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On a Role of Viscosity in Phenomena of Mass Transfer Caused by Nonlinear Periodic Waves Propagating Over the Liquid Surface

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1. Introduction

In the 1847 G. Stokes pointed out in his work [G.G. Stokes, 1847] that the propagation of periodic gravitational waves on the horizontal surface of a liquid induces the total drift of liquid particles at a velocity lower than the phase velocity of waves, which became known as the Stokes drift. Stokes considered the problem of calculation of the velocity field in an infinitely deep inviscid incompressible liquid with a gravitational wave propagating over its surface, the wave amplitude being much smaller than the wavelength. The solution of the problem in the second order approximation in the wave amplitude showed that the periodic wave perturbation of the free surface induces not only a periodic motion of liquid particles about a certain mean position in the bulk of the liquid, but also a horizontal motion of the mean position in the direction of wave propagation.
Stokes derived the following expression for the absolute value of the velocity of this drift [G.G. Stokes, 1847, 1880]:

$$w_S = A^2 k\omega_0 \exp(-2kd). \tag{1}$$

Here, A is the wave amplitude, k is the wavenumber, ω_0 is the circular frequency of oscillatory motion, and d is the depth at which the drift velocity is calculated. Formula (1) was derived for the velocity of the drift induced by the gravitational wave is also valid for a capillary-gravitational wave if we use the following expression for circular frequency ω_0 [Le Blon & Mysak, 1978]:

$$\omega_0 = \sqrt{gk\left(1+\alpha^2 k^2\right)}\ ;\ \alpha = \sqrt{\frac{\gamma}{\rho g}}. \tag{2}$$

Here, α is the capillary constant of the liquid, g is the free-fall acceleration, γ is the surface tension , and ρ is the density of the liquid.
The existence of the Stokes drift was confirmed by various observations and experiments [Le Blon & Mysak 1978; Longuet-Higgens 1953, 1986]. The drift phenomenon induced by a small-amplitude periodic traveling wave is second-order effect in wave amplitude. The rough analysis of the problem in the linear (first-order) approximation in the wave amplitude

reveals only a circulating of liquid particles in the vertical plane around a certain stationary position with period $T = 2\pi/\omega_0$. The radius of the circular trajectory of liquid particles relating to the free surface is estimated as the wave amplitude A. But second-order approximation shows that during the period T liquid particle draws trajectory which is not an exact circle but a disconnected loop. The lower part of the trajectory must be shorter than the upper one by a certain quantity much smaller than the amplitude, for particle motion decays with increasing depth. Consequently, after time T, the particle does not return to the initial position but is slightly shifted relative to this position in the direction of propagation of the wave. With each new period, this displacement is systematically accumulated and adds up into average drift with the velocity defined by formula (1). Liquid particles located not on the free surface but at a certain depth perform analogous movements. With increasing depth, the amplitude of periodic movements, as well as the mean drift velocity, decreases.

In various applications dealing with wave motion on the free surface of a liquid, it is important to take into account the drift flow emerging as a result of propagation of surface waves, which can be responsible for the transport of a surfactants or electric charge distributed on the liquid surface. It is important to note that in the general case, the dynamics of distribution of a certain substance over the free surface of a liquid is controlled by viscous shear stresses [Belonozhko & Grigor'ev, 2004; Belonozhko et. al, 2005] which are disregarded in the Stokes drift model.

In 1953 M.S. Longuet-Higgins supposed an improved model of the mass-transport induced by surface progressive waves propagating over the free surface of a low viscous liquid and currently this model is the main tool for making a various estimations concerning the drift phenomena in a viscous liquid [Longuet-Higgins, 1953]. The base of the model is several auxiliary assumptions corresponding to properties of viscous boundary layer located in the vicinity of the free surface. Thereby the supposed reasoning is suitable only for a low-viscosity limit. The practical employment of Longuet-Higgins's approach is essentially complicated, for the model has somewhat artificiality and a cumbersome structure.

Analytic description of the influence of arbitrary viscous forces on the structure of the drift flow caused by the propagation of waves has not been obtained for more than 150 years after the formulation of the problem because of the absence of an appropriated nonlinear solution to the problem of the arbitrary viscosity influence on the propagation of periodic capillary-gravitational wave. The suited solution was obtained in works [Belonozhko & Grigor'ev, 2003, 2004] only at beginning of XXI century and analytical analysis of the problem had become possible.

2. Determination of the mean drift caused by nonlinear periodic waves propagating over the surface of viscous liquid

We will consider analytic calculations of the velocity field in an infinitely deep and unbounded in horizontal direction incompressible viscous liquid with a periodic capillary-gravitational wave propagating over its free horizontal surface. The solution will be constructed in the second order of smallness in wave amplitude. The main attention will be paint to the details of the solution associated with the appearance of the drift terms. The notations used in the procedure of the solution permit to concentrate efforts only to definition of the drift part of the flow without calculating total expression for the velocity field. The suggested approach makes it possible to effectively analyze more complicated questions concerning to a medium drift induced by the wave motion, for instance an analytic calculation of the mean surface drift of surfactant or redistributing surface electric charge.

2.1 Decomposition of the problem of the velocity field calculating on the problem of the first and second order of smallness in wave amplitude

Let us suppose that an incompressible Newtonian liquid of kinematic viscosity ν, density ρ, and surface tension γ in a Cartesian system of coordinates with the z axis directed vertically upwards fills the half-space $z<0$ in the gravity field $\mathbf{g}$. We disregard the physical properties of the medium above the liquid and consider a periodic capillary-gravitational wave propagating over the free surface of the liquid along the horizontal x axis, assuming that the wave amplitude is much smaller than the wavelength. We also assume for simplicity that the flow of the liquid is independent of horizontal coordinate y. Let us determine the mean velocity of the horizontal drift of the liquid, induced by the propagation of a periodic wave with known amplitude and wavenumber.

We denote by $u=u(t,x,z)$ and $v=v(t,x,z)$ the horizontal and vertical components of the velocity field of the liquid; $\mathbf{e}_x$ and $\mathbf{e}_z$ are the unit vectors along the x and z axis. The deviation of free surface $\xi=\xi(t,x)$ of the liquid from equilibrium state $z=0$, which is associated with the wave motion, and velocity field $\mathbf{U}=u\,\mathbf{e}_x+v\,\mathbf{e}_z$ induced in the liquid satisfy the familiar set of hydrodynamic equations for an incompressible Newtonian liquid and the corresponding boundary conditions [Le Blon & Mysak, 1978; Le Méhauté, 1976]:

$$z<\xi:\quad \frac{\partial \mathbf{U}}{\partial t}+(\mathbf{U}\cdot\nabla)\mathbf{U}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{U};\quad \nabla\cdot\mathbf{U}=0;$$

$$z=\xi:\quad \frac{\partial\xi}{\partial t}+u\frac{\partial\xi}{\partial x}=v;\quad p-2\rho\nu\,\mathbf{n}(\mathbf{n}\cdot\nabla)\mathbf{U}=-\gamma\frac{\partial^2\xi}{\partial x^2}\left(1+\left(\frac{\partial\xi}{\partial x}\right)^2\right)^{-3/2};\qquad (3)$$

$$\boldsymbol{\tau}\cdot\big((\mathbf{n}\cdot\nabla)\mathbf{U}\big)+\mathbf{n}\cdot\big((\boldsymbol{\tau}\cdot\nabla)\mathbf{U}\big)=0;$$

$$z\to-\infty:\quad u\to 0;\quad v\to 0;\quad \nabla\equiv\frac{\partial}{\partial x}\mathbf{e}_x+\frac{\partial}{\partial z}\mathbf{e}_z.$$

Here, $p=p(t,x,z)$ is the pressure in the liquid; $\mathbf{n}=\mathbf{n}(t,x)$ is the unit vector of the outward normal to the free surface constructed from the point of the surface with horizontal coordinate x at instant t; $\boldsymbol{\tau}=\boldsymbol{\tau}(t,x)$ is the unit vector of the tangent to the free surface.

Instead of the initial conditions, which determine in the general case the spectrum of modes of the wave motion generated at the initial instant, we will follow the considerations of the simplest spectral composition of the sought solution. The appropriate approach is traditionally used in similar problems to obtain the least cumbersome solution fittest for analytic description and qualitative analysis [Le Blon & Mysak, 1978; Le Méhauté, 1976; Belonozhko & Grigor'ev 2003, 2004].

Following the standard procedure in the nonlinear theory of periodic waves of small but finite amplitude [Le Blon & Mysak, 1978; Le Méhauté, 1976], we will construct the solution to problem (3) in the form of power expansions of unknown quantities in the small parameter equal to the product of the wave amplitude and wavenumber $\varepsilon=kA$. We will seek unknown quantities confining our analysis to the second approximation in parameter ε in the form of asymptotic expansions:

$$\begin{aligned} &\mathbf{U} = \mathbf{U}_1 + \mathbf{U}_2 + O\left(\varepsilon^3\right);\ \mathbf{U}_j = O\left(\varepsilon^j\right); \\ &p = p_0 + p_1 + p_2 + O\left(\varepsilon^3\right);\ p_j = O\left(\varepsilon^j\right); \quad j = 1,2\,. \\ &\xi = \xi_1 + \xi_2 + O\left(\varepsilon^3\right);\ \xi_j = O\left(\varepsilon^j\right); \end{aligned} \tag{4}$$

Here, O is the symbol of the order of magnitude. In expressions for the velocity vector, the order of magnitude is estimated for each vector component. In final expressions, we will disclose the definition of parameter $\varepsilon = kA$, $\varepsilon^2 = k^2A^2$ and, using the traditional terminology in the theory of waves with a small but finite amplitude on the surface of an liquid, refer to variables $\mathbf{U}_j$, p_j and ξ_j as j-th order quantities in the wave amplitude, bearing in mind that the small parameter is in fact the ratio of the wave amplitude to the wavelength, which is proportional to dimensionless parameter $\varepsilon = kA$.
Substitution of expansions (4) into relations (3) and transposition of the boundary conditions to unperturbed surface $z = 0$ allow separate the problem (3) into the zeroth-, first, and second-order problems in the wave amplitude. The procedure of the separation of problem (3) in accordance with the order of magnitude is described in detail in [Le Blon & Mysak, 1978; Le Méhauté, 1976].
The analytic formulation of the first-order problem in the wave amplitude has the form

$$z < 0:\ \frac{\partial \mathbf{U}_1}{\partial t} + \frac{1}{\rho}\nabla p_1 - \nu\nabla^2\mathbf{U}_1 = 0;\ \ \nabla\cdot\mathbf{U}_1 = 0; \tag{5}$$

$$z = 0:\ \ \frac{\partial \xi_1}{\partial t} - v_1 = 0;\ \ -\rho g\xi_1 - p_1 - 2\rho\nu\frac{\partial v_1}{\partial z} + \gamma\frac{\partial^2\xi_1}{\partial x^2} = 0;\ \ \frac{\partial u_1}{\partial z} + \frac{\partial v_1}{\partial x} = 0; \tag{6}$$

$$z \to -\infty:\ \ u_1 \to 0;\ \ v_1 \to 0\,.$$

The formulation of the second-order problem consists of relation:

$$z < 0:\ \frac{\partial \mathbf{U}_2}{\partial t} + \frac{1}{\rho}\nabla p_2 - \nu\nabla^2\mathbf{U}_2 = -\left(\mathbf{U}_1\cdot\nabla\right)\mathbf{U}_1;\ \ \nabla\cdot\mathbf{U}_2 = 0; \tag{7}$$

$$\begin{aligned} z = 0:\ \ &\frac{\partial \xi_2}{\partial t} - v_2 = \xi_1\frac{\partial v_1}{\partial z} - u_1\frac{\partial \xi_1}{\partial x};\ \ -\rho g\xi_2 - p_2 - 2\rho\nu\frac{\partial v_2}{\partial z} + \gamma\frac{\partial^2\xi_2}{\partial x^2} = \xi_1\left(2\rho\nu\frac{\partial^2 v_1}{\partial z^2} - \frac{\partial p_1}{\partial z}\right); \\ &\frac{\partial u_2}{\partial z} + \frac{\partial v_2}{\partial x} = -4\frac{\partial v_1}{\partial z}\frac{\partial \xi_1}{\partial z} - \xi_1\frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial z}\right); \end{aligned} \tag{8}$$

$$z \to -\infty:\ \ u_2 \to 0;\ \ v_2 \to 0\,.$$

2.2 Solution of the first-order problem in wave amplitude

The solution to the first-order problem (5), (6) is well known and can be described by expressions of the type of a traveling wave [Belonozhko & Grigor'ev, 2003]:

$$\begin{pmatrix} \xi_1 \\ u_1 \\ v_1 \\ p_1 \end{pmatrix} = \frac{A}{2} \begin{pmatrix} 1 \\ -i\left(S+2\nu k^2\right)\exp(kz) + 2\,i\,\nu k q \exp(qz) \\ \left(S+2\nu k^2\right)\exp(kz) - 2\,i\,\nu k^2 \exp(qz) \\ -\rho S\left(\frac{S}{k}+2\nu k\right)\exp(kz) \end{pmatrix} \exp(St - ikx) + c.c.; \tag{9}$$

where

$$q = \sqrt{k^2 + \frac{S}{\nu}}; \tag{10}$$

or a superposition of the waves (9) with different wavenumbers k. For a fixed value of k, set of relations (9) is an individual mode of the wave motion. Abbreviation "c.c." means "complex-conjugate terms" and i is the imaginary unit. Parameter S is the complex frequency. It is connected with the wavenumber and other parameters of the problem via the dispersion relation :

$$\left(S+2\nu k^2\right)^2 + \omega_0^2 = 4\nu^2 k^3 \sqrt{k^2 + \frac{S}{\nu}}; \tag{11}$$

where ω_0 is defined by (2). The physical meaning can be attached not to all values of complex frequency S satisfying the dispersion relation, but only to those for which the condition holds:

$$\left(S+2\nu k^2\right)^2 + \omega_0^2 > 0.$$

In this case only, the vortex part of the velocity field, which is described in set (9) by the terms proportional to $\exp(qz)$, decays with increasing depth.
We will use in further analysis the following notation:

$$r = \mathrm{Re}(S); \quad \omega = \mathrm{Im}(S); \quad b = \mathrm{Re}(q); \quad \chi = \mathrm{Im}(q). \tag{12}$$

The absolute value of real-valued parameter r characterizes the rate of variation of the amplitude of wave motion. The value of r is smaller than zero and for this reason $|r|$ is the damping decrement of the wave motion. Real-valued quantity ω has the meaning of the circular frequency of wave motion in the viscous liquid.
In accordance with the results of work [Belonozhko & Grigor'ev, 2004] in the limit of a low viscosity dispersion relation (11) reduces to the asymptotic expansions for quantities r and ω as well as for auxiliary parameters b and χ:

$$\frac{r}{\omega_0} = -2N^2 + O\left(N^3\right) \approx -2\nu k^2; \quad \frac{\omega}{\omega_0} = 1 + O\left(N^3\right) \approx 1;$$
$$\frac{b}{k} = \frac{1}{N\sqrt{2}} - \frac{1}{2\sqrt{2}}N + O\left(N^3\right) \approx \frac{1}{k}\sqrt{\frac{\omega_0}{2\nu}}; \quad \frac{\chi}{k} = \frac{1}{N\sqrt{2}} + \frac{1}{2\sqrt{2}}N - N^2 + O\left(N^3\right) \approx \frac{1}{k}\sqrt{\frac{\omega_0}{2\nu}}; \tag{13}$$
$$N = k\sqrt{\frac{\nu}{\omega_0}} << 1.$$

In the right-hand sides of the relations (13) after the symbol "$\approx$" the expressions are written witch retain only the principal terms of the expansions in the viscosity.

Expressions (9) were deliberately written in complete form, although these expressions do not contain the drift terms explicitly, for the following two reasons. First, to solve the second-order problem in the wave amplitude, we must calculate the right-hand sides of first relation (7) and relations (8) containing quantities u_1 v_1, ξ_1 and p_1. Second, it will be shown below that the expressions for quantities u_1 v_1 play an important role in calculating the velocity of the mean drift flow.

2.3 Structure of expressions for the velocity field in the second order in wave amplitude

In contrast to first-order problem (5), (6), the solution to second-order problem (7), (8) contains the component describing the explicit drift of the liquid along the x axis. To simplify our analysis, we will confine ourselves to determining the form of this particular part of the solution. It will be shown below that the rest part of the solution is not used in constructing the expression for the velocity of total drift.

We assume that the solution to the starting problem (3) in the first approximation in the wave amplitude is described by only one mode of the wave motion, viz., a set of relations of type (9) with a specified wavenumber k. Using set (9), we can write the right-hand side of the first equation in system (7) in a more detailed form:

$$z<0: \quad \frac{\partial \mathbf{U}_2}{\partial t}+\frac{1}{\rho}\nabla p_2 - \nu\nabla^2\mathbf{U}_2 = V_x\mathbf{e}_x + V_z\mathbf{e}_z;$$

$$V_x = A^2\left(-2k^3\omega\nu\exp(2bz)+\left\{-\frac{ikS}{2}\left(S^*+2\nu k^2\right)\exp\left((k+q)z\right)+c.c.\right\}+\right.$$
$$\left.+\left\{\frac{ikS^*}{2}\left(S+2\nu k^2\right)\exp\left((k+q^*)z\right)+c.c.\right\}\right)\exp(2rt)+A^2\Pi(2\theta)\exp(2rt);$$

$$V_z = A^2\left(-\left|S+2\nu k^2\right|^2 k\exp(2kz)-4\nu^2k^4\,b\exp(2bz)+\right.$$
$$+\left\{\nu k^2(k+q)\left(S^*+2\nu k^2\right)\exp\left((k+q)z\right)+c.c.\right\}+$$
$$\left.+\left\{\nu k^2\left(k+q^*\right)\left(S+2\nu k^2\right)\exp\left((k+q^*)z\right)+c.c.\right\}\right)\exp(2rt)+A^2\Pi(2\theta)\exp(2rt);$$

$$\theta = \omega t - kx. \tag{14}$$

The asterisk in the superscript indicates complex conjugation. Here and below, symbol $\Pi(\Theta)$ is used as the general notation for various sums consisting of terms proportional to $\cos(\Theta)$ and $\sin(\Theta)$ with constant coefficients of proportionality or with coefficients depending only on coordinate z. For the column of quantities each of which is the sum of this type will be denoted by bold symbol $\mathbf{\Pi}(\Theta)$.

The expressions obtained for V_x and V_z describe the right-hand side of the first equation in system (7) explicitly. This allows us to use the method of undetermined coefficients and find partial solution (7) u_2^a, v_2^a, p_2^a:

$$\begin{pmatrix} u_2^a \\ v_2^a \\ p_2^a \end{pmatrix} = A^2 \left(\left(\begin{pmatrix} 0 \\ 0 \\ \dfrac{-\rho\left|S+2\nu k^2\right|}{2} \end{pmatrix} \exp(2kz) + \left(\begin{pmatrix} -\dfrac{2\nu k b\chi}{r-2\nu b^2} \\ 0 \\ -2\rho\nu^2 k^4 \end{pmatrix} \exp(2bz) + \right.\right.\right.$$
$$\left.\left.\left. + \left\{ \begin{pmatrix} \dfrac{ikS\left(S^*+2\nu k^2\right)}{2\left(2r-\nu(k+q)^2\right)} \\ 0 \\ \rho\nu k^2\left(S^*+2\nu k^2\right) \end{pmatrix} \exp((k+q)z) + c.c. \right\} \right) \right) \exp(2rt) + A^2\Pi(2\theta)\exp(2rt). \tag{15}$$

The values u_2^a, v_2^a, p_2^a are auxiliary. They satisfy (7) and can therefore be used to construct the substitution:

$$\begin{pmatrix} u_2 \\ v_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} u_2^a \\ v_2^a \\ p_2^a \end{pmatrix} + \begin{pmatrix} u_2^b \\ v_2^b \\ p_2^b \end{pmatrix}.$$

which reduces (7), (8) to the problem with homogeneous equations and nonhomogeneous boundary conditions

$z<0$:

$$\begin{aligned} &\frac{\partial u_2^b}{\partial t} + \frac{1}{\rho}\frac{\partial p_2^b}{\partial x} - \nu\left(\frac{\partial^2 u_2^b}{\partial x^2} + \frac{\partial^2 v_2^b}{\partial z^2}\right) = 0; \\ &\frac{\partial v_2^b}{\partial t} + \frac{1}{\rho}\frac{\partial p_2^b}{\partial z} - \nu\left(\frac{\partial^2 v_2^b}{\partial x^2} + \frac{\partial^2 v_2^b}{\partial z^2}\right) = 0; \\ &\left(\frac{\partial u_2^b}{\partial x} + \frac{\partial v_2^b}{\partial z}\right) = 0; \end{aligned} \tag{16}$$

$z = 0$:

$$\frac{\partial \xi_2}{\partial t} - v_2^b = \xi_1\frac{\partial v_1}{\partial z} - u_1\frac{\partial \xi_1}{\partial x} + v_2^a; \tag{17}$$

$$p_2^b - 2\rho\nu\frac{\partial v_2^b}{\partial z} + \gamma\frac{\partial^2 \xi_2}{\partial x^2} = \xi_1\left(2\rho\nu\frac{\partial^2 v_1}{\partial z^2} - \frac{\partial p_1}{\partial z}\right) - p_2^a + 2\rho\nu\frac{\partial v_2^a}{\partial z}; \tag{18}$$

$$\frac{\partial u_2^b}{\partial z} + \frac{\partial v_2^b}{\partial x} = -4\frac{\partial v_1}{\partial z}\frac{\partial \xi_1}{\partial z} - \xi_1\frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial z}\right) - \frac{\partial u_2^a}{\partial z} - \frac{\partial v_2^a}{\partial x}; \tag{19}$$

$z \to -\infty$:

$$u_2^b \to 0;\quad v_2^b \to 0. \tag{20}$$

Taking into account the explicit expressions (9) for quantities ξ_1, u_1 v_1, p_1 and (15) for u_2^a, v_2^a, p_2^a, we can easily establish even without detailed calculations that the structure of the right-hand side of each boundary condition from (17)-(19) defined at the level of $z = 0$ can be described by the formula $A^2(F(t)+\Pi(2\theta)\exp(2rt))$, where $F(t)$ is a certain function of time for each condition from $(17)-(19)$. This means that the unknown quantities (in particular, $u_2^b = u_2^b(t,x,z)$ and $p_2^b = p_2^b(t,x,z)$ should be sought in the form of a term independent of coordinate x and a term proportional to $A^2\Pi(2\theta)\exp(2rt)$:

$$u_2^b = A^2(w(t,z)+\Pi(2\theta)\exp(2rt)); \quad p_2^b = A^2(h(t,z)+\Pi(2\theta)\exp(2rt)). \tag{21}$$

Here, $w(t,z)$ and $h(t,z)$ are the functions to be determined.
Using relations (9), (15) we can easily calculate the right-hand side of boundary condition (19):

$z = 0$:

$$\frac{\partial u_2^b}{\partial z} + \frac{\partial v_2^b}{\partial x} = A^2\Lambda\exp(2rt) + A^2\Pi(2\theta)\exp(2rt); \tag{22}$$

$$\begin{aligned}\Lambda = &-\frac{1}{2}ik\left(2i\omega k + 6i\nu k^2\chi + \nu\left(q^3 - \left(q^*\right)^3\right)\right) + \\ &+\frac{1}{2}\nu k\left(\frac{4k^2\omega b}{r+b^2} + \left\{\frac{i(q-k)(q+k)^2\left(S^* + \nu k^2\right)}{2r+(q+k)^2} + c.c.\right\}\right).\end{aligned} \tag{23}$$

Substituting relations (21) into the first equation of system (16) and conditions (20), (22) and noting that $\left(\partial p_2^b/\partial x\right) \sim \Pi(2\theta)$, we can easily separate the component independent of parameter θ in these expressions and obtain the individual problem for determining function $w = w(t,z)$:

$$\begin{aligned} z<0: &\qquad \frac{\partial w}{\partial t} - \nu\frac{\partial^2 w}{\partial z^2} = 0; \\ z=0: &\qquad \frac{\partial w}{\partial x} = \Lambda\exp(2rt); \\ z\to-\infty: &\qquad w \to 0. \end{aligned} \tag{24}$$

In accordance with relations (21), value $A^2 w = A^2 w(t,z)$ at fixed z is velocity of the horizontal drift for all liquid particles that reside on the level $z = \text{constant}$.
The solution to problem (24) has the form [Polyanin, 2002]:

$$\begin{aligned} w = w(t,z) = &\Lambda\sqrt{\frac{\nu}{\pi}}\int_0^t \exp\left(-\frac{z^2}{4\nu(t-\eta)}\right)\frac{\exp(2r\eta)}{\sqrt{t-\eta}}d\eta + \\ &+\Lambda\sqrt{\frac{1}{\pi\nu t}}\int_0^\infty\left\{\exp\left(-\frac{(z-\zeta)^2}{4\nu t}\right) + \exp\left(-\frac{(z+\zeta)^2}{4\nu t}\right)\right\}\Psi(\zeta)d\zeta; \end{aligned} \tag{25}$$

$$\psi(z) \equiv w(0,z). \tag{26}$$

For the vertical component of the velocity field, we have, analogously to relations (21), $v_2^b = A^2\left(\beta(t,z)+\Pi(2\theta)\exp(2rt)\right)$ where $\beta(t,z)$ is the function to be determined. Substituting the expression for v_2^b and the second relation from (21) into last equation (16), we can easily see that $(\partial\beta/\partial z)=0$ and hence $\beta(t,z)=F(t)$ is a function of time. We must set $F(t)\equiv 0$, for the liquid is not moves with time along the z axis as a "whole". Thereby $v_2^b = A^2\Pi(2\theta)\exp(2rt)$.

The above calculations show that when a periodic capillary-gravitational wave with wavenumber k propagates over the horizontal surface of an infinitely deep liquid, a velocity field emerging in the liquid has following structure of the components in the second approximation in wave amplitude:

$$u = u_1 + A^2 w(t,z) + U_2^a + A^2\Pi(2\theta)\exp(2rt); \tag{27}$$

$$v = v_1 + A^2\Pi(2\theta)\exp(2rt); \tag{28}$$

$$u_1 = \frac{A}{2}\left(-i\left(S+2\nu k^2\right)\exp(kz) + 2\,i\,\nu kq\exp(qz)\right)\exp(St - ikx) + \text{c.c.}; \tag{29}$$

$$v_1 = \frac{A}{2}\left(\left(S+2\nu k^2\right)\exp(kz) - 2\,i\,\nu k^2\exp(qz)\right)\exp(St - ikx) + \text{c.c.}; \tag{30}$$

$$U_2^a = A^2\left(-\frac{2\nu k b\chi}{r-2\nu b^2}\exp(2bz) + \left\{\frac{ikS\left(S^*+2\nu k^2\right)}{2\left(2r-\nu(k+q)^2\right)}\exp\left((k+q)z\right) + c.c.\right\}\right)\exp(2rt)\,. \tag{31}$$

Here U_2^a is the part of the expression for u_2^a without the terms proportional to $\Pi(2\theta)$.

The function $w(t,z)$ is given by (25), (26) with parameter Λ defined by (23). In limit of low viscosity $r=\mathrm{Re}(S)$, $\omega=\mathrm{Im}(S)$, $b=\mathrm{Re}(q)$, $\chi=\mathrm{Im}(q)$ are described by asymptotic relations (13) and in this regard asymptotic expressions for Λ and U_2^a containing only the principle on the viscosity terms take on the form:

$$\Lambda \approx 2k^2\omega_0^2; \tag{32}$$

$$U_2^a \approx A^2\,k\omega_0\cos\left(\frac{z}{\delta}\right)\exp\left(\left(k+\delta^{-1}\right)z\right)\exp\left(-4\nu k^2 t\right); \tag{33}$$

$$\delta = \sqrt{\frac{2\nu}{\omega_0}}\,. \tag{34}$$

The value of parameter δ is rough estimation for the thickness of viscous boundary layer in vicinity of free surface. At the depth δ the amplitude of the vortex motion decreases in e time compare to value of the amplitude at the surface [Longuet-Higgins, 1953]. There is improving estimation 4δ for the boundary layer thickness [Belonozhko & Grigor'ev, 2008]. On this depth the amplitude of the vortex motion decreases to only several percent from value of the amplitude at the surface. The liquid flow is essentially rotational inside the boundary layer and can be considered as nearly irrotational outside the layer.

2.4 Construction of expression for the velocity of the mean horizontal drift taking viscosity into account

Expressions (27), (28) describe the velocity field of a liquid flow in the Euler representation [Longuet-Higgens, 1986)]. To calculate the velocity of an individual particle, we must pass from the description of the velocity field in the Euler variables

$$\mathbf{U} \equiv \mathbf{U}(t,\mathbf{r}) = U(t,x,z) = u(t,x,z)e_x + v(t,x,z)\mathbf{e}_z \, ;$$

to the description in the Lagrange form:

$$\mathbf{U}_L \equiv \mathbf{U}_L(t,\mathbf{r}) = U_L(t,x,z) = u_L(t,x,z)e_x + v_L(t,x,z)\mathbf{e}_z \, .$$

In both cases, the argument of the quantities considered here is the radius vector $\mathbf{r}$ of a point with coordinates x and z. In the Euler representation, $\mathbf{r}$ defines the position of a stationary point in space, through which various liquid particles pass in time t with velocity $\mathbf{U}(t,\mathbf{r})$. In the Lagrange representation, $\mathbf{r}$ fixes the position of an individual liquid particle at instant $t=0$, while vector $\mathbf{U}_L(t,r)$ characterizes the time variation of the velocity of namely this liquid particle.

It was shown in [Le Blon & Mysak L, 1978; Lokenath, 1994] that in the problem of propagation of a periodic small-amplitude capillary-gravitational wave over the horizontal surface of a liquid in the second approximation in amplitude of the wave motion, the velocity vector in the Lagrange representation can be expressed via components of velocity vector in the Euler representation:

$$\mathbf{U}_L(t,\mathbf{r}) = \mathbf{U}(t,\mathbf{r}) + \left(\left(\int_0^t \mathbf{U}(\eta,\mathbf{r})d\eta \right) \nabla \right) \mathbf{U}(t,\mathbf{r}).$$

For the horizontal velocity component, we have

$$u_L(t,x,z) = u(t,x,z) + \left\{ \left(\int_0^t u(\eta,x,z)d\eta \right) \frac{\partial u(t,x,z)}{\partial x} \right\} + \left\{ \left(\int_0^t v(\eta,x,z)d\eta \right) \frac{\partial u(t,x,z)}{\partial z} \right\} \tag{35}$$

Substituting relations (27), (28) into (35) and retaining the terms of the order not higher than second in wave amplitude we can easily derive the expression for the horizontal velocity of the individual liquid particle, which was at the point with coordinates x and z at $t=0$:

$$\begin{aligned} u = {} & u_1 + A^2 w(t,z) + \\ & + U_2^a + \left\{ \left(\int_0^t u_1(\eta,x,z)d\eta \right) \frac{\partial u_1(t,x,z)}{\partial x} \right\} + \left\{ \left(\int_0^t v_1(\eta,x,z)d\eta \right) \frac{\partial u_1(t,x,z)}{\partial z} \right\} + \\ & + A^2 \Pi(2\theta)\exp(2rt). \end{aligned} \tag{36}$$

To derive the formula (36) we have taken into account that in the second approximation in wave amplitude the asymptotic relations hold:

$$\left(\int_0^t u(\eta,x,z)d\eta \right) \frac{\partial u(t,x,z)}{\partial x} \approx \left(\int_0^t u_1(\eta,x,z)d\eta \right) \frac{\partial u_1(t,x,z)}{\partial x} \, ;$$

$$\left(\int_0^t v(\eta,x,z)d\eta\right)\frac{\partial u(t,x,z)}{\partial z} \approx \left(\int_0^t v_1(\eta,x,z)d\eta\right)\frac{\partial u_1(t,x,z)}{\partial z}.$$

Expressions for u_1, w and U_2^a are known. The explicit form for part of expression (36) enclosed in braces is derived by substitution (29) (30) in the braces. But total form of the expression is not needed. A significant part of the terms on the right-hand side of expression (36) is sums consisting of terms proportional to $\cos(\theta)$ and $\sin(\theta)$ or $\cos(2\theta)$ and $\sin(2\theta)$ ($\theta = \omega t - kx$). Obviously they are responsible for an apptoximatly circular motion of liquid particle relative to a certain average position. Let call them the cyclic components of the velocity. According to (29) and definition of symbol Π the terms u_1, $A^2\Pi(2\theta)\exp(2rt)$ are cyclic components of the velocity. All cyclic terms in (36) must be drooped in order to write the expression for mean horizontal velocity of liquid particle. Specifically for the part (36) in braces it is sufficient to obtain an explicit expression only for noncyclic terms without making complete calculations. As result relation (36) is transformed to expression for velocity of mean drift caused by periodical capillary-gravitational wave propagating over the surface of a viscous liquid:

$$u_d = A^2 w(t,z) + u_S(t,z); \tag{37}$$

$$u_S(t,z) = U_B(t,z) + U_2^a(t,z). \tag{38}$$

Here U_B denotes the noncyclic part of relation enclosed in braces in the expression (36). Taking into account the explicit relations for u_1 and v_1 (see (9)) the expression for U_B may be written in the form:

$$U_B(t,z) = \frac{A^2}{|S|}\Big(M\exp(2kz) + N\exp(2bz) + \\ + \nu k\big(G\cos(\chi z) + H\sin(\chi z)\big)\exp\big((k+b)z\big)\Big)\exp(2rt); \tag{39}$$

$$M = \omega k\left(\omega^2 + \left(r + 2\nu k^2\right)\right)^2; \quad N = 4\nu^2 k^3 b(b\omega + \chi r);$$

$$G = -2\Big(\omega(k+b)(\chi\omega + kr) + b\chi r^2 + \nu k^2\Big(\omega\big((k+b)^2 - \chi^2\big) + 2b\chi r\Big)\Big);$$

$$H = 2k\omega(\chi r - b\omega) + \left(\chi^2 - b^2\right)|S|^2 + 2\nu k^3(2\chi\omega + kr) + \\ + k^2\Big(r\big(r + 2\nu\left(\chi^2 - b^2\right)\big) - \omega(\omega - 4\nu\chi b)\Big).$$

With help of asymptotic expression (13) we can proceed to the limit of low viscosity and write asymptotic expressions for U_B containing only the principle on the viscosity terms:

$$U_B \approx A^2 k\omega_0 \exp(2kz)\exp\left(-4\nu k^2 t\right)\left(1 - \exp\left(\frac{z}{\delta} - kz\right)\cos\left(\frac{z}{\delta}\right)\right). \tag{40}$$

The expression (40) follows from (39) on the basis of (13). The parameter δ is defined in (34).

In what follows one term $A^2w(t,z)$ in the sum (37) will be called velocity of "Additional drift" and another term $u_S(t,z)$ will be referred to velocity of "Modified Stokes drift". Reasons for using the above terminology and individual notation $u_S(t,x)$ for sum of $U_B(t,x)$ and $U_2^a(t,x)$ will be clarified below.

2.5 Modified Stokes drift

The drift component (38) with $U_B(t,x)$ defined by (39) ((40) in limit of low viscosity) and $U_2^a(t,x)$ defined by (31) ((33) in limit of low viscosity) is called Modified Stokes drift due to closeness of its properties with features of the classical Stokes drift defined by (1). The best agreement of the properties is archived when one takes into account both terms $U_B(t,x)$ and $U_2^a(t,x)$ together.

The fig.1 show the behavior only term $U_B(t,x)$ calculated at different times (line 1-3) compare to the classical Stokes drift (line 4). For instance we assumed that the drift is caused by the propagation of a periodic wave with a length 5 sm over the surface of water ($\rho = 1\, kg/m^3$; $\gamma = 72\cdot10^{-3}\, N/m$; $\nu = 10^{-6}\, m^2/s$). It was thought that at initial time moment the wave amplitude is equal to 1 mm.

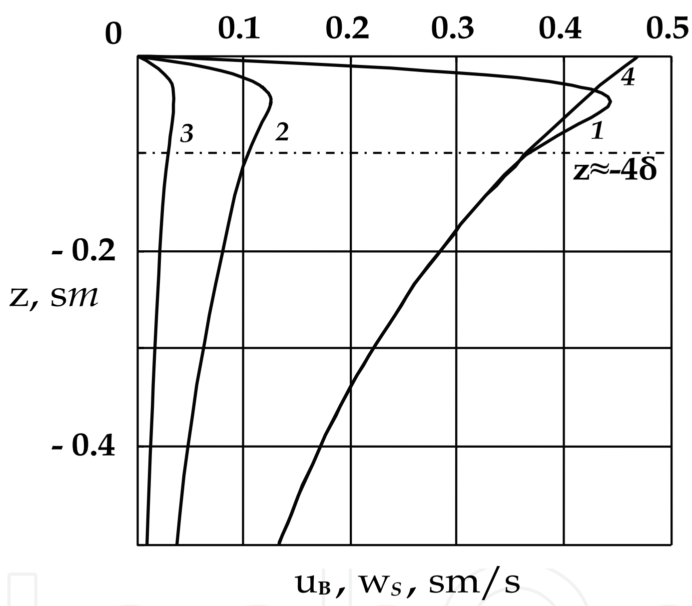


Fig. 1. The behavior of the drift velocity component $U_B(t,x)$ from (38) at different depth is plotted. Calculations are performed for the drift caused by periodic wave with wavelength 5 sm and amplitude 1 mm propagating over horizontal water surface. The curves 1,2,3 specify profile of $U_B(t,z)$ at moment times: $t = 0\,s$; $t = 20\,s$; $t = 40\,s$. The curve 1 show behavior of the classical Stokes Drift (1). The horizontal dotted line held at the lower border of the surface boundary layer $z = -4\delta$

In the course of time a monotonic profile of the classical Stokes drift is not changed since an approximation of inviscid liquid is working. Fig 1 show that the velocity of the component $U_B(t,x)$ at all depths decreases with a time proportionally to factor $\exp(2rt)$ where $r < 0$ (at low viscosity $r \approx -4\nu k^2$). The damping of the liquid motion in time is natural since the influence of viscous dissipation was taken into account. A characteristic shape of the profile $U_B(t,x)$ remains same for all time but horizontal extent of the profile shrinks with the course of time.

The characteristic shape of the profile $U_B(t,x)$ qualitative distinguishes from profile of the classical Stokes drift. In contrast to classical model the profile of velocity $U_B(t,x)$ is not monotonic near the liquid surface but one has a maximum on some level inside viscous boundary layer. Above the level corresponding to the maximum of velocity the values $U_B(t,x)$ dramatically decrease with decreasing a depth and tend to zero at the liquid surface. But it should be noted that pure classical drift (1) (see curve 4 at fig.1) rigorously monotonic on all depth and one reaches its maximum exactly at the liquid surface.

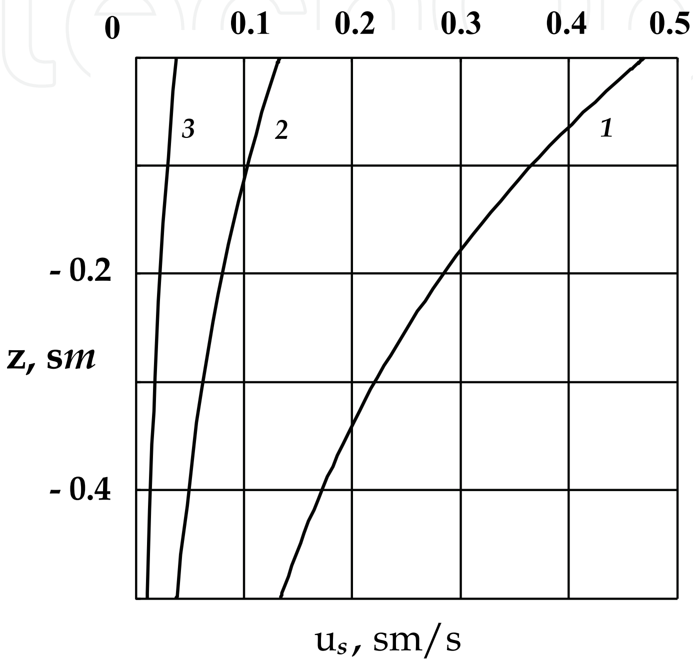


Fig. 2. The behavior of the Modified Stokes Drift $u_S(t,z)$ (see (38)) at different depth is plotted. The physical conditions are same that was used for plotting fig.1. The curves 1,2,3 specify profile of $u_S(t,z)$ at moment times: $t = 0$ s; $t = 20$ s; $t = 40$ s

Analysis of expression (40) presenting the limit values of $U_B(t,x)$ at a low viscosity show that the discovered difference between $U_B(t,x)$ and classical Stokes drift dos not vanish even for an arbitrary small viscosity. The decreases of viscosity leads only to a narrowing of the field near the liquid surface within which the deference between velocities (1) and (40) is essential. Directly on the surface the value of the difference at limit of zero viscosity tends to $A^2k\omega$ and not equal to zero. The observed discrepancy between behavior of $U_B(t,x)$ and drift (1) in vicinity of liquid surface suggests that the component $U_B(t,x)$ alone can not be regarded as naturally generalization of the classical Stokes drift.
We have combined the components $U_B(t,x)$ and $U_2^a(t,z)$ of drift velocity in sum (38) in order to produce expression for special part of drift velocity in viscous liquid $u_S(t,x)$ that is named by velocity of the Modified Stokes drift. Values $U_2^a(t,z)$ are essential only in vicinity of the surface and negligible in region below the viscous boundary layer. The component $U_2^a(t,z)$ is added to $U_B(t,x)$ in order to correct the properties of component $U_B(t,x)$ inside the viscous boundary layer. Profile of the sum in vicinity of liquid surface has no any drastic changes inherent in to the separate term $U_B(t,x)$. At least at low viscosity the shape of profile of the Modified Stokes drift $u_S(t,x)$ (sum $U_B(t,x)$ and $U_2^a(t,z)$) is rather close to profile of the Classical Stokes drift.

The fig. 2 is similar to fig.1 but the profiles of velocity $u_S(t,z) = U_B(t,x) + U_2^a(t,z)$ are depicted instead of the profiles related to the component $U_B(t,x)$. One can see that Modified Stokes drift at all depths behaves almost like the classical Stokes drift and it is only horizontal extent of the profile $u_S(t,x)$ that shrinks with time due to viscous dissipation.

2.6 Additional drift flow

Let consider the component $A^2w(t,z)$ of total mean drift (37) that was called the Additional drift flow. To find the reasons for the emergence of the Additional drift flow, let us consider an example of calculation of together evolution of horizontal velocity of the Modified Stokes drift $u_S(t,z)$ and the Additional drift velocity $A^2w(t,z)$ at a level of $z=0$. To definiteness let suppose that initially velocity of the Additional drift equal to zero $A^2w(0,z)=0$.
The time variation of velocities $u_S(t,z)$ and $A^2w(t,z)$ of the drift flows induced by a periodic capillary-gravitational wave having a wavelength of 5 cm and an amplitude of 1 mm and propagating on the surface of water are plotted in the fig 3.

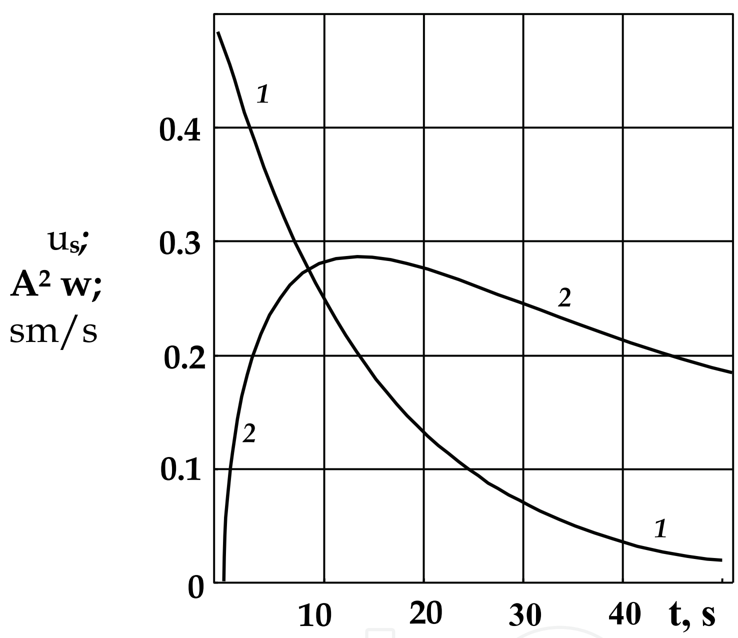


Fig. 3. The dependences of two velocity components of mean horizomtal drift on the water surface in the directions of propagation of the periodic capillary-gravitational wave. The values of all physical parameter same as used for plotting fig. 1 and fig. 2. Curve 1 - velocity $u_S(t,0)$ of the Modified Stokes drift (formula (37)); curve 2 – velocity $A^2w(t,0)$ of the Additional drift (formula (25) under condition $A^2w(0,z)=0$)

It can be seen that at the initial stage of the flow, the Modified Stokes drift (curve 1) is the main part of the total horizontal drift. The drift flow almost follows the laws for an inviscid liquid, and it is only the magnitude of the drift velocity that weakly decreases with time due to viscous dissipation.
At the same time, owing to components of viscous forces associated with the drift component of the flow, other types of changes occur in the velocity field. Viscous stresses emerging in the liquid have components periodic in the horizontal coordinate as well as aperiodic components. The aperiodic horizontal component of viscous stresses appears due to the drift component of the flow. Since horizontal drift velocity $u_S(t,z)$ varies with the

depth (see fig. 2) and the inequality $\left(\partial u_S(t,z)/\partial z\right)<0$ holds, horizontal viscous shear stresses $-\rho\nu\left(\partial u_S(t,z)/\partial z\right)$ appear between adjacent horizontal layers in accordance with the properties of a Newtonian liquid [Faber, 1997]. Under the action of these shear stresses, the liquid is involved into an additional horizontal drift flow with a velocity described by function $A^2w(t,z)$ (curve 2).

For chosen initial condition $A^2w(0,z)=0$, the Additional drift velocity increases monotonically with time from zero at $t=0$ to the velocity of the Modified Stokes drift. For values of physical parameters used for construction of the fig. 3, the velocities of both drifts become equal approximately 8.5 s after the beginning of the flow. The Modified Stokes drift entrains the liquid due to horizontal viscous stresses distributed from the surface to the deep layers. After leveling out the drift velocities, the velocity of the Modified Stokes drift continues to decrease exponentially and becomes smaller than the velocity of Additional drift. At now the Modified Stokes drift decelerates the Additional drift due to the same viscous stresses, which are acting now in the opposite direction. The deceleration of the Additional drift takes a certain time (about 3 s in the figure), after which its velocity attains the maximal value and then decreases together with the decreasing velocity of the Modified Stokes drift.

At fig. 4 the depths profiles of drift component $u_S(t,z)$ and $A^2w(t,z)$ are plotted at instant $t=8.5$ s when velocities of the both component are equal (the physical conditions are same that used for plotting fig. 1-3). One can see the velocity of the Additional drift $A^2w(t,z)$ decreases with depth faster than velocity of the Modified Stokes drift. This means that Additional drift decreases with depth faster than on exponential law.

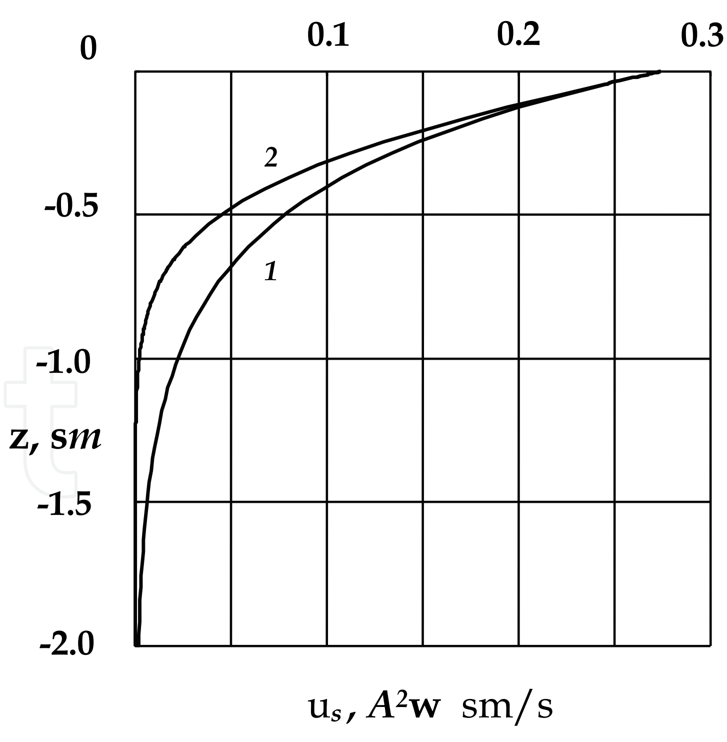


Fig. 4. The profiles of drift component $u_S(t,z)$ and $A^2w(t,z)$ are plotted at $t=8.5$ s after launching of the Additional drift ($A^2w(0,z)=0$). The values of all physical parameter are same as used for plotting fig.1-3. Curve 1 – the profile of the Modified Stokes drift (formula (37)); curve 2 – profile of the Additional drift (formula (25) under condition $A^2w(0,z)=0$)

2.7 Example: qualitative investigation of influence of surface electric charge on the drift flow caused by periodic capillary-gravitational waves

Let consider the problem similar to what was discussed in paragraph 2.1-2.4, but with an electric charge distributed over the liquid surface. We suppose that liquid is an ideal conductor and the upper half-space is filled with a dielectric medium having a permittivity approximately equal to unit (like water and air). Let upper medium has a negligible density and not affects on the motion in the lower liquid. We assume that in the equilibrium state the electric charge uniformly distributes at the plane surface with surface density σ_0.

Now in order to calculate the liquid motion caused by capillary-gravitational wave we must take into account an effect of electric forces on the free surface. In electro-hydrodynamic approximation [Melcher, 1963] the formulation of the problem (3) should be supplemented by the equation for the electric potential φ in the field above liquid and appropriate conditions at the surface and at the infinity:

$$z > \xi: \ \Delta\varphi = 0 ; \ z = \xi: \ \varphi = \text{const} ; \ z \to \infty: \ -\nabla\varphi = 4\pi\sigma_0^2 \mathbf{e}_z . \tag{41}$$

In addition the effect of the electric ponderomotive force $4\pi\sigma_0^2$ should be included in the condition for pressure:

$$z = \xi: \qquad p - 2\rho\nu\,\mathbf{n}(\mathbf{n}\cdot\nabla)\mathbf{U} + 4\pi\sigma_0^2 = -\gamma\frac{\partial^2\xi}{\partial x^2}\left(1+\left(\frac{\partial\xi}{\partial x}\right)^2\right)^{-3/2} . \tag{42}$$

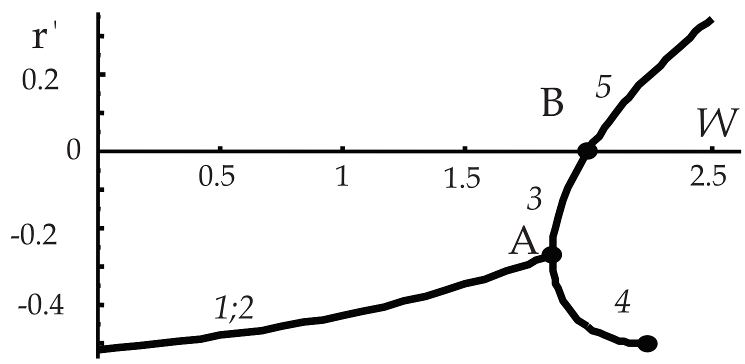
We offer to draw attention on a one way of qualitative investigation of properties of the drift in depending on value of some parameter (in our case parameter is surface charge). In the introduction it was indicate on the physical mechanism responsible for the creation of the mean drift. A liquid particle makes approximately circular motion but returns not to initial position but is shifted a little in the direction of wave propagation. The shift is occurred due to that the lower part of the trajectory is shorter than upper since the motion decays with depth. In the viscous liquid these mechanism is responsible for initiating main drift or Modified Stokes drift (see 2.4). The Modified Stokes drift initiates Additional drift (see 2.6). Thereby the presence of cyclical movements of liquid particles is an important factor contributing to the formation of the mean drift. Of course in order to obtain the expressions for drift components we have omitted the cyclic component of velocity of a liquid particle. But it is important remember that the noncyclic particle's velocity components that have been saved previously had been expressed via cyclic components of the velocity field in the liquid. The key role of the cyclic motion manifests itself in fact that in the low viscosity approximation all drift components are proportional to the circular frequency of cyclic motion (see (32), (33), (40)). More detailed calculations showed that the marked regularity holds for arbitrary viscosity too. In general, a drift of a liquid can be created through a variety of circumstance but we emphasize that our conclusion relate to the mean drift generated by propagation of the capillary-gravitational wave along the surface of a liquid.

From the above we can conclude that qualitative investigation of the effect of a physical parameter on the drift is reduced to the question about influence of the parameter at the frequency of the cyclic motion. In other words it is necessary to study the behavior of the roots of a dispersion equation depending on value of the parameter.

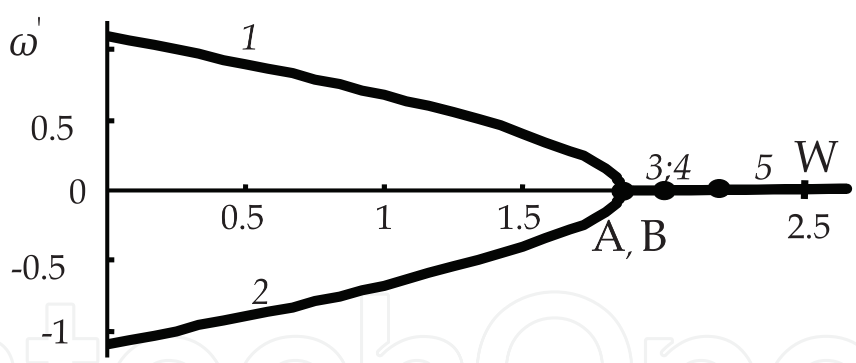
The problem (3), (41), (42) easy reduce to the problem of the first order of smallness in wave amplitude and then dispersion equation is found by standard method. For considered problem the dispersion equation has already obtained (see for example [Belonozhko & Grigor'ev (2004)]) and has same view as equation (11), only the parameter ω_0 is defined by the new formula:

$$\omega_0 = \sqrt{gk\left(1+\alpha^2k^2 - \alpha k W\right)}\,;\; W = \frac{4\pi\sigma_0^2}{\sqrt{\rho g \gamma}}\,. \tag{43}$$

Here α as before is the capillary constant of a liquid. The dimensionless parameter W can be regarded as the square of the dimensionless electric charge density in equilibrium state when a wave motion at the free surface is absent. On the other hand parameter W is proportional to ratio of electric and capillary forces at the crest of the wave with wave number $k = \alpha^{-1}$ propagating at the free surface.



(a)



(b)

Fig. 5. Dependences of real (a) and imaginary (b) parts of dimensionless complex frequency on parameter W calculated at dimensionless values wave number $k' = 1$ and at dimensionless viscosity $\nu' = 0.5$

Typical dependencies of real and imaginary parts of the complex frequency on parameter W, calculated by the dispersion equation (11) with taking into account relation (43) are plotted at fig. 5a,b. We used dimensionless variable

$$r' = \frac{r}{r_*} = \frac{\mathrm{Re}(S)}{r_*}\,;\; \omega' = \frac{\omega}{\omega_*} = \frac{\mathrm{Im}(S)}{\omega_*}\,;\; k' = \frac{k}{k_*}\,;\; \nu' = \frac{\nu}{\nu_*}\,.$$

with characteristic scale

$$r_* = \omega_* = \sqrt[4]{\frac{\rho g^3}{\gamma}};\ \ k_* = \frac{1}{\alpha} = \sqrt{\frac{\rho g}{\gamma}};\ \ \nu_* = \sqrt[4]{\frac{\gamma^3}{\rho^3 g}}.$$

According to work [Belonozhko & Grigor'ev, 2008] the position of the reference points W_A and W_B depicted at the fig. 5 are defined by formulas:

$$W_A \equiv W_A(k) = W_B - (\nu' k')^2 = W_B - \frac{(\nu k)^2}{g\alpha};\ \ W_B \equiv W_B(k) = \frac{1}{k'} + k' = \frac{1}{\alpha k} + \alpha k. \qquad (44)$$

The fig. 5 show that for every value of wave number k there is a critical value of the surface charge density corresponding to condition $W = W_B(k)$. If the surface charge density larger than the critical value ($W > W_B$), complex frequency is pure real and cyclic motion of liquid particle is absent (see curve 5: if $W > W_B$ then $r' > 0$ and $\omega' = 0$). The liquid motion under these conditions can not be a wave. The real part of the complex frequency describes increment of well-known instability of the charged liquid surface relative to excess of a surface charge [Taylor, 1965; Baily, 1974]. Aperiodic growth over time is only an initial stage of the instability appearing owing to what electric forces on the free surface dominate over capillary and gravity forces. The experiments have shown that the instability leads to what a strongly nonlinear conical projections (Taylor's cones) are formed on the charged liquid surface [Taylor & McEwan, 1965]. From the peak of the cones the emission of small strongly charged droplet is occurring. Analyses of expression for $W_B(k)$ shows that if condition $W > 2$ is valid, there is a range of values k for which $W > W_B(k)$ and appropriated small wave perturbations (for example thermal fluctuating) are involved in formation of the Taylor's cones. This phenomenon is well-known and underlies the work of different devices for electrodispersion of various liquid [Baily, 1974].
Thereby if $W > 2$, we can not speak about any drift flow caused by the propagation of the capillary gravitational waves.
If $W_A < W < W_B$, real and imaginary parts of roots of the dispersion equation described by curves 3,4. It can be see that under these conditions there are two modes of liquid motion. Both modes are aperiodic damping ($r' < 0$) and differ only in the damping rate. The cyclic motion of the liquid particle is absent ($\omega' < 0$). For both modes initially deformation of the liquid surface monotonically diminishes over time till complete disappearing. Liquid particles participate only in vertical motion and there is not any horizontal drift.
If $W < W_B(k)$, there are two modes of wave liquid motion with same damping rate and $\omega' \neq 0$ (see curves 1,2 at fig.5). One mode $\omega'_1 > 0$ (curve 1) correspond to wave that is propagating in direction Ox and another mode $\omega'_2 = -\omega'_1 < 0$ (curve 2) describes the wave that is traveling in the opposite direction. One can see that for the capillary-gravitational wave with wave number k the dimensionless circular frequency ω' (in absolute value) decreasing with increasing value of W and vanishes if $W \geq W_B(k)$. As was notice above the drift velocity is proportional to the frequency and consequently behaves as it.
Summarizing the above-said we can conclude that horizontal drift caused by propagation of the periodic capillary-gravitational wave with wave number k is possible only under conditions $W < 2$ and $W < W_B(k)$ where W and $W_B(k)$ is defined by (43) and (44). The first condition ensures that liquid surface is not subjected to instability with respect to an excess of surface electric charge. The second condition provides circumstances under which frequency of cyclical motion of the liquid is not zero. The velocity of the drift flow decreasing with increasing of surface charge density and vanishes if values of surface charge density is reached quantity so that $W = W_B(k)$.

We have demonstrated the possibilities of qualitative investigation of behavior of the mean drift caused by propagation of capillary-gravitational wave over surface of a viscous liquid in dependence on values of certain parameter responsible for supplementary physical effect. In any more complicated physical circumstance the qualitative investigation can be carried out by similarly.

3. Conclusion

Viscous forces play an important role in formation and evolution of the mean horizontal drift induced by periodic capillary-gravitational waves propagating over liquid surface. There are two components of the mean drift flow caused by the propagation of waves: main part that is called the Modified Stokes drift and supplementary part that is named the Additional flow or Additional drift.

At low but nonzero viscosity the Modified Stokes drift behaves almost like classical Stokes drift (model of drift phenomenon without a viscosity) at all depths there is only exponential decrease in the rate of flow over time due to viscous dissipation. The physical mechanism responsible for an appearance of the Modified Stokes drift is same as that of the classical Stokes drift. For period of the wave motion a liquid particle makes approximately circular motion but returns not to initial position but is shifted a little in the direction of wave propagation. The shift is occurred due to that the lower part of the trajectory is shorter than upper since the motion decays with depth.

A considerable contribution to the total drift flow comes from Additional drift into which the liquid is entrained by horizontal viscous stresses acting along the direction of propagation of the Modified Stokes drift. The horizontal viscous shear stresses appear between adjacent horizontal layers since velocity of Modified Stokes drift decreases with the depth. The phenomenon of the Additional drift appears exclusively in the model of a viscous liquid and is ruled out by the laws of an inviscid liquid flow.

Velocities of the Modified Stokes drift and the Additional drift are values of the second order of smallness in wave amplitude. In present work we have offered an analytical procedure of calculation of both drift components.

Expression for velocity of the Modified Stokes drift consists of two terms. The first term is calculated in the same way as in the case of the classical Stokes drift and it is a result of special manipulation with products of values of first order in wave amplitude. Thereby, first term of the Modified Stokes drift is expressed only via quantities that are found as result of calculating in the first order in wave amplitude velocity field caused by propagation of a capillary-gravitational wave over liquid surface.

The second term of the Modified Stokes drift is a special particular solution of the problem of calculating of second order in wave amplitude corrections for the velocity field caused by propagation of a capillary-gravitational wave over liquid surface. This term is essential only in vicinity of liquid surface in the narrow field of viscous boundary layer and negligible in deeper layers. In the limit of almost vanishing viscosity the thickness of the surface viscous boundary layer becomes nearly zero. At the upper bounder of this layer the first term tends to zero and the second term takes care of the correct description of the drift. The best agreement of the properties classical and Modified drifts is archived only when one takes into account both terms.

Expression for the Additional drift is derived as a special part of solution of the problem of calculating of second order in wave amplitude corrections for the velocity field caused by propagation of a capillary-gravitational wave over liquid surface.

The procedure developed here for calculating the drift is especially important for applications associated with phenomena on the free surface of a liquid with the participation of viscous stresses. In particular, the approach proposed here makes it possible to calculate analytically the velocity of surface drift caused by wave perturbation for various surface substances (surface charge, surface-active substances, etc.) distributed over the free surface. If there is a total motion of the upper liquid along the interface between liquids (like wind along surface of ocean) then is interesting to study influence of the total upper liquid velocity on the arising of drift flow in lower liquid. Especially interesting if total upper liquid velocity is sufficient to excite an oscillatory instability which in the case of ideal liquids is known as Kelvin-Helmholtz instability. For all cases the qualitative preliminary analysis is helpful which is based on what the velocity of drift caused by the wave propagation is proportional to frequency of the wave motion and hence supporting information about behavior of the drift is contained in dispersion equation of the analyzed problem.

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