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# Steady-State Heat Transfer and Thermo-Elastic Analysis of Inhomogeneous Semi-Infinite Solids

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## 1. Introduction

The advancement in efficient modeling and methodology for thermoelastic analysis of structure members requires a variety of the material characteristics to be taken into consideration. Due to the critical importance of such analysis for adequate determination of operational performance of structures, it presents a great deal of interest for scientists in both academia and industry. However, the assumption that the material properties depend on spatial coordinates (material inhomogeneity) presents a major challenge for analytical treatment of relevant heat conduction and thermoelasticity problems. The main difficulty lies in the need to solve the governing equations in the differential form with variable coefficients which are not pre-given for arbitrary dependence of thermo-physical and thermo-elastic material properties on the coordinate. Particularly, for functionally graded materials, whose properties vary continuously from one surface to another, it is impossible, except for few particular cases, to solve the mentioned problems analytically (Tanigawa, 1995). The analytical, semi-analytical, and numerical methods for solving the heat conduction and thermoelasticity problems in inhomogeneous solids attract considerable attention in recent years. The overview of the relevant literature is given in our publications (Tokovyy & Ma, 2008, 2008a, 2009, 2009a). On the other hand, determination of temperature gradients, stresses and deformations is usually an intermediate step of a complex engineering investigation. Therefore analytical methods are of particular importance representing the solutions in a most convenient form. The great many of existing analytical methods are developed for particular cases of inhomogeneity (e.g., in the form of power or exponential functions of a coordinate, etc.). The methods applicable for wider ranges of material properties are oriented mostly on representation the inhomogeneous solid as a composite consisting of tailored homogeneous layers. However, such approaches are inconvenient for applying to inhomogeneous materials with large gradients of inhomogeneity due to the weak convergence of the solution with increasing the number of layers.

A general method for solution of the elasticity and thermoelasticity problems in terms of stresses has been developed by Prof. Vihak (Vigak) and his followers in (Vihak, 1996; Vihak

et al., 1998, 2001, 2002; Vigak, 1999; Vigak & Rychagivskii, 2000, 2002). The method consists in construction of analytical solutions to the problems of thermoelasticity based on direct integration of the original equilibrium and compatibility equations. Originally the equilibrium equations are in terms of stresses, and they do not depend on the physical stress-strain relations, as well as on the material properties. At the same time the general equilibrium relates all the stress-tensor components. This enables one to express all the stresses in terms of the governing stresses. The compatibility equations in terms of strain are then reduced to the governing equations for the governing stresses. When these equations are solved, all the stress-tensor components can be found by means of the aforementioned expressions. In addition, the method enables the derivation of: *a*) fundamental integral equilibrium and compatibility conditions for the imposed thermal and mechanical loadings and the stresses and strains; *b*) one-to-one relations between the stress-tensor and displacement-vector components. Therefore, when the stress-tensor components are found, then the displacement-vector components are also found automatically. Such relations have been derived for the case of one-dimensional problem for a thermoelastic hollow cylinder (Vigak, 1999a) and plane problems for elastic and thermoelastic semi-plane (Vihak & Rychahivskyy, 2001; Vigak, 2004; Rychahivskyy & Tokovyy, 2008).

Since application of this method rests upon the direct integration of the equilibrium equations, the proposed solution scheme offers ample opportunities for efficient analysis of inhomogeneous solids. In contrast to homogeneous materials, the compatibility equations in terms of stresses are with variable coefficients. This causes that the governing equations, obtained on the basis of the compatibility ones, appear as integral equations of Volterra type. By following this solution strategy, the one-dimensional thermoelasticity problem for a radially-inhomogeneous cylinder has been analyzed (Vihak & Kalyniak, 1999; Kalyniak, 2000). In the same manner, the two-dimensional elasticity and thermoelasticity problems for inhomogeneous cylinders, strips, planes and semi-planes were solved in (Tokovyy & Rychahivskyy, 2005; Tokovyy & Ma, 2008, 2008a, 2009, 2009a). The same method has also been extended for three-dimensional problems (Tokovyy & Ma, 2010, 2010a). Application of this method for analysis of inhomogeneous solids exhibits several advantages. First of all, this method is unified for various kinds of inhomogeneity and different shape of domain and it does not impose any restriction on the material properties. Moreover, when applying the resolvent-kernel algorithm for solution of the governing Volterra-type integral equation, the solutions of corresponding elasticity and thermoelasticity problems for inhomogeneous solids appear in the form of explicit functional dependences on the thermal and mechanical loadings, which makes them to be rather usable for complex engineering analysis.

Herein, we consider an application of the direct integration method for analysis of thermoelastic response of an inhomogeneous semi-plane within the framework of linear uncoupled thermoelasticity (Nowacki, 1962). The solution of this problem consists of two stages: *i*) solution of the in-plane steady-state heat conduction problem under certain boundary conditions, and *ii*) solution of the plane thermal stress problem due to the above determined temperature field and appropriate boundary conditions. Solution of both problems is reduced to the governing Volterra integral equation. By making use of the resolvent-kernel solution technique, the governing equation is solved and the solution of the original problem is presented in an explicit form. Due to the later result, the one-to-one relationships are set up between the tractions and displacements on the boundary of the inhomogeneous semi-plane. Using these relations and the solution in terms of stresses, we find solutions for the boundary value problems with displacements or mixed conditions

imposed on the boundary. It is shown that these solutions are correct if the tractions satisfy the necessary equilibrium conditions, the displacements meet the integral compatibility conditions, and the heat sources and heat flows satisfy the integral condition of thermal balance.

## 2. Analysis of the steady-state heat conduction problem in an inhomogeneous semi-plane

In this section, we consider an application of the direct integration method for solution of the in-plane steady-state (stationary) heat conduction problem for a semi-plane whose thermal conductivity is an arbitrary function of the depth-coordinate. Having applied the Fourier integral transformation to the differential heat conduction equation with variable coefficients, this equation is reduced to the Volterra-type integral equation, which then is solved by making use of the resolvent-kernel technique. As a result, the temperature distribution is found in an explicit functional form that can be efficiently used for analysis of thermal stresses and displacements in the semi-plane.

### 2.1 Formulation of the heat conduction problem

Let us consider the two-dimensional heat conduction problem for semi-plane  $D = \{(x, y) \in (-\infty, \infty) \times [0, \infty)\}$  in the dimensionless Cartesian coordinate system  $(x, y)$ . In assumption of the isotropic material properties, the problem is governed by the following heat conduction equation (Hetnarski & Reza Eslami, 2008)

$$\frac{\partial}{\partial x} \left( k(y) \frac{\partial T(x, y)}{\partial x} \right) + \frac{\partial}{\partial y} \left( k(y) \frac{\partial T(x, y)}{\partial y} \right) = -q(x, y), \quad (1)$$

where  $T(x, y)$  is the steady-state temperature distribution,  $k(y)$  stands for the coefficient of thermal conduction, and  $q(x, y)$  denotes the quantity of heat generated by internal heat sources in semi-plane  $D$ . When the coefficient of thermal conduction is constant, then equation (1) presents the classical equation of quasi-static heat conduction (Nowacki, 1962; Carslaw & Jaeger, 1959)

$$\Delta T(x, y) = -W(x, y), \quad (2)$$

where  $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$  and  $W(x, y) = q(x, y) / k$  denotes the density of internal sources of heat. In the steady-state case, the temperature  $T(x, y)$  can be determined from equation (1) for  $k(y)$  or (2) for  $k = \text{const}$  under certain boundary condition imposed at  $y = 0$  (Nowacki, 1962). We consider the boundary condition in one of the following forms:

a. the temperature distribution is prescribed on the boundary

$$T(x, y) = T_0(x), \quad y = 0; \quad (3)$$

b. the heat flux over the limiting line  $y = 0$  is prescribed on the boundary

$$\frac{\partial T(x, y)}{\partial y} = \Phi_0(x), \quad y = 0; \quad (4)$$

c. the heat exchange condition is imposed on the boundary

$$\frac{\partial T(x, y)}{\partial y} + \alpha_0 T(x, y) = \beta_0, \quad y = 0. \quad (5)$$

Here  $\alpha_0$  and  $\beta_0$  are constants,  $T_0(x)$  and  $\Phi_0(x)$  are given functions. Assuming that the temperature field, heat fluxes, and the density of heat sources vanish with  $|x|, y \rightarrow \infty$ , we consider finding the solution to equation (1) or (2) under either of the boundary conditions (3) – (5) and the supplementary conditions of integrability of the functions in question in their domain of definition.

## 2.2 Solution of the stated heat conduction problem by reducing to the Volterra-type integral equation

In the case when  $k = \text{const}$ , it has been shown (Rychahivskyy & Tokovyy, 2008) that for construction of a correct solution to equation (2) with boundary condition (4), the following necessary condition

$$\iint_D W(x, y) dx dy = \int_{-\infty}^{\infty} \Phi_0(x) dx \quad (6)$$

is to be fulfilled. This condition of thermal balance postulates that the resultant heat flux through the boundary  $y = 0$  is equal to the resultant action of internal heat sources within domain  $D$ . In the case of boundary conditions (3) or (5), the right-hand side of condition (6) should be replaced by the integral of the heat flux at  $y = 0$ , which is determined by the temperature. Due to this reason, condition (6) can be regarded as an efficient tool for verification of the solution correctness.

Note that condition (6) is natural for steady-state thermal processes in bounded solids. However, it is not intuitive for non-stationary thermal regimes since then only certain distribution of the temperature field is possible inside the solid implying that the heating of the solid until an average temperature is not achievable. Thus, condition (6) for a semi-plane is a consequence of application of solid mechanics to the oversimplified geometrical model.

By denoting

$$\Phi_x(x, y) = k(y) \frac{\partial T(x, y)}{\partial x}, \quad \Phi_y(x, y) = k(y) \frac{\partial T(x, y)}{\partial y}$$

in equation (1), when  $k = k(y)$ , and following the strategy presented in (Rychahivskyy & Tokovyy, 2008), it can be shown that condition (6) holds for the case of inhomogeneous material. In addition, the resultant of the temperature is necessarily equal to zero

$$\iint_D T(x, y) dx dy = 0 \quad (7)$$

for both homogeneous and inhomogeneous cases.

Let us construct the solution to equation (1) under boundary conditions (3), (4), or (5) by taking conditions (6) and (7) into consideration. Having applied the Fourier integral transformation (Brychkov & Prudnikov, 1989)

$$\bar{f}(y;s) = \int_{-\infty}^{\infty} f(x,y) \exp(-isx) dx \quad (8)$$

to the aforementioned equation and boundary conditions, we arrive at the following second order ODE

$$\frac{d^2 \bar{T}(y;s)}{dy^2} - s^2 \bar{T}(y;s) = -\frac{1}{k(y)} \left( \bar{q}(y;s) + \frac{dk(y)}{dy} \frac{d\bar{T}(y;s)}{dy} \right) \quad (9)$$

that is accompanied with one of the following boundary conditions:

$$\bar{T}(y;s) = \bar{T}_0(s), \quad y = 0; \quad (10)$$

$$\frac{d\bar{T}(y;s)}{dy} = \bar{\Phi}_0(s), \quad y = 0; \quad (11)$$

$$\frac{d\bar{T}(y;s)}{dy} + \alpha_0 \bar{T}(y;s) = \beta_0, \quad y = 0. \quad (12)$$

Here  $s$  is a parameter of the integral transformation,  $i^2 = -1$ ,  $f(x,y) \in L(D)$ . For the sake of brevity, the parameter  $s$  will be omitted from the arguments of functions in the following text.

A general solution to equation (9) in semi-plane  $D$  can be given in the form

$$\begin{aligned} \bar{T}(y) = & C \exp(-|s|y) + \frac{1}{2|s|} \int_0^\infty \frac{\bar{q}(\eta)}{k(\eta)} \exp(-|s||y-\eta|) d\eta \\ & + \frac{1}{2|s|} \int_0^\infty \frac{1}{k(\eta)} \frac{dk(\eta)}{d\eta} \frac{d\bar{T}(\eta)}{d\eta} \exp(-|s||y-\eta|) d\eta, \end{aligned} \quad (13)$$

where  $C$  is a constant of integration and  $|\cdot|$  denotes the absolute-value function. By applying the integration by parts to the last integral in equation (13), we can obtain the following Volterra-type integral equation of second kind:

$$\bar{T}(y) = \left( C - \frac{\bar{T}(0)}{2|s|k(0)} \frac{dk(0)}{dy} \right) \exp(-|s|y) + q^*(y) + \int_0^\infty \bar{T}(\eta) K(y,\eta) d\eta. \quad (14)$$

Here

$$q^*(y) = \frac{1}{2|s|} \int_0^\infty \frac{\bar{q}(\eta)}{k(\eta)} \exp(-|s||y-\eta|) d\eta,$$

$$K(y,\eta) = -\frac{1}{2|s|} \frac{d}{d\eta} \left( \frac{1}{k(\eta)} \frac{dk(\eta)}{d\eta} \exp(-|s||y-\eta|) \right)$$



$$= \frac{1}{2|s|} \left( \left( \frac{1}{k(\eta)} \frac{dk(\eta)}{d\eta} - |s| \operatorname{sgn}(y - \eta) \right) \frac{dk(\eta)}{d\eta} - \frac{d^2 k(\eta)}{d\eta^2} \right) \frac{\exp(-|s||y - \eta|)}{k(\eta)}. \quad (15)$$

To solve integral equation (14), different algorithms can be employed, for instance, the Picard's process of successive approximations (Tricomi, 1957; Kalynyak, 2000; Tokovyy & Ma, 2008a), the operator series method (Bartoshevich, 1975), the Bubnov-Galerkin method (Fedotkin et al., 1983), a numerical procedure based on trapezoidal integration and a Newton-Raphson method (Frankel, 1991), iterative-collocation method (Hăcia, 2007), discretization method, special kernels method and projection-iterative method (Domke & Hăcia, 2007), spline approximations (Kushnir et al., 2002), the quadratic-form method (Belik et al. 2008), the greed methods (Peng & Li, 2010). Herein we employ the resolvent-kernel algorithm (Pogorzelski, 1966; Porter & Stirling, 1990) in the same manner as it has been done in (Tokovyy & Ma, 2008, 2009a). This method allows us to obtain the explicit-form analytical solution that is convenient for analysis. As a result, the transformation of temperature appears as

$$\bar{T}(y) = \left( C - \frac{\bar{T}(0)}{2|s|k(0)} \frac{dk(0)}{dy} \right) \varphi(y) + \theta(y), \quad (16)$$

where

$$\varphi(y) = \exp(-|s|y) + \int_0^\infty \exp(-|s|\eta) \mathfrak{R}(y, \eta) d\eta, \quad (17)$$

$$\theta(y) = q^*(y) + \int_0^\infty q^*(\eta) \mathfrak{R}(y, \eta) d\eta, \quad (18)$$

and the resolvent-kernel is determined by the recurring kernels as

$$\mathfrak{R}(y, \eta) = \sum_{n=0}^{\infty} K_{n+1}(y, \eta), \quad (19)$$

$$K_1(y, \eta) = K(y, \eta), \quad K_{n+1}(y, \eta) = \int_0^\infty K(y, \zeta) K_n(\zeta, \eta) d\zeta, \quad n = 1, 2, \dots$$

Note that expression (16) is advantageous in comparison with the analogous solutions constructed by means of the aforementioned techniques for solution of the Volterra integral equations. First of all, solution (16) is obtained in explicit functional form. This fact can be efficiently used for complex analysis involving solution of thermoelasticity problem. Next, the resolvent (19) is expressed only through the kernel (15) of integral equation (14) ("intrinsic" properties of an integral equation) and is non-dependent of the free term ("external" properties of an integral equation). Consequently, being computed once for certain kernel (which means for certain material properties, obviously), resolvent (19) can be employed for various kinds of thermal loading.

To determine the unknown constant  $C$  in equation (16), one of conditions (10) – (12) should be employed. Insertion of (16) into condition (10) yields

$$C = \frac{\bar{T}_0}{\varphi(0)} \left( 1 + \frac{\varphi(0)}{2|s|k(0)} \frac{dk(0)}{dy} \right) - \frac{\theta(0)}{\varphi(0)},$$

and then the temperature can be given as

$$\bar{T}(y) = \frac{\bar{T}_0 - \theta(0)}{\varphi(0)} \varphi(y) + \theta(y). \quad (20)$$

In the case of boundary condition (11), the constant  $C$  appears as

$$C = \bar{q}_0 \left( \frac{d\varphi(0)}{dy} \right)^{-1} + \frac{\bar{T}(0)}{2|s|k(0)} \frac{dk(0)}{dy} - \frac{d\theta(0)}{dy} \left( \frac{d\varphi(0)}{dy} \right)^{-1}.$$

Then the temperature can be given as

$$\bar{T}(y) = \left( \bar{q}_0 - \frac{d\theta(0)}{dy} \right) \left( \frac{d\varphi(0)}{dy} \right)^{-1} \varphi(y) + \theta(y). \quad (21)$$

In the case of boundary condition (12), the constant  $C$  takes the form

$$C = \left( \beta_0 - \alpha_0 \theta(0) - \frac{d\theta(0)}{dy} \right) \left( \alpha_0 \varphi(0) + \frac{d\varphi(0)}{dy} \right)^{-1} + \frac{\bar{T}(0)}{2|s|k(0)} \frac{dk(0)}{dy},$$

and, consequently,

$$\bar{T}(y) = \left( \beta_0 - \alpha_0 \theta(0) - \frac{d\theta(0)}{dy} \right) \left( \alpha_0 \varphi(0) + \frac{d\varphi(0)}{dy} \right)^{-1} \varphi(y) + \theta(y). \quad (22)$$

Having determined the expressions for the temperature field in the form (20), (21), or (22) and applying the formula

$$f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(y; s) \exp(isx) ds \quad (23)$$

of inverse Fourier transformation (Brychkov & Prudnikov, 1989), we can obtain the expressions for temperature field in semi-plane  $D$ .

Note that according to the resolvent-kernel theory (Verlan & Sizikov, 1986), the recurring kernels  $K_{n+1}$  tend to zero as  $n \rightarrow \infty$ . Thus, for practical computations, the series in expression (19) can be truncated. Consequently,

$$\Re(y, \eta) \approx \Re_N(y, \eta) = \sum_{n=0}^N K_{n+1}(y, \eta), \quad (24)$$

where  $N$  is a natural number which depends on required accuracy of calculation.



### 2.3 Numerical analysis

To verify the obtained solution to the heat conduction problem, let us examine the case, when the semi-plane is heated by a single concentrated internal heat source

$$q(x, y) = q_0 \delta(x) \delta(y - y_0). \quad (25)$$

Meanwhile the boundary  $y = 0$  remains of the constant temperature,  $T_0 = 0$ . Here  $q_0$  is a constant dimensional parameter;  $\delta(\cdot)$  denotes the Dirac delta-function. In this case, the temperature should be computed on the basis of expression (20). The coefficient of thermal conductivity is assumed to be in the following form

$$k(y) = k_0 \exp(\gamma y), \quad (26)$$

where  $k_0$  and  $\gamma$  are constants. Note that for  $\gamma = 0$ , the thermal conductivity in the form (26) is constant, that corresponds to the case of homogeneous material. Then, on the basis of expression (19), the resolvent  $\Re(y, \eta) \equiv 0$  and thus expression (20) presents an exact analytical solution

$$\frac{\bar{T}(y)k_0}{q_0} = \frac{1}{2|s|} (\exp(-|s||y - y_0|) - \exp(-|s|(y + y_0))). \quad (27)$$

Application of the Fourier inversion (23) to formula (27) yields the expression for the temperature in the homogeneous semi-plane, as follows:

$$\frac{k_0}{q_0} T(x, y) = \frac{1}{4\pi} \ln \frac{x^2 + (y + y_0)^2}{x^2 + (y - y_0)^2}. \quad (28)$$

The full-field distributions of the temperature (28) and the components of corresponding heat flux are depicted in Fig. 1 for  $y_0 = 1$ . Distribution of the temperature (28) versus the

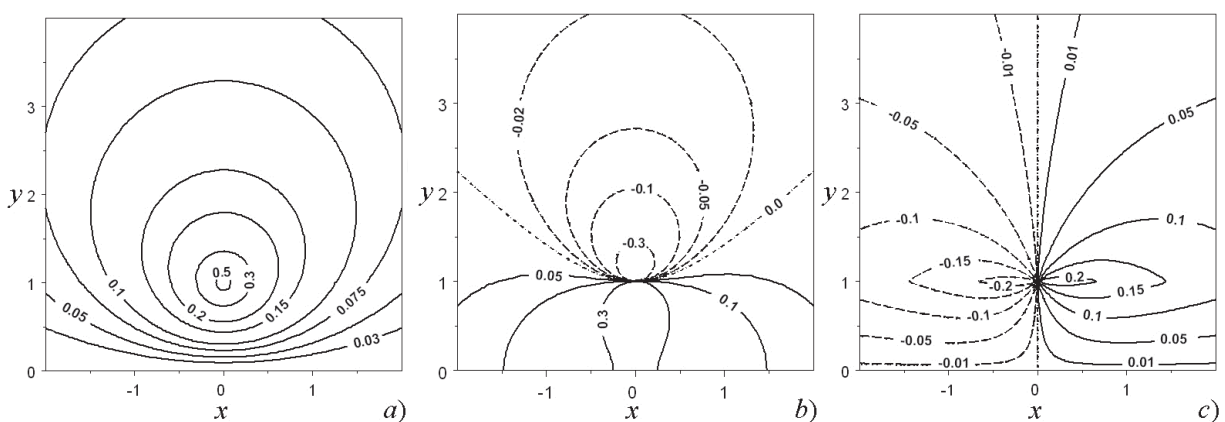


Fig. 1. Full-field distributions of (a) the dimensionless temperature  $\frac{k_0 T(x, y)}{q_0}$ , and (b) trans-

versal  $\frac{k_0}{q_0} \frac{\partial T(x, y)}{\partial y}$  and (c) longitudinal  $\frac{k_0}{q_0} \frac{\partial T(x, y)}{\partial x}$  components of the heat flux for  $y_0 = 1$

variable  $y$  is shown in Fig. 2 for  $x = 0.0; 0.5$  and different values  $y_0 = 1.0; 2.0; 3.0; 4.0$ . As we can observe in both figures, the thermal state is symmetrical with respect to the line  $x = 0$ . The temperature vanish when mowing away from the location of the heat source  $(0, y_0)$ . When approaching the boundary  $y = 0$ , the temperature vanish faster than in the opposite direction (due to satisfaction of the boundary condition). When the location of the heat source is moving away from the boundary, then the thermal state tends to one symmetrical with respect to the line  $y = y_0$  (Fig. 2) due to the lowering influence of the boundary (in analogy to the case of an infinite plane).

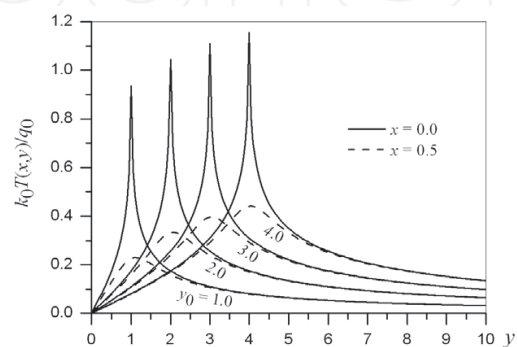


Fig. 2. Distribution of the temperature (28) versus coordinate  $y$  for  $x = 0.0; 0.5$

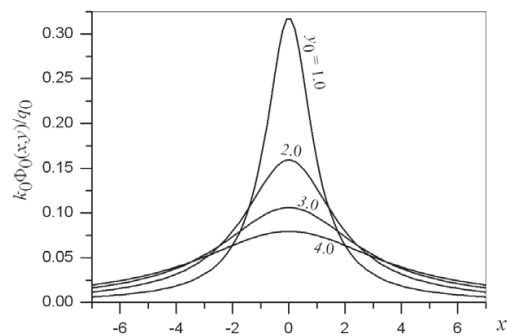


Fig. 3. The heat flux (29) for different values of  $y_0 = 1.0; 2.0; 3.0; 4.0$

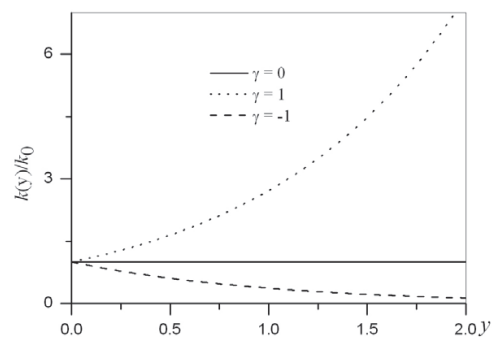


Fig. 4. Dependence of the thermal conductivity on the coordinate  $y$  for different values of  $\gamma$

For the obtained temperature, the heat flux trough the boundary  $y = 0$  can be found as

$$\frac{k_0 \Phi_0}{q_0} = \frac{y_0}{(x^2 + y_0^2)\pi}. \quad (29)$$

By taking formulae (25) and (29) into consideration, it is easy to see that condition (6) is satisfied for the considered case. Distribution of the heat flux (29) for different values of  $y_0$  is shown in Fig. 3. As we can see, the heat flux over the boundary is locally distributed with the maximum value at  $x = 0$  which decreases as the heat source is further from the boundary.

Now let  $\gamma \neq 0$  in (26). For this case, the exact solution can be constructed by following the technique presented in (Ma & Lee, 2009; Ma & Chen, 2011). According to this technique, the exact solution to the problem (1), (3), (25) can be found in the form

$$\frac{k_0}{q_0} \bar{T}(y) = -\frac{q_0}{\sqrt{\gamma^2 - 4s^2}} \times \begin{cases} \left( \exp(y_0 \gamma^+ / 2) - \exp(-y_0 \gamma^- / 2) \right) \exp(-y \gamma^+ / 2), & y \geq y_0, \\ \left( \exp(y \gamma^- / 2) - \exp(-y \gamma^+ / 2) \right) \exp(-y_0 \gamma^- / 2), & y < y_0, \end{cases} \quad (30)$$

where  $\gamma^\pm = \sqrt{\gamma^2 - 4s^2} \pm \gamma$ . To obtain the distribution of temperature due to Fourier transform (30), the inversion formula (23) can be applied. The distributions of obtained temperature and corresponding heat flux are examined for different values of the parameter of inhomogeneity:  $\gamma = 1$ ,  $\gamma = -1$ , and, for comparison with above-discussed homogeneous case,  $\gamma = 0$  (Fig. 4). For  $\gamma = 1$ , the thermal conductivity grows exponentially from 1 to infinity; for  $\gamma = -1$  it decreases from 1 to 0.

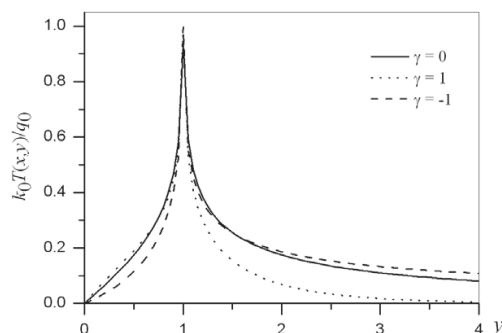


Fig. 5. Distribution of the temperature due to transformant (30) versus coordinate  $y$  at  $x = 0.0$  for  $\gamma = 0, \pm 1$

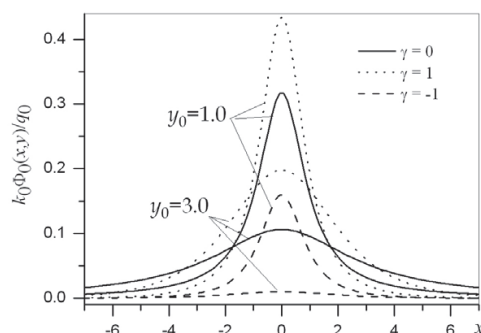


Fig. 6. Distribution of the heat flux across the boundary  $y = 0$  for  $y_0 = 1.0; 3.0$ ,  $\gamma = 0; \pm 1$

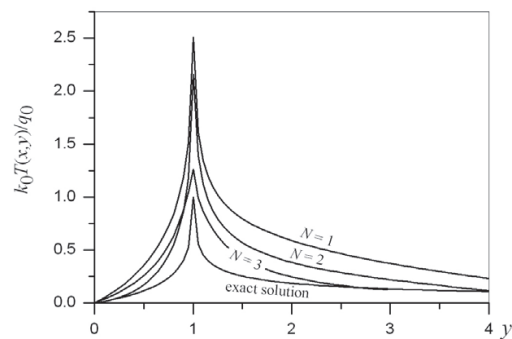


Fig. 7. Distribution of the heat flux for  $y_0 = 1.0; 3.0$ ,  $\gamma = 0, \pm 1$

The effect of inhomogeneity on the temperature and heat flux over the boundary  $y = 0$  is shown in Figs. 5 and 6, respectively. As we observe in Fig. 5, the temperature vanishes with  $y \rightarrow \infty$  as faster as the parameter of inhomogeneity  $\gamma$  is greater, whereas for  $0 \leq y < y_0$  vice-verse. Consequently, the heat flux over the boundary  $y = 0$  is greater for greater values of  $\gamma$  (Fig. 6).

Now we consider application of formula (20) for computation of the temperature in the inhomogeneous semi-plane. We employ formula (24) instead of (19) in (17) and (18). Distribution of the temperature computed by formula (20) for different values of parameter  $N$  in (24) is shown in Fig. 7. With growing  $N$ , the result naturally tends to the exact solution (30) and for  $N = 5$  they coincide. This result shows that expression (24) provides sufficiently good approximation for the resolvent  $\Re(y, \eta)$  by holding few terms only.

### 3. Analysis of thermal stresses in an inhomogeneous semi-plane

In this section, the technique for solving the plane thermoelasticity problem for an isotropic inhomogeneous semi-plane with boundary conditions for stresses or displacements, as well as mixed boundary conditions, is developed by establishing one-to-one relations between boundary tractions and displacements. This technique is based on integration of the Cauchy relations to express displacements in terms of strains. Then, by taking the physical strain-stress relations into consideration, the displacements are expressed through the stress-tensor components. Finally, by making use of the explicit-form analytical solution to the corresponding problem with boundary tractions, the displacements on the boundary can be expressed through the tractions. The technique for establishment of the one-to-one relations between the tractions and displacements on the boundary, as well as for deriving the necessary equilibrium and compatibility conditions in the case of homogeneous semi-plane has been developed in (Rychahivskyy & Tokovy, 2008).

#### 3.1 Formulation of the problem

Let us consider the plane quasi-static thermoelasticity problem in inhomogeneous semi-plane  $D$ . In absence of body forces, this problem is governed (Nowacki, 1962) by the equilibrium equations

$$\frac{\partial \sigma_x(x, y)}{\partial x} + \frac{\partial \tau_{xy}(x, y)}{\partial y} = 0, \quad \frac{\partial \tau_{xy}(x, y)}{\partial x} + \frac{\partial \sigma_y(x, y)}{\partial y} = 0, \quad (31)$$

the compatibility equation in terms of strains

$$\frac{\partial^2 \varepsilon_x(x, y)}{\partial y^2} + \frac{\partial^2 \varepsilon_y(x, y)}{\partial x^2} = \frac{\partial^2 \gamma_{xy}(x, y)}{\partial x \partial y}, \quad (32)$$

the physical thermoelasticity relations

$$\begin{aligned} \varepsilon_x(x, y) &= \frac{1}{E^*(y)} \sigma_x(x, y) - \frac{\nu^*(y)}{E^*(y)} \sigma_y(x, y) + \alpha^*(y) T(x, y), \\ \varepsilon_y(x, y) &= \frac{1}{E^*(y)} \sigma_y(x, y) - \frac{\nu^*(y)}{E^*(y)} \sigma_x(x, y) + \alpha^*(y) T(x, y), \end{aligned} \quad (33)$$

$$\gamma_{xy}(x, y) = \frac{1}{G(y)} \tau_{xy}(x, y),$$

and the geometrical Cauchy relations

$$\varepsilon_x(x, y) = \frac{\partial u(x, y)}{\partial x}, \quad \varepsilon_y = \frac{\partial v(x, y)}{\partial y}, \quad \gamma_{xy} = \frac{\partial u(x, y)}{\partial y} + \frac{\partial v(x, y)}{\partial x}. \quad (34)$$

Here  $\sigma_x, \sigma_y, \tau_{xy}$  and  $\varepsilon_x, \varepsilon_y, \gamma_{xy}$  denote the stress- and strain-tensor components, respectively;

$$E^*(y) = \begin{cases} \frac{E(y)}{1 - \nu^2(y)}, & \text{plane strain,} \\ E(y), & \text{plane stress,} \end{cases} \quad \nu^*(y) = \begin{cases} \frac{\nu(y)}{1 - \nu(y)}, & \text{plane strain,} \\ \nu(y), & \text{plane stress,} \end{cases}$$

$$\alpha^*(y) = \begin{cases} \alpha(y)(1 + \nu(y)), & \text{plane strain,} \\ \alpha(y), & \text{plane stress,} \end{cases}$$

$E(y)$  denotes the Young modulus,  $\nu(y)$  stands for the Poisson ratio;  $G(y) = \frac{E(y)}{2(1 + \nu(y))}$  is the shear modulus,  $\alpha(y)$  denotes the coefficient of linear thermal expansion;  $u(x, y)$  and  $v(x, y)$  are the dimensionless displacements;  $T(x, y)$  is the temperature field that is given or determined in the form (20), (21), or (22) by means of the technique proposed in the previous section.

We shall construct the solutions of the set of equations (31)–(34) for each of the three versions of boundary conditions prescribed on the line  $y = 0$ :

a. in terms of stresses

$$\sigma_y(x, y) = -p(x), \quad \tau_{xy}(x, y) = q(x), \quad y = 0; \quad (35)$$

b. in terms of displacements

$$u(x, y) = u_0(x), \quad v(x, y) = v_0(x), \quad y = 0; \quad (36)$$

c. mixed conditions, when one of the following couples of relations

$$\begin{aligned}\sigma_y(x, y) &= -p(x), v(x, y) = v_0(x), y = 0; \\ \sigma_y(x, 0) &= -p(x), v(x, 0) = v_0(x), y = 0; \\ \tau_{xy}(x, 0) &= q(x), u(x, 0) = u_0(x), y = 0; \\ \tau_{xy}(x, 0) &= q(x), v(x, 0) = v_0(x), y = 0\end{aligned}\quad (37)$$

is imposed on the boundary. The boundary tractions and displacements, those are mentioned in conditions (35)–(37), as well the temperature field, vanish with  $|x| \rightarrow \infty$ ,  $y \rightarrow \infty$ . We consider finding the solutions (stresses and displacements) of the stated boundary value problems.

### 3.2 Construction of the solutions

#### 3.2.1 Case A: Boundary condition in terms of external tractions

Let us consider the construction of solution to the problem (31) – (34) under boundary conditions (35) with given tractions  $p(x)$  and  $q(x)$ . The boundary displacements  $u_0(x)$  and  $v(x)$  are unknown and, thus, they should be determined in the process of solution. By following the solution strategy (Tokovyy & Ma, 2009), the stress-tensor components can be expressed through the in-plane total stress  $\bar{\sigma} = \bar{\sigma}_x + \bar{\sigma}_y$  as

$$\begin{aligned}\bar{\sigma}_x &= \bar{\sigma} - \bar{\sigma}_y, \\ \bar{\sigma}_y &= -\bar{p} \exp(-|s|y) + \frac{|s|}{2} \int_0^\infty \bar{\sigma}(\xi) (\exp(-|s||y - \xi|) - \exp(-|s|(y + \xi))) d\xi, \\ \bar{\tau}_{xy} &= \frac{i|s|}{s} \bar{p} \exp(-|s|y) - \frac{is}{2} \int_0^\infty \bar{\sigma}(\xi) (\exp(-|s||y - \xi|) \operatorname{sgn}(y - \xi) - \exp(-|s|(y + \xi))) d\xi.\end{aligned}\quad (38)$$

In turn, the total stress can be found as a solution of the Volterra-type integral equation of second kind:

$$\bar{\sigma}(y) = E^*(y) (\bar{p}P(y) + A \exp(-|s|y) - \alpha^*(y)\bar{T}(y)) + \int_0^\infty \bar{\sigma}(\eta) M(y, \eta) d\eta, \quad (39)$$

where

$$M(y, \eta) = -\frac{E^*(y)}{8} \int_0^\infty \frac{d^2}{d\xi^2} \left( \frac{1}{G(\xi)} \right) (\exp(-|s|(|\xi - \eta| + |y - \xi|)) - \exp(-|s|(\xi + \eta + |y - \xi|))) d\xi,$$

$$P(y) = \frac{1}{4|s|} \int_0^\infty \frac{d^2}{d\xi^2} \left( \frac{1}{G(\xi)} \right) \exp(-|s|(\xi + |y - \xi|)) d\xi,$$

and the constant of integration  $A$  is to be found from the following integral condition

$$\int_0^\infty \bar{\sigma}(y) \exp(-|s|y) dy = -\frac{\bar{p}}{|s|} - \frac{i\bar{q}}{s}.$$



To solve equation (39), we employ the resolvent-kernel solution technique (Tokovyy & Ma, 2009a) with the following resolvent

$$\aleph(y, \eta) = \sum_{n=0}^{\infty} N_{n+1}(y, \eta),$$

$$N_1(y, \eta) = M(y, \eta), \quad N_{n+1}(y, \eta) = \int_0^{\infty} M(y, \zeta) N_n(\zeta, \eta) d\zeta, \quad n = 1, 2, \dots$$

As a result, the in-plane total stress appears in the form

$$\bar{\sigma}(y) = \bar{p}\Pi(y) + \Theta(y) + Af_A(y), \quad (40)$$

where

$$A = -\frac{1}{f_A(0)} \left( \bar{p} \left( \frac{1}{|s|} + \Pi(0) \right) + \frac{i\bar{q}}{s} + \Theta(0) \right),$$

$$\Pi(y) = E^*(y)P(y) + \int_0^{\infty} E^*(\eta)P(\eta)\aleph(y, \eta)d\eta,$$

$$\Theta(y) = -\alpha^*(y)E^*(y)\bar{T}(y) - \int_0^{\infty} \alpha^*(\eta)E^*(\eta)\bar{T}(\eta)\aleph(y, \eta)d\eta,$$

$$f_A(y) = E^*(y)\exp(-|s|y) + \int_0^{\infty} E^*(\eta)\exp(-|s|\eta)\aleph(y, \eta)d\eta.$$

Having determined the total stress  $\bar{\sigma}$  by formula (40), the stress-tensor components can be computed by means of formulae (38). The displacement-vector components  $u(x, y)$  and  $v(x, y)$ , as well as the boundary displacement  $u_0(x)$  and  $v_0(x)$ , can be also determined by the stresses by means of correct integration of the Cauchy relations (34).

### 3.2.2 Integration of the Cauchy relations and determination of the displacement-vector components in the inhomogeneous semi-plane due to the given boundary tractions

By taking the boundary conditions (36) with unknown boundary displacements  $u_0(x)$  and  $v_0(x)$  into account, the first and second relations of (34) yield

$$\begin{aligned} u(x, y) &= \frac{1}{2} \int_{-\infty}^{\infty} \varepsilon_x(\xi, y) \operatorname{sgn}(x - \xi) d\xi, \\ v(x, y) &= \frac{v_0}{2} + \frac{1}{2} \int_0^{\infty} \varepsilon_y(x, \eta) \operatorname{sgn}(y - \eta) d\eta. \end{aligned} \quad (41)$$

By letting  $x \rightarrow \pm\infty$  in the first equation of (41), we derive the integral condition

$$\int_{-\infty}^{\infty} \varepsilon_x(x, y) dx = 0, \quad (42)$$

which is necessary for compatibility of strains. Analogously, by letting  $y = 0$  in the second equation of (41), the condition

$$\int_0^\infty \varepsilon_y(x, y) dy = -v_0(x) \quad (43)$$

can be obtained. By substitution of expressions (41) into the third formula of (34), we derive following equation

$$\begin{aligned} 2\gamma_{xy}(x, y) - \frac{dv_0(x)}{dx} = & \int_{-\infty}^\infty \frac{\partial \varepsilon_x(\xi, y)}{\partial y} \operatorname{sgn}(x - \xi) d\xi \\ & + \int_0^\infty \frac{\partial \varepsilon_y(x, \eta)}{\partial x} \operatorname{sgn}(y - \eta) d\eta, \end{aligned} \quad (44)$$

which presents the condition of compatibility for strains. It is easy to see that by differentiation by variables  $x$  and  $y$ , equation (44) can be reduced to the classical compatibility equation (32). However, for the equivalence of these two equations, the following fitting condition

$$\frac{dv_0(x)}{dx} = \gamma_{xy}(x, 0) - \frac{1}{2} \int_{-\infty}^\infty \frac{\partial \varepsilon_x(\xi, 0)}{\partial y} \operatorname{sgn}(x - \xi) d\xi \quad (45)$$

is to be fulfilled. This condition is obtained by integration of equation (32) over  $x$  and  $y$  with conditions (36) and (43) in view and comparison of the result to equation (44).

To determine the displacement-vector components, we can employ formulae (41) with conditions (42), (43), and (45) in view. Having applied the Fourier transformation (8) to the mentioned equations, we arrive at the formulae

$$\begin{aligned} \bar{u}(y) &= -\frac{i}{s} \bar{\varepsilon}_x(y), \\ \bar{v}(y) &= \frac{1}{2} \int_0^\infty \bar{\varepsilon}_y(\eta) (\operatorname{sgn}(y - \eta) - 1) d\eta. \end{aligned} \quad (46)$$

Putting the first and second physical relations of (33) along with (38) and (40) into the obtained formulae yields the following expressions:

$$\begin{aligned} \bar{u}(y) &= \bar{p}\Pi_u(y) + \Theta_u(y) + Af_u(y), \\ \bar{v}(y) &= \bar{p}\Pi_v(y) + \Theta_v(y) + Af_v(y), \end{aligned} \quad (47)$$

where

$$\begin{aligned} \Pi_u(y) &= -\frac{i}{s} \left( \frac{\Pi(y)}{E^*(y)} - \frac{|s|}{4G(y)} \int_0^\infty \Pi(\xi) (\exp(-|s||y - \xi|) - \exp(-|s|(y + \xi))) d\xi + \frac{\exp(-|s|y)}{2G(y)} \right), \\ \Theta_u(y) &= -\frac{i}{s} \left( \frac{\Theta(y)}{E^*(y)} - \frac{|s|}{4G(y)} \int_0^\infty \Theta(\xi) (\exp(-|s||y - \xi|) - \exp(-|s|(y + \xi))) d\xi + \alpha^*(y) \bar{T}(y) \right), \end{aligned}$$

$$\begin{aligned}
f_u(y) &= -\frac{i}{s} \left( \frac{f_A(y)}{E^*(y)} - \frac{|s|}{4G(y)} \int_0^\infty f_A(\xi) (\exp(-|s||y-\xi|) - \exp(-|s|(y+\xi))) d\xi \right), \\
\Pi_v(y) &= \frac{1}{2} \int_0^\infty (\operatorname{sgn}(y-\eta) - 1) \left( -\frac{\exp(-|s|\eta)}{2G(\eta)} - \frac{\nu^*(\eta)}{E^*(\eta)} \Pi(\eta) \right. \\
&\quad \left. + \frac{|s|}{4G(y)} \int_0^\infty \Pi(\xi) (\exp(-|s||\eta-\xi|) - \exp(-|s|(\eta+\xi))) d\xi \right) d\eta, \\
\Theta_v(y) &= \frac{1}{2} \int_0^\infty (\operatorname{sgn}(y-\eta) - 1) \left( \alpha^*(\eta) \bar{T}(\eta) - \frac{\nu^*(\eta)}{E^*(\eta)} \Theta(\eta) \right. \\
&\quad \left. + \frac{|s|}{4G(y)} \int_0^\infty \Theta(\xi) (\exp(-|s||\eta-\xi|) - \exp(-|s|(\eta+\xi))) d\xi \right) d\eta, \\
f_v(y) &= -\frac{1}{2} \int_0^\infty (\operatorname{sgn}(y-\eta) - 1) \left( \frac{\nu^*(\eta)}{E^*(\eta)} f_A(\eta) \right. \\
&\quad \left. - \frac{|s|}{4G(y)} \int_0^\infty f_A(\xi) (\exp(-|s||\eta-\xi|) - \exp(-|s|(\eta+\xi))) d\xi \right) d\eta.
\end{aligned}$$

Formulae (47) present the expression for determination of the displacement-vector components in the inhomogeneous semi-plane due to given external tractions  $\bar{p}$  and  $\bar{q}$ , and the temperature field  $\bar{T}(y)$ .

### 3.2.3 One-to-one relations between the tractions and displacements on the boundary

Putting  $y = 0$  into (45) and (46), we obtain the relations

$$\begin{aligned}
\bar{u}_0 &= -\frac{i}{s} \bar{\varepsilon}_x(0), \\
is\bar{v}_0 &= \bar{\varepsilon}_{xy}(0) + \frac{i}{s} \frac{d\bar{\varepsilon}_x(0)}{dy}.
\end{aligned}$$

Having substituted the corresponding physical relations (33) into the latter relations, we arrive at the following one-to-one relations

$$\begin{aligned}
\bar{u}_0 &= a_{11}\bar{p} + a_{12}\bar{q} + b_1, \\
\bar{v}_0 &= a_{21}\bar{p} + a_{22}\bar{q} + b_2
\end{aligned} \tag{48}$$

between the tractions and displacements on the boundary of semi-plane  $D$ . Here

$$a_{11} = \frac{i}{s} \left( \frac{1}{|s|E^*(0)} - \frac{1}{2G(0)} \right), \quad a_{12} = -\frac{1}{s^2 E^*(0)}, \quad b_1 = -\frac{i}{s} \alpha^*(0) \bar{T}(0),$$

$$\begin{aligned}
a_{21} = & \frac{1}{s^2} \frac{d}{dy} \left( -\frac{\Pi(y)}{E^*(y)} + \left( \frac{1}{|s|} + \Pi(0) \right) \frac{f_A(y)}{f_A(0)E^*(y)} - \frac{\exp(-|s|y)}{2G(y)} \right. \\
& \left. + \frac{|s|}{4G(y)} \int_0^\infty \left( \Pi(\xi) - \left( \frac{1}{|s|} + \Pi(0) \right) \frac{f_A(\xi)}{f_A(0)} \right) (\exp(-|s||y-\xi|) - \exp(-|s|(y+\xi))) d\xi \right) \Bigg|_{y=0}, \\
a_{21} = & \frac{i}{sG(0)} + \frac{i}{s^2 f_A(0)} \frac{d}{dy} \left( \frac{1}{s} \frac{f_A(y)}{E^*(y)} \right. \\
& \left. - \frac{|s|}{4sG(y)} \int_0^\infty f_A(\xi) (\exp(-|s||y-\xi|) - \exp(-|s|(y+\xi))) d\xi \right) \Bigg|_{y=0}, \\
b_2 = & \frac{1}{s^2} \frac{d}{dy} \left( \frac{\Theta(0)}{E^*(y)} \frac{f_A(y)}{f_A(0)} - \frac{\Theta(y)}{E^*(y)} + \alpha^*(y) \bar{T}(y) \right. \\
& \left. + \frac{|s|}{4G(y)} \int_0^\infty \left( \Theta(\xi) - \frac{\Theta(0)}{f_A(0)} f_A(\xi) \right) (\exp(-|s||y-\xi|) - \exp(-|s|(y+\xi))) d\xi \right) \Bigg|_{y=0}.
\end{aligned}$$

The obtained expressions of (48) allow us to determine the displacements on the boundary through the given tractions, and vice-versa.

### 3.2.4 Case B: Boundary condition in terms of displacement

Consider the problem of thermoelasticity (31) - (34), (36), where the boundary displacements  $u_0(x)$  and  $v_0(x)$  are given, meanwhile, the corresponding boundary tractions  $p(x)$  and  $q(x)$  are to be determined. By solving (48) with respect to  $\bar{p}$  and  $\bar{q}$ , we find the transforms of tractions on the boundary through the displacements as

$$\begin{aligned}
\bar{p} = & \frac{a_{22}}{\omega} \bar{u}_0 - \frac{a_{12}}{\omega} \bar{v}_0 + \frac{a_{12}b_2 - a_{22}b_1}{\omega}, \\
\bar{q} = & -\frac{a_{21}}{\omega} \bar{u}_0 + \frac{a_{11}}{\omega} \bar{v}_0 + \frac{a_{21}b_1 - a_{11}b_2}{\omega},
\end{aligned} \tag{49}$$

where  $\omega = a_{11}a_{22} - a_{12}a_{21}$ . Having determined the boundary tractions (49), we can find the stress-tensor components by formulae (38), (40), and the displacement-vector components by formulae (47).

### 3.2.5 Case C: Solution of the problem with mixed boundary conditions

Finally, we consider the thermoelasticity problem (31) - (34) in the semi-plane  $D$ , when mixed boundary conditions of either the type (37) are imposed on the boundary. For four versions of the mixed boundary conditions (37), by making use of one of the relations (48), we express the Fourier transform of the unknown traction in terms of the given functions on the boundary and the temperature; inserting the expression into (38) and (40), we calculate the stresses and eventually the displacements by formula (47).

#### 4. Conclusions

Using the method of direct integration, the explicit-form analytical solutions have been found for the equations of in-plane heat conduction and plane thermoelasticity problems in an inhomogeneous semi-plane provided the tractions, displacement, and mixed conditions are prescribed on the boundary. Due to the fact that the application of technique for reducing the aforementioned equations to the governing Volterra-type integral equations with further employment of the resolvent-kernel solution algorithm provides us with the explicit-form solutions of the thermoelasticity problems, the one-to-one relations between the tractions and the displacements on the boundary of the semi-plane are derived. Making use of these relations, we have reduced quasi-static boundary value problems of the plane theory of thermoelasticity with displacement or mixed boundary conditions to the solution of the problem when the tractions are prescribed on the boundary. Application of this technique does not impose any restrictions for the functions prescribing the material properties (besides existence of corresponding derivatives, at least, in generalized sense). But from mechanical point of view, it can be concluded, that the material properties should be in agreement with the model of continua mechanics assuring strain-energy within the necessary restrictions.

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## **Heat Conduction - Basic Research**

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The content of this book covers several up-to-date approaches in the heat conduction theory such as inverse heat conduction problems, non-linear and non-classic heat conduction equations, coupled thermal and electromagnetic or mechanical effects and numerical methods for solving heat conduction equations as well. The book is comprised of 14 chapters divided into four sections. In the first section inverse heat conduction problems are discussed. The first two chapters of the second section are devoted to construction of analytical solutions of nonlinear heat conduction problems. In the last two chapters of this section wavelike solutions are attained. The third section is devoted to combined effects of heat conduction and electromagnetic interactions in plasmas or in pyroelectric material elastic deformations and hydrodynamics. Two chapters in the last section are dedicated to numerical methods for solving heat conduction problems.

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