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The Group Theory and Non-Euclidean Superposition Principle in Quantum Mechanics

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1. Introduction

One of the unsolved problem in theoretical physics during some decades remains a construction of the complete and closed theory in which quantum mechanics and special relativity would be consistent without divergences and renormalization (Dirac, 1978). It may be assumed that divergences show conservation laws violation in the conventional theories, and a cause for it may be in turn violation of the group-theoretic principles in these theories, in accordance with the Noether theorems. A success of renormalization allows one to believe that the theory without divergences is possible.

This paper is devoted to consideration of possibility to develop the consistent group-theoretic scheme of the quantum mechanics merely. It consists of Introduction, three parts, and Conclusion.

The requirements which allow one to consider the quantum mechanics as a consistent group-theoretic theory are formulated in Introduction.

The Noether theorems set one-to-one correspondence between conservation laws of the variables to be measured, i.e. observables (Dirac, 1958), and groups of symmetries of the solutions transformations of equations for complex wave functions, spinors, matrices and so on in which the space-time properties appear (Olver, 1986). These solutions do not obey to be an observables but the last ones may be constructed as the Hermitian forms corresponding to these observables on their basis. The mathematical tool to express the space-time symmetry properties is the group theory.

Two circumstances connected with the stated above attract attention in the generally accepted schemes of the quantum mechanics.

The exact conservation laws fulfilment is inconceivable in any theoretical scheme under absence of the *complete* set of the Hermitian forms, based on the main equations solutions and its derivatives, each of them would be corresponded to the observables. Some of these Hermitian forms have to be conserved, another have to be changed but all of them have to satisfy to some completeness condition expressed mathematically. The last subject has exceptional significance since if only some part of the unknown complete set of observables really existing is included into the theory, then both physical interpretation and conservation laws would be dependent on the Hermitian forms which are excluded from the theory. Of course, such theory can not be recognized to be the consecutive, complete and closed theory.

One of the impressive consequence of the observables complete set and corresponding completeness condition absence is the well known question on the hidden parameters

(Einstein et al., 1935) being discussed up to now (Goldstein, 1998) including experimental investigations (Greenstein & Zajonc, 2006). Moreover, absence of the completeness condition for observables immediately relates to the physical contents of the wave function and its probabilistic interpretation. These subjects are discussed during many years but the uniqueness of the last one has not been proved up to now (Feynman & Hibbs, 1965).

The second question is not so obvious and discussed, the author had not seen the papers on the subject.

The logical foundations of the physical theory having been consistent with causality, require to act of the consequent transformation on the result of the previous one. The mathematical description of this requirement is expressed by means of the operators product in the same order as they act in the physical process to be described.

Transformation operators describing different physical processes map the space-time properties, and the successive products define the binary operation over the transformations set, it is the multiplication, its result depends on the operators order in general case. Therefore, transformations set have to be multiplicative non-commutative groups in the fundamental physical theories.

The elements of the multiplicative non-commutative groups are nonequivalent under the group operation but the physical phenomena, similar to the interference, require to include the non-commutative group elements compositions in which its elements would be equivalent. The results of such kinds compositions have not be dependent on the order of the group elements in it, and have to belong to the same group as two elements entered the compositions at the same time.

In the ordinary superposition principle used in a great number of physical theories the pairwise permutable composition is expressed as the sum of the elements, in particular the elements of the multiplicative non-commutative groups. It means that the second binary operation, the sum, over the group elements is introduced, besides of the multiplication.

Meanwhile, the group is the monoid, i.e. the set with only one binary operation, in accordance with its definition (Zhelobenko & Shtern, 1983). Therefore, the theories in which two binary operations are used over the set of transformations can not be recognized to be the group-theoretic theory. For example, all elements of the unitary group $SU(2)$ describing rotations are unimodular. If one will sum two any elements of the group, the result would not be unimodular, then it does not belong to the group. As a consequence of the Noether theorems it may lead to violation of the conservation laws. The consistent group-theoretic physical theory, in particular quantum mechanics, may be carried out only under fulfilment of all the group definition requirements. So as associativity, existence of the unit and the inverse elements, and, of course, the multiplication as the only binary operation over its elements.

In accordance with the stated above, such theory has to contain at least the pairwise permutable composition over any elements of the non-Abelian Lie groups. Of course, such composition has to turn into the ordinary superposition principle under correspondent parameters area.

Oddly enough that the non-commutativity was not to be a cause of refusal to construct quantum mechanics as the group-theoretic theory, it was only complicating factor (Feynman & Hibbs, 1965). For example, one has only commutative propagators in the double-slit experiment in homogeneous medium, they are multiplied along successive path segments. Nevertheless, even if non-commutativity does not create any difficulties since all propagators belong to the commutative subgroup of the $SU(1,1)$ group, an alternative propagators are added together accordingly to the ordinary superposition principle (Feynman et al., 1963).

Of course, using only multiplication both for successive and alternative propagators, and considering the only Hermitian form $\rho = \chi\chi^*$ as an observable, one can not obtain an "interference" pattern since $\rho = const$ everywhere.
Nevertheless, the experimental pattern may be obtained without addition of the second operation, as it would be shown below.
Therefore the inclusion of two binary operations over the set of transformations in quantum mechanics (see for example (Landau & Lifshitz, 1963)) means the groundless rejection to construct the last one as the consistent group-theoretic physical theory.
The section 2 contains consideration of the complete set of observables for the stationary Schroedinger equation (Lunin, 1998; 1999). It consists of four bilinear Hermitian forms, being together they satisfy to some identity which means the completeness condition at the same time. Therefore only three of them are independent. Its geometric interpretation in the Euclidean space is proposed. All conservation laws are considered for the free particle described with the Schroedinger equation, it is shown that the successive points where these laws are fulfilled form the spiral line in general case. Transformations of such lines are considered under some simplest potentials. The qualitative explanation of the double-slit experiment when particles go from the source up to detector one by one, and the experimental pattern is formed by isolated point-wise traces is proposed there.
The section 3 contains the most important part of the paper: a short presentation of the non-Euclidean superposition principle deduction. At first there are established the metric of the propagators logarithms space for the stationary Schroedinger equation, it is the Lobachevsky space. Then, mapping the group elements onto the Lobachevsky plane together with the group operation one establishes the additive representation of the $SU(1,1)$ group in the curved space. Geometric consideration of the subject allows one to develop the symmetric binary composition which is invariable with respect to permutation of two non-commutative group elements and which belongs to the same group as these ones entered the composition (Lunin, 1994). Geometric investigation of this composition with respect to discrete symmetries had also lead to three other compositions, all of them form the non-Euclidean superposition principle, which turns into the ordinary, i.e. the Euclidean, superposition principle in the vicinity of the identity, and applicable up to the Lie groups of arbitrary dimension (Lunin, 1998; 2002). The geometric deduction of all four compositions establishes their geometric contents at the same time.
This section contains also a comparison of these two different rules of the propagators composition for the experiment with two slits arranged at the two different media boundary. It is shown there that the non-Euclidean superposition principle leads to fulfilment of conservation laws everywhere whereas the Euclidean one leads to the same only in some areas. This conclusion is valid also in the case of the homogeneous medium.
The section 4 contains an example of application of the non-Euclidean superposition principle to the physically significant problem of the irreversibility in quantum mechanics (Ginzburg, 1999; Kadomtzev, 2003). All transformations of the time-dependent Schroedinger equation solutions are reversible due to its reversibility, it means that all propagators turn into the inverse ones under time inversion. However, the non-Euclidean superposition principle contains also two binary compositions which do not turn into the inverse ones under inversion of both propagators entered them. This circumstance allowed one to include irreversible processes into the scheme of quantum mechanics. The reversibility of the equation is occurred to be only necessary condition but not quite sufficient for the reversible evolution of the closed physical system.

Thus, the non-Euclidean superposition principle allows coexistence of the reversible and irreversible processes in the closed systems described with only reversible equations (Lunin & Kogan, 2004; 2009).

2. Completeness of observables

To introduce the main and necessary notions for solution the problem mentioned above in the simplest but sufficient way, let us consider the unidimensional Schroedinger equation with real potential for the particle above barrier. According to (Kolkunov, 1969; 1970), and also (Lontano & Lunin, 1991), we shall start with the equation under corresponding conditions at the initial point z_0

$$\frac{d^2\chi(z)}{dz^2} + k^2(z)\chi(z) = 0, \quad \chi(z_0) = \chi_0, \quad \chi'(z_0) = \chi'_0, \tag{1}$$

where $k^2(z) = (2m/\hbar^2)[E - U(z)], E$ and $U(z)$ are energy and real potential respectively. Going over to the pair of first order equations for complex functions

$$\Phi_\pm(z) = \frac{k^{1/2}}{\sqrt{2}}[\chi \pm \frac{1}{ik}\chi^{'}] \tag{2}$$

with corresponding conditions at the initial point z_0, one has the following matrix equation for Φ= column $\|\Phi_+, \Phi_-\|$

$$\Phi^{'}(z) = [ik(z)\sigma_3 + \frac{k^{'}(z)}{2k(z)}\sigma_1]\Phi(z), \tag{3}$$

where σ_s are Pauli matrices including identity one σ_0, $s = 0, 1, 2, 3$. Let us notice that equation (2.3) may be also obtained by means of staircase approximation (Kolkunov, 1969; 1970). Dividing axis z into segments Δz_i with $k_i = const$ and steps Δk_i at its common points, requiring continuity of χ_i, χ'_i there, and going over to $\Delta z_i \to 0$, one has also the equation (2.3). Therefore propagator Q (see below) includes continuity of χ, χ' everywhere.
A solution of (2.3) may be written in the form $\Phi(z) = Q(z, z_0)\Phi(z_0)$, where Q is a propagator, i.e. matrix, transforming $\Phi(z_0)$ into $\Phi(z)$,

$$Q(z, z_0) = T \exp \int_{z_0}^{z} [ik\sigma_3 + \frac{k'}{2k}\sigma_1]dz. \tag{4}$$

Matrix Q is named as a product integral (Gantmakher, 1988), it is a limit of product of the infinitesimal matrix transformations, in general case they are non-commutative.
Let us consider four bilinear Hermitian forms with respect to Φ, Φ^+,

$$j_s(z) = \Phi^+(z)\sigma_s\Phi(z). \tag{5}$$

They satisfy to the identity

$$j_0^2 = j_1^2 + j_2^2 + j_3^2 \tag{6}$$

independently if they are solutions of equation (2.3) or not, therefore only three of them are independent. Let us introduce, accordingly to the direct product definition (Lankaster, 1969), Hermitian matrix

$$J = \|\Phi^*_+, \Phi^*_-\| \bigotimes \left|\left| \begin{matrix} \Phi_+ \\ \Phi_- \end{matrix} \right|\right| = \left|\left| \begin{matrix} \Phi^*_+\Phi_+ & \Phi^*_-\Phi_+ \\ \Phi^*_+\Phi_- & \Phi^*_-\Phi_- \end{matrix} \right|\right| = \frac{1}{2}\left|\left| \begin{matrix} j_0 + j_3 & j_1 - ij_2 \\ j_1 + ij_2 & j_0 - j_3 \end{matrix} \right|\right|. \tag{7}$$

Its determinant is equal to zero due to the identity (2.6), it satisfies to relation $J^2 = j_0 J$ which under normalization condition $j_0 = 1$ coincides with definition of the idempotent matrix, therefore the matrix J is similar to the density matrix of pure states (Feynman, 1972). Differentiating expression (2.7) and using equation (2.3) together with its Hermitian conjugate, one obtains

$$J' = ik\{\Phi^+ \bigotimes \sigma_3 \Phi - \Phi^+ \sigma_3 \bigotimes \Phi\} + \frac{k'}{2k}\{\Phi^+ \bigotimes \sigma_1 \Phi + \Phi^+ \sigma_1 \bigotimes \Phi\}, \quad (8)$$

which is equivalent to four equations for j_s:

$$j_0' = \frac{k'}{k} j_1, \quad j_1' = 2kj_2 + \frac{k'}{k} j_0, \quad j_2' = -2kj_1, \quad j_3' = 0. \quad (9)$$

Differentiating the identity (2.6) for j_s and taking equations (2.9) into account, we derive the identity also for j_s and j_s'.

Let us notice that two Hermitian forms, $\rho = \chi\chi^*$ and $j_3 = i(\chi\chi'^* - \chi^*\chi')$, are considered, as a rule, as observables named the density and the current in the generally accepted schemes of quantum mechanics. They are a compositions of only χ and χ', along with the complex conjugate ones, of course. But there are exist also other its real compositions based on *only* these variables. We introduce here into consideration four Hermitian forms expressed by means of only these variables

$$\begin{array}{ll} j_0 = k\chi\chi^* + (\chi')(\chi^{*'})/k, & j_1 = k\chi\chi^* - (\chi')(\chi^{*'})/k, \\ j_2 = \chi\chi^{*'} + \chi^*\chi', & j_3 = i(\chi\chi^{*'} - \chi^*\chi'), \end{array} \quad (10)$$

therefore all four of them satisfy to the identity (2.6) and may also be considered as observables.Taking into account (2.2) and comparing equations (2.5), (2.10) one may see that both quadruples, (2.5) and (2.10), are the same. Therefore both quadruples of j_s may be considered as an observables in the same way as ρ and j_3 mentioned above.

It means that four Hermitian forms j_s form the complete set of observables due to the completeness condition (2.6), only three of them are independent. Besides, the Schroedinger equation (2.1), its spinor representation (2.3) and relations (2.2) allow one to derive equations (2.9), leading not only to conservation law for current j_3, but also to the consistent variations of the Hermitian forms complete set at the same time.

Let us consider the group-theoretic properties of propagators in the spinor description. The last equation in (2.9), $j_3' = 0$, means that the real scalar Hermitian form $j_3 = \Phi^+\sigma_3\Phi$ is a constant. Let Q is a matrix transforming $\Phi(z_0)$ into $\Phi(z)$, i.e. $\Phi(z) = Q(z, z_0)\Phi(z_0)$. Substituting this expression into the conservation condition $j_3 = const$ under arbitrary $\Phi(z_0)$, one has the relation

$$Q^+\sigma_3 Q = \sigma_3, \quad (11)$$

which means that matrix Q belongs to the group $Q \in SU(1,1)$ (Lontano & Lunin, 1991) with the properties $detQ = 1, Q_{22}^* = Q_{11}, Q_{21}^* = Q_{12}$. Of course, this conclusion can also be drawn from the expression for the product integral (2.4), which is a solution to equation (2.3).

The Schroedinger equation describes spatial behavior both free particle and also particle in potential. It defines also all conservation laws for observables at the same time (Malkin & Man'ko, 1979). Therefore it is quiet clear that the ordered sequence of the points where all necessary conservation laws are fulfilled forms the line which may be considered as the particle trajectory. It means that a free particle described with the Schroedinger equation does not obey to move along Euclidean straight line *under any conditions*, as it takes place in classical mechanics. Although all variables in the Schroedinger equation depend on only z in our case, however conservation laws fulfilment for the Hermitian forms under arbitrary conditions at the initial point may lead to another such line spatial behavior, where all necessary conservation laws are fulfilled, as it would be shown below.

Therefore the first our task is to define the spatial configuration of the line where all exact conservation laws are fulfilled for the free particle under arbitrary conditions at the initial point.

The stationary Schroedinger equation is the second order equation over a set of complex functions. The wave function and its derivative at the initial point have to be set independently, therefore they are defined by four real parameters. Connection of the theory with experiment requires, in particular, to define initial conditions from measurements. It means that these conditions would be expressed as Hermitian forms which are consistent with observables to be measured, and vice versa. The complete set of Hermitian forms contains four ones, and three of them are independent. Therefore two Hermitian forms, ρ and j_3, which are considered in a generally accepted schemes of quantum mechanics, can not be recognized sufficient for construction of a complete and closed theory.

In an accepted schemes of quantum mechanics the vector $\boldsymbol{j} = i(\chi\nabla\chi^* - \chi^*\nabla\chi)$ is associated with particle momentum (Landau & Lifshitz, 1963), its amplitude coincides with j_3 in the unidimensional case, therefore we shall also connect j_3 with momentum. It would be expected that all other j_s have a similar sense due to the identity (2.6). One may suppose that an energy is also included in the set of j_s on account of its completeness, but due to the circumstance that the complete set of Hermitian forms includes more variables then it is considered in the accepted forms of quantum mechanics, a connection between energy and momentum here does not coincide with this one in the ordinary schemes of quantum mechanics. They coincide only in the case of $j_1 = j_2 = 0$. It may be shown that a wave function has a form of plane wave under these conditions, j_0 and j_3 are constant everywhere and they have no periodical behavior, although the particle de Broglie wave exists.

An energy and momentum of free particle are reserved both in classical and in quantum mechanics. It is quiet clear that, keeping succession, we have to associate an energy with the Hermitian form j_0, which is positive defined at the same time, as it seen from (2.5). Such incomplete knowledge on j_s is sufficient for our aim here, explicit its identification is more appropriate under more evident alignment of this scheme and the non-Euclidean superposition principle with special relativity where the group-theoretic requirements are especially important.

All exact differential conservation laws are fulfilled on the line to be defined, and the identity (2.6) is also fulfilled there. Moreover, it is the only law containing all observables, on the one hand, and it is fulfilled independently if these Hermitian forms are constructed on the base of the Schroedinger equation solution or not, on the other hand. A similar significance and structure has only the consequence of the Euclidean metric, which under parametric representation of line $X(t), Y(t), Z(t)$ may be written in form $S'^2(t) = X'^2(t) + Y'^2(t) + Z'^2(t)$, where $S(t)$ is a curve length depending on monotonic parameter t. Requiring consistence of the identity (2.6) with the consequence of the Euclidean metric, we shall accept a following correspondence: $j_0 \sim S', j_1 \sim X', j_2 \sim Y', j_3 \sim Z'$.

Let $X(t), Y(t), Z(t)$ are coordinates of the points where all conservation laws are fulfilled. To define the line which is formed by ordered sequence of these points, one may use the fact that a spatial curve is uniquely defined, up to orientation in space, by its curvature and torsion.

Nonnegative curvature is defined by the first and the second its derivative with respect to parameter, and the torsion depends also on the third derivative (Poznyak & Shikin, 1990).
Thus, we obtain a following conclusion on the line where all conservation laws are fulfilled: the quantum particle trajectory is defined uniquely under fulfilling of all exact conservation laws following from the Schroedinger equation excluding its space orientation, i.e. up to insignificant circumstance of a coordinate system choice. If some theory based on such equation does not lead to such trajectory, then it means that the theory does not contain all necessary observables and (or) some conservation laws are violated.
This line is defined by parameters j_s, j_s', j_s''. If $j_3 = const$ the curvature K_1 and the torsion K_2 are expressed as (Lunin, 2008)

$$K_1 = \frac{\sqrt{j_0^2(j_1'^2 + j_2'^2) - (j_1 j_1' + j_2 j_2')^2}}{j_0^3}, \quad K_2 = \frac{j_3(j_1' j_2'' - j_2' j_1'')}{j_0^2(j_1'^2 + j_2'^2) - (j_1 j_1' + j_2 j_2')^2}. \tag{12}$$

The group-theoretic properties of transformations under quantum particle motion most clearly appear in the spinor representation of the Schroedinger equation (2.3). Taking a spinor in its most general form we have

$$\Phi = \left\| \begin{matrix} ae^{i\alpha} \\ be^{i\beta} \end{matrix} \right\| = e^{i\frac{(\beta+\alpha)}{2}} \left\| \begin{matrix} ae^{-i\frac{(\beta-\alpha)}{2}} \\ be^{i\frac{(\beta-\alpha)}{2}} \end{matrix} \right\| \tag{13}$$

with its Hermitian forms

$$\begin{aligned} j_0 &= a^2 + b^2,\ j_1 = 2ab\cos(\beta-\alpha), \\ j_3 &= a^2 - b^2,\ j_2 = 2ab\sin(\beta-\alpha). \end{aligned} \tag{14}$$

It is quite clear that they are defined by three independent real parameters $a, b, (\beta - \alpha)$ and satisfy to the identity (2.6). Relations (2.13), (2.14) and (2.2) allow one to express χ, χ', and also $\Phi_\pm$ by means of j_s.
If the parameter k^2 in (2.1) is constant, $k'(z) = 0$, the term $(k'/2k)\sigma_1$ in (2.4) is vanished together with non-commutativity, and $Q(z, z_0) = \exp[ik(z - z_0)\sigma_3]$. Then the propagator Q satisfies to $Q^+\sigma_0 Q = \sigma_0$ which means conservation j_0 in addition to j_3. As far as $Q^+ = Q^{-1}$, then Q belongs to the unitary commutative subgroup of the group $SU(1, 1)$.
It is clear from equations (2.14) that a and b are constant for the free particle, then spinor components under arbitrary conditions at z_0 may be written at any point z as

$$\Phi_+ = a_0 e^{i[k(z-z_0) - \frac{\beta_0-\alpha_0}{2}]}, \quad \Phi_- = b_0 e^{-i[k(z-z_0) - \frac{\beta_0-\alpha_0}{2}]}, \tag{15}$$

therefore one has free particle observables under correspondent parameters at the z_0

$$\begin{aligned} j_0 &= a_0^2 + b_0^2,\ j_1 = 2a_0 b_0 \cos[2kz - (\beta_0 - \alpha_0)], \\ j_3 &= a_0^2 - b_0^2,\ j_2 = 2a_0 b_0 \sin[2kz - (\beta_0 - \alpha_0)]. \end{aligned} \tag{16}$$

The expressions for K_1 and K_2 are simpler in this case

$$K_1 = \frac{\sqrt{j_1'^2 + j_2'^2}}{j_0^2}, \quad K_2 = \frac{j_3(j_1' j_2'' - j_2' j_1'')}{j_0^2(j_1'^2 + j_2'^2)}. \tag{17}$$

Taking into account equations (2.16) and (2.17) under condition $k(z) = const$ one may see that $K_1(z)$ and $K_2(z)$ satisfy to the following conditions

$$\begin{array}{c} K_1(z) = 2k\frac{\sqrt{j_1^2(z)+j_2^2(z)}}{j_0^2(z)} = 2k\frac{\sqrt{j_1^2(0)+j_2^2(0)}}{j_0^2(0)} = const, \\ K_2(z) = -2k\frac{j_3(z)}{j_0^2(z)} = -2k\frac{j_3(0)}{j_0^2(0)} = const. \end{array} \tag{18}$$

Thus, both the curvature and the torsion of *free* quantum particle are constant and, being dependent on j_s at the initial point, may have arbitrary values. Only the spiral lines have such properties. If $K_1 = 0$, i.e. $j_1^2 + j_2^2 = 0$, then trajectory is the straight line; if $K_2 = 0$, i.e. $j_3 = 0$, then it is situated at the plane, and $K_1 = 2k/j_0$. The sign minus in K_2 means that the spinor components (2.15) and its observables correspond to the left-hand spiral line. The action of the inversion operator σ_1 (Lunin, 2002), i.e. permutation of the spinor components, change the torsion sign, and the left spiral line converts into the right one.

Integrating the expressions (2.16) under corresponding constants choice, then excluding integration variable z and go over to the particle Z-coordinate, we have the following expressions for particle coordinates and its path length

$$\begin{array}{c} X(Z) = -\frac{\sqrt{j_1^2(0)+j_2^2(0)}}{2k}\cos[2(k/j_3(0))Z + \arctan(j_1(0)/j_2(0))], \\ Y(Z) = \frac{\sqrt{j_1^2(0)+j_2^2(0)}}{2k}\sin[2(k/j_3(0))Z + \arctan(j_1(0)/j_2(0))], \\ Z = Z, \quad S(Z) = [j_0(0)/j_3(0)]Z. \end{array} \tag{19}$$

Let us consider the main peculiarities of free-particle trajectories. The requirement $2(k/j_3(0))Z_{st} = 2\pi$ defines the spiral line step Z_{st}. The first two expressions in (2.19) lead to its radius R: $Z^2 + Y^2 = [j_1^2(0) + j_2^2(0)]/(4k^2) \equiv R^2 = const$. Particle path length along one step is $S_{st} = \pi j_0(0)/k$. Going over to the de Broglie wavelength $\lambda = 2\pi/k$ the trajectory parameters may be expressed as (Lunin, 2008)

$$Z_{st} = [j_3(0)/2]\lambda, \quad R = \frac{\sqrt{j_1^2(0) + j_2^2(0)}}{4\pi}\lambda, \quad S_{st} = [j_0(0)/2]\lambda. \tag{20}$$

It is seen from equations (2.20) that the free quantum particle described with the Schroedinger equation contains also a transverse components of its motion depending on the de Broglie wavelength. All components of such motion are proportional to this wavelength but they are also dependent upon the observables j_s at the initial point. The last circumstance leads, for example, to the same S_{st} under different combinations of $j_s(0)$.

Let notice that variable k entered the Schroedinger equation and defining the de Broglie wavelength may be expressed as $k(z) = -j_2'/(2j_1)$ due to equations (2.9). Unrolling surface of the cylinder onto a plane and applying the Pythagorean theorem to the triangle formed by legs Z_{st} and $2\pi R$, and hypotenuse S_{st}, one obtains the equality $Z_{st}^2 + (2\pi R)^2 = S_{st}^2$, which leads to the identity (2.6) due to the conditions (2.20). The angle between an element of the cylinder directed along the axis Z and the tangent to the spiral line is determined by $\tan\theta = (2\pi R)/Z_{st} = \sqrt{j_1^2 + j_2^2}/j_3$. It coincides with the ratio of the curvature of the spiral line to its torsion.

Potential variations lead, according to equations (2.9), to variations of j_0, j_1, j_2, they change, in turn, the curvature and the torsion, i.e. trajectory. Let the particle beginning motion at $z = 0$ under arbitrary conditions, moves in area $o \leq z \leq a$ under $k_1 = const$, then passing through the potential step at $z = a\,k_1$ goes to $k_2 = const$. The propagator expression calculated according (2.4) (Kolkunov, 1969; 1970) is expressed in this case as

$$Q(z,0) = \exp(iM\sigma_3)\exp(L\sigma_1)\exp(iN\sigma_3), \tag{21}$$

where $N = k_1 a, M = k_2(z-a), L = (1/2)\ln(k_2/k_1)$ are real parameters. At both sides of the step the particle trajectories are spiral lines with different parameters. Therefore, since only transformation of motion is interesting in this case, let us put $N = M = 0$ in (2.21), then $Q = \exp(L\sigma_1)$ and $Q^+ = Q$. Such matrix satisfies to conditions $Q^+\sigma_2 Q = \sigma_2$ and $Q^+\sigma_3 Q = \sigma_3$, i.e. j_2 and j_3 are conserved. One has the following transformations in this case

$$J_0 = \cosh(2L)j_0 + \sinh(2L)j_1, \quad J_1 = \sinh(2L)j_0 + \cosh(2L)j_1, \quad J_2 = j_2, \quad J_3 = j_3, \tag{22}$$

then

$$\begin{aligned} Z_{st}(k_2) &= \pi J_3/k_2 = \pi j_3/k_2, \\ R(k_2) &= \frac{\sqrt{J_1^2+J_2^2}}{2k_2} = \frac{\sqrt{j_1^2+j_2^2+\sinh(4L)j_0 j_1+\sinh^2(2L)(j_0^2+j_1^2)}}{2k_2}, \\ S_{st}(k_2) &= \pi J_0/k_2 = \pi[\cosh(2L)j_0+\sinh(2L)j_1]/k_2. \end{aligned} \tag{23}$$

It is seen from expressions (2.23) that there are exist conditions dependent on the value L leading to $R = 0$. It means that, as far as an arbitrary element of the group $SU(1,1)$ is representable in the form (2.21), it is possible a transformation of the spiral particle trajectory with $R \neq 0$ into the Newtonian free particle trajectory, and vice versa.
Similar consideration of particle motion above right angle potential barrier shows that there are exist conditions under which all j_s in front of the barrier go to the same behind it (Lunin, 2008). These conditions coincide with the same ones when the reflection coefficient is zero in the ordinary forms of quantum mechanics.
Let us notice here a similarity of transformations (2.22) to those in the special relativity.

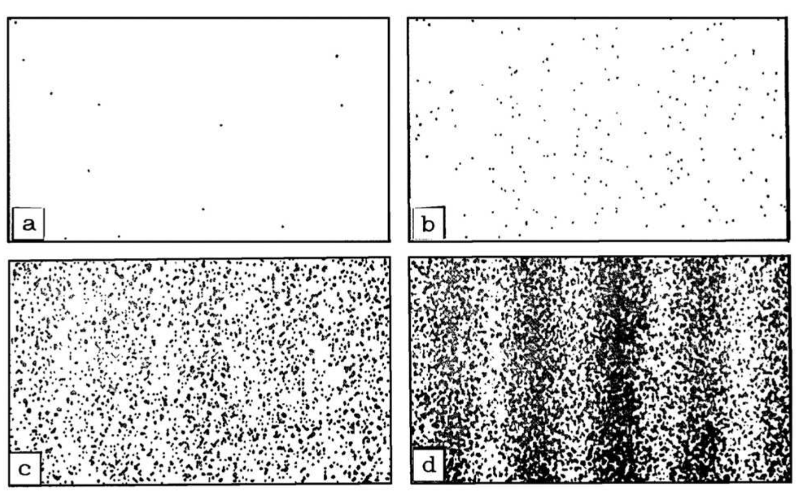

Fig. 1. Double-slit experiment with a low-intensity source of electrons under different expositions (Tonomura et al., 1989).

Free particle spiral-like trajectories allow one to propose a qualitative explanation of the double-slit experiment with single electrons which does not require a particle dualism and a wave function collapse (Kadomtzev, 2003). Figure 1 shows the results of the double-slit experiment (Tonomura et al., 1989) under individual electrons when the next particle leaves

the source after the previous one has already been registered and disappeared. It is seen two peculiarities there. The first and the main one is the fact that each electron produces only one point-wise trace, and the second one is the periodic spatial distribution of the traces density appearing only under enough long expositions.
As it is shown above the question on completeness of observables is not solved both in the theory and in the experiment. Therefore it is necessary to make some assumptions, especially on the free particle transverse motion, i.e. on j_1 and j_2 (Lunin, 2008).
Let us assume that $j_1^2 + j_2^2 \neq 0$ are equal for all particles, i.e. their R are also the same. However j_1 and j_2 may be different at the same time, and we assume that they have random values. Figure 2 shows results of simulation for the experiment under this assumption.

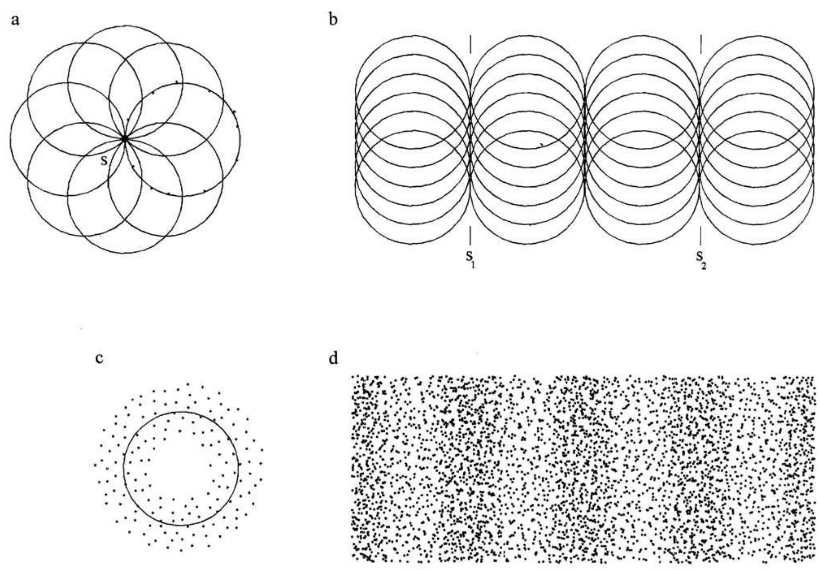

Fig. 2. Simulation of the double-slit experiment for particles moving along helical lines.

There are shown some circles in figure 2a which are cross sections of the cylinder surfaces where spiral trajectories are situated. The points on one of them show a random positions of different particles, and only those of particles form a traces on the photographic plate which go through the point-wise slit S. Therefore *one circle* leads to *one trace*, and another circle leads to another trace and so on, but all of them will create increased traces density near the circumscribed circle of all previous circles.
It may be said that the isolated spiral lines set one-to-one mapping the point-wise source (or slit) to the points of the detector plane. This circumstance explain the point-wise traces on the photographic plate.
The stretched slit is the set of point-wise ones. Figure 2b shows two slits S_1, S_2 and a set of corresponding circles described above under the assumption that the distance between slits is close to twice diameter of the spiral curve. Let us note that the particles having velocity projections almost parallel to the slits direction go through the slits in relatively more number then those having perpendicular projections.
Comparing the simulation with the experiment one would take into account the main experimental factors: a particles source dimensions and angle distribution of particles velocities. These factors lead to smoothing of the interference-like picture but they can not lead to disappearance of *point-like* traces, see fig.2c.

Combining the simulation results shown in fig. 2b and fig.2c one will get the fig.2d, then comparing it with the experimental ones in fig.1, one may see a qualitative similarity of them. It is necessary to emphasize once more that this result has only qualitative character.
Combining in turn fig.2a and fig.2c it may be also explained the old experiment on the scattering of individual electrons on the hole (Biberman et al., 1949), also only qualitatively, of course.
It may be said that the systems of small holes or slits are the particles transverse motion analyzers, good or bad.

3. Non-Euclidean superposition principle

This subject should be considered to be the key question for the group-theoretic structure of quantum mechanics. Keeping in mind the Feynman scheme, we shall attempt to develop a similar construction, however taking into account the group-theoretic requirements for non-commutative propagators, and observables j_s complete set which are arbitrary at the initial point of particle motion.
Let the particle is described by the Schroedinger equation $\nabla^2\chi(\mathbf{r}) + k^2(\mathbf{r})\chi(\mathbf{r}) = 0$. In the case of spatial dependent potential, let us connect an initial and final points $\mathbf{r}_i$ and $\mathbf{r}_f$ with arbitrary piecewise smooth line n defined by tangent unit vector $\mathbf{u}_n(\mathbf{r})$ with initial and final ones $\mathbf{u}_i$ and $\mathbf{u}_f$. Projecting all vector variables onto this line and keeping in mind an infinite set of unidimensional equations along such paths, one has the following form of the product integral along n-th path

$$Q_n(\mathbf{r}_f, \mathbf{u}_f; \mathbf{r}_i, \mathbf{u}_i) = T\exp\int_{\mathbf{r}_i}^{\mathbf{r}_f}[i(\mathbf{k}\mathbf{u}_n)\sigma_3 + \frac{(\mathbf{u}_n\nabla k)}{2k}\sigma_1]dl, \tag{24}$$

where all variables depend on path length l. We shall call it as n-th partial propagator, it has the same group-theoretic properties as (2.4), i.e. matrices Q_n belong to the same non-commutative group $SU(1,1)$ (Lunin, 2002).
To construct the complete propagator taking all paths into account, it is necessary to find at least the composition of two such non-commutative matrices, which belongs to the same multiplicative group and unchanging under these matrices permutation. Let us define a metric of the propagators logarithms space (Lunin, 2002). As far as the product integrals in (2.4) and (3.1) have the same structure and therefore they define the same groups, we shall use for simplicity the first one. Considering integrand in (2.4) as vector in of the space to be defined in orthogonal basis σ_s (Casanova, 1976), one makes up the first quadratic form as $ds^2 = -k^2dz^2 + dk^2/(4k^2)$. This expression defines the plane (k,z) with the Gaussian curvature $C_G = -4$, i.e. the Lobachevsky plane. Going over to variables $u = 1/(2ik), v = z$, one gets the integrand ds and the Kleinian metric form of this plane ds^2 with the same Gaussian curvature

$$ds = \frac{dv\sigma_3 - du\sigma_1}{2u}, \quad ds^2 = \frac{du^2 + dv^2}{4u^2}. \tag{25}$$

As far as equations of kind (2.1) describe a number of physical phenomena, let us investigate the significance of this curvature value. If we multiply (2.3) by dz and go over to variables u, v, we get the expression $d\Phi = [(dv\sigma_3 - du\sigma_1)/(2u)]\Phi$, where the integer 2 defines $C_G = -4$. Let replace this integer by an arbitrary constant R and return to variables k, z. Then one has an equation $(R/2)\Phi' = [ik\sigma_3 + k'/(2k)\sigma_1]\Phi$ instead of (2.3). Taking (2.2) into account under conservation R and returning to equation for χ, we have

$$\chi'' + k^2(z)\chi + (\frac{2}{R} - 1)[(ik + \frac{k'}{2k})\chi' + ik(-ik + \frac{k'}{2k})\chi] = 0.$$

It is quiet clear that the last equation goes over to (2.1) only under $R = 2$. As far as a great number of physical phenomena obey to the equations of the spatial stationary Schroedinger equation kind, so as the Helmholtz one, and the Gaussian curvature $C_G = -4$ is its consequence, this curvature value has the exceptional role compared with the role of such kind equations.

Having determined the propagators logarithms space, or the space of the Lie algebra of the group $SU(1,1)$, which is the Lobachevsky plane, it is needed to map the group into this space. It is necessary for it to map the group elements there as the geometrical objects, and to find the operation under these objects corresponding to the group operation.

The metric in (3.2) maps the hyperbolic plane onto the upper Euclidean half-plane $u > 0$ as the conformal mapping in semi-geodesic orthogonal coordinate system (the Poincare map) (Bukreev, 1951). Any group element from $SU(1,1)$ may be expressed in form (2.21), and also as $Q = \exp(\boldsymbol{a\sigma})$, then one has the following equality

$$Q = e^{iM\sigma_3} e^{L\sigma_1} e^{iN\sigma_3} = e^{\boldsymbol{a\sigma}} = \cosh a + (\boldsymbol{a\sigma})(\sinh a/a), \tag{26}$$

where $(\boldsymbol{a\sigma}) = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3, a^2 = a_1^2 + a_2^2 + a_3^2$ with real a_1, a_2 and imaginary a_3, $\boldsymbol{a} = \boldsymbol{n}_a a$.

It should be noted that the geodesic lines (straight lines) on the Lobachevsky plane in its representation on the Poincare map are the semicircles with its centers on the horizontal axis v (see figure 3 below) and euclidean straight lines parallel to axis u. Following to (Lunin, 1994; 1998; 2002), taking an arbitrary point on the Poincare map as initial one, let us map the matrix $\exp(iN\sigma_3)$ as the oriented segment with length N along any geodesic line outgoing from the initial point. Note that the geodesic vector length a is defined by the matrix trace, as it follows from (3.3). Then we map the matrix $\exp(L\sigma_1)$ as the next geodesic segment with the initial point at the end of previous segment and length L along the perpendicular geodesic line. The matrix $\exp(iM\sigma_3)$ is mapped in the similar way.

Let us connect the initial point and the end of the last segment with the geodesic line on the Poincare map. Then we shall obtain the plane figure named as *bi-rectangle*, the fourth its side corresponds to the geodesic vector $\boldsymbol{a}$ in (3.3). Equalities for matrix elements in (3.3) allow one to obtain all elements of the bi-rectangles or triangles (if N or M is equal to zero).

Thus, the group $SU(1,1)$ element is mapped as the oriented segment of the geodesic line, or geodesic vector, on the Poincare map. It is quite clear that the successive addition of the geodesic vectors corresponds to the group operation of successive matrices multiplication at the same time. This circumstance explains also the sense of the term "propagator logarithms space" used above.

To make more clear the geometric sense of the group operation, let us multiply two arbitrary matrices:

$$\begin{aligned}\exp(\boldsymbol{c\sigma}) = \exp(\boldsymbol{b\sigma})\exp(\boldsymbol{a\sigma}) = \cosh b \cosh a + (\boldsymbol{n}_b \boldsymbol{n}_a)\sinh b \sinh a +\\ +\boldsymbol{\sigma}\{\boldsymbol{n}_b \sinh b \cosh a + \boldsymbol{n}_a \sinh a \cosh b + i[\boldsymbol{n}_b \boldsymbol{n}_a]\sinh b \sinh a\}.\end{aligned} \tag{27}$$

One may see from (3.4) that the resulting geodesic vector $\boldsymbol{c}$ contains the orthogonal component to the plane defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and its length c may be obtained from the expression $\cosh c = \cosh b \cosh a + (\boldsymbol{n}_b \boldsymbol{n}_a)\sinh b \sinh a$. The non-commutativity of the matrices $\exp(\boldsymbol{a\sigma})$

and $\exp(\boldsymbol{b\sigma})$ is defined by this orthogonal component. It is also seen that the geodesic vectors of the commutative matrices are situated on the same geodesic line due to $[\boldsymbol{n}_b \boldsymbol{n}_a] = 0$.
These facts make clear the geometric sense of non-commutativity. Let us note here that the group $SU(1,1)$ logarithms space involves the complete three-dimensional Lobachevsky space. It may be said that the multiplicative non-commutative three-parameter group $SU(1,1)$ is isomorphically represented as additive group in the Lobachevsky space with the constant negative Gaussian curvature being similar to the map of the group $SU(2)$ on the unit sphere.
Let us find the binary commutative composition over the non-commutative group SU(1,1). Let the matrix Q_M is the result of the composition to be find of two arbitrary equivalent non-commutative matrices Q_A and Q_B. Let us formulate the requirements for Q_M:

$$\begin{gathered} a) Q_M \in SU(1,1); \\ b) Q_M \to Q_A,\ if\ Q_B \to 1\ and\ Q_M \to Q_B,\ if\ Q_A \to 1; \\ c) Q_M \to Q_M,\ if\ Q_A \to Q_B\ and\ Q_B \to Q_A. \end{gathered}$$

In accordance with (a) all these matrices are representable as $Q_A = \exp(\boldsymbol{a\sigma})$, $Q_B = \exp(\boldsymbol{b\sigma})$, $Q_M = \exp(\boldsymbol{m\sigma})$. All geodesic vectors have the common initial point O on the Poincare map due to (b), see figure 3.

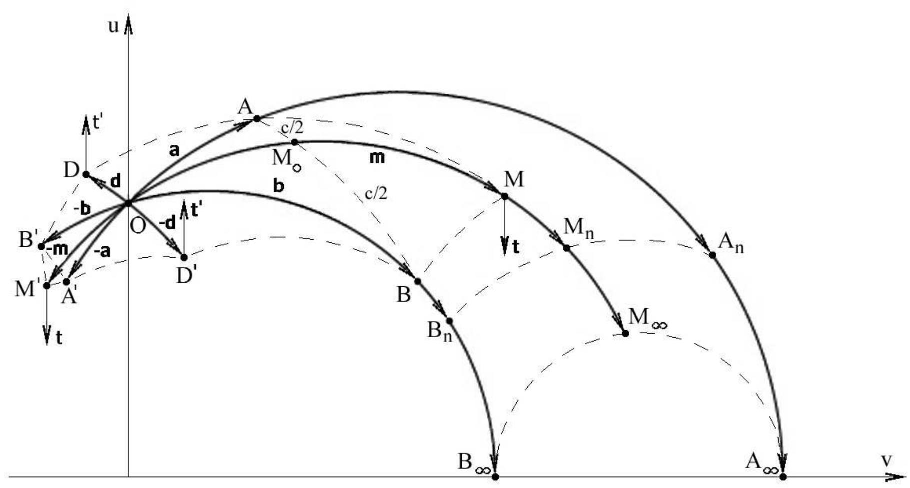

Fig. 3. The Poincare map. Geodesic lines are semicircles with centers on v-axis.

The requirement in (c) would be fulfilled if the vector $\boldsymbol{m}$ goes through the hyperbolic middle M_0 of the oriented segment $AB = \boldsymbol{c}$ connecting the ends of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ there. Our task now is to obtain the geodesic vector $\boldsymbol{m}$, finding at first the triangle OAB median OM_0 outgoing from the initial point O. Taking into account that $\boldsymbol{m}$ and $\boldsymbol{c}$ intersect in their midpoints M_0, one has the following relations for triangles OAB, OAM_0, and OM_0B respectively:

$$\begin{gathered} \exp(\boldsymbol{c\sigma}) \exp(\boldsymbol{a\sigma}) = \exp(\boldsymbol{b\sigma}), \\ \exp(\boldsymbol{c\sigma}/2) \exp(\boldsymbol{a\sigma}) = \exp(\boldsymbol{m\sigma}/2), \\ \exp(\boldsymbol{c\sigma}/2) \exp(\boldsymbol{m\sigma}/2) = \exp(\boldsymbol{b\sigma}). \end{gathered}$$

These relations lead to the expression to be find

$$\begin{aligned} \exp(\boldsymbol{m\sigma}) &= \{[\exp(\boldsymbol{a\sigma}) \exp(-\boldsymbol{b\sigma})]^{1/2} \exp(\boldsymbol{b\sigma})\}^2 = \\ &= \{[\exp(\boldsymbol{b\sigma}) \exp(-\boldsymbol{a\sigma})]^{1/2} \exp(\boldsymbol{a\sigma})\}^2. \end{aligned} \tag{28}$$

Obviously that $\exp(\boldsymbol{m\sigma}) = \exp(\boldsymbol{b\sigma})$ if $\boldsymbol{a} = 0$ and $\exp(\boldsymbol{m\sigma}) = \exp(\boldsymbol{a\sigma})$ if $\boldsymbol{b} = 0$.
Since the products and their real powers do not change the group belonging, then the matrix $\exp(\boldsymbol{m\sigma})$ also belongs to the same group as both $\exp(\boldsymbol{a\sigma})$ and $\exp(\boldsymbol{b\sigma})$. Therefore the expressions (3.5) set the commutative binary composition over non-Abelian group SU(1,1).
If $\exp(\boldsymbol{a\sigma})$ and $\exp(\boldsymbol{b\sigma})$ are commutative then $\boldsymbol{m} \equiv \boldsymbol{a} + \boldsymbol{b}$. As long as the group SU(1,1) is the topological one, then one can expand (3.5) into series under conditions of small $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{m}$ is also small in this case. Taking into account the smallness second order, one has $\boldsymbol{m} \cong \boldsymbol{a} + \boldsymbol{b}$ and, independently, $\boldsymbol{m}^2 \cong \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2(\boldsymbol{ab})$. Therefore the composition rule (3.5) goes to the ordinary superposition principle up to the smallness second order. Representing the geodesic vectors as $\boldsymbol{a} = \boldsymbol{n}_a a$ and so on, we have the following expressions for the vector $\boldsymbol{m}$:

$$\boldsymbol{n}_m = \frac{\boldsymbol{n}_a \sinh a + \boldsymbol{n}_b \sinh b}{p}, \quad \tanh(m/2) = \frac{p}{\cosh a + \cosh b}, \tag{29}$$

where $p^2 = \sinh^2 a + 2(\boldsymbol{n}_a \boldsymbol{n}_b) \sinh a \sinh b + \sinh^2 b$.
The composition rule (3.5) may be extended up to the square nonsingular matrices of any order and also up to an arbitrary Lie groups under condition of existence their matrix representations:

$$M = \{[AB^{-1}]^{1/2}B\}^2 = \{[BA^{-1}]^{1/2}A\}^2.$$

Extremely important role belongs to the discrete symmetries in physics, especially in quantum mechanics. Beforehand we mean here the inversion and permutations. Such symmetries become geometrically apparent and contain particularly rich capabilities in the binary compositions of the propagators.
It is clear that $\boldsymbol{a} \to -\boldsymbol{a}$ leads to $Q_A \to Q_A^{-1}$. Let us consider the geometric properties of the binary composition (3.5) on the Poincare map, figure 3. If O is the common point of both geodesic vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\boldsymbol{m}$ is the diagonal of the Lobachevsky parallelogram $OAMB$. Let us prolong the corresponding geodesic lines to the left hand of the point O, then we shall get vectors $-\boldsymbol{a}$ and $-\boldsymbol{b}$, they define the inverse matrices Q_A^{-1} and Q_B^{-1}. The vector $-\boldsymbol{m}$ corresponds to the inversed parallelogram $OA'M'B'$ diagonal OM', then one has the inversed composition $M^{-1} = \{[A^{-1}B]^{1/2}B^{-1}\}^2$. Analogically, if one replaces only one vector $\boldsymbol{b}$ by $-\boldsymbol{b}$, then we shall have the parallelogram $OADB'$ with its diagonal $\boldsymbol{d}$. It leads to the composition $D = \{[AB]^{1/2}B^{-1}\}^2$, which goes also to the inverse one under inversion both A and B. If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are small then $\boldsymbol{d} \cong \boldsymbol{a} - \boldsymbol{b}$. Let us emphasize that all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$, and $\boldsymbol{d}$ (see fig.3) are situated on the same Lobachevsky plane, all of them do not contain an orthogonal components to their Lobachevsky plane. It is quit clear from fig.3 that permutation of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ leads to $\boldsymbol{d} \to -\boldsymbol{d}$. In the matrix terms it means that $D \to D^{-1}$ under permutation of A and B.
We have investigated all discrete symmetries mentioned above which may be represented in the Lobachevsky plane. However it is not the complete investigation of the geometric properties of the $SU(1,1)$ group in the complete Lobachevsky space, it is necessary to go outside of the Lobachevsky plane to obtain the complete geometric description of non-commutativity.
For this aim it is necessary to obtain the composition which includes only the term proportional to the $[\boldsymbol{n}_b \boldsymbol{n}_a]$ in its exponential expression, as it is clear from the expression (3.4). Omitting cumbersome geometric tracings and also cumbersome algebraic calculations, we shall bring the results. The composition to be defined has two forms: $T = (AB^{-2}A)^{1/2}A^{-1}B$ and $T' = (AB^2A)^{1/2}A^{-1}B^{-1}$ (here prime means only notation, without any other sense). Let

us find the geometric sense of the composition T. Representing all matrices in exponential form one obtains

$$T = \exp(\boldsymbol{t\sigma}) = \cosh t + (\boldsymbol{n}_t\boldsymbol{\sigma}) \sinh t = (e^{\boldsymbol{a\sigma}} e^{-2\boldsymbol{b\sigma}} e^{\boldsymbol{a\sigma}})^{1/2} e^{-\boldsymbol{a\sigma}} e^{\boldsymbol{b\sigma}},$$

then the parameters of the vector $\boldsymbol{t}$ are expressed as (Lunin & Kogan, 2004; 2009)

$$\boldsymbol{n}_t = i\frac{[\boldsymbol{n}_b\boldsymbol{n}_a]}{\sqrt{1-(\boldsymbol{n}_b\boldsymbol{n}_a)^2}}, \quad \tanh t = \frac{\sqrt{1-(\boldsymbol{n}_b\boldsymbol{n}_a)^2}\tanh b \tanh a}{1-(\boldsymbol{n}_b\boldsymbol{n}_a)\tanh b \tanh a}. \tag{30}$$

It is seen from formulae (3.7) that vector $\boldsymbol{n}_t$ is orthogonal to the plane of both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{t}$ is equal to zero if they are collinear, i.e. T is the identity matrix under A and B commutativity. One has to imagine the vectors $\boldsymbol{t}$ and $\boldsymbol{t}'$ to be perpendicular to the Lobachevsky plane of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e. to the Poincare map in this case, figure 3. The geodesic vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ form the triangle on the Lobachevsky plane. Taking into account the Cagnoli formula expressing the triangle area via its two sides and the angle between them and comparing it with (3.7), one may see that $\boldsymbol{t}$ defines oriented parallelogram $OAMB$ area. Of course, there are exist connections between an area value and angle defect δ: $\tanh t = \sin\delta$, the vector $\boldsymbol{t}$ is also connected with the Berry phase. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are small then $\boldsymbol{t} \cong i[\boldsymbol{ba}] \cong -\boldsymbol{t}'$, i.e. parallelograms areas are the same.

Let us investigate the properties of compositions T and T' with respect to the discrete symmetries. It is seen from (3.7) that permutation of matrices A and B leads to $\boldsymbol{n}_t \to -\boldsymbol{n}_t$, i.e. to $T \to T^{-1}$. If we shall replace both vectors $\boldsymbol{a} \to -\boldsymbol{a}$ and $\boldsymbol{b} \to -\boldsymbol{b}$, then both expressions in (3.7) would not be changed. Geometrically these replacements lead to the transformation of the parallelogram $OAMB$ into one $OA'M'B'$, fig.3, with the same orientation and area, i.e. $T \to T$.

The replacement of only one vector $\boldsymbol{b} \to -\boldsymbol{b}$ leads to the parallelogram $OADB'$ with contrary directed unit vector $\boldsymbol{n}_t$ and with changed area value. Note that this replacement transforms $T \to T'$ at the same time, then T and T' have the similar symmetry properties, of course.

Opposite directions of $\boldsymbol{n}_t$ and $\boldsymbol{n}_{t'}$ for adjacent areas express the saddle character of the planes with negative Gaussian curvature.

One may see that addition of the binary compositions T and T' to M and D extends the geometry contents of the binary compositions over the group $SU(1,1)$ up to the complete three-dimensional Lobachevsky space.

The symmetry properties of all binary compositions obtained above in the geometric way may be also verified by means of the ordinary algebraic calculations (Lunin, 2002; Lunin & Kogan, 2009).

All the binary compositions mentioned above may be considered as the *non-Euclidean superposition principle*:

$$\begin{aligned} M &= \{[AB^{-1}]^{1/2}B\}^2 = \{[BA^{-1}]^{1/2}A\}^2, \\ D &= \{[AB]^{1/2}B^{-1}\}^2, \\ T &= [AB^{-2}A]^{1/2}A^{-1}B, \quad T' = [AB^2A]^{1/2}A^{-1}B^{-1}, \end{aligned} \tag{31}$$

applicable to the multiplicative non-Abelian Lie groups of any order. All these compositions belong to the same groups as both A and B, since real powers do not change the group belonging. These compositions have the following properties with respect to the discrete symmetries under non-commutative group elements A and B

$$if \quad A \to B, \quad B \to A, \quad then \quad M \to M, \quad D \to D^{-1}, \quad T \to T^{-1}, \quad T' \to (T')^{-1};$$

$$if \quad A \to A^{-1}, \quad B \to B^{-1}, \quad then \quad M \to M^{-1}, \quad D \to D^{-1}, \quad T \to T, \quad T' \to T'.$$

These compositions go over to the ordinary superposition principle in the vicinity of identity with the same symmetry properties. The group elements A and B may also belong to the commutative group, the compositions M and D conserve their symmetry properties, both T and T' are the identity in this case. The last circumstance allowed one to consider the compositions T and T' as the commutators over the multiplicative groups.

In the simple cases of some subgroups of the group $SL(2, C)$, such as $SU(2), SU(1, 1)$, the non-Euclidean superposition principle has the geometric interpretation in the spaces with nonzero Gaussian curvature. Such groups may be mapped as additive groups into such spaces with quit clear geometric sense of the group elements, the operation over the group, and the compositions discussed above.

It is extremely important to compare the ordinary superposition principle used in a great number of physical phenomena up to now, and the non-Euclidean superposition principle. We shall consider the double-slit experiment for this aim.

At first, it is needed to consider the factors which may be different or the same in the ordinary consideration and proposed here. These factors may be separated with respect to the experimental and also theoretical ones.

If we are interesting now only to compare two composition rules, we have to set the same experimental conditions, and to take the common initial theoretical principles, where it is possible.

From the experimental view point, we regard that stretched slits, as it is usually supposed, lead to loss of subject clarity. It is clear that different pairs of points along stretched slits, one or both, may bring any phase shifts at any detector surface fixed points, and this circumstance has to be taken into account.The last one is not included into the ordinary calculations, it is carried out only for individual path pairs (Feynman & Hibbs, 1965). Therefore we shall consider only two point-wise slits here.

The double-slit experiment is supposed to involve all enigmas of quantum mechanics (Feynman, 1965). However, the ordinary consideration of the experiment does not contain the propagators non-commutativity, as a rule. As long as this circumstance is one of the fundamental peculiarity of quantum mechanics, we shall include this one locating two point-wise slits onto the two media boundary, then the non-commutativity will appear immediately. Nevertheless, excluding the boundary in the final expressions one may compare the composition rules under the same conditions.

Relating to the theoretical distinctions it is necessary to take into account a number of factors. They are following: the Hermitian forms to be compared in the framework of only theory under its incompleteness in the ordinary schemes; the observables have to be compared with the experimental results; the scalar or matrix expressions of the propagators in both approaches; and, of course, the composition rules itself, which have to be roughly consistent with respect to some limiting cases.

Since it is senseless to compare some part of unknown Hermitian forms set with the complete one, we shall accept the complete set in both cases.

It is accepted in the ordinary schemes of quantum mechanics to demonstrate only one observable, the "probability density" $\rho = \chi\chi* = (j_0 + j_1)/(2k)$, with interference pattern. We regard restriction with only one observable to be insufficient due to reasons discussed in the second part of the paper, therefore we shall include all observables into consideration.

The scalar character of the propagators in the ordinary schemes, for example in the Feynman one, we suppose also to be insufficient, then we are forced to use the matrix one.

The last factor we shall discuss below.
Taking all these assumptions into account, let us consider the double-slit experiment when two point-like slits are arranged at the two media boundary (Lunin, 1998; 2002). The propagators along different paths may be written, in accord with the expression (2.21), as

$$\begin{aligned} A &= \exp(\boldsymbol{a\sigma}) = \exp\left(iM_A\sigma_3\right)\exp\left(L\sigma_1\right)\exp\left(iN_A\sigma_3\right), \\ B &= \exp(\boldsymbol{b\sigma}) = \exp\left(iM_B\sigma_3\right)\exp\left(L\sigma_1\right)\exp\left(iN_B\sigma_3\right), \end{aligned} \tag{32}$$

where $N_A = k_1 s_A, N_B = k_1 s_B, M_A = k_2 r_A, M_B = k_2 r_B, L = (1/2)\ln(k_2/k_1)$, k_1 and k_2 are reciprocals of waves before and behind slits respectively, s_A and s_B are path lengths from the source up to slits under k_1, r_A and r_B are the same from slits up to the common point of the detector surface under k_2.
Now let us consider the last factor mentioned above. It is quit clear that one needs at first to compare the composition M from (3.8) and the sum of A and B. Let us note that the coincidence of two point-wise slits, i.e. shift one of them to the position of another, and shutting down one of them have to lead to the same propagator. On the one hand, if we shall displace the slit B to the position of A we shall have the propagator $M_A = \exp(\boldsymbol{2a\sigma})$ in the case of the non-Euclidean superposition principle. Under the Euclidean one, using the sum of propagators, one has $2\exp(\boldsymbol{a\sigma})$, and these matrices have different determinants. On the other hand, if we shut down the slit B, both propagators would be the same, $\exp(\boldsymbol{a\sigma})$. The geometric investigation of this subject (we have no place to prove it here, see (Lunin, 1994)) shows that the composition of propagators would be the first order hyperbolic moment on the Lobachevsky plane, or the geometric mean, in this and similar cases. It means that the non-Euclidean complete propagator M_{NE} for double-slit experiment has to be taken as $[AB^{-1}]^{1/2}B$ with the same group-theoretic properties. The Euclidean one M_E would be the arithmetic mean at the same time, $(A+B)/2$. Now both propagators are roughly to be consistent in the double slit experiment, besides the group-theoretic requirements, of course.
Omitting some calculation details, we shall bring the following expressions for them

$$M_{NE} = \frac{1}{2}\cdot\frac{e^{iM_A\sigma_3}e^{L\sigma_1}e^{iN_A\sigma_3} + e^{iM_B\sigma_3}e^{L\sigma_1}e^{iN_B\sigma_3}}{\sqrt{\cos^2\frac{(N_1-N_2)+(M_1-M_2)}{2} - \sinh^2 L\sin(N_1-N_2)\sin(M_1-M_2)}},$$

$$M_E = \frac{e^{iM_A\sigma_3}e^{L\sigma_1}e^{iN_A\sigma_3} + e^{iM_B\sigma_3}e^{L\sigma_1}e^{iN_B\sigma_3}}{2}.$$

Since matrices in the numerators of both expressions are the same, and since the observables are the bilinear Hermitian forms, all observables calculated by means of two composition rules are distinguished only by factor depending on the problem parameters. Then we have

$$j_s(E) = [\cos^2\frac{(N_1-N_2)+(M_1-M_2)}{2} - \sinh^2 L\sin(N_1-N_2)\sin(M_1-M_2)]j_s(NE).$$

As far as $j_3(NE)$ is constant everywhere due to fulfilment of the group-theoretic requirements to the composition M from (3.8), then $j_3(E) \neq const$, in particular it depends on coordinates as it is seen from the expression above. It means that the Euclidean superposition principle leads to violation of some conservation laws excluding the points where expression in brackets is equal to unit.
We note here that the calculation of the interference pattern for more number of point-wise slits requires to obtain the hyperbolic first order moment over corresponding number of

non-collinear geodesic vectors on the Lobachevsky plane. For example, if one has three slits it is necessary to find at least the composition of three non-commutative matrices which belongs to its group and which does not change under permutation of any pair of them.
As far as we do not know any theoretical or experimental results devoted to the double-slit experiment under double-media conditions, we shall restrict with the homogeneous medium when $k_1 = k_2$, i.e. $L = 0$. Therefore we shall bring two connections between $j_s(NE, M), j_s(E, M)$ and $j_s(NE, D), j_s(E, D)$, where the first pair corresponds to the symmetric composition M, and the second one corresponds to the antisymmetric composition D:

$$\begin{aligned} j_s(E, M) &= [\cos^2 \tfrac{(N_1-N_2)+(M_1-M_2)}{2}] j_s(NE, M), \\ j_s(E, D) &= [\cos^2 \tfrac{(N_1+N_2)+(M_1+M_2)}{2}] j_s(NE, D). \end{aligned} \tag{33}$$

We remind that $j_3(NE, D)$, $j_0(NE, D)$ are constant in homogeneous medium in just the same way as $j_3(NE, M)$, $j_0(NE, M)$, therefore $j_3(E, D)$, $j_0(E, D)$ and $j_3(E, M)$, $j_0(E, M)$ are not constant. It means that last observables calculated by means of the ordinary superposition principle lead to violation of the conservation laws, excluding the points where

$$\begin{aligned} (N_1 - N_2) + (M_1 - M_2) &= \pm 2\pi n \quad for \quad M, \quad n = 0, 1... \\ (N_1 + N_2) + (M_1 + M_2) &= \pm 2\pi m \quad for \quad D, \quad m = 0, 1... \end{aligned} \tag{34}$$

The first expressions in (3.10) and (3.11) show that $j_s(E, M)$ are equal to $j_s(NE, M)$ at the points where two paths length difference is multiply to the wave length, i.e. at the points of peaks in interference pattern.
Two superposition rules are rather compared, now we shall briefly discuss the consequence of the non-Euclidean superposition principle concerning with the double-slit experiment in homogeneous medium restricting with symmetric and antisymmetric compositions M and D. Two observables, j_3 and j_0, are conserved for both compositions whereas j_1 and j_2 at the final point F are dependent upon them at the initial point I as

$$\begin{aligned} j_1(F, M) &= \cos[(N_1 + M_1) + (N_2 + M_2)] j_1(I) + \sin[(N_1 + M_1) + (N_2 + M_2)] j_2(I), \\ j_2(F, M) &= -\sin[(N_1 + M_1) + (N_2 + M_2)] j_1(I) + \cos[(N_1 + M_1) + (N_2 + M_2)] j_2(I) \end{aligned} \tag{35}$$

for composition M, and for composition D as

$$\begin{aligned} j_1(F, D) &= \cos[(N_1 + M_1) - (N_2 + M_2)] j_1(I) + \sin[(N_1 + M_1) - (N_2 + M_2)] j_2(I), \\ j_2(F, D) &= -\sin[(N_1 + M_1) - (N_2 + M_2)] j_1(I) + \cos[(N_1 + M_1) - (N_2 + M_2)] j_2(I). \end{aligned} \tag{36}$$

The expressions (3.12), (3.13) and (2.19), (2.20) define two spiral lines with the same radii and step but having different torsion. It is interesting to note that the line defined by (3.12) does not depend on paths permutation whereas another one changes the torsion at the same time. These two spiral lines have also some other peculiarities, for example all $j_s(F, D)$ are conserved under condition $(N_1 + M_1) = (N_2 + M_2) \pm 2\pi n$.

4. Irreversibility in quantum mechanics

This problem is considered to be unsolved (Ginzburg, 1999; Kadomtzev, 2003) due to the fact that equations describing a physical phenomena, in particular the Schroedinger one, in a closed systems are reversible, they describe such phenomena highly satisfactory, but an entropy is increasing at the same time. Therefore it seems that a problem of irreversibility is

first of all the mathematical one, and the reversible equations have to be accepted as the initial condition.
It is quite clear that the irreversibility may be coupled with interactions. It is also quite clear that an interactions lead to non-commutativity of a propagators describing processes. We shall assume that the mathematical explanation of irreversibility may be carried out on the base of the non-commutative properties of transformations which are contained in the reversible equations, and the reversibility of equations is only necessary condition for the closed system reversible evolution, but perfectly insufficient for that. It would be meant that a reversible equations contain irreversibility in general case. Further we shall follow to the (Lunin & Kogan, 2004; 2009) where the subject is set forth in more details.
Transformations of solutions for the time-dependent Schroedinger equation in its spinor representation belong to the $SL(2,C)$ group. It describes a most general spinors transformations up to unessential scalar factor - matrix determinant. The last one for the $SL(2,C)$-group matrix representation is equal to the unit.
Reversibility of the equations means in particular that any transformation has the inverse one, in just the same way as any group element has the inverse one. In other words, the equation reversibility and the group description of the transformations are closely connected.
The process is reversible if the system goes through the same sequence of states in reverse order under time inversion as it went in the straightforward one. It means that all conservation laws are the same in both processes, i.e. both ones are described by the same group. Interchange of lower and upper integration limits in the product integral leads to the propagator inversion $Q \to Q^{-1}$ (Gantmakher, 1988). In other words, if Q corresponds to the process $t_1 \to t_2$, then Q^{-1} corresponds to the process $t_2 \to t_1$.
As far as irreversibility is the experimental fact, we shall use the density matrix of pure states J from (2.7) based on observables j_s, it has no the inverse one. Then the irreversibility investigation means to investigate the consequences $J(t_0)...J(t_1)...J(t_2)...J(t)$ for times $t_0...t_1...t_2...t$ under inversion of the last consequence.
Let us assume $\Phi(t_1), \Phi(t_2)$ and $J(t_1), J(t_2)$ are to be the spinors and the density matrices correspondingly for arbitrary times t_1, t_2. Let these spinors are connected by matrix $Q(t_2, t_1)$ from the group $SL(2,C)$ as $\Phi(t_2) = Q(t_2, t_1)\Phi(t_1)$.Then one has

$$J(t_1) = \frac{1}{2}\sum_{s=0}^{3} \sigma_s\{\Phi^+(t_1)\sigma_s\Phi(t_1)\}, J(t_2) = \frac{1}{2}\sum_{s=0}^{3} \sigma_s\{\Phi^+(t_1)Q^+(t_2,t_1)\sigma_s Q(t_2,t_1)\Phi(t_1)\}. \quad (37)$$

All propagators in (3.8), excluding T, T', go to inverse ones under time inversion, they do not contain irreversibility. Let us consider one of two last compositions from (3.8) under inversion of both matrices entered it, and prove that $T(A^{-1}, B^{-1}) = T(A, B)$, i.e. $(A^{-1}B^2A^{-1})^{1/2}AB^{-1} = (AB^{-2}A)^{1/2}A^{-1}B$. Multiplying this equality on the right subsequently by B, A^{-1} and raising it to the second power one has

$$A^{-1}B^2A^{-1} = (AB^{-2}A)^{1/2}A^{-1}B \cdot BA^{-1} \cdot (AB^{-2}A)^{1/2}A^{-1}B \cdot BA^{-1} =$$

$$= (AB^{-2}A)^{1/2}(AB^{-2}A)^{-1}(AB^{-2}A)^{1/2}A^{-1}B^2A^{-1} = A^{-1}B^2A^{-1},$$

i.e. $T \to T$ under $A \to A^{-1}$ and $B \to B^{-1}$. The composition T' has the same properties.
Thus, we have the following transformations for propagators compositions in time-depending process $t_1 \to t_2 \to t_1$: $1 \to Q \to 1$, if Q is any reversible propagator, and $1 \to T \to T^2$ for irreversible composition T (or T').

Considering process $t_1 \to t_2 \to t_1$ and replacing Q in the second expression in (4.1) by T^2 one has finally the following expression for the matrix J

$$J(t_1 \to t_2 \to t_1) = \frac{1}{2}\sum_{s=0}^{3} \sigma_s\{\Phi^+(t_1)(T^2)^+\sigma_s T^2\Phi(t_1)\},$$

which does not coincide with $J(t_1)$ there. It means that the process is irreversible in general case. However, even this matrix may lead to a reversible process. A comparison of the last expression with $J(t_1)$ in (4.1) shows that such process is also reversible under conditions $(T^2)^+\sigma_s T^2 = \sigma_s, s = 0,1,2,3$, which lead to $T^2 = \sigma_0$, or $T = \pm\sigma_s$. As far as $T = \sigma_0$ under A and B commutativity, one may see that interaction is the necessary condition for irreversibility, but insufficient.

As an example of the system in which irreversibility may take place let us consider the double-slit experiment where point-wise slits are arranged at the two media boundary. A propagators for it were calculated in (3.9).

The reversibility condition, $t = 0$, as it is seen from (3.7), leads to the requirement $\sqrt{1-(\boldsymbol{n}_b\boldsymbol{n}_a)^2}\tanh b \tanh a = 0$. It means that the process is reversible if at least one vector $\boldsymbol{a}$ or $\boldsymbol{b}$ is equal to zero, or they are collinear.

Using expressions (3.9) the parameters of the vector **a** may be expressed as

$$\cosh a = \cosh L\cos(N_A + M_A), \quad n_{a1}\sinh a = \sinh L\cos(N_A - M_A),$$
$$n_{a2}\sinh a = \sinh L\sin(N_A - M_A),\ n_{a3}\sinh a = i\cosh L\cos(N_A + M_A),$$

and similar for the vector **b**.

If media are identical, i.e. $k_1 = k_2, L = 0$, interaction is absent, the propagators A and B are commutative, then the matrix $T = \sigma_0$. Therefore only reversible processes take place in homogeneous media.

If media are inhomogeneous but the propagators satisfy to the condition $\cosh L\cos(N_A + M_A) = \pm 1$ or $\cosh L\cos(N_B + M_B) = \pm 1$, then $T = \sigma_0$, i.e. one has also reversibility.

Irreversibility takes place for the points where these conditions are violated.

Irreversibility of some process taking place in a closed system has to become apparent to an observer. It means that some observables, i.e. some Hermitian forms, have to be influenced by irreversible process.

Let some process in a closed system is irreversible along $t_1 \to t_2 \to t_1$, and A and B are two corresponding non-commutative propagators from $SU(1,1)$ group representable as $A = \exp[(\boldsymbol{n}_a\boldsymbol{\sigma})a], B = \exp[(\boldsymbol{n}_b\boldsymbol{\sigma})b]$. We shall also assume for definiteness that $(1/2)TrA > 1, (1/2)TrB > 1$, the lengths of vectors **a** and **b** are real under these conditions.

Let the system evolution is a repeating process mentioned above, and if $\Delta t = t_2 - t_1$ then time duration of n-multi-periodic process is $2n\Delta t$ and the lengths of vectors **a** and **b** are also increased by $2n$ times.

Thus, irreversibility has to be appeared as dependence of some observables calculated by means of the composition T on number of cycles n.

The value $(\boldsymbol{n}_b\boldsymbol{n}_a) \neq \pm 1$ due to A and B non-commutativity, then the length of the vector $\boldsymbol{t}$ is positive. The length $\tilde{t}$ of the vector $\boldsymbol{t}$ after n-multiple repetitions of the process will be defined by

$$\tanh\tilde{t} = \sqrt{1-(\boldsymbol{n}_b\boldsymbol{n}_a)^2}\,\frac{\tanh 2nb\tanh 2na}{1-(\boldsymbol{n}_b\boldsymbol{n}_a)\tanh 2nb\tanh 2na}. \tag{38}$$

Calculating correspondent matrices on the base of composition T we shall obtain the expression for $\tilde{j}_0$ after n-times repetitions of the process:

$$\tilde{j}_0 = \frac{\{1 + (1 + 2\frac{t_3^2}{t^2}) \tanh^2 \tilde{t}\} j_0 + 2\frac{t_1 j_1 + t_2 j_2}{t} \tanh \tilde{t} + 2t_3 \frac{t_1 j_2 - t_2 j_1}{t^2} \tanh^2 \tilde{t}}{1 - \tanh^2 \tilde{t}}, \tag{39}$$

where j_s are the observables at the beginning of process. It was taken into account here that all t_s/t do not depend on n, and, as far as a_1, a_2 are real and a_3 is imaginary and the same for b_s , it is also accepted here $t_3 \to it_3$, so that t_3 in (4.3) is real.
The value $\tilde{j}_0 = \Phi^+\Phi$ is positive defined, the value $\tilde{j}_0$ coincides with j_0 at the beginning of process. It is seen from (4.2) that $\tilde{t}$ is restricted under $n \to \infty$, then $\tilde{j}_0$ in (4.3) is positive, increases and also restricted under this condition. Besides, it is the only positive defined functional.
There were carried out the geometric analysis of irreversibility, and also the functional $\tilde{j}_0$ in (Lunin & Kogan, 2009). It was shown there that the functional is closely connected with the area of triangle defined by vectors **a** and **b** on the Lobachevsky plane. This area is coupled in turn with the Berry phase. Such consideration allows also to show that the functional grows more quickly under interaction increase.
It may be assumed that this functional may be coupled with an entropy.

5. Conclusion

Three subjects connected with quantum mechanics considered above allow one to make some conclusions. Two of them, the observables set completeness and the superposition principle, lie in the foundations of quantum mechanics, the third one, an irreversibility, is its essential consequence.
The first topic of the paper is devoted to an analysis of the conventional quantum mechanics structure from the view point of requirements of the observables set completeness and fulfilment of the conservation laws for them. Both last subjects are closely connected among themselves, and with the group theory, of course.
As long as different observables may be connected with each other in accordance with the uncertainty relations in the conventional forms of quantum mechanics, then the observables completeness obtains an exceptional sharpness. If one has no complete set of them then it is impossible to prove that the theory includes all similar relations, even for the known observables.
Considering a stationary Schroedinger equation it was defined the complete set of the Hermitian forms based only on the complex wave function and its derivative. It may be said that the complete set is a consequence of only the equation and combinatorial analysis. These Hermitian forms contain only the same variables which are used for probability density and its current in the ordinary forms of quantum mechanics.
The complete set includes four Hermitian forms, they satisfy to some identity in any case, therefore it may be considered as the completeness condition, and only three of them are independent. The set is also applicable to the time-dependent Schroedinger equation as far as the last one contains only the first order time derivative.
Since the stationary Schroedinger equation is similar to the Helmholtz one, the complete set of the Hermitian forms is also similar to the Stokes parameters, they satisfy to the same identity. Obviously, that the *complete* set contains the parameters used in quantum mechanics now, and also the hidden parameters discussed there. As far as the set of the Stokes parameters

is complete and known during many decades, the complete set of the Hermitian forms connected with the Schroedinger equation and described here is similar to them, one may say that there are an unused parameters in quantum mechanics but not at all a hidden ones.
The complete set of observables is assumed to have a spatial interpretation. An analysis of the free particle conservation laws fulfilment under arbitrary initial conditions based on the complete set of observables shows that a spatial line where all necessary conservation laws are fulfilled is the spiral line in a general case, such line may be named as the free particle trajectory. Obviously, that even free quantum particle has a transversal motion components in this case.
Consideration of the trajectories transformations under some simplest potentials shows that the spiral line may turn to the straight line under some conditions, and vice versa.
The observables transformation on the step-wise potential which is similar to the Lorentz one allows one to suspect that such transformation may play a role of a bridge between quantum mechanics and special relativity.
Combination of the complete set of observables with its spatial interpretation allows one to say that the quantum particle position is defined uniquely by initial conditions and conservation laws. An ordinary probabilistic interpretation in quantum mechanics is assumed to be connected with some unused, and unmeasured of course, parameters containing transversal components of a particle motion.
The observables completeness or its absence influences also on the wave function interpretation. The observables at the initial point have to define the wave function and its derivative there. If some part of observables is unknown, i.e. is not measured or is not considered at all, then the wave function can not be defined uniquely, even taking into account a phase factor, therefore any interpretation of the wave function, including probabilistic one, can not be proved. Such situation takes place now in the conventional quantum mechanics.
In the opposite case, when the complete set of observables is included into the theory, a quantum particle position is assumed to be uniquely defined. Any interpretation of the wave function is not necessary in this case although the last one may be expressed via observables, as well as its derivative.
The observable complete set leads to a definite position of quantum particle. Obviously, to prove an ordinary probabilistic interpretation in quantum mechanics it is necessary to prove in turn that it is necessary to exclude from consideration some Hermitian forms which are constructed on the basis of the *same* variables, ψ and $\nabla\psi$, as used for construction of ρ and $\boldsymbol{j}$ in the conventional schemes, and which define a transverse components of quantum particle motion.
This approach has led to the uniquely defined trajectories of quantum particle on the one hand, and to the unclassical their configuration, the spiral lines, on the other hand. These two circumstances has led to an explanation of the point-wise traces on the one hand, and to a qualitative one of their distribution on the other hand in the double-slit experiment with a single-particles source without use of a wave function collapse and a particle-wave dualism.
The second topic of the paper is a consideration of the superposition principle in quantum mechanics from the point of view of the Noether theorems. These theorems require the rigorous group-theoretic construction of the fundamental physical theories due to the necessary requirement of the conservation laws fulfilment. The last one is the consequence of the space symmetries.
The approach proposed in the paper has led to the non-Euclidean superposition principle which allows one to fulfill these requirements.

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A successive matrix transformations of solutions for the Schroedinger, the Helmholtz and other similar wave equations are non-commutative in a general case. Such transformations may be geometrically mapped into a curved spaces, in particular into the Lobachevsky space with the Gaussian curvature $C_G = -4$ as it was shown above. The problems similar to interference ones require to use some composition rule for alternative transformations, and a use of the ordinary superposition principle leads to the compositions on the complex Euclidean plane, i.e. in the flat space. Therefore one has the situation when we need to compose the same objects (transformations or solutions) either in the one, curved, space or in the other, flat, space.

The non-Euclidean superposition principle allows one to compose all transformations, successive and alternative, in the common space with the Gaussian curvature defined by the equation.

To compare the ordinary superposition principle and the non-Euclidean one it was considered the double-slit experiment when both slits are arranged at the two-media boundary. The approach assumes to consider also a homogeneous medium.

As far as the case with a boundary independently calculated on the base of the ordinary superposition principle is not known to author, consideration of the conservation laws fulfilment was carried out on the base of the partial propagators calculated by means of the product integral, and subsequent comparison of two different rules of their compositions, in accordance with the ordinary and non-Euclidean superposition principles. Such comparison was carried out also for the case of the homogeneous medium.

It was shown that the non-Euclidean superposition principle leads to fufilment of the conservation laws everywhere under presence or absence of a boundary.

The ordinary superposition principle leads to its fulfilment only at the points of peaks of the interference pattern, and to their violation in the other points.

Two compositions entered the non-Euclidean superposition principle, symmetric and antisymmetric with respect to permutations, are considered to see a differences to which they may lead. It may be assumed that these compositions may be connected with bosons and fermions correspondingly, in particular under conditions of the double-slit experiment with such kinds particles.

Taking into account expressions (3.12) and (3.13) one may see that they having different permutation properties lead to different spatial behavior of the j_1 and j_2 in both cases. The experiments with particles of different kinds mentioned above, particularly with polarized ones, i.e. $j_1 \simeq 0$ or $j_2 \simeq 0$, may demonstrate in principle these differences.

It may be assumed that a differences of similar kind are contained also in the ordinary forms of quantum mechanics, for example differences for the central peak in interference pattern for bosons and fermions.

Here it is necessary to take into account that the central peak in the interference pattern is the same for bosons and fermions in accordance with point of view accepted now (Feynman, 1965).

Such kind experiments in combination with expression $\rho = \chi\chi* = (j_0 + j_1)/(2k)$, which does not contain $j_2(F)$, and expressions (3.12), (3.13) for polarized particles may be found also useful to compare the probability interpretation of the density ρ in the quantum mechanics ordinary forms and complete set of observables proposed here experimentally.

Obviously that more rich opportunities appear in the case of the double-slit experiment arranged at the two-media boundary with polarized particles.

The last topic considered in the paper concerns with the irreversibility in quantum mechanics. The problem consists of the circumstance that all the main equations, in particular the Schroedinger one, are reversible, and they describe a physical phenomena satisfactorily excluding irreversible processes. A known attempts to solve the problem contain a proposals to introduce different modifications into existing theory which may lead to the unacceptable changes concerning with reversible processes taking place simultaneously with irreversible ones in the closed physical systems.

The approach proposed in this paper and based on the non-Euclidean superposition principle comes from the reversible Schroedinger equation which includes interactions. Any partial propagators are reversible in this case, all of them belong to some group therefore any propagator has the inverse one. Any such propagator turns to the inverse one under time inversion, as well as some of their compositions entered the non-Euclidean superposition principle. It means that they do not contain irreversibility, and reversible processes described with the reversible Schroedinger equation take place in the closed systems even under interactions.

However, two binary compositions entered the non-Euclidean superposition principle, T and T', do not turn into the inverse ones under time inversion, for example $T \to T$ under such time transformation. It means that such kind binary composition is transformed as $1 \to T \to T^2$ under $t_1 \to t_2 \to t_1$ in general case, and such composition may contain irreversibility.

Thus, a reversibility of the Schroedinger equation is only the necessary condition for a closed physical system reversible evolution but not the sufficient one, on the one hand. On the other hand, it is obviously that inclusion into the Schroedinger equation of some irreversible terms may lead only to the irreversibility for any processes their.

In an opposite way, the non-Euclidean superposition principle assumes coexistence of both reversible and irreversible processes simultaneously in the closed physical systems described with the only the reversible Schroedinger equation.

Let us note two circumstances connected with the opportunity to include irreversibility into the quantum mechanics scheme.

The first one is following: none partial (single) propagators do not contain irreversibility, it is necessary to find at least some their binary compositions. The second one necessarily implying interactions in a system, leads to mapping all propagators and their compositions into the Lobachevsky space, i.e. into the curved space.

It is interesting to compare these circumstances with two conclusions from
(Prigogine & Stengers, 1994) which are the following:

a) Irreversibility expressed by the time arrow is a statistical property. It can not be introduced in terms of individual paths or wave functions. Therefore it demands a radical withdrawal from the Newtonian mechanics or from orthodox quantum mechanics based on concepts of the individual path or wave function;

b) The main assumption that we have to introduce here is the statement that the space with zero Gaussian curvature, similar to the Minkowski space, does not contain entropy,
which are cited unfortunately only in the reverse translation from Russian.

It would be recognized that these expressions formulate the really necessary conditions of irreversibility as it was shown above.

The approach stated above allows one to express the following general point of view on the structure of the fundamental theories.

Taking the exceptional role of the group theory and the Noether theorems in such physical theories into account the last ones may be split into two classes. The first one consists of

the theories constructed before the Noether theorems establishment, and the second ones constructed later.
Evidently that it is difficult to assume the consecutive group-theoretic construction of the first class theories. In opposite case, the theories of the second class would be assumed to be the group-theoretic ones, since the Noether theorems were known to the time of their development.
Therefore it seems to be useful to carry out the group-theoretic analysis of the foundations of the first class theories, whereas a similar consideration of the second class theories seems to be unnecessary.
Besides, it would be considered in both cases if the ordinary (Euclidean) superposition principle, if it used there, is sufficient for the aims of the theory, or insufficient.
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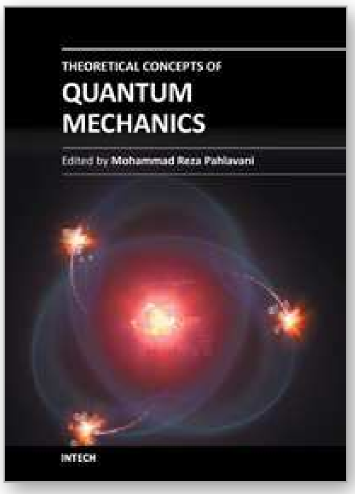

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Quantum theory as a scientific revolution profoundly influenced human thought about the universe and governed forces of nature. Perhaps the historical development of quantum mechanics mimics the history of human scientific struggles from their beginning. This book, which brought together an international community of invited authors, represents a rich account of foundation, scientific history of quantum mechanics, relativistic quantum mechanics and field theory, and different methods to solve the Schrodinger equation. We wish for this collected volume to become an important reference for students and researchers.

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