{p}} \psi', \tag{66}$$

we obtain

$$Z_{\alpha=0}^{(0)} = i \left( \frac{-1}{\sqrt{2\pi}} \right)^{N^2} \left( \int_0^\varepsilon d\mathcal{N}_B \frac{1}{\mathcal{N}_B^{1+\frac{1}{p}}} e^{-\frac{1}{2}\mathcal{N}_B^2} \right) \int d\Omega_B \int d^{N^2} \phi' e^{-iN \text{tr}(\Omega_B g_p \phi'^p)} \\ \times \int d^{N^2} \psi \int d\bar{\varepsilon} d^{N^2-1} \bar{\psi}' e^{-N \text{tr}[\bar{\psi}' g_p \sum_{\ell=0}^{p-1} \phi'^\ell \psi \phi'^{p-\ell-1}]} \left[ 1 + \mathcal{O}(\varepsilon^{1/p}) \right], \quad (67)$$

where the measure of the  $B$ -integral was expressed in terms of polar coordinates in  $\mathbf{R}^{N^2}$  as

$$d^{N^2} B = \prod_{a=1}^{N^2} \frac{dB^a}{\sqrt{2\pi}} = \left( \frac{1}{2\pi} \right)^{\frac{N^2}{2}} \mathcal{N}_B^{N^2-1} d\mathcal{N}_B d\Omega_B, \quad (68)$$

and  $\Omega_B \equiv \frac{1}{\mathcal{N}_B} B$  represents a unit vector in  $\mathbf{R}^{N^2}$ . Since the  $\bar{\varepsilon}$ -integral vanishes while the integration of  $\mathcal{N}_B$  becomes singular at the origin,  $Z_{\alpha=0}^{(0)}$  takes an indefinite form ( $\infty \times 0$ ). When  $W'(\phi)$  is linear ( $p = 1$ ), the  $\bar{\psi}$ -integrals in (65) yield

$$Z_{\alpha=0}^{(0)} = i \left( \frac{-1}{|g_1|} \right)^{N^2} \int_{\|B\| < \varepsilon} \left( \prod_{a=1}^{N^2} dB^a \right) \frac{1}{\mathcal{N}_B} e^{-\frac{1}{2}\mathcal{N}_B^2} \prod_{a=1}^{N^2} \delta(B^a) \\ \times \int d^{N^2} \psi \int d\bar{\varepsilon} d^{N^2-1} \bar{\psi} e^{-N \text{tr}(\bar{\psi} g_1 \psi)}, \quad (69)$$

which is also of indefinite form – the  $B$ -integrals diverge while  $\int d\bar{\varepsilon}$  trivially vanishes. The indefinite form reflects that  $Z_{\alpha=0}^{(0)}$  possibly takes a nonzero value if it is evaluated in a well-defined manner.

### 3.1.1 Unnormalized expectation values

Next, let us consider the unnormalized expectation values of  $\frac{1}{N} \text{tr} B^n$  ( $n \geq 1$ ):

$$\left\langle \frac{1}{N} \text{tr} B^n \right\rangle' \equiv \int d^{N^2} B \left( \frac{1}{N} \text{tr} B^n \right) \Xi_{\alpha=0}(B). \quad (70)$$

Since contribution from the region  $\|B\| \geq \varepsilon$  is shown to be zero by the change of variables (55), we focus on the  $B$ -integration around the origin ( $\|B\| < \varepsilon$ ).

When  $W'(\phi)$  is a polynomial (40) of degree  $p \geq 2$ , after the rescaling (66) we obtain

$$\left\langle \frac{1}{N} \text{tr} B^n \right\rangle' = i \left( \int_0^\varepsilon d\mathcal{N}_B \mathcal{N}_B^{n-1-\frac{1}{p}} e^{-\frac{1}{2}\mathcal{N}_B^2} \right) Y_N \left[ 1 + \mathcal{O}(\varepsilon^{1/p}) \right], \\ Y_N \equiv \left( \frac{-1}{\sqrt{2\pi}} \right)^{N^2} \int d\Omega_B \frac{1}{N} \text{tr}(\Omega_B^n) \int d^{N^2} \phi' e^{-iN \text{tr}(\Omega_B g_p \phi'^p)} \\ \times \int d^{N^2} \psi \int d\bar{\varepsilon} d^{N^2-1} \bar{\psi}' e^{-N \text{tr}[\bar{\psi}' g_p \sum_{\ell=0}^{p-1} \phi'^\ell \psi \phi'^{p-\ell-1}]}]. \quad (71)$$

The  $\mathcal{N}_B$ -integral is finite, and it is seen that  $Y_N$  definitely vanishes. Thus, the change of variables (55) is possible for any  $B$  in evaluating  $\left\langle \frac{1}{N} \text{tr} B^n \right\rangle'$  to give the result

$$\left\langle \frac{1}{N} \text{tr} B^n \right\rangle' = 0 \quad (n \geq 1). \quad (72)$$

When  $W'(\phi)$  is linear,  $\left\langle \frac{1}{N} \text{tr} B^n \right\rangle'$  has the same expression as the RHS of (69) except the integrand multiplied by  $\frac{1}{N} \text{tr} B^n$ . It leads to a finite result of the  $B$ -integration for  $n \geq 1$ , and (72) is also obtained.

Furthermore, it can be similarly shown that the unnormalized expectation values of multi-trace operators  $\prod_{i=1}^k \frac{1}{N} \text{tr} B^{n_i}$  ( $n_1, \dots, n_k \geq 1$ ) vanish:

$$\left\langle \prod_{i=1}^k \frac{1}{N} \text{tr} B^{n_i} \right\rangle' = 0. \quad (73)$$

### 3.1.2 Localization to $W'(\phi) = 0$ , and localization versus Vandermonde

Since (73) means

$$\left\langle e^{-N \text{tr} \left( \frac{u-1}{2} B^2 \right)} \right\rangle' = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -N^2 \frac{u-1}{2} \right)^n \left\langle \left( \frac{1}{N} \text{tr} B^2 \right)^n \right\rangle' = \langle 1 \rangle' = Z_{\alpha=0}^M \quad (74)$$

for an arbitrary parameter  $u$ , we may compute  $\left\langle e^{-N \text{tr} \left( \frac{u-1}{2} B^2 \right)} \right\rangle'$  to evaluate the partition function  $Z_{\alpha=0}^M$ . It is independent of the value of  $u$ , so  $u$  can be chosen to a convenient value to make the evaluation easier.

Taking  $u > 0$  and integrating  $B$  first, we obtain

$$Z_{\alpha=0}^M = (-1)^{N^2} \int d^{N^2} \phi \left( \frac{1}{u} \right)^{\frac{N^2}{2}} e^{-N \text{tr} \left[ \frac{1}{2u} W'(\phi)^2 \right]} \int \left( d^{N^2} \psi d^{N^2} \bar{\psi} \right) e^{-N \text{tr} [\bar{\psi} Q W'(\phi)]}. \quad (75)$$

Then, let us consider the  $u \rightarrow 0$  limit. Localization to  $W'(\phi) = 0$  takes place because

$$\lim_{u \rightarrow 0} \left( \frac{1}{u} \right)^{\frac{N^2}{2}} e^{-N \text{tr} \left[ \frac{1}{2u} W'(\phi)^2 \right]} = (2\pi)^{\frac{N^2}{2}} \prod_{a=1}^{N^2} \delta(W'(\phi)^a). \quad (76)$$

It is important to recognize that  $W'(\phi)^a = 0$  for all  $a$  implies localization to a continuous space. Namely, if this condition is met,  $W'(U^\dagger \phi U)^a = 0$  for  $\forall U \in SU(N)$ . Thus the original  $SU(N)$  gauge symmetry in the matrix model makes the localization continuous in nature. This is characteristic of SUSY matrix models.

The observation above suggests that in order to localize the path integral to discrete points, we should switch to a description in terms of gauge invariant quantities. This motivates us to change the expression of  $\phi$  to its eigenvalues and  $SU(N)$  angles as

$$\phi = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} U^\dagger, \quad U \in SU(N). \quad (77)$$

This leads to an interesting situation, which is peculiar to SUSY matrix models and is not seen in the (discretized) SUSY quantum mechanics. For a polynomial  $W'(\phi)$  given by (40), the partition function (75) becomes

$$Z_{\alpha=0}^M = \left( \frac{1}{u} \right)^{\frac{N^2}{2}} \int d^{N^2} \phi e^{-N \text{tr} \left[ \frac{1}{2u} W'(\phi)^2 \right]} \det \left[ \sum_{k=1}^p g_k \sum_{\ell=0}^{k-1} \phi^\ell \otimes \phi^{k-\ell-1} \right], \quad (78)$$

after the Grassmann integrals. Note that the  $N^2 \times N^2$  matrix  $\sum_{k=1}^p g_k \sum_{\ell=0}^{k-1} \phi^\ell \otimes \phi^{k-\ell-1}$  has the eigenvalues  $\sum_{k=1}^p g_k \sum_{\ell=0}^{k-1} \lambda_i^\ell \lambda_j^{k-\ell-1}$  ( $i, j = 1, \dots, N$ ). Thus, the fermion determinant can be expressed as

$$\det \left[ \sum_{k=1}^p g_k \sum_{\ell=0}^{k-1} \phi^\ell \otimes \phi^{k-\ell-1} \right] = \prod_{i,j=1}^N \left[ \sum_{k=1}^p g_k \sum_{\ell=0}^{k-1} \lambda_i^\ell \lambda_j^{k-\ell-1} \right] \\ = \left( \prod_{i=1}^N W''(\lambda_i) \right) \prod_{i>j} \left( \frac{W'(\lambda_i) - W'(\lambda_j)}{\lambda_i - \lambda_j} \right)^2. \quad (79)$$

The measure  $d^{N^2} \phi$  given in (51) can be also recast to

$$d^{N^2} \phi = \tilde{C}_N \left( \prod_{i=1}^N d\lambda_i \right) \Delta(\lambda)^2 dU, \quad (80)$$

where  $\Delta(\lambda) = \prod_{i>j} (\lambda_i - \lambda_j)$  is the Vandermonde determinant, and  $dU$  is the  $SU(N)$  Haar measure normalized by  $\int dU = 1$ .  $\tilde{C}_N$  is a numerical factor depending only on  $N$  determined by

$$\frac{1}{\tilde{C}_N} = \int \left( \prod_{i=1}^N d\lambda_i \right) \Delta(\lambda)^2 e^{-N \sum_{i=1}^N \frac{1}{2} \lambda_i^2}. \quad (81)$$

Plugging these into (78), we obtain

$$Z_{\alpha=0}^M = \tilde{C}_N \int \left( \prod_{i=1}^N d\lambda_i \right) \left( \prod_{i=1}^N W''(\lambda_i) \right) \left\{ \prod_{i>j} \frac{1}{u} \left( W'(\lambda_i) - W'(\lambda_j) \right)^2 \right\} \\ \times \left( \frac{1}{u} \right)^{\frac{N}{2}} e^{-N \sum_{i=1}^N \frac{1}{2u} W'(\lambda_i)^2}. \quad (82)$$

In this expression, the factor in the second line forces eigenvalues to be localized at the critical points of the superpotential as  $u \rightarrow 0$ , while the last factor in the first line, which is proportional to the square of the Vandermonde determinant of  $W'(\lambda_i)$ , gives repulsive force among eigenvalues which prevents them from collapsing to the critical points. The dynamics of eigenvalues is thus determined by balance of the attractive force to the critical points originating from the localization and the repulsive force from the Vandermonde determinant. This kind of dynamics is not seen in the (discretized) SUSY quantum mechanics.

To proceed with the analysis, let us consider the situation of each eigenvalue  $\lambda_i$  fluctuating around the critical point  $\phi_{c,i}$ :

$$\lambda_i = \phi_{c,i} + \sqrt{u} \tilde{\lambda}_i \quad (i = 1, \dots, N), \quad (83)$$

where  $\tilde{\lambda}_i$  is a fluctuation, and  $\phi_{c,1}, \dots, \phi_{c,N}$  are allowed to coincide with each other. Then, the partition function (82) takes the form

$$Z_{\alpha=0}^M = \tilde{C}_N \sum_{\phi_{c,i}} \int \left( \prod_{i=1}^N d\tilde{\lambda}_i \right) \prod_{i=1}^N W''(\phi_{c,i}) \prod_{i>j} \left( W''(\phi_{c,i}) \tilde{\lambda}_i - W''(\phi_{c,j}) \tilde{\lambda}_j \right)^2 \\ \times e^{-N \sum_{i=1}^N \frac{1}{2} W''(\phi_{c,i})^2 \tilde{\lambda}_i^2 + \mathcal{O}(\sqrt{u})}. \quad (84)$$

Although only the Gaussian factors become relevant as  $u \rightarrow 0$ , there remain  $N(N - 1)$ -point vertices originating from the Vandermonde determinant of  $W'(\lambda_i)$  which yield a specific effect of SUSY matrix models.

In the case of  $W'(\phi) = g_1\phi$ , where the corresponding scalar potential  $\frac{1}{2}W'(\phi)^2$  is Gaussian, the critical point is only the origin:  $\phi_{c,1} = \dots = \phi_{c,N} = 0$ . Then, (84) is reduced to

$$Z_{\alpha=0}^M = \tilde{C}_N \int \left( \prod_{i=1}^N d\tilde{\lambda}_i \right) g_1^{N^2} \prod_{i>j} (\tilde{\lambda}_i - \tilde{\lambda}_j)^2 e^{-N \sum_{i=1}^N \frac{1}{2} g_1^2 \tilde{\lambda}_i^2}, \tag{85}$$

where no  $\mathcal{O}(\sqrt{u})$  term appears since  $W'(\phi)$  is linear. By using (81) we obtain

$$Z_{\alpha=0}^M = (\text{sgn}(g_1))^{N^2} = (\text{sgn}(g_1))^N. \tag{86}$$

For a general superpotential, we change the integration variables as

$$\tilde{\lambda}_i = \frac{1}{W''(\phi_{c,i})} y_i, \tag{87}$$

then the integration of  $\tilde{\lambda}_i$  becomes  $\int_{-\infty}^{\infty} d\tilde{\lambda}_i \dots = \frac{1}{|W''(\phi_{c,i})|} \int_{-\infty}^{\infty} dy_i \dots$ . In the limit  $u \rightarrow 0$ , (84) is computed to be

$$\begin{aligned} Z_{\alpha=0}^M &= \sum_{\phi_{c,i}} \prod_{i=1}^N \frac{W''(\phi_{c,i})}{|W''(\phi_{c,i})|} \left\{ \tilde{C}_N \int_{-\infty}^{\infty} \left( \prod_{i=1}^N dy_i \right) \Delta(y)^2 e^{-N \sum_{i=1}^N \frac{1}{2} y_i^2} \right\} \\ &= \sum_{\phi_{c,i}} \prod_{i=1}^N \text{sgn}(W''(\phi_{c,i})) \\ &= \left[ \sum_{\phi_c: W'(\phi_c)=0} \text{sgn}(W''(\phi_c)) \right]^N. \end{aligned} \tag{88}$$

Note that the last factor in the first line of (88) is nothing but the partition function of the Gaussian case with  $g_1 = 1$ . The last line of (88) tells that the total partition function is given by the  $N$ -th power of the degree of the map  $\phi \rightarrow W'(\phi)$ .

Furthermore, we consider a case that the superpotential  $W(\phi)$  has  $K$  nondegenerate critical points  $a_1, \dots, a_K$ . Namely,  $W'(a_I) = 0$  and  $W''(a_I) \neq 0$  for each  $I = 1, \dots, K$ . The scalar potential  $\frac{1}{2}W'(\phi)^2$  has  $K$  minima at  $\phi = a_1, \dots, a_K$ . When  $N$  eigenvalues are fluctuating around the minima, we focus on the situation that

the first  $\nu_1 N$  eigenvalues  $\lambda_i$  ( $i = 1, \dots, \nu_1 N$ ) are around  $\phi = a_1$ ,  
 the next  $\nu_2 N$  eigenvalues  $\lambda_{\nu_1 N + i}$  ( $i = 1, \dots, \nu_2 N$ ) are around  $\phi = a_2$ ,  
 $\dots$ ,

and the last  $\nu_K N$  eigenvalues  $\lambda_{\nu_1 N + \dots + \nu_{K-1} N + i}$  ( $i = 1, \dots, \nu_K N$ ) are around  $\phi = a_K$ ,

where  $\nu_1, \dots, \nu_K$  are filling fractions satisfying  $\sum_{I=1}^K \nu_I = 1$ . Let  $Z_{(\nu_1, \dots, \nu_K)}$  be a contribution to the total partition function  $Z_{\alpha=0}^M$  from the above configuration. Then,

$$Z_{\alpha=0}^M = \sum_{\nu_1 N, \dots, \nu_K N=0}^N \frac{N!}{(\nu_1 N)! \dots (\nu_K N)!} Z_{(\nu_1, \dots, \nu_K)}. \tag{89}$$

(The sum is taken under the constraint  $\sum_{I=1}^K \nu_I = 1$ .) Since  $Z_{(\nu_1, \dots, \nu_K)}$  is equal to the second line of (88) with  $\phi_{c,i}$  fixed as

$$\begin{aligned} \phi_{c,1} &= \dots = \phi_{c,\nu_1 N} = a_1, \\ \phi_{c,\nu_1 N+1} &= \dots = \phi_{c,\nu_1 N+\nu_2 N} = a_2, \\ &\dots \\ \phi_{c,\nu_1 N+\dots+\nu_{K-1} N+1} &= \dots = \phi_{c,N} = a_K, \end{aligned} \quad (90)$$

we obtain

$$Z_{(\nu_1, \dots, \nu_K)} = \prod_{I=1}^K Z_{G,\nu_I}, \quad Z_{G,\nu_I} = (\text{sgn}(W''(a_I)))^{\nu_I N}. \quad (91)$$

$Z_{G,\nu_I}$  can be interpreted as the partition function of the Gaussian SUSY matrix model with the matrix size  $\nu_I N \times \nu_I N$  describing contributions from Gaussian fluctuations around  $\phi = a_I$ .

### 3.2 $\alpha \neq 0$ case

In the presence of the external field  $\alpha$ , let us consider  $\Xi_\alpha(B)$  in (58) with the action (62) obtained after the change of variables (55). Using the explicit form of the measure (59) and (60), we obtain

$$\begin{aligned} \Xi_\alpha(B) &= (e^{i\alpha} - 1) \frac{(-1)^{N^2-1}}{\mathcal{N}_B^2} \int d^{N^2} \tilde{\phi} \left( d^{N^2} \psi d^{N^2} \tilde{\psi} \right) e^{-N \text{tr}[\frac{1}{2} B^2 + i B W'(\tilde{\phi}) + \tilde{\psi} Q W'(\tilde{\phi})]} \\ &\quad \times N \text{tr}(B \tilde{\psi}) N \text{tr}(B \psi) e^{-(e^{i\alpha}-1) N \text{tr}(\tilde{\psi} \psi)}, \end{aligned} \quad (92)$$

which is valid for  $B \neq 0$ . It does not vanish in general by the effect of the twist  $e^{i\alpha} - 1$ . This suggests that the localization is incomplete by the twist. Although we can proceed the computation further, it is more convenient to invoke another method based on the Nicolai mapping we will present in the next section.

## 4. $(e^{i\alpha} - 1)$ -expansion and Nicolai mapping

In the previous section, we have seen that the change of variables is useful to localize the path integral, but in the  $\alpha \neq 0$  case the external field makes the localization incomplete and the explicit computation somewhat cumbersome. In this section, we instead compute  $Z_\alpha^M$  in an expansion with respect to  $(e^{i\alpha} - 1)$ . For the purpose of examining the spontaneous SUSY breaking, we are interested in behavior of  $Z_\alpha^M$  in the  $\alpha \rightarrow 0$  limit. Thus it is expected that it will be often sufficient to compute  $Z_\alpha^M$  in the leading order of the  $(e^{i\alpha} - 1)$ -expansion for our purpose.

### 4.1 Finite $N$

Performing the integration over fermions and the auxiliary field  $B$  in (50) with  $W'(\phi)$  in (40), we have

$$Z_\alpha^M = \int d^{N^2} \phi \det \left( (e^{i\alpha} - 1) \mathbf{1} \otimes \mathbf{1} + \sum_{k=1}^p g_k \sum_{\ell=0}^{k-1} \phi^\ell \otimes \phi^{p-\ell-1} \right) e^{-N \text{tr} \frac{1}{2} W'(\phi)^2}. \quad (93)$$

Hereafter, let us expand this with respect to  $(e^{i\alpha} - 1)$  as

$$Z_\alpha^M = \sum_{k=0}^{N^2} (e^{i\alpha} - 1)^k Z_{\alpha,k}, \quad (94)$$

and derive a formula in the leading order of this expansion. The change of variable  $\phi$  as (77) recasts (93) to

$$Z_\alpha^M = \tilde{C}_N \int \left( \prod_{i=1}^N d\lambda_i \right) \Delta(\lambda)^2 \prod_{i,j=1}^N \left( e^{i\alpha} - 1 + \sum_{k=1}^p g_k \sum_{\ell=0}^{k-1} \lambda_i^\ell \lambda_j^{p-\ell-1} \right) e^{-N \sum_{i=1}^N \frac{1}{2} W'(\lambda_i)^2}, \quad (95)$$

after the  $SU(N)$  angles are integrated out. Crucial observation is that we can apply the Nicolai mapping (Nicolai, 1979) for each  $i$  even in the presence of the external field

$$\Lambda_i = (e^{i\alpha} - 1)\lambda_i + W'(\lambda_i), \quad (96)$$

in terms of which the partition function is basically expressed as an unnormalized expectation value of the Gaussian matrix model

$$Z_\alpha^M = \tilde{C}_N \int \left( \prod_{i=1}^N d\Lambda_i \right) \prod_{i>j} (\Lambda_i - \Lambda_j)^2 e^{-N \sum_i \frac{1}{2} \Lambda_i^2} e^{-N \sum_i (-A\Lambda_i \lambda_i + \frac{1}{2} A^2 \lambda_i^2)}, \quad (97)$$

where  $A = e^{i\alpha} - 1$ . However, there is an important difference from the Gaussian matrix model, which originates from the fact that the Nicolai mapping (96) is not one to one. As a consequence,  $\lambda_i$  has several branches as a function of  $\Lambda_i$  and it has a different expression according to each of the branches. Therefore, since the last factor of (97) contains  $\lambda_i(\Lambda_i)$ , we have to take account of the branches and divide the integration region of  $\Lambda_i$  accordingly. Nevertheless, we can derive a rather simple formula at least in the leading order of the expansion in terms of  $A$  owing to the Nicolai mapping (96). In the following, let us concentrate on the cases where

$$\Lambda_i \rightarrow \infty \quad \text{as} \quad \lambda_i \rightarrow \pm\infty, \quad \text{or} \quad \Lambda_i \rightarrow -\infty \quad \text{as} \quad \lambda_i \rightarrow \pm\infty, \quad (98)$$

i.e. the leading order of  $W'(\phi)$  is even. In such cases, we can expect spontaneous SUSY breaking, in which the leading nontrivial expansion coefficient is relevant since the zeroth order partition function vanishes:  $Z_{\alpha=0}^M = Z_{\alpha,0} = 0$ . Namely, in the expansion of the last factor in (97)

$$e^{-N \sum_{i=1}^N (-A\Lambda_i \lambda_i + \frac{1}{2} A^2 \lambda_i^2)} = 1 - N \sum_{i=1}^N \left( -A\Lambda_i \lambda_i + \frac{1}{2} A^2 \lambda_i^2 \right) + \dots, \quad (99)$$

the first term "1" does not contribute to  $Z_\alpha^M$ . It can be understood from the fact that it does not depend on the branches and thus the Nicolai mapping becomes trivial, i.e. The mapping degree is zero. Notice that the second term also gives a vanishing effect. For each  $i$ , we have the unnormalized expectation value of  $N \left( A\Lambda_i \lambda_i - \frac{1}{2} A^2 \lambda_i^2 \right)$ , where the  $\Lambda_j$ -integrals ( $j \neq i$ ) are independent of the branches leading to the trivial Nicolai mapping. Thus, in order to get a nonvanishing result, we need a branch-dependent piece in the integrand for any  $\Lambda_i$ . This immediately shows that in the expansion (94),  $Z_{\alpha,k} = 0$  for  $k = 0, \dots, N-1$  and that the first possibly nonvanishing contribution starts from  $\mathcal{O}(A^N)$  as

$$Z_{\alpha,N} = \tilde{C}_N N^N \int \left( \prod_{i=1}^N d\Lambda_i \right) \prod_{i>j} (\Lambda_i - \Lambda_j)^2 e^{-N \sum_{i=1}^N \frac{1}{2} \Lambda_i^2} \prod_{i=1}^N (\Lambda_i \lambda_i) \Big|_{A=0}. \quad (100)$$

Note that the  $A (= e^{i\alpha} - 1)$ -dependence of the integrand comes also from  $\lambda_i$  as a function of  $\Lambda_i$  through (96). Although the integration over  $\Lambda_i$  above should be divided into the branches, if we change the integration variables so that we will recover the original  $\lambda_i$  with  $A = 0$  (which we call  $x_i$ ) by

$$\Lambda_i = W'(x_i), \quad (101)$$

then by construction the integration of  $x_i$  is standard and runs from  $-\infty$  to  $\infty$ . Therefore, we arrive at

$$Z_{\alpha,N} = \tilde{C}_N N^N \int_{-\infty}^{\infty} \left( \prod_{i=1}^N dx_i \right) \prod_{i=1}^N (W''(x_i)W'(x_i)x_i) \prod_{i>j} (W'(x_i) - W'(x_j))^2 \times e^{-N \sum_{i=1}^N \frac{1}{2} W'(x_i)^2}, \quad (102)$$

which does not vanish in general. For example, taking  $W'(\phi) = g(\phi^2 - \mu^2)$  we have for  $N = 2$

$$Z_{\alpha,2} = 10g^2 \tilde{C}_2 I_0^2 \left[ \frac{I_4}{I_0} - \frac{9}{5} \left( \frac{I_2}{I_0} \right)^2 \right], \quad (103)$$

where

$$I_n \equiv \int_{-\infty}^{\infty} d\lambda \lambda^n e^{-g^2(\lambda^2 - \mu^2)^2} \quad (n = 0, 2, 4, \dots). \quad (104)$$

In fact, when  $g = 1, \mu^2 = 1$  (double-well scalar potential case) we find

$$I_0 = 1.97373, \quad \frac{I_4}{I_0} - \frac{9}{5} \left( \frac{I_2}{I_0} \right)^2 = -0.165492 \neq 0, \quad (105)$$

hence  $Z_{\alpha,2}$  actually does not vanish. In the case of the discretized SUSY quantum mechanics, we have seen in (35) that the expansion of  $Z_{\alpha}^M$  with respect to  $(e^{i\alpha} - 1)$  terminates at the linear order for any  $T$ . Thus, the nontrivial  $\mathcal{O}(A^N)$  contribution of higher order can be regarded as a specific feature of SUSY matrix models.

We stress here that, although we have expanded the partition function in terms of  $(e^{i\alpha} - 1)$  and (102) is the leading order one, it is an exact result of the partition function for any finite  $N$  and any polynomial  $W'(\phi)$  of even degree in the presence of the external field. Thus, it provides a firm ground for discussion of spontaneous SUSY breaking in various settings.

#### 4.2 Large- $N$

As an application of (102), let us discuss SUSY breaking/restoration in the large- $N$  limit of our SUSY matrix models. From (102), introducing the eigenvalue density

$$\rho(x) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i) \quad (106)$$

rewrites the leading  $\mathcal{O}(A^N)$  part of  $Z_{\alpha}^M$  as

$$Z_{\alpha,N} = N^N \int \left( \prod_{i=1}^N dx_i \right) \exp(-N^2 F), \quad (107)$$

$$F \equiv - \int dx dy \rho(x) \rho(y) \log |W'(x) - W'(y)| + \int dx \rho(x) \frac{1}{2} W'(x)^2 - \frac{1}{N^2} \log \tilde{C}_N - \frac{1}{N} \int dx \rho(x) \log(W''(x)W'(x)x). \quad (108)$$

In the large- $N$  limit,  $\rho(x)$  is given as a solution to the saddle point equation obtained from  $\mathcal{O}(N^0)$  part of  $F$  as

$$0 = \int dy \rho(y) \frac{W''(x)}{|W'(x) - W'(y)|} - \frac{1}{2} W'(x) W''(x). \quad (109)$$

Plugging a solution  $\rho_0(x)$  into  $F$  in (108), we get  $Z_\alpha^M$  in the large- $N$  limit in the leading order of  $(e^{t\alpha} - 1)$ -expansion as

$$\begin{aligned} Z_{\alpha,N} &\rightarrow N^N \exp(-N^2 F_0), \\ F_0 &= - \int dx dy \rho_0(x) \rho_0(y) \log |W'(x) - W'(y)| + \int dx \rho_0(x) \frac{1}{2} W'(x)^2 \\ &\quad - \frac{1}{N^2} \log C_N, \end{aligned} \quad (110)$$

where  $C_N$  is a factor dependent only on  $N$  which arises in replacing the integration over  $\phi$  by the saddle point of its eigenvalue density, thus including  $\tilde{C}_N$ . From consideration of the Gaussian matrix model (85),  $C_N$  is calculated in appendix B in ref. (Kuroki & Sugino, 2010) as

$$C_N = \exp \left[ \frac{3}{4} N^2 + \mathcal{O}(N^0) \right]. \quad (111)$$

In (110) we notice that, if we include  $\mathcal{O}(1/N)$  part of  $F$  (the last term in (108)) in deriving the saddle point equation, the solution will receive an  $\mathcal{O}(1/N)$  correction as  $\rho(x) = \rho_0(x) + \frac{1}{N} \rho_1(x)$ . However, when we substitute this into (108),  $\rho_1(x)$  will contribute to  $F$  only by the order  $\mathcal{O}(1/N^2)$ , because  $\mathcal{O}(1/N)$  corrections to  $F_0$  under  $\rho_0(x) \rightarrow \rho_0(x) + \frac{1}{N} \rho_1(x)$  vanish as a result of the saddle point equation at the leading order (109) satisfied by  $\rho_0(x)$ .

### 4.3 Example: SUSY matrix model with double-well potential

For illustration of results in the previous subsection, let us consider the SUSY matrix model with  $W'(\phi) = \phi^2 - \mu^2$ . The saddle point equation (109) becomes

$$\int dy \frac{\rho(y)}{x-y} + \int dy \frac{\rho(y)}{x+y} = x^3 - \mu^2 x. \quad (112)$$

Let us consider the case  $\mu^2 > 0$ , where the shape of the scalar potential is a double-well  $\frac{1}{2} (x^2 - \mu^2)^2$ .

#### 4.3.1 Asymmetric one-cut solution

First, we find a solution corresponding to all the eigenvalues located around one of the minima  $\lambda = +\sqrt{\mu^2}$ . Assuming the support of  $\rho(x)$  as  $x \in [a, b]$  with  $0 < a < b$ , the equation (112) is valid for  $x \in [a, b]$ .

Following the method in ref. (Brezin et al., 1978), we introduce a holomorphic function

$$F(z) \equiv \int_a^b dy \frac{\rho(y)}{z-y}, \quad (113)$$

which satisfies the following properties:

1.  $F(z)$  is analytic in  $z \in \mathbf{C}$  except the cut  $[a, b]$ .

2.  $F(z)$  is real on  $z \in \mathbf{R}$  outside the cut.
3. For  $z \sim \infty$ ,  

$$F(z) = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right).$$
4. For  $x \in [a, b]$ ,  

$$F(x \pm i0) = F(-x) + x^3 - \mu^2 x \mp i\pi\rho(x).$$

Note that, if we consider the combination (Eynard & Kristjansen, 1995)

$$F_-(z) \equiv \frac{1}{2}(F(z) - F(-z)), \quad (114)$$

then the properties of  $F_-(z)$  are

1.  $F_-(z)$  is analytic in  $z \in \mathbf{C}$  except the two cuts  $[a, b]$  and  $[-b, -a]$ .
2.  $F_-(z)$  is odd ( $F_-(-z) = -F_-(z)$ ), and real on  $z \in \mathbf{R}$  outside the cuts.
3. For  $z \sim \infty$ ,  

$$F_-(z) = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^3}\right).$$
4. For  $x \in [a, b]$ ,  

$$F_-(x \pm i0) = \frac{1}{2}(x^3 - \mu^2 x) \mp i\frac{\pi}{2}\rho(x).$$

These properties are sufficient to fix the form of  $F_-(z)$  as

$$F_-(z) = \frac{1}{2}(z^3 - \mu^2 z) - \frac{1}{2}z\sqrt{(z^2 - a^2)(z^2 - b^2)} \quad (115)$$

with

$$a^2 = -2 + \mu^2, \quad b^2 = 2 + \mu^2. \quad (116)$$

Since  $a^2$  should be positive, the solution is valid for  $\mu^2 > 2$ . The eigenvalue distribution is obtained as

$$\rho_0(x) = \frac{x}{\pi} \sqrt{(x^2 - a^2)(b^2 - x^2)}. \quad (117)$$

From (117), we see that

$$\lim_{\alpha \rightarrow 0} \left( \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \text{tr} \phi \right\rangle_\alpha \right) = \int_a^b dx x \rho_0(x) \quad (118)$$

is finite and nonsingular, differently from the situation in (19). It can be understood that the tunneling between separate broken vacua is suppressed by taking the large- $N$  limit, and thus the superselection rule works. Note that the large- $N$  limit in the matrix models is analogous to the infinite volume limit or the thermodynamic limit of statistical systems. In fact, this will play an essential role for restoration of SUSY in the large- $N$  limit of the matrix model with a double-well potential.

Using (117), we compute the expectation value of  $\frac{1}{N} \text{tr} B$  as

$$\lim_{\alpha \rightarrow 0} \left( \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \text{tr} B \right\rangle_\alpha \right) = \int_a^b dx (x^2 - \mu^2) \rho_0(x) = 0. \quad (119)$$

Furthermore, all the expectation values of  $\frac{1}{N} \text{tr} B^n$  are proven to vanish:

$$\lim_{\alpha \rightarrow 0} \left( \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \text{tr} B^n \right\rangle_\alpha \right) = 0 \quad (n = 1, 2, \dots). \quad (120)$$

(For a proof, see appendix C in ref. (Kuroki & Sugino, 2010).) Also, the large- $N$  free energy (110) vanishes. These evidences convince us that the SUSY is restored at infinite  $N$ .

### 4.3.2 Two-cut solutions

Let us consider configurations that  $\nu_+ N$  eigenvalues are located around one minimum  $\lambda = +\sqrt{\mu^2}$  of the double-well, and the remaining  $\nu_- N (= N - \nu_+ N)$  eigenvalues are around the other minimum  $\lambda = -\sqrt{\mu^2}$ .

First, we focus on the  $Z_2$ -symmetric two-cut solution with  $\nu_+ = \nu_- = \frac{1}{2}$ , where the eigenvalue distribution is supposed to have a  $Z_2$ -symmetric support  $\Omega = [-b, -a] \cup [a, b]$ , and  $\rho(-x) = \rho(x)$ . The equation (112) is valid for  $x \in \Omega$ . Due to the  $Z_2$  symmetry, the holomorphic function  $F(z) \equiv \int_{\Omega} dy \frac{\rho(y)}{z-y}$  has the same properties as  $F_-(z)$  in section 4.3.1 except the property 4, which is now changed to

$$F(x \pm i0) = \frac{1}{2} (x^3 - \mu^2 x) \mp i\pi\rho(x) \quad \text{for } x \in \Omega. \quad (121)$$

The solution is given by

$$F(z) = \frac{1}{2} (z^3 - \mu^2 z) - \frac{1}{2} z \sqrt{(z^2 - a^2)(z^2 - b^2)}, \quad (122)$$

$$\rho_0(x) = \frac{1}{2\pi} |x| \sqrt{(x^2 - a^2)(b^2 - x^2)}, \quad (123)$$

where  $a, b$  coincide with the values of the one-cut solution (116). It is easy to see that, concerning  $Z_2$ -symmetric observables, the expectation values are the same as the expectation values evaluated under the one-cut solution. In particular, we have the same conclusion for the expectation values of  $\frac{1}{N} \text{tr} B^n$  and the large- $N$  free energy vanishing.

It is somewhat surprising that the end points of the cut  $a, b$  and the large- $N$  free energy coincide with those for the one-cut solution, which is recognized as a new interesting feature of the supersymmetric models and can be never seen in the case of bosonic double-well matrix models. In bosonic double-well matrix models, the free energy of the  $Z_2$ -symmetric two-cut solution is lower than that of the one-cut solution, and the endpoints of the cuts are different between the two solutions (Cicutta et al., 1986; Nishimura et al., 2003).

Next, let us consider general  $Z_2$ -asymmetric two-cut solutions (i.e., general  $\nu_{\pm}$ ). We can check that the following solution gives a large- $N$  saddle point:

The eigenvalue distribution  $\rho_0(x)$  has the cut  $\Omega = [-b, -a] \cup [a, b]$  with  $a, b$  given by (116):

$$\rho_0(x) = \begin{cases} \frac{\nu_+}{\pi} x \sqrt{(x^2 - a^2)(b^2 - x^2)} & (a < x < b) \\ \frac{\nu_-}{\pi} |x| \sqrt{(x^2 - a^2)(b^2 - x^2)} & (-b < x < -a). \end{cases} \quad (124)$$

This is a general supersymmetric solution including the one-cut and  $Z_2$ -symmetric two-cut solutions. The expectation values of  $Z_2$ -even observables under this saddle point coincide with those under the one-cut solution, and the expectation values of  $\frac{1}{N} \text{tr} B^n$  and the large- $N$  free energy vanish, again. Thus, we can conclude that the SUSY matrix model with the double-well potential has an infinitely many degenerate supersymmetric saddle points parametrized by  $(\nu_+, \nu_-)$  at large  $N$  for the case  $\mu^2 > 2$ .

### 4.3.3 Symmetric one-cut solution

Here we obtain a one-cut solution with a symmetric support  $[-c, c]$ . As before, let us consider a complex function

$$G(z) \equiv \int_{-c}^c dy \frac{\rho(y)}{z-y}, \quad (125)$$

and further define

$$G_-(z) \equiv \frac{1}{2}(G(z) - G(-z)). \quad (126)$$

Then  $G_-(z)$  has following properties:

1.  $G_-(z)$  is odd, analytic in  $z \in \mathbf{C}$  except the cut  $[-c, c]$ .
2.  $G_-(x) \in \mathbf{R}$  for  $x \in \mathbf{R}$  and  $x \notin [-c, c]$ .
3.  $G_-(z) \rightarrow \frac{1}{z} + \mathcal{O}(\frac{1}{z^3})$  as  $z \rightarrow \infty$ .
4.  $G_-(x \pm i0) = \frac{1}{2}(x^2 - \mu^2)x \mp i\pi\rho(x)$  for  $x \in [-c, c]$ .

They lead us to deduce

$$G_-(z) = \frac{1}{2}(z^2 - \mu^2)z - \frac{1}{2}\left(z^2 - \mu^2 + \frac{c^2}{2}\right)\sqrt{z^2 - c^2} \quad (127)$$

with

$$c^2 = \frac{2}{3}\left(\mu^2 + \sqrt{\mu^4 + 12}\right), \quad (128)$$

from which we find that

$$\rho_0(x) = \frac{1}{2\pi}\left(x^2 - \mu^2 + \frac{c^2}{2}\right)\sqrt{c^2 - x^2}, \quad x \in [-c, c]. \quad (129)$$

The condition  $\rho_0(x) \geq 0$  tells us that this solution is valid for  $\mu^2 \leq 2$ , which is indeed the complement of the region of  $\mu^2$  where both the two-cut solution and the asymmetric one-cut solution exist. (129) is valid also for  $\mu^2 < 0$ . Given  $\rho_0(x)$ , it is straightforward to calculate the large- $N$  free energy as

$$F_0 = \frac{1}{3}\mu^4 - \frac{1}{216}\mu^8 - \frac{1}{216}(\mu^6 + 30\mu^2)\sqrt{\mu^4 + 12} - \log(\mu^2 + \sqrt{\mu^4 + 12}) + \log 6, \quad (130)$$

which is positive for  $\mu^2 < 2$ . Also, the expectation value of  $\frac{1}{N} \text{tr} B$  is computed to be

$$\left\langle \frac{1}{N} \text{tr} B \right\rangle = -i \left[ \frac{c^4}{16}(c^2 - \mu^2) - \mu^2 \right] \neq 0 \quad \text{for } \mu^2 < 2. \quad (131)$$

These are strong evidence suggesting the spontaneous SUSY breaking. Also, the  $\mu^2$ -derivatives of the free energy,

$$\lim_{\mu^2 \rightarrow 2-0} F_0 = \lim_{\mu^2 \rightarrow 2-0} \frac{dF_0}{d(\mu^2)} = \lim_{\mu^2 \rightarrow 2-0} \frac{d^2F_0}{d(\mu^2)^2} = 0, \quad \lim_{\mu^2 \rightarrow 2-0} \frac{d^3F_0}{d(\mu^2)^3} = -\frac{1}{2}, \quad (132)$$

show that the transition between the SUSY phase ( $\mu^2 \geq 2$ ) and the SUSY broken phase ( $\mu^2 < 2$ ) is of the third order.

## 5. Summary and discussion

In this chapter, firstly we discussed spontaneous SUSY breaking in the (discretized) quantum mechanics. The twist  $\alpha$ , playing a role of the external field, was introduced to detect the SUSY breaking, as well as to regularize the supersymmetric partition function (essentially equivalent to the Witten index) which becomes zero when the SUSY is broken. Differently from spontaneous breaking of ordinary (bosonic) symmetry, SUSY breaking does not require cooperative phenomena and can take place even in the discretized quantum mechanics with a finite number of discretized time steps. There is such a possibility, when the supersymmetric partition function vanishes. In general, some non-analytic behavior is necessary for spontaneous symmetry breaking to take place. For SUSY breaking in the finite system, it can be understood that the non-analyticity comes from the vanishing partition function.

Secondly we discussed localization in SUSY matrix models without the external field. The formula of the partition function was obtained, which is given by the  $N$ -th power of the localization formula in the  $N = 1$  case ( $N$  is the rank of matrix variables). It can be regarded as a matrix-model generalization of the ordinary localization formula. In terms of eigenvalues, localization attracts them to the critical points of superpotential, while the square of the Vandermonde determinant originating from the measure factor gives repulsive force among them. Thus, the dynamics of the eigenvalues is governed by balance of the attractive force from the localization and the repulsive force from the Vandermonde determinant. It is a new feature specific to SUSY matrix models, not seen in the (discretized) SUSY quantum mechanics. For a general superpotential which has  $K$  critical points, contribution to the partition function from  $\nu_I N$  eigenvalues fluctuating around the  $I$ -th critical point ( $I = 1, \dots, K$ ), denoted by  $Z_{(\nu_1, \dots, \nu_K)}$ , was shown to be equal to the products of the partition functions of the Gaussian SUSY matrix models  $Z_{G, \nu_1} \cdots Z_{G, \nu_K}$ . Here,  $Z_{G, \nu_I}$  is the partition function of the Gaussian SUSY matrix model with the rank of matrix variables being  $\nu_I N$ , which describes Gaussian fluctuations around the  $I$ -th critical point. It is interesting to investigate whether such a factorization occurs also for various expectation values.

Thirdly, the argument of the change of variables leading to localization can be applied to  $\alpha \neq 0$  case. Then, we found that  $\alpha$ -dependent terms in the action explicitly break SUSY and makes localization incomplete. Instead of it, the Nicolai mapping, which is also applicable to the  $\alpha \neq 0$  case, is more convenient for actual calculation in SUSY matrix models. In the case that the supersymmetric partition function (the partition function with  $\alpha = 0$ ) vanishes, we obtained an exact result of a leading nontrivial contribution to the partition function with  $\alpha \neq 0$  in the expansion of  $(e^{i\alpha} - 1)$  for finite  $N$ . It will play a crucial role to compute various correlators when SUSY is spontaneously broken. Large- $N$  solutions for the double-well case  $W(\phi) = \phi^2 - \mu^2$  were derived, and it was found that there is a phase transition between the SUSY phase corresponding to  $\mu^2 \geq 2$  and the SUSY broken phase to  $\mu^2 < 2$ . It was shown to be of the third order.

For future directions, this kind of argument can be expected to be useful to investigate localization in various lattice models for supersymmetric field theories which realize some SUSYs on the lattice. Also, it will be interesting to investigate localization in models constructed in ref. (Kuroki & Sugino, 2008), which couple a supersymmetric quantum field theory to a certain large- $N$  matrix model and cause spontaneous SUSY breaking at large  $N$ .

Finally, we hope that similar analysis for super Yang-Mills matrix models (Banks et al., 1997; Dijkgraaf et al., 1997; Ishibashi et al., 1997), which have been proposed as nonperturbative

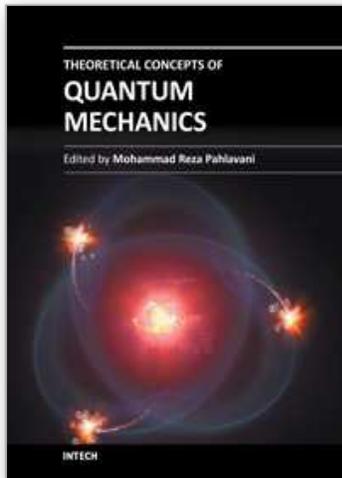
definitions of superstring/M theories, will shed light on new aspects of spontaneous SUSY breaking in superstring/M theories.

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