Entanglement in Two and Three Quantum Mechanical Systems

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1. Introduction

Entanglement is one of most peculiar characteristic of Quantum Mechanics (probably the other most mentioned is the particle tunneling through a barrier) because it entails correlations that supersedes classical correlations. It is conceived as the genuine quantum characteristic by many researchers and quite recently it has emerged as a physical resource to produce non-classical task, like quantum teleportation, quantum cryptography and quantum information. Entanglement could be distributed (efficient quantum communication is equivalent to efficient entanglement distribution (Plenio & Virmani, 2007)), concentrated (given an amount of entanglement, equal to the full content of entanglement in n pairs of identical entangled pure state systems, it is possible, using local operations on each system, to concentrate the total amount of entanglement into a smaller number m, $m < n$, of maximally pure entangled state system (Bennett, et al.; 1996)) and used to perform many quantum information tasks useful to overcome technical restrictions present on classical communication.

In this chapter, we review the entanglement of quantum mechanical systems. First, in Section 2, we give a characterization of entanglement in terms of its special features as resource and its mathematical structure. Then, in section 3, we review some of the Bell inequalities. In Sections 4 and 5, we review some of the most important entanglement measures published for two and three entangled systems. After that, in Section 6, we give a characterization of quantum gates and operators by its entanglement power, i. e. the amount of entanglement that they can produce when acting on an arbitrary state. Finally, in Section 7, we briefly review the experimental detection of entanglement.

2. What is entanglement?

In this Section we define entanglement by its mathematical structure and talk about the operational entanglement definition. We show some important issues of the mathematical description of entanglement, in terms of density operator, that serves as a base to the development of the following sections of the chapter. The question: What is entanglement? is one of the most quite difficult and subtle one that we can ask, then it is a naive thought to think that it is an easy task to answer it. Then, what we propose as an answer to that question is just a characterization of entanglement by both its mathematical structure and its operational features in terms of nonlocal properties.

Entanglement could be mathematically or operationally defined. Operational definition of entanglement uses the concept of *Local Operations and Classical Communication (LOCC)*

(Bennett, et al.; 1996). LOCC is a restriction to use only local operation and classical communications; that is to say, it is not allowed to exchange any quantum system nor to perform any nonlocal operation (Bennett, et al.; 1996). In this case, entanglement is conceived as the resource that allows to overcome the restriction of LOCC. Keeping this line of thought, classical correlations are those generated by LOCC operations and the quantum ones are those that cannot be simulated classically, see (Plenio & Virmani, 2007) page 3. The operational definition of entanglement is quite useful because it allows both to understand entanglement as a physical resource and gives the background to define some measurements of entanglement, like the entropy of entanglement.

Formally, mathematically, but roughly speaking, we can define entanglement as quantum states that can not be writing as the product of two wave functions, that is, for two two-state systems we have, for example:

$$|\psi\rangle_{12} = |\phi_0\rangle_1 \left(c_1|\xi_0\rangle_2 + c_2|\xi_1\rangle_2\right) \rightarrow un-entangled, \tag{1}$$

$$|\varphi\rangle_{12} = c\left(|\phi_0\rangle_1|\xi_0\rangle_2 + |\phi_1\rangle_1|\xi_1\rangle_2\right) \rightarrow entangled. \tag{2}$$

Where $|\phi_0\rangle$, $|\xi_0\rangle$, $|\phi_1\rangle$ and $|\xi_1\rangle$ are basis states, and c_1, c_2 and c are normalization constant. The subindices refer to system 1 and system 2. ***We can consider, also, that system 1 and system 2 are situated far away in two different places***. Usually, in quantum information theory, this is stated as: System 1 belongs to Alice and system 2 belongs to Bob. Physically we can say regarding Eq. (1) that each quantum system have its own state, whereas in Eq. (2) each systems does not have its own state, but share (although they are far away) a global state. However, it is possible to assign a density matrix with each system, this could be seen more clearly if we look at the density operator of the state given by Eq. (2):

$$\hat{\rho}_{12} = |\varphi\rangle_{12}\quad {}_{12}\langle\varphi| = |c|^2\Big\{|\phi_0\rangle_{11}\langle\phi_0|\quad |\xi_0\rangle_{22}\langle\xi_0| + |\phi_0\rangle_{11}\langle\phi_1|\quad |\xi_0\rangle_{22}\langle\xi_1| +$$
$$|\phi_1\rangle_{11}\langle\phi_0|\quad |\xi_1\rangle_{22}\langle\xi_0| + |\phi_1\rangle_{11}\langle\phi_1|\quad |\xi_1\rangle_{22}\langle\xi_1|\Big\}. \tag{3}$$

Now, in order to obtain the density operator for one system, we can take the partial trace over one system of the density operator given in Eq. (3), for instance taking the trace over system 2 we obtain the density operator for system one, as follow:

$$\hat{\rho}_1 = Tr_2\{\hat{\rho}_{12}\} = |c|^2\{|\phi_0\rangle_{11}\langle\phi_0| + |\phi_1\rangle_{11}\langle\phi_1|\}, \tag{4}$$

Eq. (4) is a mixed state. This mean that although systems 1 and 2 does not have its own wave function, as it is stated in Eq. (2), yet it is possible to assign a density operator with each system. In other words, and contrary to Eq. (1), an entangled state seems to each party to be a mixed state Bennett (et al.; 1996).

In fact, any entangled two two-state systems could be writing in terms of its Schmidt form:

$$|\varphi\rangle = \sum_{i=1} \sqrt{\lambda_i}|e_i\rangle \otimes |h_i\rangle \tag{5}$$

where, $|e_i\rangle$ and $|h_i\rangle$ are ortogonal vectors in the Hilbert spaces of each system, with dimensions m and n, respectively. Hence, the density matrix for each system could be written as, for a proof see (Hughston, et al.; 1993):

$$\hat{\rho}_1 = \sum_{i=1}^{m} \lambda_i|e_i\rangle\langle e_i|, \qquad \hat{\rho}_2 = \sum_{i=1}^{n} \lambda_i|h_i\rangle\langle h_i|. \tag{6}$$

On the other hand, if $\hat{\rho}_x$ correspond to a product state, i.e. non-entangled, then

$$\hat{\rho}_x^2 = \hat{\rho}_x, \tag{7}$$

where $x = 1, 2$. That is to say, for two systems an entangled state does not fulfill Eq. (7). You can check this using Eqs. (1) and (3), for a general proof see (Popescu & Rohrlich, 1998).
Many intriguing phenomena come from the peculiarities of Eq. (2). For example, in classical mechanics it is not possible to influence the result of a measurement in one system by acting in another system that no longer interacts with the former system, however entangled states make this possible for quantum physical systems. This is the *non-locality* feature of quantum mechanical systems, although entanglement is not the same that non-locality Méthot & Scarani (2007).

3. J. S. Bell insight on the physical properties of entangled systems

The physical properties of entangled states represents a high departure from the properties of classical systems. Besides of being the building blocks to construct many non-clasical task, entanglement serves as theoretical tool to understand and prove many genuine quantum properties. Then, in this section, we review some of the first noticed physical peculiarities of entanglement based in the insight of J. S. Bell, who first notices the way to test classical correlations against quantum correlations.
In the year 1935 Einstein, Podolsky and Rosen (Einstein, et al.; 1935), introduced an argument that, if is taken seriously, questioned the quantum mechanical description of physical phenomena. This argument is widely know as the EPR paradox. For almost thirty years the EPR argument remains as a philosophical issue, until J. S. Bell deduced some predictions that could be experimentally tested. These predictions were based on the great insight that Bell developed about the quantum correlations that entangled states have.
Briefly, the EPR argument was based on two quite reasonable assumptions (Einstein, et al.; 1935):

1. *If, without in any way disturbing a system, we can predict with certainty (i. e., with probability equal to unity) the value of a physical quantity, then there exist an element of physical reality corresponding to this physical quantity.*
2. *... the absence of an interaction between the two systems.*

The first assumption was aimed at establishing that a theory should be complete; of course it had have strong influence on the conceptions of the physical reality; that is to say, that physical variables have predetermined values independent of measuring them or not. The second, intended to consider as valid only physical process in which the measurement in one quantum system does not affect the result of measurement done on another quantum system when they do not interact and are far away from each other. Based on these assumptions, an in the analysis of measurement's results made on entangled states correlated in position and momentum, Einstein et. al. conclude (Einstein, et al.; 1935): " *While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such description exist. We believe, however, that such a theory is possible.* " This conclusion has been interpreted by physicists, if it is true, as implying that quantum mechanics is not a complete theory and that it must be supplemented with some hidden variables. In short, what Einstein, et al., were looking for was to determine whether or not the measurement result, on two correlated systems, of one observable in system B could be valid predicted using the measurement result of one observable on system A, when the

observables, i. e. the operators representing them, does not commute and when there is not any interaction between the systems A and B.
In his work, Bell first show that previously published arguments concerning the impossibility of hidden variables theories were not well supported (Bell, 1966). Then, he assume some desirable requirements, as the condition of locality, that such theory should have (Bell, 1964) -although he show that such theory should be non-local to reproduce the quantum mechanical statistic results-. These requirements produces some predictions, regarding the statistical correlations of observables, that could be experimentally tested (Bell, 1964). These predictions are established as inequalities equations. Below we write down the Clauser, et al. inequality because it has been one of the most influencing inequalities (Clauser, et al.; 1969),

$$|P(a,b) - P(a,c)| + P(b',b) + P(b',c) \leq 2, \tag{8}$$

in Eq. (8), the correlation $P(i,j) \equiv \int A(i,\lambda)B(j,\lambda)\rho(\lambda)d\lambda$; $A(i,\lambda)$ and $B(j,\lambda)$ are the result of the measurement (with values ± 1). The average is taken over many experiment measurements, i. e. inside the total λ space. Locality requires that $A(i,\lambda)$ must be independent of j and $B(j,\lambda)$ independent of i. Inequality given by Eq. (8) could be deduced by appealing to local hidden variables (Clauser, et al.; 1969) or, as was done by Eberhard (Eberhard, 1977), using just local causes. What Bell shown was that quantum mechanics statistical correlations violate an inequality equation similar to that given by Eq. (8).
As it was stated above, Eq. (8) has been deduced under the assumption of local hidden variable theories. A recent study (Gröblacher, et al.; 2007) obtains a similar inequality for a class of non-local hidden variable theories (which is a generalization of one proposed by Leggett (Leggett, 2003)). This group experimentally test the violation of this inequalities, therefore they shown that a wide varieties of non-local hidden variables theories are in contradiction with quantum mechanics predictions.

4. Physical properties and entanglement measures of two quantum mechanical system

In the current years, there is an agreement between many researcher about the distinguishing features of the physical properties and entanglement measures for just two quantum mechanical system. We can say that this agreement is a universal one. Then, in this Section we review some of the most important entanglement measures for two states systems.
After passing the years of philosophical debates, the most studied entangled state correspond to two-entangled systems. This entangled states were experimentally produced and used to test Bell's inequalities. Some quantum states were named after Bell, nowadays widely know as Bell states:

$$\begin{aligned}
|\Psi^+\rangle &= \frac{1}{2}\left(|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2\right), \\
|\Psi^-\rangle &= \frac{1}{2}\left(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\right), \\
|\Phi^+\rangle &= \frac{1}{2}\left(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\right), \\
|\Phi^-\rangle &= \frac{1}{2}\left(|0\rangle_1|0\rangle_2 - |1\rangle_1|1\rangle_2\right).
\end{aligned} \tag{9}$$

We can say that this are probably the most studied and understood states between the entangled states. They are though as the maximally two-entangled states and they could be used as a basis.

The physical properties of this states are remarkable, it allows teleportation of quantum states between separate locations (Bennett, et al.; 1993), were the background to test Bells inequalities (Popescu & Rohrlich, 1998) and they serve as a reference tool to understand many facets of quantum information theory.

4.1 Two state entanglement measures

Nowadays, there is a general consensus about Entanglement measures for two systems, specially for two two-states systems. In fact, there are some proposed desired requirements for these measurements (Horodecki, et al.; 2000; Vedral, et al.; 1997; Vedral & Plenio, 1998). In this subsection we only refer to three entanglement measures: the entropy of entanglement, the concurrence and the geometric measurement of entanglement. For more complete review, see (Horodecki, et al.; 2009; Plenio & Virmani, 2007).

4.1.1 Entropy of entanglement

One of the first proposed entanglement measures was the entropy of entanglement, which for two entangled quantum mechanical system is defined as the von Neumann entropy of either system, i. e. $\hat{\rho}_1$ or $\hat{\rho}_2$, (Bennett, et al.; 1996):

$$E = -Tr\hat{\rho}_1 \log_2 \hat{\rho}_1. \tag{10}$$

E gives the amount of entanglement for a given pure quantum entangled state, it is one for maximum entangled states and zero for separable states. For a mixed state, it is possible to use E as a departure to measure its amount of entanglement. In this case the measure of entanglement is called *entanglement of formation*, $E_f(\hat{\rho})$, and it is given by (Bennett, et al.; 1996b):

$$E_f(\hat{\rho}) = \min \sum_j p_j E(\Phi_j), \tag{11}$$

where the minimum must be taken over all pure state decomposition of $\hat{\rho}$. The operational meaning of the entanglement of formation is that the $E_f(\hat{\rho})$ given by Eq. (11) gives the number of singlet states required to create $\hat{\rho}$ (Plenio & Virmani, 2007; Wootters, 2001). Also, it is worth to mention that neither E given by Eq. (10) nor $E_f(\hat{\rho})$ given by Eq. (11) do not increase under local operations and classical communications (Bennett, et al.; 1996b).

4.1.2 The concurrence

Perhaps, one of the most widely recognized entanglement measure is the concurrence (Wootters, 2001) and (Wootters, 1998). The concurrence $C(\Phi)$ is defined as (Wootters, 2001):

$$C(\Phi) = |\langle\Phi|\widetilde{\Phi}\rangle|, \tag{12}$$

where $|\widetilde{\Phi}\rangle$ is the spin-flip operation $|\widetilde{\Phi}\rangle = (\sigma_y \otimes \sigma_y)\,|\Phi^*\rangle$, and $|\Phi^*\rangle$ is the complex conjugate of $|\Phi\rangle$. An easy way to calculate the concurrence is by using the magic states, which are defined in terms of the bell states given in Eq. (9) (Hill, 1997):

$$\begin{aligned}
|e_3\rangle &= i|\Psi^+\rangle = \frac{i}{2}\left(|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2\right), \\
|e_4\rangle &= -|\Psi^-\rangle = -\frac{1}{2}\left(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\right), \\
|e_1\rangle &= |\Phi^+\rangle = \frac{1}{2}\left(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\right), \\
|e_2\rangle &= -i|\Phi^-\rangle = \frac{i}{2}\left(-|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\right).
\end{aligned} \tag{13}$$

in terms of the states defined in Eq. (13), the concurrence is defined as (Hill, 1997):

$$C(\Phi) = \left| \sum_i \alpha_i^2 \right| \tag{14}$$

where α_i are the expansion coefficients of $|\Psi\rangle = \sum_i \alpha_i |e_i\rangle$. Notice that what is squared is the α_i itself, not their absolute value.

4.1.3 The Geometric Measurement of Entanglement (GME)

The Geometric Measurement of Entanglement (GME) was defined by Shimony (Shimony, 1993). The GME uses ideas of Hilbert space geometry (Shimony, 1993; Wei & Goldbart, 2003) and it is related with the relative entropy of entanglement (Wei, et al; 2004). Given an arbitrary bipartite state $|\psi\rangle$, the GME is defined as (Shimony, 1993):

$$E(\psi) = \frac{1}{2} \min || \, |\psi\rangle - |\phi\rangle \, ||^2 , \tag{15}$$

where $|\phi\rangle$ is a normalized product state and the minimum is taken over the set of normalized product state (Shimony, 1993). The Eq. (15) determines the entanglement content of $|\psi\rangle$ by calculating the minimum distance between $|\psi\rangle$ and the nearest separable state $|\phi\rangle$. Eq. (15) could be interpreted as: the more amount of entanglement in a given state, then further away it will be from its nearest un-entangled approximant (Wei & Goldbart, 2003). The GME reaches its greatest value $\frac{1}{2}$ for a maximal entangled state and zero for un-entangled states.

5. Physical properties and entanglement measures of three quantum mechanical systems

In this Section we review some of the important features of three quantum mechanical entangled systems. First we talk about the Bell's theorem for the three party case, then we review the generalizations of Gisin's theorems for three entangled states. After that, we briefly review some of the most important entanglement measures for three quantum mechanical systems. It is not possible to give a complete review of all the entanglement measures, so we left out some others important entanglement measures like the negativity.

The multi-particle entanglement is more complex than two party entanglement, even for the case of three party quantum mechanical systems. For example, just for the three party entangled state case, there are quantum correlations that could not be explained even in the case of perfect correlations (Greenberger, et al.; 1990); in other words, the three entangled states present features not usually encounter in two entangled states which make them the simplest nontrivial entangled states showing fundamental and strong differences compared to that of two entangled systems. Additionally, whereas for two entangled states it is possible to define entanglement measures by mean of a single number[1] giving the amount of entanglement present in the state, such a single number does not exist for three entanglement measures. In fact, to quantify the amount of entanglement present in a given three quantum mechanical system is yet an open problem.

[1] When the state of two quantum mechanical systems is expressed using the Schmidt decomposition, if it is entangled, then, it contains only two different Schmidt number, i. e. $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ that are related by $\lambda_2 = 1 - \lambda_1$.

5.1 Bell's theorems for three quantum mechanical systems

It was shown by Greenberger, et. al. (Greenberger, et al.; 1990) that three quantum entangled mechanical systems reveal inconsistencies between the EPR premises. In particular the expectation value $E(\phi_1, \phi_2, \phi_3)$ of the outcomes of a measurement of three observables made in the following state:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|a\rangle|b\rangle|c\rangle + |a^*\rangle|b^*\rangle|c^*\rangle \right), \tag{16}$$

is given by (Greenberger, et al.; 1990):

$$E(\phi_1, \phi_2, \phi_3) = \sin\left(\phi_1 + \phi_2 + \phi_3\right), \tag{17}$$

this expectation values predicts the following perfect correlations:

$$\begin{aligned} E(\phi_1, \phi_2, \phi_3) &= 1 \quad for \quad \phi_1 + \phi_2 + \phi_3 = \frac{\pi}{2}, \\ E(\phi_1, \phi_2, \phi_3) &= -1 \quad for \quad \phi_1 + \phi_2 + \phi_3 = \frac{3\pi}{2}, \end{aligned} \tag{18}$$

these correlations are inconsistent with the premises of EPR (Greenberger, et al.; 1990). To show this, set $\phi_1 = \frac{\pi}{2}$ and $\phi_2 = \phi_3 = 0$; then there could exist variables $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ in a classical theory such that for a suitable election of apparatus (Żukowski, et al.; 1998):

$$\begin{aligned} A(\pi/2)B(0)C(0) &= 1, \\ A(0)B(0)C(\pi/2) &= 1, \\ A(0)B(\pi/2)C(0) &= 1, \\ A(\pi/2)B(\pi/2)C(\pi/2) &= -1. \end{aligned} \tag{19}$$

the allowed values of the observables $A, B; C$ are ± 1, therefore multiplying the left hand side of Eqs. (19) we obtain 1, however the right hand side multiplication gives -1 (Żukowski, et al.; 1998). Therefore (Greenberger, et al.; 1990; Żukowski, et al.; 1998), the EPR assumption are inconsistent with quantum mechanics even for the special case of perfect correlations given in Eq. (18).

5.1.1 Gisin's theorem for three qubits

In 1991, N. Gisin (Gisin, 1991) proves that any non-product state, i.e. any entangled state, of two particle systems violate Bell's inequality. Essentially what he proves was that the Clauser, et al. (Clauser, et al.; 1969) inequality is violated for any entangled two-state system. That is to say, using the Schmidt decomposition given in Eq. (5) and the fact that for an entangled state there is at least two different $\sqrt{\lambda_i}$, i. e. $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$, then:

$$|P(a,b) - P(a,b^*)| + P(a^*,b) + P(a^*,b^*) = 2\left(1 + 4|\sqrt{\lambda_1}\sqrt{\lambda_2}|\right)^{-1/2}, \tag{20}$$

Eq. (20) is strictly greater that 2 for entangled states. This is Gisin's theorem.
The next question is whether a similar proof can be given for more than two entangled state systems. Gisin himself argue that such a result does not holds. However, as far as we know, there are at least two restrict generalizations of Gisin's theorem for three quantum entangled systems. The first one was given by Chen, et al. (Chen, et al.; 2004); first they study cases for the generalized GHZ states, i.e. $|\psi\rangle_{GHZ} = \cos(\xi)|000\rangle + \sin(\xi)|111\rangle$ and work out an

inequality of the following type:

$$B(Bell) \geq \frac{1}{2} + \frac{1}{3}\sqrt{1+\sin^2(2\xi)}. \tag{21}$$

There is an essential difference between Eq. (20) and Eq. (21), it is that in the case of Gisin's theorem $P(a,b)$ refers to expectation value whereas in the case of Eq. (21) $B(Bell)$ is a probability. Hence, what Chen, et. al. prove is the theorem: *All generalized GHZ states of three-qubit system violates a probability Bell's inequality*.

There are other proof of Bell's inequalities for three entanglement quantum mechanical systems. In fact, the search for Bell's type inequalities is an active research field . For example, Wu et. al. (Wu, et al.; 2008) presented a new Bell's inequality in terms of correlation functions and according to the authors this inequality is violated by any pure entangled state of three qubits. Also, Li and Fei (Li & Fei, 2010) present Bell type inequalities for multipartite states of arbitrary dimensions. Additionally, see the paper by Żukowski, et al. (Żukowski, et al.; 2002).
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Finally, Ghose, et. al., (Ghose, et. al.; 2009) have shown that the generalized GHZ state does not violate the Svetlichnys inequality for $\tau \leq 1/2$, where τ is the 3-tangle defined below. Then, showing a way to test for genuine tripartite nonlocal correlations.

5.2 Three entanglement measures

Three entangled quantum mechanical systems differ from two entangled systems in many features. One of them is that whereas for two entangled systems it is possible to find local operations (LOCC) that interconverts one entangled state to another such process is not possible for three entangled systems. When it is possible to obtain an entangled state $|\psi\rangle_{12}$ from other entangled state $|\phi\rangle_{12}$ using only LOCC, then it is agree that these two entangled states are equivalent because it is possible to carry out the same task with them. In fact, for two two-state entangled quantum mechanical systems it is possible to find such interconversion. However, this is not true for three entangled quantum mechanical systems. In this case it is well know that there are at least two inequivalent kinds of tripartite entanglement (Dür, et al.; 2000). The first one correspond to the GHZ state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right), \tag{22}$$

the second one correspond to the W state:

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right). \tag{23}$$

There is no way to convert $|GHZ\rangle$ in $|W\rangle$ using LOCC and viceversa (Dür, et al.; 2000). This fact is related to the problem of the generalization from two quantum states measurement of entanglement to three entanglement measurement. What it is found in the literature is that there is not a single universal measurement of entanglement for the case of three entangled quantum mechanical systems.

5.2.1 Residual entanglement

When generalizing the concurrence $C(\Phi)$ to three quantum mechanical systems Coffman et. al. (Coffman, et al.; 2000) define the *residual entanglement*, called *the 3-tangle*, as:

$$\tau_{ABC} = C^2_{B(AC)} - C^2_{BC} - C^2_{BA}, \tag{24}$$

in Eq. (24), C_{BC}^2 is the concurrence between the systems B and C, C_{BA}^2 is the concurrence between the systems B and A and $C_{B(AC)}^2$ is the concurrence between the system B and the system formed by systems A and C, the last taken as if were a single system AC. Far from be an universal measurement of entanglement, the residual entanglement gives 1 for the GHZ states and 0 for the W state (Coffman, et al.; 2000).

5.2.2 Geometric Measurement of Entanglement (GME)

The Geometric Measurement of Entanglement (GME) can be extended to define an entanglement measure for three quantum mechanical systems (Wei & Goldbart, 2003). For example, consider a three pure entangled state given by :

$$|\psi\rangle_{123} = \sum_{j_1,j_2,j_3} |e^1_{j_1} e^2_{j_2} e^3_{j_3}\rangle. \tag{25}$$

Then, the GME is obtained by the following procedure (Wei & Goldbart, 2003), first minimize $E(\psi)$

$$E(\psi) = \min \mid |\psi\rangle_{123} - |\phi\rangle \mid^2, \tag{26}$$

where $|\phi\rangle = \otimes_{i=1}^3 |\phi^i\rangle$ and $|\phi^i\rangle = \sum_{j_i} c^i_{j_i} |e^i_{j_i}\rangle$ (Wei & Goldbart, 2003). Then, find the *entanglement eigenvalue* Λ_{max}. Finally, the GME for three entangled system is given by:

$$E_{\sin^2} = 1 - \Lambda^2_{max}. \tag{27}$$

In some cases it is possible to use the symmetry of the entangled state to alleviate the difficulty of calculations Wei & Goldbart (2003). For the GHZ state the $E_{\sin^2} = 1/2$ and for the W state it is $E_{\sin^2} = 5/9$ Wei & Goldbart (2003).

5.2.3 Schmidt measurement

The Schmidt measurement of entanglement was defined by Eisert and Briegel (Eisert & Briegel, 2001), it is based in the Schmidt representation of quantum states. That is, consider a three state quantum system with parties A_1, A_2 and A_3, then its state is given by:

$$|\psi\rangle = \sum_i^R \alpha_i |\psi^{(i)}_{A_1}\rangle \otimes |\psi^{(i)}_{A_2}\rangle \otimes |\psi^{(i)}_{A_3}\rangle, \tag{28}$$

where $|\psi^{(i)}_{A_j}\rangle \in \mathbb{C}^{d_j}$, $j = 1, 2, 3$; $\alpha_i \in \mathbb{C}$; $i =, 1, ..., R$. Let r be the minimal number of product terms R in the decomposition of $|\psi\rangle$, then the Schmidt measure is defined as (Eisert & Briegel, 2001):

$$P(|\psi\rangle) = \log_2 r. \tag{29}$$

This entanglement measure is a generalization of the concept of Schmidt rank of density matrix for more than two quantum mechanical systems (Sanpera, et al.; 2001). It satisfy almost all the desired properties of a entanglement measure (Eisert & Briegel, 2001). Inclusive, contrary to the 3-tangle, it discriminates between the GHZ and W states, for the former $P(|\psi\rangle) = 1$ and for the latter $P(|\psi\rangle) = \log_2 3$ (Eisert & Briegel, 2001). See also, (Sperling & Vogel, 2011).

6. Entanglement power of operators and quantum gates

In this section we show that an operator or quantum gate could be characterized by calculating the entanglement produced by the operator or gate when acting in a product state. In Kraus and Cirac (Kraus& Cirac, 2001), it was stated that it is possible to give a two-qubit gate characterization in terms of the entanglement that it can produce when acting on a separable state. The Kraus & Cirac goal were the determination of the best separable input state that gives as much entanglement as possible when acting on them by a given operator represented by an unitary operator. One of the operator that they study was $U_d = e^{-i\alpha S_x} = \cos(\alpha) - i\sin(\alpha)\sigma_x \otimes \sigma_x$. they found that the best input state to produce the maximal entangled state, as measured by the concurrence, is:

$$
\begin{aligned}
|\phi\rangle_{AA^*} &= c_a|0\rangle_A|0\rangle_{A^*} + s_a|1\rangle_A|1\rangle_{A^*}, \\
|\psi\rangle_{BB^*} &= s_b|0\rangle_B|0\rangle_{B^*} + c_b|1\rangle_B|1\rangle_{B^*}.
\end{aligned} \tag{30}
$$

Another possibility to characterize a quantum gate is by calculating the entanglement produced by that gate on an arbitrary separable state. This was done by García Quijas, et. al. (García, et al.; 2011). Additionally, García Quijas and Arévalo Aguilar (García, et al.; 2010) characterize, both the control Z phase gate and the *relative phase gate* for two qubits. Here is shown the amount of entanglement produced by the relative phase gate for three qubits, in order to characterize this gate. Supose that the relative phase gate acts on an arbitrary three qubit given by $(\alpha|0\rangle_1 + \beta|1\rangle_1) \times (\delta|0\rangle_2 + \gamma|1\rangle_2) \times (\mu|0\rangle_2 + \nu|1\rangle_2)$. The relative phase gate for three qubits produces the following change:

$$
\begin{aligned}
U_r^\theta|0\rangle_1|0\rangle_2|0\rangle_3 &\to |0\rangle_1|0\rangle_2|0\rangle_3, \\
U_r^\theta|0\rangle_1|0\rangle_2|1\rangle_3 &\to e^{i\theta}|0\rangle_1|0\rangle_2|1\rangle_3, \\
U_r^\theta|0\rangle_1|1\rangle_2|0\rangle_3 &\to e^{i\theta}|0\rangle_1|1\rangle_2|0\rangle_3, \\
U_r^\theta|0\rangle_1|1\rangle_2|1\rangle_3 &\to |0\rangle_1|1\rangle_2|1\rangle_3, \\
U_r^\theta|1\rangle_1|0\rangle_2|0\rangle_3 &\to e^{i\theta}|1\rangle_1|0\rangle_2|0\rangle_3, \\
U_r^\theta|1\rangle_1|0\rangle_2|1\rangle_3 &\to |1\rangle_1|0\rangle_2|1\rangle_3, \\
U_r^\theta|1\rangle_1|1\rangle_2|0\rangle_3 &\to |1\rangle_1|1\rangle_2|0\rangle_3, \\
U_r^\theta|1\rangle_1|1\rangle_2|1\rangle_3 &\to e^{i\theta}|1\rangle_1|1\rangle_2|1\rangle_3.
\end{aligned} \tag{31}
$$

Then, when acting in the arbitrary three qubit state, it gives the following entangled state:

$$
\begin{aligned}
&U_r^\theta\,(\alpha|0\rangle_1 + \beta|1\rangle_1) \times (\delta|0\rangle_2 + \gamma|1\rangle_2) \times (\mu|0\rangle_2 + \nu|1\rangle_2) \to \\
&\{\alpha\delta\mu|0\rangle_1|0\rangle_2|0\rangle_3 + \alpha\delta\nu e^{i\theta}|0\rangle_1|0\rangle_2|1\rangle_3 + \alpha\gamma\mu e^{i\theta}|0\rangle_1|1\rangle_2|0\rangle_3 \\
&\alpha\gamma\nu|0\rangle_1|1\rangle_2|1\rangle_3 + \beta\delta\mu e^{i\theta}|1\rangle_1|0\rangle_2|0\rangle_3 + \beta\delta\nu e^{i\theta}|1\rangle_1|0\rangle_2|1\rangle_3 \\
&+\beta\gamma\mu|1\rangle_1|1\rangle_2|0\rangle_3 + \beta\gamma\nu e^{i\theta}|1\rangle_1|1\rangle_2|1\rangle_3\},
\end{aligned} \tag{32}
$$

for $\theta = \pi$, Eq. (32) produces the following relative phase state:

$$
\begin{aligned}
&U_r^\pi\,(\alpha|0\rangle_1 + \beta|1\rangle_1) \times (\delta|0\rangle_2 + \gamma|1\rangle_2) \times (\mu|0\rangle_2 + \nu|1\rangle_2) \to \\
&(\alpha|0\rangle_1 - \beta|1\rangle_1) \times (\delta|0\rangle_2 - \gamma|1\rangle_2) \times (\mu|0\rangle_2 - \nu|1\rangle_2).
\end{aligned} \tag{33}
$$

To calculate the amount of entanglement in the state given by Eq. (32), we calculate the 3-tangle defined above. This measure is given by:

$$\tau_{123} = 4 \mid d_1 - 2d_2 + 4d_3 \mid, \tag{34}$$

where:

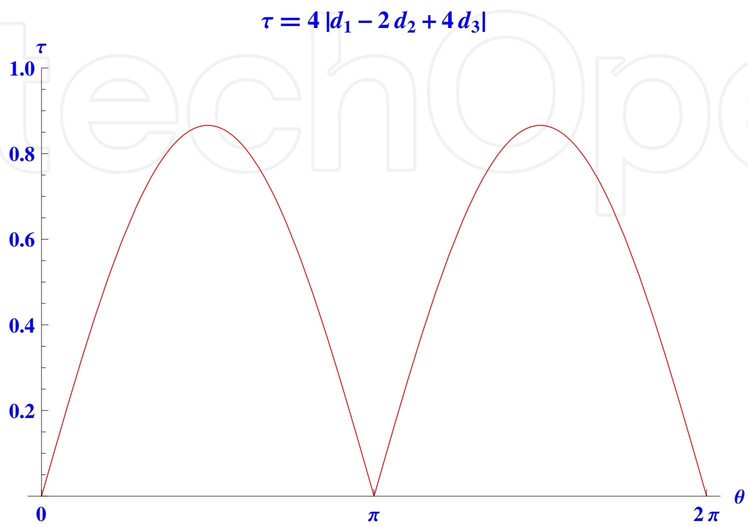

Fig. 1. Plot of formula (36), which gives the 3-tangle measure of entanglemet.

$$\begin{aligned} d_1 &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2; \\ d_2 &= a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} + a_{000}a_{111}a_{110}a_{001} \\ &\quad + a_{011}a_{100}a_{101}a_{010} + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001}; \\ d_3 &= a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}, \end{aligned} \tag{35}$$

and the coefficients a_{ijk} are the expansion of the three party entangled states in the standard basis. Comparing with Eq. (32) the coefficients are given by $a_{000} = \alpha\delta\mu$, $a_{111} = \beta\gamma\nu e^{i\theta}$and so on. If we take $\alpha = \beta = \delta = \gamma = \mu = \nu = 1/\sqrt{2}$, then, using Eq. 32, Eq. 34 and Eq. 35, the residual entanglement for this case is:

$$\tau_{123} = \frac{1}{16}\sqrt{96 - 128\cos 2\theta + 32\cos 4\theta} \tag{36}$$

A plot of Eq. (36) is given in Fig. 1. We see that the residual entanglement of the relative entangled state for three qubits, given by Eq. (36), reachs its maximum value twice in the interval $(0, 2\pi)$. This correspond to $\theta = \pi/2$ and $\theta = 3\pi/2$, this agree with the maximum value for the relative entangled state for two qubits (**?**).

Therefore, the characterization of a quantum gate (or and arbitrary operator) could be done by calculating the amount of entanglement produced by it when acting on an arbitrary state.

7. Experimental determination of entanglement

The experimental detection of entanglement requires the measurement of many physical properties, making this process quite difficult. Then, it is necessary to reduce the quantity of measurements to increase the efficiency of such process. In this section we briefly review

the effort made for many research groups to reduce the quantity of measurements to detect entanglement, both by deducing new definitions for some entanglement measures given in previous section (like concurrence) and by the new approach of proposing entanglement measures in terms of uncertainties of operators.
Nowadays, entanglement is considered a physical resource. Therefore, to use this resource in its potential applicabilities in quantum information theory, it is necessary to realize the following three tasks: i) to produce it, ii) to manipulate it, and iii) to detect it. Then, it is necessary to develop efficient methods to carry these tasks. Recently, there has been a huge effort in the task of detecting entanglement (Audenaert & Plenio, 2006; Gühne, et al.; 2009).
Usually, the principal procedures to experimentally detect entanglement is through Bell's inequalities. Therefore, to experimentally determine entanglement, it is necessary to make measurement in many variables, especially when the amount of entanglement is measured using one of the entanglement measures stated in section 5. An additional problem is the fact that some entanglement measures involves non-physical process, such is the case of the concurrence $C|\Phi\rangle$ defined by (Hill, 1997) where it is defined using the complex conjugate $\langle\Psi^*|$ which is an unphysical operation according with Walborn, et. al. (Walborn, et al.; 2006). For this reason, it was necessary to deduce new definitions of entanglement measures or to generalize the previous one taking into account the experimental settings.
To experimentally measure entanglement Walborn, et. al., uses an alternative definition of concurrence given by Mintert, et. al., (Mintert, et. al.; 2005), they use two copies of the qubit and found that the expectation value of an operator $\hat{A}$, with respect to these two copies, can capture entanglement properties. Then, the concurrence is defined as (Mintert, et. al.; 2005):

$$c(\Psi) = \sqrt{\langle\Psi| \otimes \langle\Psi|\hat{A}|\Psi\rangle \otimes |\Psi\rangle}. \tag{37}$$

In their setting, Walborn, et. al., store a copy of the state $|\Psi\rangle$ in the polarization and momentum degrees of freedom of a single photon. Entangled polarization states were created by pumping two perpendicular nonlinear crystals. The concurrence of $|\Psi\rangle$ is determined by the probability of observing the first photon in the state:

$$|\psi^-\rangle = \frac{1}{2}\left(|H\rangle|b\rangle - |V\rangle|a\rangle\right). \tag{38}$$

The probability P_A is then given by the count rate for the observation of $|\psi^-\rangle$ normalized by the sum of the count rates of the others Bell states.
Another approach to the experimental quantification of entanglement is based on uncertainty relations, this approach is motivated by the goal of detecting entanglement with just few measurements (Hofmann & Takeuchi, 2003). Then, making the detection process more efficient. This approach was highly influenced by the work of Hofmann and Takeuchi (Hofmann & Takeuchi, 2003), who showed that there are uncertainty limits that are violated by entangled states. In particular, the uncertainty of an observable $\hat{A}_i$ is given by the statistical variance as follow:

$$\left(\delta\hat{A}_i\right)^2 = \langle\hat{A}_i^2\rangle - \langle\hat{A}_i\rangle^2. \tag{39}$$

The uncertainty given by Eq. (39) is zero if the wave function is an eigenfunction of $\hat{A}_i$. Then, for a set of commuting observables $\{\hat{A}_i\}$ there are common eigenfunctions such that $\delta\hat{A}_i = 0$. This follows when the expectation values are taken with respect to this common eigenfunction, therefore $\sum_i \left(\delta\hat{A}_i\right)^2 = 0$. On the other hand, if the set $\{\hat{A}_i\}$ is a set of non commuting observables, then there must be a lower limit $U > 0$ such that (Hofmann &

Takeuchi, 2003) $\sum_i \left(\delta \hat{A}_i\right)^2 \geq U$. The generalization of this equation for two systems is given by:

$$\sum_i \left(\delta(\hat{A}_i + \hat{B}_i)\right)^2 \geq U_A + U_B. \tag{40}$$

A quantitative amount of entanglement can be defined as (Hofmann & Takeuchi, 2003):

$$C_{LUR} = 1 - \frac{\sum_i \left(\delta(\hat{A}_i + \hat{B}_i)\right)^2}{2(U_A + U_B)}. \tag{41}$$

C_{LUR} measures the amount of entanglement verified by the violation of local uncertainty (Hofmann & Takeuchi, 2003).

Some criteria must be satisfied by any entanglement measure (Horodecki, et al.; 2000), in particular it must be invariant under local unitary transformation. To apply this criteria to uncertainty relations Samuelsson and Björk (Samuelsson & Björk, 2006) and Kothe and Björk (Kothe & Björk, 2006) defined a new measure of entanglement based on the covariance, i. e. $C(\hat{A}_i, \hat{B}_i) = \langle \hat{A}_i \hat{B}_i \rangle - \langle \hat{A}_i \rangle \langle \hat{B}_i \rangle$. This entanglement measure is given by (Kothe & Björk, 2006):

$$G = \sum_{i,j=1}^{3} C^2(\hat{\sigma}_i^A, \hat{\sigma}_j^B), \tag{42}$$

where $\hat{\sigma}_i^A$ is the $i - th$ Pauli operator for system A and $\hat{\sigma}_j^B$ for system B. The covariance given by Eq. (42) was used by Wang, et. al. Wang (et al.; 2007), to experimentally quantify the entanglement of photons produced by BBO crystals.

A quite related approach was followed by Klyachko 2006, et. al., (Klyachko, et al.; 2006) and Klyachko, et. al., (Klyachko, et al.; 2007) to propose an experimental entanglement measure. In particular they shown that the concurrence is related to the variance by the following relation (Klyachko, et al.; 2007) and (Klyachko, et al.; 2006):

$$C^2(\Psi) = \sqrt{\frac{V(\Psi) - V_{min}}{V_{max} - V_{min}}}, \tag{43}$$

where, $V(\Psi) = C_H - \langle \psi | X_\psi | \psi \rangle$, $X_\psi = \sum_\alpha \langle \psi | X_\alpha | \psi \rangle X_\alpha$ (Klyachko, et al.; 2007) and $C_H = \sum_\alpha X_\alpha^2$.

On the other hand, this approach has been extended to cover the entropic uncertainty relations, see Li, et. al. (Li, et al.; 2010), and Ghune and Lewenstein(Gühne, et al.; 2004). Finally, Fei, et. al., (Fei, et. al.; 2009) has proposed that the experimental measurement of concurrence can be done by a single measurement of the expectation value of tensor products of local observables. In particular, they have shown that the square of the concurrence for two qubit states could be measured just by measuring certain expectation value of the Pauli matrixes.

8. Conclusion

In this chapter we have reviewed the concept of entanglement and some issues regarding it. Specially, we have focused in some of the more common entanglement's measures, like the concurrence. Also, we reviewed some interesting issues of the Bell inequalities. Additionally, we showed a way to characterize quantum gates and operators. Finally, we briefly reviewed the experimental procedure to measure the entanglement content of an entangled state.

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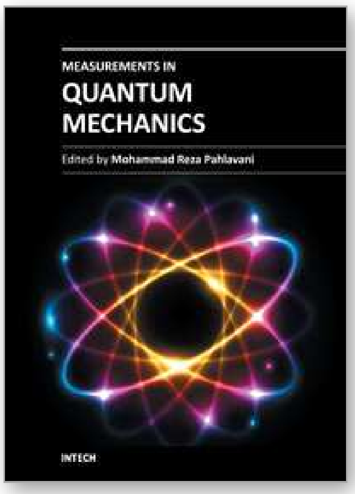

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Perhaps quantum mechanics is viewed as the most remarkable development in 20th century physics. Each successful theory is exclusively concerned about "results of measurement". Quantum mechanics point of view is completely different from classical physics in measurement, because in microscopic world of quantum mechanics, a direct measurement as classical form is impossible. Therefore, over the years of developments of quantum mechanics, always challenging part of quantum mechanics lies in measurements. This book has been written by an international invited group of authors and it is created to clarify different interpretation about measurement in quantum mechanics.