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Generalized Variational Principle for Dissipative Hydrodynamics: Shear Viscosity from Angular Momentum Relaxation in the Hydrodynamical Description of Continuum Mechanics

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1. Introduction

A system of hydrodynamic equations for a viscous, heat conducting fluid is usually derived on the basis of the mass, the momentum and the energy conservation laws (Landau & Lifshitz, 1986). Certain assumptions about the form of the viscous stress tensor and the energy density flow vector are made to derive such a system of equations for the dissipative viscous, heat conductive fluid. The system of equations based on the mass, the momentum and the energy conservation laws describes adequately a large set of hydrodynamical phenomena. However, there are some aspects which suggest that this system is only an approximation.

For example, if we consider propagation of small perturbations described by this system, then it is possible to separate formally the longitudinal, shear and heat or entropy waves. The coupling of the longitudinal and heat waves results in their splitting into independent acoustic-thermal and thermo-acoustic modes. For these modes the limits of phase velocities tends to infinity at high frequencies so that the system is in formal contradiction with the requirements for a finite propagation velocity of any perturbation which the medium can undergo. Thus it is possible to suggest that such a hydrodynamic equation system is a mere low frequency approximation. Introducing the effects of viscosity relaxation (Landau & Lifshitz, 1972), guarantees a limit for the propagation velocity of the shear mode, and the introduction of the heat relaxation term (Deresiewicz, 1957; Nettleton, 1960; Lykov, 1967) in turn ensures finite propagation velocities of the acoustic-thermal and thermo-acoustic modes. However, the introduction of such relaxation processes requires serious effort with motivation.

Classical mechanics provides us with the Lagrange's variational principle which allows us to derive rigorously the equations of motion for a mechanical system knowing the forms of kinetic and potential energies. The difference between these energies determines the form of the Lagrange function. This approach translates directly into continuum mechanics by introduction of the Lagrangian density for non-dissipative media. In this approach the dissipation forces can be accounted for by the introduction of the dissipation function derivatives into the corresponding equations of motion in accordance with Onsager's principle of symmetry of kinetic coefficients (Landau & Lifshitz, 1964). There is an established opinion that for a dissipative system it is impossible to formulate the variational principle analogously to the least action principle of Hamilton (Landau & Lifshitz, 1964). At the same time there are successful approaches (Onsager, 1931a, 1931b; Glensdorf & Prigogine, 1971; Biot, 1970; Gyarmati, 1970; Berdichevsky, 2009) in which the variational principles for heat conduction theory and for irreversible thermodynamics are applied to account explicitly for the dissipation processes. In spite of many attempts to formulate a variational principle for dissipative hydrodynamics or continuum mechanics (see for example (Onsager, 1931a, 1931b; Glensdorf & Prigogine, 1971; Biot, 1970; Gyarmati, 1970; Berdichevsky, 2009) and references inside) a consistent and predictive formulation is still absent. Therefore, there are good reasons to attempt to formulate the generalized Hamilton's variational principle for dissipative systems, which argues against its established opposition (Landau & Lifshitz, 1964). Thus the objective of the chapter is a new formulation of the generalized variational principle (GVP) for dissipative hydrodynamics (continuum mechanics) as a direct combination of Hamilton's and Osager's variational principles. The first part of the chapter is devoted to formulation of the GVP by itself with application to the well-known Navier-Stokes hydrodynamical system for heat conductive fluid. The second part of the chapter is devoted to the consistent introduction of viscous terms into the equation of fluid motion on the basis of the GVP. Two different approaches are considered. The first one dealt with iternal degree of freedom described in terms of some internal parameter in the framework of Mandelshtam – Leontovich approach (Mandelshtam & Leontovich, 1937). In the second approach the rotational degree of freedom as independent variable appears additionally to the mean mass displacement field. For the dissipationless case this approach leads to the well-known Cosserat continuum (Kunin, 1975; Novatsky, 1975; Erofeev, 1998). When dissipation prevails over angular inertion this approach describes local relaxation of angular momentum and corresponds to the sense of internal parameter. Finally, it is shown that the nature of viscosity phenomenon can be interpreted as relaxation of angular momentum of material points on the kinetic level.

2. Generalized variational principle for dissipative hydrodynamics

2.1 Hamilton's variational principle

The non-dissipative case of Hamilton's variational principle can be formulated for a continuous medium in the form of the extreme condition for the action functional $\delta S = 0$:

$$S = \int_{t_1}^{t_2} dt \int_V d\vec{r} L \ , \qquad (1)$$

which is an integral over the time interval (t_1, t_2) and the initial volume V of a given mass of a continuum medium in terms of Lagrangian's coordinates. From the principles of particle mechanics the Lagrangian density L is represented as the difference between the kinetic K and potential U energies:

$$L(\dot{\vec{u}}, \nabla\vec{u}) = K(\dot{\vec{u}}) - U(\nabla\vec{u}) \ . \qquad (2)$$

Expression (2) implies that the Lagrangian can be considered as a function of the velocities of the displacements $\dot{\vec{u}} = \frac{\partial \vec{u}}{\partial t}$ and deformations $\nabla\vec{u} = div(\vec{u})$.

The motion equations derived from variational principles (1), (2) have the following form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{u}}} + \nabla \frac{\partial L}{\partial \nabla \vec{u}} = 0 . \tag{3}$$

In the simplest case, when the kinetic and potential energies are determined by the quadratic forms

$$2K(\dot{\vec{u}}^2) = \rho_0 \dot{\vec{u}}^2 , \quad 2U = \lambda \varepsilon_{ll}^2 + 2\mu \varepsilon_{ik}^2 , \quad \varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \tag{4}$$

the well-known equation of motion for an elastic medium (Landau & Lifshitz, 1972) can be derived:

$$\rho_0 \frac{d}{dt}\dot{\vec{u}} - \mu \Delta \vec{u} - (\lambda + \mu)\nabla(\nabla \vec{u}) = 0 , \tag{5}$$

where ρ_0 is the density of the medium, and λ and μ are the Lamé's constants.

2.2 Onsager's variational principle

If we consider quasi-equilibrium systems, then the Onsager's variational principle for least energy dissipation can be formulated (Onsager, 1931a, 1931b). This principle is based on the symmetry of the kinetic coefficients and can be formulated as the extreme condition for the functional constructed as the difference between the rate of increase of entropy, $\dot{s}$, and the dissipation function, D. Here the entropy s is considered as a function of some thermodynamic relaxation process α, and the dissipation function D as a function of the rate of change of α, i.e.

$$\delta_{\dot{\alpha}} \left[\dot{s}(\alpha) - D(\dot{\alpha}) \right] = 0 . \tag{6}$$

The kinetic equation can then be derived from variational principle (6) to describe the relaxation of a thermodynamic system to its equilibrium state, i.e.:

$$\frac{d}{dt} s(\alpha) = 2D(\dot{\alpha}) . \tag{7}$$

The above equation satisfies strictly the symmetry principle for the kinetic coefficients (Landau & Lifshitz, 1986).

2.3 Variational principle for mechanical systems with dissipation

As was mentioned above, the generalization of the equation of motion (3) in the presence of dissipation is obtained by introducing the derivative of the dissipation function with respect to the velocities into the right hand side of the equation (3). Therefore, in accordance with Onsager's symmetry principle for the kinetic coefficients (Landau & Lifshitz, 1964) we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{u}}} + \nabla \frac{\partial L}{\partial \nabla \vec{u}} = -\frac{\partial D}{\partial \dot{\vec{u}}} . \tag{8}$$

Now it is possible to show, that the equation of motion can be derived in the form of equation (8) if Hamilton's variational principle is adapted with the following form of the Lagrangian function:

$$L(\dot{\vec{u}},\nabla\vec{u}) = K(\dot{\vec{u}}) - U(\nabla\vec{u}) - \int_0^t D(\dot{\vec{u}})dt' , \tag{9}$$

where the time integral of the dissipation function is introduced into equation (2). The initial time in integral (9) denoted for simplicity equal to 0 corresponds to the time t_1 in functional (1). It needs, however, to pay attention that at variation of dissipative term in such approach an additional item appears, which has to be neglected by hands. Indeed, variation of the last term in (9) leads us to result

$$\delta\int_0^t D(\dot{\vec{u}})dt' = \int_0^t \frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\delta\dot{\vec{u}}\,dt' = \int_0^t \frac{d}{dt'}\left(\frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\delta\vec{u}\right)dt' - \int_0^t \frac{d}{dt'}\left(\frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\right)\delta\vec{u}dt' \tag{10a}$$

If to neglect by the last item in this expression

$$\delta\int_0^t D(\dot{\vec{u}}(t'))dt' = \frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\delta\vec{u}(t) - \int_0^t \frac{d}{dt'}\left(\frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\right)\delta\vec{u}dt' \approx \frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\delta\vec{u}(t) , \tag{10b}$$

then the result gives us the same term $\frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}$, which we need artificially introduce in the motion equation (8) for account of dissipation. From the one hand this approach can be considered as some rule at variation of integral term, because it leads us to the required form of the motion equation (8). From the other hand the following supporting basement can be proposed. Variation of action containing all terms in Lagrangian (9) with account of initial and boundary conditions can be written in the form

$$\int_{t_1}^{t_2} dt\int dV\left\{\left(-\frac{d}{dt}\frac{\partial K(\dot{\vec{u}})}{\partial\dot{\vec{u}}} + \nabla\frac{\partial U(\nabla\vec{u})}{\partial\nabla\vec{u}} - \frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\right)\delta\vec{u} + \int_0^t \frac{d}{dt'}\left(\frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\right)\delta\vec{u}dt'\right\} = \tag{11a}$$

It is seen from (11a) that the required form of the motion equation with dissipation arises due to zero value of coefficient at arbitrary variation of the displacement field $\delta\vec{u}$. The last additional item, containing variation $\delta\vec{u}$ under integral, prevent to the strict conclusion in the given case. Nevertheless, if to rewrite the first term in (11a) in the same integral form as the last term

$$= \int_{t_1}^{t_2} dt\int dV\int_0^t dt'\left\{\delta(t-t')\left(-\frac{d}{dt'}\frac{\partial K(\dot{\vec{u}})}{\partial\dot{\vec{u}}} + \nabla\frac{\partial U(\nabla\vec{u})}{\partial\nabla\vec{u}} - \frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\right) + \frac{d}{dt'}\left(\frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\right)\right\}\delta\vec{u} \tag{11b}$$

then due to the same reason of arbitrary variation $\delta\vec{u}$ the multiplier in brackets at this variation has to be equal to zero. It is possible to see now, that, if the function $\frac{d}{dt'}\left(\frac{\partial D(\dot{\vec{u}})}{\partial\dot{\vec{u}}}\right)$ is not singular in the point $t' = t$, then its contribution can be neglected in this point in comparison with singular contribution from delta-function. The presented arguments can be considered as a basis for variation rule of integral term in Lagrangian.

In particular, if the dissipation function is considered as a quadratic form of the deformation velocities, i.e.:

$$2D(\nabla\dot{\vec{u}}) = \eta'\left(\frac{\partial \dot{u}_i}{\partial x_k} + \frac{\partial \dot{u}_k}{\partial x_i}\right)^2 + \varsigma'\left(\frac{\partial \dot{u}_l}{\partial x_l}\right)^2, \tag{12}$$

then the derived equation of motion with account of (4) corresponds to the linearized Navier–Stokes equation:

$$\rho_0 \frac{d}{dt}\dot{\vec{u}} - (\lambda + \mu)\Delta\vec{u} - \lambda\nabla(\nabla\vec{u}) = \eta\Delta\dot{\vec{u}} + \left(\varsigma + \frac{\eta}{3}\right)\nabla(\nabla\dot{\vec{u}}), \tag{13}$$

where the shear and volume viscosities, η and ς respectively are given by $\eta'/2$ and $\varsigma' + \frac{4}{3}\eta'$ respectively, from the constants in (12).

2.4 Independent variables

When GVP is formulated in the form (9) we need to determine variables in which terms the Lagrange's function has to be expressed. To answer on this question let's return to the hydrodinamics equations and look at variables for their description.

In absence of dissipation, as it easy to see, these variables are velocity, density, pressure and entropy $\vec{v}, \rho, P, s$. For the dissipationless case the entropy holds to be constant for given material point, hence a pressure can be considered, for example, as a function of solely density $P(\rho, s = const)$. The density of the given mass of continuum is expressed in terms of its volume. Hence variation of density can be expressed in terms of variation of volume or through divergence of the displacement field $\rho = \rho(div\vec{u})$. In particular, linearization of the continuity equation leads to relation

$$\rho = \rho_0(1 - div\vec{u}) \tag{14}$$

Velocity by definition is a time derivative from displacement $\vec{v} = \dot{\vec{u}}$. Thus, the displacement field $\vec{u}$ can be considered as the principal hydrodinamical variable for the dissipationless case.

In the presence of dissipation, the hydrodynamic equations also involve the temperature T, implying in the following set of variables: $\vec{v}, \rho, P, s, T$. If pressure and entropy depend on density and temperature $P(\rho, T)$, $s(\rho, T)$ in accordance to the state equation, then the fields of displacements and temperatures: $\vec{u}, T$ can be considered as the principal hydrodynamical variables.

Further, we will adopt the idea of Biot (Biot, 1970), and introduce some vector field $\vec{u}_T$ (some vector potential), called the heat displacement, as independent variable instead temperature, so that the relative deviation of temperature T from its equilibrium state T_0 is determined by the divergence of the field $\vec{u}_T$. Namely in analogy with (14)

$$T = T_0(1 - \theta\, div\vec{u}_T) \tag{15a}$$

where θ is some dimensionless constant which is specially introduced in definition (15a) for simplification of the expression for the dissipation function. Thus, the divergence of the heat displacement field $\vec{u}_T$ determines temperature deviation from its equilibrium level

$$\frac{T - T_0}{T_0} = -\theta \nabla \vec{u}_T \, . \tag{15b}$$

2.5 Generalized variational principle (GVP) for dissipative hydrodynamics

The above example (12), (13) of the derivation of the equation of motion for dissipative systems on the basis of Hamilton's variational principle with the Lagrange's function (9) suggests the possibility of formulating a generalized variational principle for dissipative hydrodynamical systems. This formulation can be obtained by a simple combination of Hamilton's variational principle (eqs. (1) and (2)) and Onsager's variational principle (eq. (6)), if the latter is integrated over time and multiplied by a temperature term (Maximov , 2008, 2010, originally formulated by Maximov , 2006). The Lagrangian density in this case can be written in the following form:

$$L = K - E + T\left[s - \int_0^t D dt' \right] = K - F - T \int_0^t D dt' \, , \tag{16}$$

where E and F are the internal energy (potential for the dissipationless case) and the free energy respectively. For the non-dissipative case, the Lagrangian depends on the time and spatial derivatives of the mean mass displacement field $\vec{u}$, which is a basic independent variable in this formulation. For the dissipative case, the temperature should be considered as an additional independent variable for a complete description. Hence, a free energy and dissipation function should also depend on the temperature variations. But temperature by itself is not a convenient variable here. Instead it is more convenient to consider the heat displacements $\vec{u}_T$, introduced in previous section, of which the divergence will give us temperature.

In this case the generalized Lagrangian can be written in the following form:

$$L(\dot{\vec{u}}, \nabla \vec{u}, \nabla \vec{u}_T) = K(\dot{\vec{u}}) - F(\nabla \vec{u}, \nabla \vec{u}_T) - T_0 \int_0^t D(\dot{\vec{u}}, \dot{\vec{u}}_T) dt' \, . \tag{17}$$

It is important to note here that the opportunity to formulate the variational principle for a dissipative system arises due to the energy conservation for two interacting fields: the mean mass displacement $\vec{u}$ and the heat displacement $\vec{u}_T$. The dissipation function only plays a role in the transformation rate between these fields.

In this way the motion equations derived by variation of action with the Lagrangian (17), can be expressed in the following forms

$$\begin{aligned} &\frac{d}{dt}\frac{\partial K}{\partial \dot{\vec{u}}} - \nabla \frac{\partial F}{\partial \nabla \vec{u}} = -T_0 \frac{\partial D}{\partial \dot{\vec{u}}} \, , \\ &T_0 \frac{\partial D}{\partial \dot{\vec{u}}_T} - \nabla \frac{\partial F}{\partial \nabla \vec{u}_T} = 0 \, . \end{aligned} \tag{18}$$

Taking into account that the kinetic energy is given by quadratic form (4), the free energy is given by its usual expression for thermo-elasticity in quadratic form (Landau & Lifshitz, 1972):

$$2F(\nabla\vec{u},T) = 2\mu\varepsilon_{ik}^2 + \lambda\varepsilon_{ll}^2 + \tilde{\kappa}\left(\frac{T-T_0}{\theta T_0}\right)^2 + 2\tilde{\alpha}\varepsilon_{ll}\left(\frac{T-T_0}{\theta T_0}\right), \tag{19a}$$

or with substitution of expression (13) instead of the temperature terms:

$$2F(\nabla\vec{u},\nabla\vec{u}_T) = 2\mu\varepsilon_{ik}^2 + \lambda\varepsilon_{ll}^2 + \tilde{\kappa}\left(\nabla\vec{u}_T\right)^2 + 2\tilde{\alpha}\varepsilon_{ll}\left(\nabla\vec{u}_T\right), \tag{19b}$$

The dissipation function is the square of the difference between the mean mass and the heat displacements

$$2D(\dot{\vec{u}},\dot{\vec{u}}_T) = \beta(\dot{\vec{u}} - \dot{\vec{u}}_T)^2 . \tag{20}$$

The meanings of the coefficients $\tilde{\kappa}$, $\tilde{\alpha}$ and β in quadratic forms (19), (20) will be defined in the next section by comparison with the classical Navier-Stokes hydrodynamical system of equations.
In this case the motion equations for the mean displacement field and for the temperature field derived on the basis of the generalized variational principle are just equivalent (at $\mu = 0$) to the linearized traditional system of hydrodynamics equations:

$$\rho_0 \frac{d}{dt}\dot{\vec{u}} - \mu\Delta\vec{u} - (\lambda + \mu + \tilde{\alpha})\nabla(\nabla\vec{u}) = (\tilde{\alpha} + \tilde{\kappa}) / (\theta T_0)\nabla T \tag{21}$$

$$\beta(\dot{T} - T_0\theta\nabla\dot{\vec{u}}) - \tilde{\kappa}\Delta T = \tilde{\alpha}T_0\theta\Delta\nabla\vec{u} . \tag{22}$$

2.6 Comparison with the system of hydrodynamics equations

Coefficients of the quadratic forms in equations (19) and (20) can be determined by comparison between the system of equation (21) and (22) and the linearized system of hydrodynamics equations (Landau & Lifshitz, 1986) considering the variables $\vec{u},T$:

$$\rho = \rho_0(1 - \nabla\vec{u}) , \tag{23}$$

$$\rho_0 \frac{d^2\vec{u}}{dt^2} - \rho_0 c_0^2 \Delta\vec{u} = -\rho_0\alpha\nabla T + \eta\Delta\dot{\vec{u}} + \left(\zeta + \frac{\eta}{3}\right)\nabla\left(\nabla\dot{\vec{u}}\right), \tag{24}$$

$$\rho_0 C_V \frac{dT}{dt} + \rho_0 T_0 \alpha\nabla\dot{\vec{u}} - \kappa\Delta T' = 0 . \tag{25}$$

where c_0 is the isothermal sound velocity, C_V - the heat capacity at constant volume, κ the heat conductivity coefficient, and α the thermal expansion coefficient. In the absence of viscosity $\eta = 0$ and $\varsigma = 0$, which was not taken into account in the dissipation function (20), the structure of equations (21), (22) nearly coincides with the second (24) and the third (25) equations in the system of hydrodynamics equations (Landau & Lifshitz, 1986). The only

difference is the additional term in the right part of equation (22) in comparison with (25). We note here briefly that the reason for the introduction of this term is related to a generalized form of the Fourier law for heat energy flow. Besides the term of the temperature gradient in the Fourier law, an additional density or pressure gradient term should appear in spite of the contradicting argument presented in (Landau & Lifshitz, 1986). The independent support of this result can be found in refs. (Martynov, 2001; Zhdanov & Roldugin 1998).
The coefficients of equations (21), (22) and (24), (25) for the fluid case ($rot(\vec{u})=0$) can be found by comparison. One needs to take into account the different dimensions of equation (22) and (25), and, hence, the presence of common dimension multiplier in the comparison of coefficients for these equations.
The parameters of the quadratic forms are expressed explicitly in terms of the physical parameters by the following expressions

$$\beta=\frac{\rho_0 c_0^2}{\chi}\left(\gamma^2-1\right),\ \theta=-\frac{\gamma-1}{\alpha T_0},\ \tilde{\alpha}=\rho_0 c_0^2(\gamma-1),\ \lambda+2\mu=\rho_0 c_0^2\gamma,\ \tilde{\kappa}=\rho_0 c_0^2\left(\gamma^2-1\right), \quad (26)$$

where γ is the specific temperature ratio, $\gamma=C_P/C_V$, and $\chi=\kappa/\rho_0 C_V$ is the temperature conductivity coefficient. It is remarkable that the coefficient in the dissipation function β is inversely proportional to the temperature conductivity coefficient.

3. Viscous terms in dissipative hydrodynamics

3.1 Account of viscosity relaxation for a fluid

To take into account fluid viscosity in the equation of motion in the framework of the generalized variational principle it is possible to introduce additional internal parameters to describe the quasi-equilibrium state of the medium, analogous to the Mandelshtam – Leontovich approach (Mandelshtam & Leontovich, 1937). As will be shown, in order to describe both the shear and the volume viscosities simultaneously, this internal parameter needs to possess the properties of a tensor. To simplify the description we consider the case when the temperature variation variable T is not essential so that the heat displacement $\vec{u}_T$ terms can be omitted. In this case the additional terms associated with the tensor internal parameter ξ_{ik}, will appear in the expression for the free energy of an elastic medium (19), and it can be written as:

$$2F(\nabla\vec{u},\ \xi_{ij})=2\mu\varepsilon_{ik}^2+\lambda\varepsilon_{ll}^2+a_1\xi_{ll}^2+a_2\xi_{ik}^2+2b_1\xi_{kk}\varepsilon_{ll}+2b_2\xi_{ik}\varepsilon_{ki}, \quad (19c)$$

where a_i and b_i are some coefficients of a positively determined quadratic form. The kinetic energy is then given by the ordinal expression (4) and the dissipative function in the absence of the temperature term can be written as the following quadratic form:

$$2D(\dot{\xi}_{ij})=\gamma_1\dot{\xi}_{ll}^2+\gamma_2\dot{\xi}_{ik}^2 \quad (27)$$

with some coefficients γ_1, γ_2.
The system of motion equations, derived on the basis of the generalized variational principle for this case can be rewritten as

$$\rho_0 \frac{d}{dt}\dot{\vec{u}} - \mu\Delta\vec{u} - (\lambda+\mu)\nabla(\nabla\vec{u}) - b_1\nabla\xi_{ll} - b_2\frac{\partial\xi_{ik}}{\partial x_k} = 0\,, \tag{28}$$

$$\gamma_1\delta_{ik}\frac{d\xi_{ll}}{dt} + \gamma_2\frac{d\xi_{ik}}{dt} + a_1\delta_{ik}\xi_{ll} + a_2\xi_{ik} + b_1\delta_{ik}\nabla\vec{u} + b_2\varepsilon_{ik} = 0\,. \tag{29}$$

Here in the first equation (28) we safe for shortness the tensor notation for vector obtained as divergence of internal parameter tensor. Equation (28) is the motion equation for an elastic medium. Equation (29) is the kinetic equation for the internal parameter tensor ξ_{ik}. Convolving the kinetic equation by indexes it is possible to obtain the separate kinetic equation for the spherical part of the internal parameter tensor ξ_{ll}:

$$\tilde{\gamma}\frac{d\xi_{ll}}{dt} + \tilde{a}\xi_{ll} + \tilde{b}\varepsilon_{ll} = 0\,, \tag{30}$$

where the coefficients with tilde have the following meaning:

$$\tilde{\gamma} = 3\gamma_1 + \gamma_2\,,\ \tilde{a} = 3a_1 + a_2\,,\ \tilde{b} = 3b_1 + b_2 \tag{31}$$

Kinetic equation (29) is an inhomogeneous ordinary differential equation of the first order. Its solution can be written as:

$$\xi_{ll} = -\frac{\tilde{b}}{\tilde{\gamma}}\int_{-\infty}^{t} e^{-\frac{\tilde{a}}{\tilde{\gamma}}(t-t')}\varepsilon_{ll}(t')\,dt' \tag{32}$$

For the other components of the internal tensor parameter ξ_{ik} we can also obtain a kinetic equation of similar form to equation (29), but with added inhomogeneous terms, i.e.

$$\gamma_2\frac{d\xi_{ik}}{dt} + a_2\xi_{ik} + b_2\varepsilon_{ik} + \tilde{a}_1\delta_{ik}\xi_{ll} + \tilde{b}_1\delta_{ik}\varepsilon_{ll} = 0 \tag{33}$$

where the following notations are introduced

$$\tilde{a}_1 = \left(a_1 - \tilde{a}\frac{\gamma_1}{\tilde{\gamma}}\right),$$

$$\tilde{b}_1 = \left(b_1 - \tilde{b}\frac{\gamma_1}{\tilde{\gamma}}\right) \tag{34}$$

Again, the solution of equation (33) has a form analogous to expression (32) with additional contributions from the terms with multipliers $\tilde{a}_1$ and $\tilde{b}_1$. Specifically,

$$\xi_{ik} = -\frac{b_2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')}\left(\varepsilon_{ik} - \delta_{ik}\varepsilon_{ll}\left(1 - \frac{\tilde{b}}{b_2}\frac{(a_1\gamma_2 - a_2\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right)\right) - \delta_{ik}\frac{\tilde{b}}{\tilde{\gamma}}\frac{(a_1\tilde{\gamma} - \tilde{a}\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\int_{-\infty}^{t} dt' e^{-\frac{\tilde{a}}{\tilde{\gamma}}(t-t')}\varepsilon_{ll} \tag{35}$$

Taking the divergence of tensor (35), we obtain the following vector

$$\frac{\partial \xi_{ik}}{\partial x_k} = -\frac{b_2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')}\left(\frac{1}{2}(\Delta\vec{u} + \nabla(\nabla\vec{u})) - \nabla(\nabla\vec{u})\left(1 - \frac{\tilde{b}}{b_2}\frac{(a_1\gamma_2 - a_2\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right)\right) - $$
$$-\frac{\tilde{b}}{\tilde{\gamma}}\frac{(a_1\tilde{\gamma} - \tilde{a}\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\int_{-\infty}^{t} dt' e^{-\frac{\tilde{a}}{\tilde{\gamma}}(t-t')}\nabla(\nabla\vec{u}) \tag{36}$$

If we substitute (36) and (32) in the motion equation (28), we can write:

$$\rho_0 \frac{d}{dt}\dot{\vec{u}} - \mu\Delta\vec{u} - (\lambda + \mu)\nabla(\nabla\vec{u}) = -\frac{\tilde{b}}{\tilde{\gamma}}\left(b_1 - b_2\frac{(a_1\tilde{\gamma} - \tilde{a}\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right)\int_{-\infty}^{t} dt' e^{-\frac{\tilde{a}}{\tilde{\gamma}}(t-t')}\nabla(\nabla\vec{u}) - $$
$$-\frac{b_2^2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')}\left(\frac{1}{2}(\Delta\vec{u} + \nabla(\nabla\vec{u})) - \nabla(\nabla\vec{u})\left(1 - \frac{\tilde{b}}{b_2}\frac{(a_1\gamma_2 - a_2\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right)\right) \tag{37}$$

In the low frequency limit, at times greater than the relaxation times $\tilde{\gamma}/\tilde{a}$ and γ_2/a_2, it is possible to derive an equation analogous to the Navier – Stokes motion equation with shear and volume viscosities:

$$\rho_0 \frac{d}{dt}\dot{\vec{u}} - \tilde{\mu}\Delta\vec{u} - (\tilde{\lambda} + \tilde{\mu})\nabla(\nabla\vec{u}) = \tilde{\eta}\Delta\dot{\vec{u}} + (\tilde{\zeta} + \frac{\tilde{\eta}}{3})\nabla(\nabla\dot{\vec{u}}) \tag{38}$$

where the effective elastic moduli $\tilde{\lambda}$ and $\tilde{\mu}$ and coefficients of shear and volume viscosities are expressed as

$$\tilde{\mu} = \mu - \frac{b_2^2}{2a_2},\ \tilde{\lambda} = \lambda + \frac{b_2^2}{2a_2} - \frac{\tilde{b}}{\tilde{a}}\left(b_1 - b_2\frac{(a_1\tilde{\gamma} - \tilde{a}\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right),\ \tilde{\eta} = \frac{1}{2}\gamma_2\frac{b_2^2}{a_2^2},$$
$$\tilde{\zeta} + \frac{\tilde{\eta}}{3} = \tilde{\gamma}\frac{\tilde{b}}{\tilde{a}^2}\left(b_1 - b_2\frac{(a_1\tilde{\gamma} - \tilde{a}\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right) - \gamma_2\frac{b_2}{a_2^2}\left(\frac{b_2}{2} - \tilde{b}\frac{(a_1\gamma_2 - a_2\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right) \tag{39}$$

It is important to note that the structure of the effective shear modulus $\tilde{\mu}$ in (39) is determined by a difference, which can be equal to zero, in which case equation (38) becomes completely equivalent to the Navier – Stokes equation for a viscous fluid. Thus the condition

$$\mu = \frac{b_2^2}{2a_2} \tag{40}$$

should be satisfied to consider a solid with shear relaxation like a viscous fluid. If $\tilde{\mu} > 0$, then we have the case of elastic medium with a shear viscosity (the Voight's model) or with relaxation in the more general case (37). Thus, in the framework of the uniform approach it is possible to describe viscous fluids and solids with visco-elastic properties.

As a final remark of this section it is possible to say several words about physical sense of the introduced internal parameter. Since in the low frequency limit the majority of gases and fluids, including the simplest of them, is described by the Navier-Stokes equation, then the only available value, which could relax in all cases, and hence could be considered as common scalar internal parameter, is the mean distance between molecules in gas or liquid. In the condensed and especially in the solid media the mutual space placement of atoms becomes to be essential, hence a space variation of their mutual positions, holding rotational invariance of a body as whole, has to be described by symmetrical tensor of the second order. Hence the corresponding internal parameter could be the same tensor. Thus, the discrete structure of medium on the kinetic level predetermines existence, at least, of mentioned internal parameters, responsible for relaxation.

3.2 Shear viscosity as a consequence of the angular momentum relaxation for the hydrodynamical description of continuum mechanics

As shown in the previous section, it is possible to derive the system of hydrodynamical equations on the GVP basis for viscous, compressible fluid in the form of Navier-Stokes equations. However for the account of terms responsible for viscosity it is required to introduce some tensor internal parameter ξ_{ik} in agreement with Mandelshtam-Leontovich approach (Mandelshtam & Leontovich, 1937). Relaxation of this internal parameter provides appearance of viscous terms in the Navier-Stokes equation. It is worth mentioning that the developed approach allowed to generalize the Navier-Stokes equation with constant viscosity coefficient to more general case accounting for viscosity relaxation in analogy to the Maxwell's model (Landau & Lifshitz, 1972). However the physical interpretation of the tensor internal parameter, which should be enough universal due to general character of the Navier-Stokes equation, requires more clear understanding. On the intuition level it is clear that corresponding internal parameter should be related with neighbor order in atoms and molecules placement and their relaxation. In the present section such physical interpretation is represented.

As was mentioned in Introduction the system of hydrodynamical equations in the form of Navier-Stokes is usually derived on the basis of conservation laws of mass M, momentum $\vec{P}$ and energy E. The correctness of equations of the traditional hydrodynamics is confirmed by the large number of experiments where it is adequate. However the conservation law of angular momentum $\vec{M}$ is absent among the mentioned balance laws laying in the basis of traditional hydrodynamics. In this connection it is interesting to understand the role of conservation law of angular momentum $\vec{M}$ in hydrodynamical description. It is worth mentioning that equation for angular momentum appeared in hydrodynamics early (Sorokin, 1943; Shliomis, 1966) and arises and develops in the momentum elasticity theory. The Cosserat continuum is an example of such description (Kunin, 1975; Novatsky, 1975; Erofeev, 1998). However some internal microstructure of medium is required for application of such approach.

In the hydrodynamical description as a partial case of continuum mechanics the definition of material point is introduced as sifficiently large ensemble of structural elements of medium (atoms and molecules) that on one hand one has to describe properties of this ensemble in statistical way and on the other one has to consider the size of material point as small in comparison with specific scales of the problem. A material point itself as closed ensemble of particles possesses the following integrals of motion: mass, momentum, energy and angular momentum.

The basic independent variables, in terms of which the hydrodynamical description should be constructed, are the values which can be determined for separate material point in

accordance with its integrals of motion: mean mass displacement vector $\vec{u}$ (velocity of this displacement $\vec{v} = \partial\vec{u} / \partial t$ is determined by integrals of motion $\vec{v} = \vec{P} / M$), rotation angle $\vec{\varphi}$ (angular velocity of rotation $\vec{\Omega} = \dot{\vec{\varphi}}$ is determined by integrals of motion $\vec{\Omega} = \vec{M} / I$, where I - inertia moment) and heat displacement $\vec{u}_T$, determining variation of temperature and related with integral of energy E.
In accordance with the set of independent field variables we can represent the kinetic K and the free F energies as corresponding quadratic forms

$$2K = \rho\dot{\vec{u}}^2 + I\dot{\vec{\varphi}}^2 \tag{41}$$

$$2F = (\lambda + 2\mu)(\nabla\vec{u})^2 + \mu[\nabla\vec{u}]^2 + 2\delta\vec{\varphi}[\nabla\vec{u}] + \sigma(\vec{\varphi})^2 + \varepsilon(\nabla\vec{\varphi})^2 + \varsigma[\nabla\vec{\varphi}]^2 \tag{42}$$

Taking into account that the dissipation dealt only with field of micro rotations, and omitting for shortness dissipation of mean displacement field, described by heat conductivity, we can write the dissipation function in the following form

$$2D = \gamma\dot{\vec{\varphi}}^2 \tag{43}$$

Equations of motion derived from GVP without temperature terms have the forms:

$$\frac{d}{dt}\frac{\partial K}{\partial\dot{\vec{u}}} - \nabla\frac{\partial F}{\partial\nabla\vec{u}} - [\nabla\frac{\partial F}{\partial[\nabla\vec{u}]}] = -\frac{\partial D}{\partial\dot{\vec{u}}} \tag{44a}$$

$$\frac{d}{dt}\frac{\partial K}{\partial\dot{\vec{\varphi}}} + \frac{\partial K}{\partial\vec{\varphi}} - \nabla\frac{\partial F}{\partial\nabla\vec{\varphi}} - [\nabla\frac{\partial F}{\partial[\nabla\vec{\varphi}]}] = -\frac{\partial D}{\partial\dot{\vec{\varphi}}} \tag{45a}$$

Without dissipation $\beta = 0$ the motion equations obtained with use of quadratic forms (41)-(43) correspond to the ones for Cosserat continuum (Kunin, 1975; Novatsky, 1975; Erofeev, 1998). Indeed for this case the equations (44) have forms:

$$\rho\frac{d}{dt}\dot{\vec{u}} - (\lambda + 2\mu)\nabla(\nabla\vec{u}) + \mu[\nabla[\nabla\vec{u}]] - \delta[\nabla\vec{\varphi}] = 0 \tag{44b}$$

$$I\frac{d}{dt}\dot{\vec{\varphi}} - \varepsilon\nabla(\nabla\vec{\varphi}) + \varsigma[\nabla[\nabla\vec{\varphi}]] + \sigma\vec{\varphi} + \delta[\nabla\vec{u}] = 0 \tag{45b}$$

The explicit form of these equations confirms that they are indeed the Cosserat continuum. If one sets formally $\delta = 0$, then equations (44b) and (45b) are split and the equation (44b) reduces to ordinal equation of the elasticity theory and the equation (45b) represents the wave equation for angular momentum.
When dissipation exists the system of equations (44)-(45) contains additional terms responsible for this dissipation

$$\rho\ddot{\vec{u}} - (\lambda + 2\mu)\nabla(\nabla\vec{u}) + \mu[\nabla[\nabla\vec{u}]] - \delta[\nabla\vec{\varphi}] = 0 \tag{44c}$$

$$I\ddot{\vec{\varphi}} - \varepsilon\nabla(\nabla\vec{\varphi}) + \varsigma[\nabla[\nabla\vec{\varphi}]] + \sigma\vec{\varphi} + \delta[\nabla\vec{u}] = -\gamma\dot{\vec{\varphi}} \tag{45c}$$

For the case $\varepsilon = 0$, $\varsigma = 0$ and $I = 0$ the second equation (45c) reduces to the pure relaxation form:

$$\dot{\vec{\varphi}} = -\frac{\sigma}{\gamma}\vec{\varphi} - \frac{\delta}{\gamma}[\nabla\vec{u}] \tag{46}$$

Its solution can be represented in the form:

$$\vec{\varphi} = -\frac{\delta}{\gamma}\int_{-\infty}^{t} dt' e^{-\frac{\sigma}{\gamma}(t-t')}[\nabla\vec{u}] \tag{47a}$$

Substitution (47a) in (44c) leads to the following result

$$\rho\ddot{\vec{u}} - (\lambda+2\mu)\nabla(\nabla\vec{u}) + \mu[\nabla[\nabla\vec{u}]] = -\frac{\delta^2}{\gamma}\int_{-\infty}^{t} dt' e^{-\frac{\sigma}{\gamma}(t-t')}[\nabla[\nabla\vec{u}]] \tag{48a}$$

For the case of large times $t\sigma/\gamma >> 1$ the upper limit of integration gives the principal contribution and equation reduces to the form

$$\rho\ddot{\vec{u}} - (\lambda+2\mu)\nabla(\nabla\vec{u}) + \left(\mu - \frac{\delta^2}{\sigma}\right)[\nabla[\nabla\vec{u}]] = \gamma\frac{\delta^2}{\sigma^2}[\nabla\dot{\vec{u}}] \tag{48b}$$

By the reason that the medium at large times should behave like a fluid then the following condition has to be satisfied

$$\mu - \frac{\delta^2}{\sigma} = 0 \tag{49}$$

Taking into account condition (49) let's make more accurate estimation of the integral, computing it by parts

$$\rho\ddot{\vec{u}} - (\lambda+2\mu)\nabla(\nabla\vec{u}) = -\frac{\delta^2}{\sigma}\int_{-\infty}^{t} dt' e^{-\frac{\sigma}{\gamma}(t-t')}[\nabla[\nabla\dot{\vec{u}}]] \tag{48c}$$

The corresponding estimation for the large time limit $t >> \gamma/\sigma$ reduces to the equation

$$\rho\ddot{\vec{u}} - (\lambda+2\mu)\nabla(\nabla\vec{u}) = \gamma\frac{\mu^2}{\delta^2}[\nabla[\nabla\dot{\vec{u}}]] \tag{48d}$$

which coincides with the structure of Navier-Stokes equation in the presence of shear viscosity.
Let's consider the case with non zero moment of inertia $I \neq 0$. For this case the second equation (45c) is also local in space and it can be resolved for the function $\vec{\varphi}$ in Fourier representation ($t \to \omega$)

$$\vec{\varphi} = \frac{-\delta}{-I\omega^2 + i\omega\gamma + \sigma}[\nabla\vec{u}] \tag{50}$$

The zeros of the denominator

$$i\omega_{1,2} = \frac{1}{2I}\left(-\gamma \pm \sqrt{\gamma^2 - 4\sigma I}\right) \tag{51}$$

determine two modes of angular momentum relaxation. Under condition $I < \gamma^2 / (4\sigma)$ both zeros are real and have the following asymptotics for small momentum of inertia $I \to 0$:

$$i\omega_1 \approx -\frac{\sigma}{\gamma} \qquad i\omega_2 \approx -\frac{\gamma}{I} \tag{52}$$

The first zero does not depend on momentum of inertia I and the second root goes to infinity when $I \to 0$. Under condition $I = \gamma^2 / (4\sigma)$ the zeros coincide and have the value $i\omega_1 \approx -2\frac{\sigma}{\gamma}$, and under the condition $I > \gamma^2 / (4\sigma)$ the zeros are complex conjugated with negative real part, which decreases with increase of I. The last case corresponds to the resonant relaxation of angular momentum.

In the time representation the solution of the equation (50) can be written in the form

$$\vec{\varphi} = -\int_{-\infty}^{t} dt' e^{-\frac{\gamma}{2I}(t-t')} [\nabla \vec{u}] \left\{ \frac{2\delta}{\sqrt{\ldots}} sh\left(\frac{\sqrt{\ldots}}{2I}(t-t') \right) \right\} \tag{47b}$$

here the notation $\sqrt{\ldots} = \sqrt{\gamma^2 - 4\sigma I}$ is used. For the case of resonant relaxation $I > \gamma^2 / (4\sigma)$ the corresponding expression has the form

$$\vec{\varphi} = -\int_{-\infty}^{t} dt' e^{-\frac{\gamma}{2I}(t-t')} [\nabla \vec{u}] \left\{ \frac{2\delta}{\sqrt{|\ldots|}} \sin\left(\frac{\sqrt{|\ldots|}}{2I}(t-t') \right) \right\} \tag{47c}$$

Substitution of the explicit expressions (47b) or (47c) in the equation (44c) gives the generalisation of the Navier – Stokes equation for a solid medium with local relaxation of angular momentum. As was mentioned above under special condition (49) and in the limiting case (52) this equation reduces exactly to the form of Navier – Stokes equation.

Thus, it is shown that relaxation of angular momentum of material points consisting a continuum can be considered as physical reason for appearance of terms with shear viscosity in Navier-Stokes equation. Without dissipation additional degree of freedom dealt with angular momentum leads to the well known Cosserat continuum.

4. Conclusion

The first part of the chapter presents an original formulation of the generalized variational principle (GVP) for dissipative hydrodynamics (continuum mechanics) as a direct combination of Hamilton's and Onsager's variational principles. The GVP for dissipative continuum mechanics is formulated as Hamilton's variational principle in terms of two independent field variables i.e. the mean mass and the heat displacement fields. It is important to mention that existence of two independent fields gives us opportunity to consider a closed mechanical system and hence to formulate variational principle. Dissipation plays only a role of energy transfer between the mean mass and the heat displacement fields. A system of equations for these fields is derived from the extreme condition for action with a Lagrangian density in the form of the difference between the kinetic and the free energies minus the time integral of the dissipation function. All mentioned potential functions are considered as a general positively determined quadratic forms of time or space derivatives of the mean mass and the heat displacement fields. The generalized system of hydrodynamical equations is then evaluated on the basis of the GVP. At low frequencies this system corresponds to the traditional Navier – Stokes equation system. It allowed us to determine all coefficients of quadratic forms by direct comparison with the Navier – Stokes equation system.

The second part of the chapter is devoted to consistent introduction of viscous terms into the equation of fluid motion on the basis of the GVP. A tensor internal parameter is used for description of relaxation processes in vicinity of quasi-equilibrium state by analogy with the Mandelshtam – Leontovich approach. The derived equation of motion describes the viscosity relaxation phenomenon and generalizes the well known Navier – Stokes equation. At low frequencies the equation of fluid motion reduces exactly to the form of Navier – Stokes equation. Nevertheless there is still a question about physical interpretation of the used internal parameter. The answer on this question is presented in the last section of the chapter.

It is shown that the internal parameter responsible for shear viscosity can be interpreted as a consequence of relaxation of angular momentum of material points constituting a mechanical continuum. Due to angular momentum balance law the rotational degree of freedom as independent variable appears additionally to the mean mass displacement field. For the dissipationless case this approach leads to the well-known Cosserat continuum. When dissipation prevails over momentum of inertion this approach describes local relaxation of angular momentum and corresponds to the sense of the internal parameter. It is important that such principal parameter of Cosserat continuum as the inertia moment of intrinsic microstructure can completely vanish from the description for dissipative continuum. The independent equation of motion for angular momentum in this case reduces to local relaxation and after its substitution into the momentum balance equation leads to the viscous terms in Navier – Stokes equation. Thus, it is shown that the nature of viscosity phenomenon can be interpreted as relaxation of angular momentum of material points on the kinetic level.

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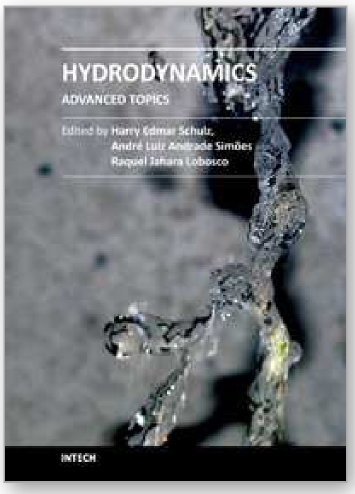

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The phenomena related to the flow of fluids are generally complex, and difficult to quantify. New approaches - considering points of view still not explored - may introduce useful tools in the study of Hydrodynamics and the related transport phenomena. The details of the flows and the properties of the fluids must be considered on a very small scale perspective. Consequently, new concepts and tools are generated to better describe the fluids and their properties. This volume presents conclusions about advanced topics of calculated and observed flows. It contains eighteen chapters, organized in five sections: 1) Mathematical Models in Fluid Mechanics, 2) Biological Applications and Biohydrodynamics, 3) Detailed Experimental Analyses of Fluids and Flows, 4) Radiation-, Electro-, Magnetohydrodynamics, and Magnetorheology, 5) Special Topics on Simulations and Experimental Data. These chapters present new points of view about methods and tools used in Hydrodynamics.