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Robustness of Feedback Linear Time-Varying Systems: A Commutant Lifting Approach

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1. Introduction

There have been numerous attempts in the literature to generalize results in robust control theory (42; 45) to linear time-varying (LTV) systems (for e.g. (10–13; 30; 33; 37; 39; 40) and references therein). In (12)(13) and (11) the authors studied the optimal weighted sensitivity minimization problem, the two-block problem, and the model-matching problem for LTV systems using inner-outer factorization for positive operators. Abstract solutions involving the computation of induced operators norms of operators are obtained. However, there is no clear indication on how to compute optimal linear LTV controllers.

In (40) the authors rely on state space techniques which lead to algorithms based on infinite dimensional operator inequalities which are difficult to solve. These methods lead to suboptimal controllers and are restricted to finite dimensional systems. An extension of these results to uncertain systems is reported in (41) relying on uniform stability concepts. In (9) both the sensitivity minimization problem in the presence of plant uncertainty, and robust stability for LTV systems in the ℓ^∞ induced norm is considered. However, their methods could not be extended to the case of systems operating on finite energy signals. In (37) the standard problem of H^∞ control theory for finite-dimensional LTV continuous-time plants is considered. It is shown that a solution to this problem exists if and only if a pair of matrix Riccati differential equations admits positive semidefinite stabilizing solutions. State-space formulae for one solution to the problem are also given.

The gap metric was introduced to study stability robustness of feedback systems. It induces the weakest topology in which feedback stability is robust (6; 7; 31; 32; 38). Extensions of the gap to time-varying systems have been proposed in (33; 34) where a geometric framework was developed. Several results on the gap metric and the gap topology were established, in particular, the concept of a graphable subspace was introduced. In (21) the problem of robust stabilization for LTV systems subject to time-varying normalized coprime factor uncertainty is considered. Operator theoretic results which generalize similar results known to hold for linear time-invariant (infinite-dimensional) systems are developed. In particular, a tight upper bound for the maximal achievable stability margin under TV normalized coprime factor uncertainty in terms of the norm of an operator with a time-varying Hankel structure is computed.

Analysis of time-varying control strategies for optimal disturbance rejection for known time-invariant plants has been studied in (2; 16). A robust version of these problems was

considered in (8; 15) in different induced norm topologies. All these references showed that for *time-invariant nominal* plants and weighting functions, time-varying control laws offer no advantage over time-invariant ones.

In this paper, we are interested in optimal disturbance rejection for (*possibly infinite-dimensional*, i.e., systems with an infinite number of states) LTV systems. These systems have been used as models in computational linear algebra and in a variety of computational and communication networks (17). This allows variable number of states which is predominant in networks which can switch on or off certain parts of the system (17), and infinite number of states as in distributed parameter systems.

Using inner-outer factorizations as defined in (3; 11) with respect of the nest algebra of lower triangular (causal) bounded linear operators defined on ℓ^2 we show that the problem reduces to a distance minimization between a special operator and the nest algebra. The inner-outer factorization used here holds under weaker assumptions than (12; 13), and in fact, as pointed in ((3) p. 180), is different from the factorization for positive operators used there.

The optimal disturbance attenuation for LTV systems has been addressed using Banach space duality theory in (20; 28). Its robust version which deals with plant uncertainty is addressed in (4; 5; 19) using also duality theory ideas. Furthermore, using the commutant lifting theorem for nest algebras the optimum is shown to be equal to the norm of a compact time-varying Hankel operator defined on the space of causal Hilbert-Schmidt operators. The latter is the "natural" analogous to the Hankel operator used in the LTI case. An operator identity to compute the optimal TV Youla parameter is also provided.

The results are generalized to the mixed sensitivity problem for TV systems as well, where it is shown that the optimum is equal to the operator induced of a TV mixed Hankel-Toeplitz operator generalizing analogous results known to hold in the linear time-invariant (LTI) case (22; 38; 43).

Our approach is purely input-output and does not use any state space realization, therefore the results derived here apply to *infinite dimensional LTV systems*, i.e., TV systems with an infinite number of state variables (33). Although the theory is developed for causal stable system, it can be extended in a straightforward fashion to the unstable case using coprime factorization techniques for LTV systems discussed in (11; 13).

The rest of the chapter is organized as follows. Section 2 the commutant lifting theorem for nest algebras is introduced. In section 3 the optimal disturbance rejection problem is formulated and solved in terms of a TV Hankel operator. A Generalization to the TV mixed sensitivity problem is carried out in section 4. Section 5 contains some concluding remarks.

Definitions and notation

- $\mathcal{B}(E,F)$ denotes the space of bounded linear operators from a Banach space E to a Banach space F, endowed with the operator norm

$$\|A\| := \sup_{x\in E,\ \|x\|\leq 1} \|Ax\|,\ \ A \in \mathcal{B}(E,F)$$

- ℓ^2 denotes the usual Hilbert space of square summable sequences with the standard norm

$$\|x\|_2^2 := \sum_{j=0}^{\infty} |x_j|^2,\ \ x := (x_0, x_1, x_2, \cdots) \in \ell^2$$

- P_k the usual truncation operator for some integer k, which sets all outputs after time k to zero.
- An operator $A \in \mathcal{B}(E,F)$ is said to be causal if it satisfies the operator equation:

$$P_k A P_k = P_k A, \ \forall k \ \text{ positive integers}$$

- $tr(\cdot)$ denotes the trace of its argument.

The subscript "$_c$" denotes the restriction of a subspace of operators to its intersection with causal (see (11; 29) for the definition) operators. "$\oplus$" denotes for the direct sum of two spaces. "$\star$" stands for the adjoint of an operator.

2. The commutant lifting theorem

The commutant lifting theorem has been proposed by Sz.Nagy and Foias (35; 36). It has been used successfully to solve several interpolation problems including H^∞ control problems for linear time invariant (LTI) systems (31; 32; 43; 44). In this chapter, we rely on a time-varying version of the commutant lifting theorem which corresponds to nest or triangular algebras. Following (3; 18) a nest $\mathcal{N}$ of a Hilbert space $\check{H}$ is a family of closed subspaces of $\check{H}$ ordered by inclusion. The triangular or nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all operators T such that $TN \subseteq N$ for every element N in $\mathcal{N}$. A representation of $\mathcal{T}(\mathcal{N})$ is an algebra homomorphism h from $\mathcal{T}(\mathcal{N})$ into the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. A representation is contractive if $\|h(A)\| \leq \|A\|$, for all $A \in \mathcal{T}(\mathcal{N})$. It is weak* continuous if $h(A_i)$ converges to zero in the weak* topology of $\mathcal{B}(\mathcal{H})$ whenever the net $\{A_i\}$ converges to zero in the weak* topology of $\mathcal{B}(\check{H})$. The representation h is said to be unital if $h(I_{\check{H}}) = I_{\mathcal{H}}$, where $I_{\check{H}}$ is the identity operator on $\check{H}$, and $I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$. The Sz. Nagy Theorem asserts that any such a representation h has a $\mathcal{B}(\check{H})$-dilation, that is, there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and a positive representation H of $\mathcal{B}(\check{H})$ such that $P_{\mathcal{H}} H(A) \mid_{\mathcal{H}} = h(A)$, where $P_{\mathcal{H}}$ is the orthogonal projection from $\mathcal{K}$ into $\mathcal{H}$ (3; 18).
We now state the commutant lifting theorem for nest algebras from (3; 18) (see also references therein).

Theorem 1. *(3; 18) Let*

$$h: \ \mathcal{T}(\mathcal{N}) \longmapsto \mathcal{B}(\mathcal{H})$$
$$h^\star: \mathcal{T}(\mathcal{N}) \longmapsto \mathcal{B}(\mathcal{H}')$$

be two unital weak* continuous contractive representations with $\mathcal{B}(\check{H})$-dilations

$$H: \ \mathcal{B}(\check{H}) \longmapsto \mathcal{B}(\mathcal{K})$$
$$H^\star: \mathcal{B}(\check{H}) \longmapsto \mathcal{B}(\mathcal{K}')$$

respectively. Assume that $X: \ \mathcal{H} \longmapsto \mathcal{H}'$ is a linear operator with $\|X\| \leq 1$, such that $Xh(A) = h'(A)X$ for all $A \in \mathcal{T}(\mathcal{N})$, that is, X intertwines h and h'. Then there exists an operator $Y: \ \mathcal{K} \longmapsto \mathcal{K}'$ such that

i) $\|Y\| \leq 1$.

ii) *Y intertwines H and H', that is, $YH(A) = H'(A)Y$ for all $A \in \mathcal{B}(\check{H})$.*

iii) *Y dilates X, that is, $Y : \mathcal{M} \longmapsto \mathcal{M}'$, and $P_{\mathcal{H}'} Y \mid_{\mathcal{M}} = X P_{\mathcal{H}} \mid_{\mathcal{M}}$, where $\mathcal{H} = \mathcal{M} \ominus \mathcal{N}$ is the orthogonal representation of $\mathcal{H}$ as the orthogonal difference of invariant subspaces for $H \mid_{\mathcal{T}(\mathcal{N})}$, and similarly for $\mathcal{H}'$.*

In the next section the optimal disturbance rejection problem is formulated and solved using this Theorem in terms of a TV Hankel operator.

3. Time-varying optimal disturbance rejection problem

In this chapter, we first consider the problem of optimizing performance for causal linear time varying systems by considering the standard block diagram for the optimal disturbance attenuation problem represented in Fig. 1, where u represents the control inputs, y the measured outputs, z is the controlled output, w the exogenous perturbations. P denotes a causal stable linear time varying plant, and K denotes a time varying controller. The

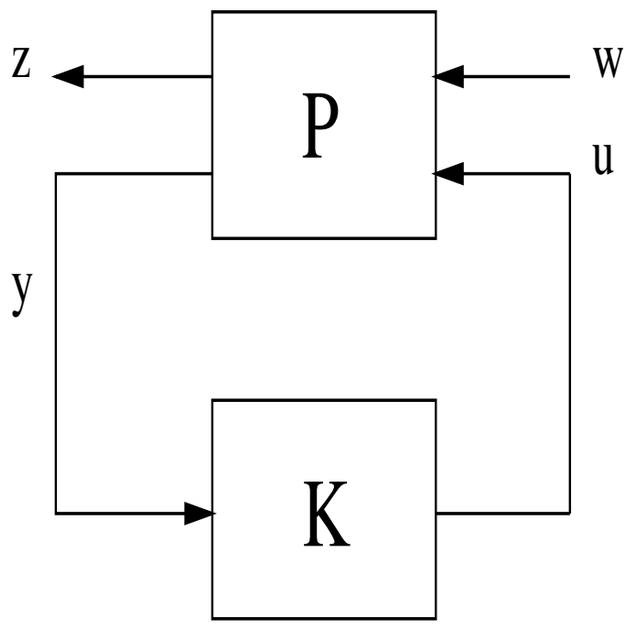


Fig. 1. Block Diagram for Disturbance Rejection

closed-loop transmission from w to z is denoted by T_{zw}. Using the standard TV Youla parametrization of all stabilizing controllers the closed loop operator T_{zw} can be written as (2; 11; 16),

$$T_{zw} = T_1 - T_2 Q T_3 \tag{1}$$

where T_1, T_2 and T_3 are stable causal LTV operators, that is, T_1, T_2 and $T_3 \in B_c(\ell^2, \ell^2)$. Here it is assumed without loss of generality that P is stable, the Youla parameter $Q := K(I + PK)^{-1}$ is then an operator belonging to $B_c(\ell^2, \ell^2)$, and is related in a one-to-one onto fashion to the controller K (29). Note that Q is allowed to be *time-varying*. If P is unstable it suffices to use the coprime factorization techniques in (11; 39) which lead to similar results. The magnitude of the signals w and z is measured in the ℓ^2-norm. The performance index which quantifies

optimal disturbance rejection can be written in the following form (20)

$$\begin{aligned}\mu &:= \inf\{\|T_{zw}\| : \ K \ \text{being robustly stabilizing linear time} - \text{varying controller}\} \\ &= \inf_{Q\in\mathcal{B}_c(\ell^2,\ell^2)} \|T_1 - T_2 Q T_3\| \end{aligned} \tag{2}$$

The performance index (2) will be transformed into a shortest distance minimization between a certain bounded linear operator and a subspace to be specified shortly. In order to do so, following (11) define a nest $\mathcal{N}$ as a family of closed subspaces of the Hilbert space ℓ^2 containing $\{0\}$ and ℓ^2 which is closed under intersection and closed span. Let $Q_n := I - P_n$, for $n = -1, 0, 1, \cdots$, where $P_{-1} := 0$ and $P_\infty := I$. Then Q_n is a projection, and we can associate to it the following nest $\mathcal{N} := \{Q_n\ell^2, \ n = -1, 0, 1, \cdots\}$. In this case the triangular or nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all operators T such that $TN \subseteq N$ for every element N in $\mathcal{N}$. That is

$$\begin{aligned}\mathcal{T}(\mathcal{N}) &= \{A \in \mathcal{B}(\ell^2,\ell^2) : \ P_n A(I - P_n) = 0, \ \forall\, n\} \\ &= \{A \in \mathcal{B}(\ell^2,\ell^2) : \ (I - Q_n)AQ_n = 0, \ \forall\, n\} \end{aligned} \tag{3}$$

Note that the Banach space $\mathcal{B}_c(\ell^2,\ell^2)$ is identical to the nest algebra $\mathcal{T}(\mathcal{N})$. For N belonging to the nest $\mathcal{N}$, N has the form $Q_n\ell^2$ for some n. Define

$$N^- = \bigvee\{N' \in \mathcal{N} : N' < N\} \tag{4}$$

$$N^+ = \bigwedge\{N' \in \mathcal{N} : N' > N\} \tag{5}$$

where $N' < N$ means $N' \subset N$, and $N' > N$ means $N' \supset N$. The subspaces $N \ominus N^-$ are called the atoms of $\mathcal{N}$. Since in our case the atoms of $\mathcal{N}$ span ℓ^2, then $\mathcal{N}$ is said to be atomic (3).
The early days of H^∞ control theory saw solutions based on the so-called inner-outer factorizations of functions belonging to the Hardy spaces H^2 and H^∞, and their corresponding matrix valued counterparts for multi-input multi-output (MIMO) systems (22; 23). Generalizations in the context of nest algebras have been proposed in (1; 3) as follows: An operator A in $\mathcal{T}(\mathcal{N})$ is called *outer* if the range projection $P(R_A)$, R_A being the range of A and P the orthogonal projection onto R_A, commutes with $\mathcal{N}$ and AN is dense in $N \cap R_A$ for every $N \in \mathcal{N}$. A partial isometry U is called *inner* in $\mathcal{T}(\mathcal{N})$ if $U^\star U$ commutes with $\mathcal{N}$ (1; 3; 11). In our case, $A \in \mathcal{T}(\mathcal{N}) = \mathcal{B}_c(\ell^2,\ell^2)$ is outer if P commutes with each Q_n and $AQ_n\ell^2$ is dense in $Q_n\ell^2 \cap A\ell^2$. $U \in \mathcal{B}_c(\ell^2,\ell^2)$ is inner if U is a partial isometry and $U^\star U$ commutes with every Q_n. Applying these notions to the time-varying operator $T_2 \in \mathcal{B}_c(\ell^2, \ \ell^2)$, we get $T_2 = T_{2i}T_{2o}$, where T_{2i} and T_{2o} are inner outer operators in $\mathcal{B}_c(\ell^2,\ell^2)$, respectively. Similarly, the operator T_3 can be factored as $T_3 = T_{3o}T_{3i}$ where $T_{3i} \in \mathcal{B}_c(\ell^2,\ell^2)$ is inner, $T_{3o} \in \mathcal{B}_c(\ell^2,\ell^2)$ is outer. The performance index μ in (2) can then be written as

$$\mu = \inf_{Q\in\mathcal{B}_c(\ell^2,\ell^2)} \|T_1 - T_{2i}T_{2o}QT_{3o}T_{3i}\| \tag{6}$$

Following the classical H^∞ control theory (22; 23; 45),we assume
(A1) that T_{2o} and T_{3o} are invertible both in $\mathcal{B}_c(\ell^2,\ell^2)$.
Assumption (A1) can be relaxed by assuming instead that the outer operators T_{2o} and T_{3o} are bounded below (see Lemma (1) p. 220). Assumption (A1) guarantees that the map $Q \longrightarrow T_{2o}\mathcal{B}_c(\ell^2,\ell^2)T_{3o}$ is bijective. Under this assumption T_{2i} becomes an isometry and T_{3i}

a co-isometry in which case $T_{2i}^{\star}T_{2i} = I$ and $T_{3i}T_{3i}^{\star} = I$. The operators T_{2o} and T_{3o} can be absorbed into the Youla operator Q, and expression (6) is then equivalent to

$$\mu = \inf_{Q \in \mathcal{B}_c(\ell^2,\ell^2)} \|T_{2i}^{\star}T_1T_{3i}^{\star} - Q\| \tag{7}$$

Expression (7) is the distance from the operator $T_{2i}^{\star}T_1T_{3i}^{\star} \in \mathcal{B}(\ell^2,\ell^2)$ to the nest algebra $\mathcal{B}_c(\ell^2,\ell^2)$. It is the shortest distance from the bounded linear operator $T_{2i}^{\star}T_1T_{3i}^{\star}$ to the space of causal bounded linear operators $\mathcal{B}_c(\ell^2,\ell^2)$, which is a subspace of $\mathcal{B}(\ell^2,\ell^2)$. In the sequel, the commutant lifting theorem is used to solve the minimization (7) in terms of a time varying version of Hankel operators.
First, let $\mathcal{C}_2$ denote the special class of compact operators on ℓ^2 called the Hilbert-Schmidt or Schatten 2-class (3; 14) under the norm,

$$\|A\|_2 := \Big(tr(A^{\star}A)\Big)^{\frac{1}{2}} \tag{8}$$

Note that $\mathcal{C}_2$ is a Hilbert space under the inner product (3)

$$(A,B) = tr(B^{\star}A), \quad \forall\ A,\ B\ \in\ \mathcal{C}_2 \tag{9}$$

Define the space

$$\mathcal{A}_2 := \mathcal{C}_2 \cap \mathcal{B}_c(\ell^2,\ell^2) \tag{10}$$

Then $\mathcal{A}_2$ is the space of causal Hilbert-Schmidt operators. This space can be viewed as the TV counterpart of the standard Hardy space H^2 in the standard H^{∞} theory. Define the orthogonal projection $\mathcal{P}$ of $\mathcal{C}_2$ onto $\mathcal{A}_2$. $\mathcal{P}$ is the lower triangular truncation, and is analogous to the standard positive Riesz projection (for functions on the unit circle) for the LTI case.
Following (27) an operator X in $\mathcal{B}(\ell^2,\ell^2)$ determines a Hankel operator H_X on $\mathcal{A}_2$ if

$$H_XA = (I-\mathcal{P})XA, \quad \text{for } A \in \mathcal{A}_2 \tag{11}$$

We shall show that the shortest distance μ is equal to the norm of a particular LTV Hankel operator using the time varying version of the commutant lifting theorem in Theorem 1, thus generalizing a similar result in the LTI setting. let H_B be the Hankel operator $(I-\mathcal{P})B\mathcal{P}$ associated with the symbol $B := T_{2i}^{\star}T_1T_{3ci}^{\star}$. The Hankel operator H_B belongs to the Banach space of bounded linear operators on $\mathcal{C}_2$, namely, $\mathcal{B}(\mathcal{C}_2,\mathcal{C}_2)$. We have then the following Theorem which relates the optimal distance minimization μ to the induced norm of the Hankel operator $H_{T_{2i}^{\star}T_1T_{3ci}^{\star}}$.

Theorem 2. *Under assumptions (A1) the following holds:*

$$\mu = \|H_{T_{2i}^{\star}T_1T_{3ci}^{\star}}\| = \|(I-\mathcal{P})T_{2i}^{\star}T_1T_{3ci}^{\star}\mathcal{P}\| \tag{12}$$

Proof. Following (3; 18) let $H_1 = \mathcal{A}_2$ and $H_2 = \mathcal{C}_2 \ominus \mathcal{A}_2$ the orthogonal complement of $\mathcal{A}_2$ in $\mathcal{C}_2$. Define the representations h and h' of $\mathcal{A}_2$ by

$$h(A) = R_A\,|_{H_1}, \quad A \in \mathcal{B}_c(\ell^2,\ell^2) \tag{13}$$

$$h'(A) = (I-\mathcal{P})R_{A|H_2}, \quad A \in \mathcal{B}_c(\ell^2,\ell^2) \tag{14}$$

where now R_A denotes the right multiplication associated to the operator A defined on the specified Hilbert space, i.e., $R_A B = BA$, $B \in \mathcal{A}_2$. The representation $h(\cdot)$ and $h'(\cdot)$ have dilations $H = H'$ given by

$$H(A) = H'(A) = R_A \text{ on } \mathcal{C}_2, \quad A \in \mathcal{B}_c(\ell^2, \ell^2) \tag{15}$$

$$\tag{16}$$

Let $M := \mathcal{B}_c(\ell^2, \ell^2)$, $N = \{0\}$, $M' := \mathcal{C}_2$, $N' := \mathcal{A}_2$, and $H_1 = M \ominus N$, $H_2 = M' \ominus N'$ are orthogonal representations of H_1 and H_2 of invariant subspaces under $H|_{\mathcal{B}_c(\ell^2,\ell^2)}$, that is, $R_A \mathcal{B}_c(\ell^2, \ell^2) \subset \mathcal{B}_c(\ell^2, \ell^2)$. Now we have to show that the operator $H_{T_{2i}^\star T_1 T_{3ci}^\star}$ intertwines h and h', that is, if $B := T_{2i}^\star T_1 T_{3ci}^\star$, then $h'(A)H_B = H_B h(A)$ holds for for all $A \in \mathcal{B}_c(\ell^2, \ell^2)$,

$$h'(A)H_B = (I - \mathcal{P})R_A(I - \mathcal{P})B \mid_{\mathcal{A}_2} = (I - \mathcal{P})R_A B \mid_{\mathcal{A}_2} \tag{17}$$

$$= (I - \mathcal{P})BR_A \mid_{\mathcal{A}_2} = (I - \mathcal{P})B\mathcal{P}R_A \mid_{\mathcal{A}_2} = H_B h(A) \tag{18}$$

Applying the Commutant Lifting Theorem for representations of nest algebras implies that H_B has a dilation $\tilde{H}_B$ that intertwines H and H', i.e., $\tilde{H}_B H(A) = H'(A)\tilde{H}_B$, $\forall A \in \mathcal{B}(\ell^2, \ell^2)$. By Lemma 4.4. in (18) $\tilde{H}_B$ is a left multiplication operator acting from $\mathcal{A}_2$ into $\mathcal{C}_2 \ominus \mathcal{A}_2$. That is, $\tilde{H}_B = L_K$ for some $K \in \mathcal{B}(\ell^2, \ell^2)$, with $\|L_K\| = \|K\| = \|\tilde{H}_X\| = \|H_B\|$ by Lemma 4.5. (18). By Lemma 4.3. (18) $K = B - Q$, $\exists Q \in \mathcal{B}_c(\ell^2, \ell^2)$ with $\|K\| = \|H_B\|$ as required.
By Theorem 2.1. (26) the Hankel operator H_B is a compact operator if and only if B belongs to the space $\mathcal{B}_c(\ell^2, \ell^2) + \mathcal{K}$, where $\mathcal{K}$ is the space of compact operators on the Hilbert space ℓ^2. A basic property of compact operators on Hilbert spaces is that they have maximizing vectors, that is, there exists at least one operator $A^o \in \mathcal{A}_2$, $\|A^o\|_2 = 1$ such that H_B achieves its induced norm at A^o. That is,

$$\|H_B A^o\|_2 = \|H_B\|\|A^o\|_2 = \|H_B\| \tag{19}$$

We can then deduce from (7) and (12) an operator identity for the minimizer, that is, the optimal TV Youla parameter Q_o as follows

$$Q_o A^o = T_{2i}^\star T_1 T_{3ci}^\star A^o - H_{T_{2i}^\star T_1 T_{3ci}^\star} A^o$$

where the unknown is Q_o.
In the next section the mixed sensitivity problems for LTV systems is formulated and solved using the commutant lifting theorem.

4. The time-varying mixed sensitivity problem

The mixed sensitivity problem for stable plants (42; 45) involves the sensitivity operator $T_1 := \begin{pmatrix} W \\ 0 \end{pmatrix}$, the complementary sensitivity operator $T_2 = \begin{pmatrix} W \\ V \end{pmatrix} P$ and $T_3 := I$ which are all assumed to belong to $\mathcal{B}_c(\ell^2, \ell^2 \times \ell^2)$, and is given by the minimization (13; 20; 46)

$$\mu_o = \inf_{Q \in \mathcal{B}_c(\ell^2,\ell^2)} \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - \begin{pmatrix} W \\ V \end{pmatrix} PQ \right\| \tag{20}$$

where $\|\cdot\|$ stands for the operator norm in $\mathcal{B}(\ell^2,\ \ell^2 \times \ell^2)$ given by

$$\|B\| = \sup_{\|x\|_2 \le 1,\ x \in \ell^2} \left(\|B_1 x\|_2^2 + \|B_2 x\|_2^2 \right)^{\frac{1}{2}}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{21}$$

The optimization problem (20) can be expressed as a shortest distance problem from the operator T_1 to the subspace $\mathcal{S} = T_2 P\ \mathcal{B}_c(\ell^2, \ell^2)$ of $\mathcal{B}(\ell^2,\ \ell^2 \times \ell^2)$.
To ensure closedness of $\mathcal{S}$, we assume that $W^\star W + V^\star V > 0$, i.e., $W^\star W + V^\star V$ as an operator acting on ℓ^2 is a positive operator. In this case, there exists an outer spectral factorization $\Lambda_1 \in \mathcal{B}_c(\ell^2, \ell^2)$, invertible in $\mathcal{B}_c(\ell^2, \ell^2)$ such that $\Lambda_1^\star \Lambda_1 = W^\star W + V^\star V$ (1; 11). Consequently, $\Lambda_1 P$ as a bounded linear operator in $\mathcal{B}_c(\ell^2, \ell^2)$ has an inner-outer factorization $U_1 G$, where U_1 is inner and G an outer operator defined on ℓ^2 (3).
Next we assume (**A2**) G is invertible, so U_1 is unitary, and the operator G and its inverse $G^{-1} \in \mathcal{B}_c(\ell^2, \ell^2)$. The assumption (A2) is satisfied when, for e.g., the outer factor of the plant is invertible. Let $R := T_2 \Lambda_1^{-1} U_1$, assumption (A2) implies that the operator $R^\star R \in \mathcal{B}(\ell^2, \ell^2)$ has a bounded inverse, this ensures closedness of $\mathcal{S}$. It follows from Corollary 2 (1), that the self-adjoint operator $R^\star R$ has a spectral factorization of the form: $R^\star R = \Lambda^\star \Lambda$, where $\Lambda,\ \Lambda^{-1} \in\ \mathcal{B}_c(\ell^2, \ell^2)$.
Define the operator $R_2 := R\Lambda^{-1}$, then $R_2^\star R_2 = I$, and $\mathcal{S}$ has the equivalent representation, $\mathcal{S} = R_2 \mathcal{B}_c(\ell^2, \ell^2)$. After "absorbing" Λ into the free parameter Q, the optimization problem (20) is then equivalent to:

$$\mu_o = \inf_{Q \in \mathcal{B}_c(\ell^2, \ell^2)} \|T_1 - R_2 Q\| \tag{22}$$

The minimization problem (22) gives the optimal mixed sensitivity with respect to controller design (as represented by Q). It is solved in terms of a projection of a multiplication operator. If the minimization (22) is achieved by a particular Q_o, we call it optimal.

Theorem 3. *Introduce the orthogonal projection* Π *as follows*

$$\Pi\ :\ \mathcal{A}_2 \oplus \mathcal{A}_2 \longmapsto (\mathcal{A}_2 \oplus \mathcal{A}_2) \ominus R_2 \mathcal{A}_2$$

Under assumptions (A2) the following holds:

$$\mu_o = \|\Pi T_1\| \tag{23}$$

Proof. Denote by $S := (\mathcal{A}_2 \oplus \mathcal{A}_2) \ominus R_2 \mathcal{A}_2$. That is, S is the orthogonal complement of the subspace $R_2 \mathcal{A}_2$ in $\mathcal{A}_2 \oplus \mathcal{A}_2$, and define the operator

$$\begin{aligned} \Xi\ &:\ \mathcal{A}_2 \longmapsto S \\ \Xi &:= \Pi T_1 \end{aligned} \tag{24}$$

We shall show with the help of the commutant lifting theorem that

$$\mu_o = \|\Xi\| \tag{25}$$

To see this we need, as before, a representation of $\mathcal{B}_c(\ell^2, \ell^2)$, that is, an algebra homomorphism, say, $h(\cdot)$ (respectively $h'(\cdot)$), from $\mathcal{B}_c(\ell^2, \ell^2)$, into the algebra $B(\mathcal{A}_2, \mathcal{A}_2)$ (respectively $\mathcal{B}_c(S, S)$), of bounded linear operators from $\mathcal{A}_2$ into $\mathcal{A}_2$ (respectively from S into S). Define the representations h and h' by

$$\begin{aligned} h:\ \ \mathcal{B}_c(\ell^2, \ell^2) &\longmapsto B(\mathcal{A}_2, \mathcal{A}_2),\ \ h' : \mathcal{B}_c(\ell^2, \ell^2)) \longmapsto \mathcal{B}_c(S, S) \\ h(A) &:= R_A,\ \ A \in \mathcal{B}_c(\ell^2, \ell^2),\ \ h'(A) := \Pi R_A,\ \ A \in \mathcal{B}_c(\ell^2, \ell^2)) \end{aligned} \tag{26}$$

where now R_A denotes the right multiplication associated to the operator A defined on the specified Hilbert space. By the Sz. Nagy dilation Theorem there exist dilations H (respectively H') for h (respectively h') given by

$$H(A) = R_A \text{ on } \mathcal{A}_2 \text{ for } A \in \mathcal{B}_c(\ell^2, \ell^2) \tag{27}$$

$$H'(A) = R_A \text{ on } \mathcal{A}_2 \oplus \mathcal{A}_2 \text{ for } A \in \mathcal{B}_c(\ell^2, \ell^2) \tag{28}$$

The spaces $\mathcal{A}_2$ and S can be written as orthogonal differences of subspaces invariant under H and H', respectively, as

$$\mathcal{A}_2 = \mathcal{A}_2 \ominus \{0\}, \quad S = \mathcal{A}_2 \oplus \mathcal{A}_2 \ominus R_2\mathcal{A}_2 \tag{29}$$

Now we have to show that the operator Ξ intertwines h and h', that is, $h'(A)\Xi = \Xi h(A)$ for all $A \in \mathcal{B}_c(\ell^2, \ell^2)$,

$$\begin{aligned} h'(A)\Xi &= \Pi R_A \Pi T_1 \mid_{\mathcal{A}_2} = \Pi R_A \Pi T_1 \mid_{\mathcal{A}_2} \\ &= \Pi R_A T_1 \mid_{\mathcal{A}_2} = \Pi T_1 R_A \mid_{\mathcal{A}_2} \\ &= \Xi h(A) \end{aligned}$$

Applying the commutant lifting theorem for representations of nest algebras implies that Ξ has a dilation Ξ' that intertwines H and H', i.e., $\Xi' H(A) = H'(A)\Xi'$, $\forall A \in \mathcal{B}(\ell^2, \ell^2)$. By Lemma 4.4. in (18) Ξ' is a left multiplication operator acting from $\mathcal{A}_2$ into $\mathcal{A}_2 \oplus \mathcal{A}_2$, and causal. That is, $\Xi' = L_K$ for some $K \in \mathcal{B}_c(\mathcal{A}_2, \ \mathcal{A}_2 \oplus \mathcal{A}_2)$, with $\|K\| = \|\Xi'\| = \|\Xi\|$. Then $\Xi = \Pi T_1 = \Pi K$, which implies that $\Pi(T_1 - K) = 0$. Hence, $(T_1 - K)f \in R_2\mathcal{A}_2$, for all $f \in \mathcal{A}_2$. That is, $(T_1 - K)f = R_2 g$, $\exists g \in \mathcal{A}_2$, which can be written as $R_2^\star(T_1 - K)f = g \in \mathcal{A}_2$. In particular, $R_2^\star(T_1 - K)f \in \mathcal{B}_c(\ell^2, \ell^2)$, for all $f \in \mathcal{B}_c(\ell^2, \ell^2)$ of finite rank. By Theorem 3.10 (3) there is a sequence F_n of finite rank contractions in $\mathcal{B}_c(\ell^2, \ell^2)$ which converges to the identity operator in the strong *-topology. By an approximation argument it follows that $R_2^\star(T_1 - K) \in \mathcal{B}_c(\ell^2, \ \ell^2)$. Letting $Q := R_2^\star(T_1 - K)$ we have $g = Qf$. We conclude that $T_1 - K = R_2 Q$, that is, $T_1 - R_2 Q = K$, with $\|K\| = \|\Xi\|$, and the Theorem is proved.
The orthogonal projection Π can be computed as

$$\Pi = I - R_2 \mathcal{P} R_2^\star \tag{30}$$

where I is the identity operator on $\mathcal{A}_2 \oplus \mathcal{A}_2$, $R_2^\star$ is the adjoint operator of R_2. To see that (30) holds note that for any $Y \in \mathcal{A}_2 \oplus \mathcal{A}_2$, we have

$$(I - R\mathcal{P}R_2^\star)^2 Y = (I - R\mathcal{P}R_2^\star)(I - R\mathcal{P}R_2^\star)Y \tag{31}$$

$$= (I - R_2\mathcal{P}R_2^\star - R_2\mathcal{P}R_2^\star + R_2\mathcal{P}R_2^\star\mathcal{P}R_2^\star)Y \tag{32}$$

but $R_2^\star R_2 = I$ and $\mathcal{P}^2 = \mathcal{P}$, therefore

$$(I - R\mathcal{P}R_2^\star)^2 Y = (I - R\mathcal{P}R_2^\star)Y \tag{33}$$

This shows that $(I - R\mathcal{P}R_2^\star)$ is a projection. The adjoint of $(I - R\mathcal{P}R_2^\star)$, $(I - R\mathcal{P}R_2^\star)^\star$, is clearly equal to $(I - R\mathcal{P}R_2^\star)$ showing that it is an orthogonal projection. Now we need to show that the null space of $(I - R\mathcal{P}R_2^\star)$ is $R_2\mathcal{A}_2$. Let $Z \in \mathcal{A}_2 \oplus \mathcal{A}_2$ such that $(I - R\mathcal{P}R_2^\star)Z = 0$, so $Z = R_2\mathcal{P}R_2^\star Z$. But $R_2^\star Z \in \mathcal{C}_2$, then $\mathcal{P}R_2^\star Z \in \mathcal{A}_2$, implying that $Z \in R_2\mathcal{A}_2$. We have showed

that the null space of the projection $(I - R\mathcal{P}R_2^\star)$ is a subset of $R_2\mathcal{A}_2$. Conversely, let $Z \in \mathcal{A}_2$, then

$$(I - R\mathcal{P}R_2^\star)R_2Z = R_2Z - R_2\mathcal{P}Z = R_2Z - R_2Z = 0 \tag{34}$$

hence R_2Z belongs to the null space of $(I - R\mathcal{P}R_2^\star)$, and (30) holds.
The operator Ξ has the following explicit form

$$\Xi = (I - R_2\mathcal{P}R_2^\star)T_1 \tag{35}$$

which leads to the explicit solution

$$\mu_o = \|(I - R_2\mathcal{P}R_2^\star)T_1\| \tag{36}$$

The expression generalizes the solution of the mixed sensitivity problem in the LTI case (25; 43; 46) to the LTV case. This result also applies to solve the robustness problem of feedback systems in the gap metric (38) in the TV case as outlined in (11; 21; 33), since the latter was shown in (11) to be equivalent to a special version of the mixed sensitivity problem (20).

5. Conclusion

The optimal disturbance rejection and the mixed sensitivity problems for LTV systems involve solving shortest distance minimization problems posed in different spaces of bounded linear operators. LTV causal and stable systems form a nest algebras, this allows the commutant lifting theorem for nest algebras to be applied and solve both problems in term of abstract TV Hankel and a TV version generalization of Hankel-Topelitz operators under fairly weak assumptions. Future work includes investigation of numerical solutions based on finite dimensional approximations, and computation of the corresponding controllers.

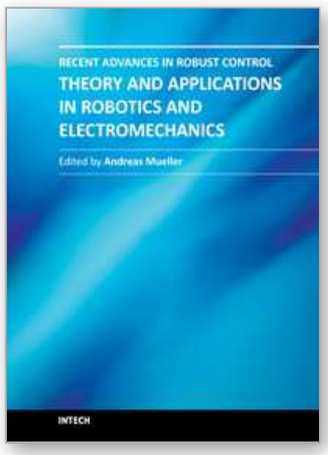
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Robust control has been a topic of active research in the last three decades culminating in H_2/H_\infty and \mu design methods followed by research on parametric robustness, initially motivated by Kharitonov's theorem, the extension to non-linear time delay systems, and other more recent methods. The two volumes of Recent Advances in Robust Control give a selective overview of recent theoretical developments and present selected application examples. The volumes comprise 39 contributions covering various theoretical aspects as well as different application areas. The first volume covers selected problems in the theory of robust control and its application to robotic and electromechanical systems. The second volume is dedicated to special topics in robust control and problem specific solutions. Recent Advances in Robust Control will be a valuable reference for those interested in the recent theoretical advances and for researchers working in the broad field of robotics and mechatronics.

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