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# Decomposition Approach for Inverse Matrix Calculation

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## 1. Introduction

Frequently, numerical algorithms are based on sequentially solution of linear set of equation $Ax=b$, applying small influences of few components of matrix $A$, which changes to a new one $A^*$. Thus, new equation set is defined, with new matrix A$^*$, which has to be solved for the current numerical iteration. Instead of solving the new equation set, it is beneficial to evaluate a new inverse matrix $A^{*-1}$, having the evaluations for the previous inverse matrix $A^{-1}$. Many control algorithms, on-line decision making and optimization problems reside on the prompt evaluation of the inverse matrix $A^{-1}$, stated as a quadratic nonsingular, e.g. $A.A^{-1}=A^{-1}.A=I$, where $I$ is identity matrix. Currently, for the evaluation of the inverse matrix $A^{-1}$ three general types of $A$ factorization are applied: LU factorization, QR – decomposition and SVD-decomposition to singular values of $A$.

*LU – factorization.* It results after the application of Gauss elimination to linear set of equations $Ax=b$ to obtain a good computational form of $A$ (Fausett, 1999). The factorization of $A$ is obtained by multiplication of two triangular matrices, upper $U$ and lower $L$ triangular, related to the initial one by $LU=A$ , or

$$\begin{vmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{vmatrix} . \begin{vmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} .$$

The LU factorization can be applied for the solution of linear set of equations to evaluate the inverse matrix of $A$: $A^{-1}$. The evaluation of $A^{-1}$ is performed on two steps, for given LU factorization of $A$, $A=LU$ ($L,U$ - given):

*First*: the matrix equation $LY=I$ is solved. The first column of matrix $Y$ is found from the linear equation system $LY(:,1)=\begin{vmatrix}1 & 0 & \dots & 0\end{vmatrix}^T$. The next columns of $Y$ are calculated by solving this linear equation set with the next columns of matrix $I$. The solution of this set of equations is found by sequential substitution from top to down, because the matrix $L$ is a lower triangular and there is no need to find the inverse $L^{-1}$.

*Second*: Using the solution matrix $Y^*$ a new linear matrix equation system is solved $U.X=Y^*$. Because $U$ is upper triangular, each column of $X$ is calculated with the corresponding

column of $Y^*$ by substitutions from bottom to top. Thus, no inverse matrix $U^{-1}$ is calculated, which speeds up the calculations.

The solution $X=A^{-1}$ is the inverse matrix of the initial one $A$. Thus, the inverse matrix $A^{-1}$ is found by LU factorization of $A$ and twice solution of triangular linear matrix equation systems, applying substitution technique.

*QR decomposition* The QR decomposition of matrix $A$ is defined by the equality $A=Q.R$, where $R$ is upper triangular matrix and $Q$ is orthogonal one, $Q^{-1}=Q^T$. Both matrices $Q$ and $R$ are real ones. As the inverse $A^{-1}$ is needed, $A^{-1}=R^{-1}.Q^{-1}$. Following the orthogonal features of $Q$, it is necessary to evaluate only $R^{-1}$, which can be done from the linear matrix system $R.Y=I$. Because $R$ is upper triangular matrix, the columns of the inverse matrix $Y=R^{-1}$ can be evaluated with corresponding columns of the identity matrix $I$ by merely substitutions from bottom to down. Hence, the inverse matrix $A^{-1}$ is found by QR factorization of A, sequential evaluation of $R^{-1}$ by substitutions in linear upper triangular matrix system and finally by multiplication of $R^{-1}$ and $Q^T$.

*SVD – decomposition to singular values* This decomposition is very powerful, because it allows to be solved system equations when $A$ is singular, and the inverse $A^{-1}$ does not exist in explicit way (Flannery, 1997). The SVD decomposition, applied to a rectangular MxN matrix $A$, represents the last like factorization of three matrices:

$$A=U.W.V^T,$$

where $U$ is MxN orthogonal matrix, $W$ is NxN diagonal matrix with nonnegative components (singular values) and $V^T$ is a transpose NxN orthogonal matrix $V$ or

$$U^TU=V^TV=I_{NxN}.$$

The SVD decomposition can always be performed, nevertheless of the singularity of the initial matrix $A$. If $A$ is a square NxN matrix, hence all the matrices $U$, $V$ and $W$ are square with the same dimensions. Their inverse ones are easy to find because $U$ and $V$ are orthogonal and their inverses are equal to the transpose ones. $W$ is a diagonal matrix and the corresponding inverse is also diagonal with components $1/w_j$ , $j=1,N$. Hence, if matrix $A$ is decomposed by SVD factorization, $A=U.W.V^T$, then the inverse one is $A^{-1}=V.[diag1/w_j].U^T$. The problem of the evaluation of the inverse $A^{-1}$ appears if a singular value $w_j$ exists, which tends to zero value. Hence, if matrix $A$ is a singular one, the SVD decomposition easily estimates this case.

Hence, the peculiarities of LU, QR and SVD factorizations determine the computational efficiency of the evaluations for finding the inverse matrix $A^{-1}$. Particularly, the simplest method, from evaluation point of view, is LU factorization followed by QR decomposition and SVD factorization. All these methods do not use peculiarities, if matrix $A^*$ slightly differs from the initial matrix $A$. The inverse of $A^*$ has to be evaluated starting with its factorization and sequentially solution of linear matrix equation systems. Hence, it is worth to find methods for evaluation of the inverse of $A^*$, which differs from $A$ in few components and $A^{-1}$ is available. The new matrix $A^*$ can contain several modified components $a_{ij}$. Hence, the utilization of components from the inverse $A^{-1}$ for the evaluation of the new inverse matrix $A^{*-1}$ can speed up considerably the numerical calculations in different control algorithms and decreases the evaluation efforts. Relations for utilization of components of $A^{-1}$ for evaluation

of a corresponding inverse of a modified matrix $A^{*\,-1}$ are derived in (Strassen, 1969). The components of the inverse matrix can be evaluated analytically.
Finding the inverse matrix is related with a lot of calculations. Instead of direct finding an inverse matrix, it is worth to find analytical relations where lower dimensions inverse matrices components are available. Here analytical relations for inverse matrix calculation are derived and the corresponding MATLAB codes are illustrated.

## 2. Analytical relations among the components of inverse matrix

Initial optimization problem is given in the form

$$\min_x \left\{ \frac{1}{2} \begin{vmatrix} x_1^T & x_2^T \end{vmatrix} \begin{vmatrix} Q_1 & 0 \\ 0 & Q_2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} + \begin{vmatrix} R_1^T & R_2^T \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \right\} \tag{1}$$

$$\begin{aligned} a_1 x_1 + a_2 x_2 &= d \\ b_1 x_1 \qquad &= C_1 \\ b_2 x_2 &= C_2 \end{aligned}$$

where the matrices dimensions are: $x_1 |_{n_1 x 1}$ ; $Q_1 |_{n_1 x n_1}$ ; $R_1 |_{n_1 x 1}$ ; $a_1 |_{m_0 x n_1}$ ; $d |_{m_0 x 1}$

$$x_2 |_{n_2 x 1} \; ; Q_2 |_{n_2 x n_2} \; ; R_2 |_{n_2 x 1} \; ; a_2 |_{m_0 x n_2} \; ; \;\; b_1 |_{m_1 x n_1} \;\; ; \;\; C_1 |_{m_1 x 1} \;\; ; \;\; b_2 |_{m_2 x n_2} \; ; \;\; C_2 |_{m_2 x 1} \; ;$$

$$Q = \begin{vmatrix} Q_1 & 0 \\ 0 & Q_2 \end{vmatrix} ; R = \begin{vmatrix} R_1 \\ R_2 \end{vmatrix}$$

Peculiarity of problem (1) is that the connected condition $a_1 x_1 + a_2 x_2 = d$ distributes a common resource $d$ while the subsystems work with own resources $C_1$ and $C_2$. For simplicity of the writing it can be put

$$A_1 |_{(m_0+m_1+m_2) x n_1} = \begin{vmatrix} a_1 \\ b_1 \\ 0 \end{vmatrix}_{\substack{m_0 x n_1 \\ m_1 x n_1 \\ m_2 x n_1}} ; A_2 |_{(m_0+m_1+m_2) x n_2} = \begin{vmatrix} a_2 \\ 0 \\ b_2 \end{vmatrix}_{\substack{m_0 x n_2 \\ m_1 x n_2 \\ m_2 x n_2}} ;$$

$$D = \begin{vmatrix} d \\ C_1 \\ C_2 \end{vmatrix}_{(m_0+m_1+m_2) x 1} ; A = \begin{vmatrix} A_1 & A_2 \end{vmatrix}_{(m_0+m_1+m_2) x (n_1 x n_2)} . \tag{2}$$

### 2.1 Decomposition of the initial problem by goal coordination

The initial problem (1) can be solved by two manners using hierarchical approach according to the hierarchical multilevel systems (Mesarovich et al, 1973; Stoilov & Stoilova, 1999): by goal coordination and by predictive one. Taking into account the substitutions

$$Q = \begin{vmatrix} Q_1 & 0 \\ 0 & Q_2 \end{vmatrix} ; R = \begin{vmatrix} R_1 \\ R_2 \end{vmatrix} ; A = \begin{vmatrix} A_1 & A_2 \end{vmatrix} ; x = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} , \tag{3}$$

the solution of (1) can be found in analytical form (Stoilova, 2010) :

$$x^{opt} = -Q^{-1}\left[R - A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}R + D)\right] \tag{4}$$

or

$$x_1^{opt} = -Q_1^{-1}R_1 + Q_1^{-1}A_1^T\left(A_1Q_1^{-1}A_1^T + A_2Q_2^{-1}A_2^T\right)^{-1}\left(A_1Q_1^{-1}R_1 + A_2Q_2^{-1}R_2 + D\right) \tag{5}$$

$$x_2^{opt} = -Q_2^{-1}R_2 + Q_2^{-1}A_2^T\left(A_1Q_1^{-1}A_1^T + A_2Q_2^{-1}A_2^T\right)^{-1}\left(A_1Q_1^{-1}R_1 + A_2Q_2^{-1}R_2 + D\right).$$

It is necessary to be known the matrices $A_1$, $A_2$, $Q_1$, $Q_2$, $R_1$, $R_2$ for evaluating the solutions (5)

*Determination of* $AQ^{-1}A^T$

Applying (3) it is obtained

$$AQ^{-1}A^T = \begin{vmatrix} A_1 & A_2 \end{vmatrix} \begin{vmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{vmatrix} \begin{vmatrix} A_1^T \\ A_2^T \end{vmatrix} = A_1Q_1^{-1}A_1^T + A_2Q_2^{-1}A_2^T \,. \tag{6}$$

Using (2) it follows

$$A_1Q_1^{-1}A_1^T = \begin{vmatrix} a_1Q_1^{-1}a_{1\,m_0xm_0}^T & a_1Q_1^{-1}b_{1\,m_0xm_1}^T & 0_{m_0xm_2} \\ b_1Q_1^{-1}a_{1\,m_1xm_0}^T & b_1Q_1^{-1}b_{1\,m_1xm_1}^T & 0_{m_1xm_2} \\ 0_{m_2xm_0} & 0_{m_2xm_1} & 0_{m_2xm_2} \end{vmatrix}, \tag{7}$$

$$A_2Q_2^{-1}A_2^T = \begin{vmatrix} a_2Q_2^{-1}a_{2\,m_0xm_0}^T & 0_{m_0xm_1} & a_2Q_2^{-1}b_{2\,m_0xm_2}^T \\ 0_{m_1xm_0} & 0_{m_1xm_1} & 0_{m_1xm_2} \\ b_2Q_2^{-1}a_{2\,m_2xm_0}^T & 0_{m_2xm_1} & b_2Q_2^{-1}b_{2\,m_2xm_2}^T \end{vmatrix}. \tag{8}$$

After substitution of (7) and (8) in (6) $AQ^{-1}A^T$ and $(AQ^{-1}A^T)^{-1}$ can be determined

$$AQ^{-1}A^T = A_1Q_1^{-1}A_1^T + A_2Q_2^{-1}A_2^T = \begin{vmatrix} a_1Q_1^{-1}a_1^T + a_2Q_2^{-1}a_{2\,m_0xm_0}^T & a_1Q_1^{-1}b_{1\,m_0xm_1}^T & a_2Q_2^{-1}b_{2\,m_0xm_2}^T \\ b_1Q_1^{-1}a_{1\,m_1xm_0}^T & b_1Q_1^{-1}b_{1\,m_1xm_1}^T & 0_{m_1xm_2} \\ b_2Q_2^{-1}a_{2\,m_2xm_0}^T & 0_{m_2xm_1} & b_2Q_2^{-1}b_{2\,m_2xm_2}^T \end{vmatrix} \tag{9}$$

$$(AQ^{-1}A^T)^{-1} = (A_1Q_1^{-1}A_1^T + A_2Q_2^{-1}A_2^T)^{-1} =$$

$$= \begin{vmatrix} a_1Q_1^{-1}a_1^T + a_2Q_2^{-1}a_{2\,m_0xm_0}^T & a_1Q_1^{-1}b_{1\,m_0xm_1}^T & a_2Q_2^{-1}b_{2\,m_0xm_2}^T \\ b_1Q_1^{-1}a_{1\,m_1xm_0}^T & b_1Q_1^{-1}b_{1\,m_1xm_1}^T & 0_{m_1xm_2} \\ b_2Q_2^{-1}a_{2\,m_2xm_0}^T & 0_{m_2xm_1} & b_2Q_2^{-1}b_{2\,m_2xm_2}^T \end{vmatrix}^{-1}. \tag{10}$$

The manner of definition of matrix $AQ^{-1}A^T$ shows that it is a symmetric one.

*Determination of* $AQ^{-1}R+D$

Using (2) and (3) it follows

$$AQ^{-1}R+D=\begin{vmatrix} a_1Q_1^{-1}R_1+a_2Q_2^{-1}R_2+d \\ b_1Q_1^{-1}R_1+C_1 \\ b_2Q_2^{-1}R_2+C_2 \end{vmatrix} \tag{11}$$

*Determination of* $(AQ^{-1}A^T)^{-1}(AQ^{-1}R+D)$

Using (10) and (11), it is obtained

$$(AQ^{-1}A^T)^{-1}(AQ^{-1}R+D)=\begin{vmatrix} a_1Q_1^{-1}a_1^T+a_2Q_2^{-1}a_2^T{}_{m_0xm_0} & a_1Q_1^{-1}b_1^T{}_{m_0xm_1} & a_2Q_2^{-1}b_2^T{}_{m_0xm_2} \\ b_1Q_1^{-1}a_1^T{}_{m_1xm_0} & b_1Q_1^{-1}b_1^T{}_{m_1xm_1} & 0_{m_1xm_2} \\ b_2Q_2^{-1}a_2^T{}_{m_2xm_0} & 0_{m_2xm_1} & b_2Q_2^{-1}b_2^T{}_{m_2xm_2} \end{vmatrix}^{-1} \begin{vmatrix} a_1Q_1^{-1}R_1+a_2Q_2^{-1}R_2+d \\ b_1Q_1^{-1}R_1+C_1 \\ b_2Q_2^{-1}R_2+C_2 \end{vmatrix}$$

$$(AQ^{-1}A^T)^{-1}(AQ^{-1}R+D)=\begin{vmatrix} \alpha_{11}{}_{m_0xm_0} & \alpha_{12}{}_{m_0xm_1} & \alpha_{13}{}_{m_0xm_2} \\ \alpha_{21}{}_{m_1xm_0} & \alpha_{22}{}_{m_1xm_1} & \alpha_{23}{}_{m_1xm_2} \\ \alpha_{31}{}_{m_2xm_0} & \alpha_{32}{}_{m_2xm_1} & \alpha_{33}{}_{m_2xm_2} \end{vmatrix} \begin{vmatrix} a_1Q_1^{-1}R_1+a_2Q_2^{-1}R_2+d \\ b_1Q_1^{-1}R_1+C_1 \\ b_2Q_2^{-1}R_2+C_2 \end{vmatrix}$$

where

$$(AQ^{-1}A^T)^{-1}=\alpha=\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}_{(m_0+m_1+m_2)x(m_0+m_1+m_2)} \tag{12}$$

Taking into account the structure of matrix $(AQ^{-1}A^T)^{-1}$ , then $\alpha_{23}=0,\quad \alpha_{32}=0$ . The manner of definition of the inverce matrix $\alpha$ shows that it is a symmetric one. Consequently, the term $(AQ^{-1}A^T)^{-1}(AQ^{-1}R+D)$ is

$$(AQ^{-1}A^T)^{-1}(AQ^{-1}R+D)=\begin{vmatrix} \alpha_{11}(a_1Q_1^{-1}R_1+a_2Q_2^{-1}R_2+d)+\alpha_{12}(b_1Q_1^{-1}R_1+C)_1+\alpha_{13}(b_2Q_2^{-1}R_2+C_2) \\ \alpha_{21}(a_1Q_1^{-1}R_1+a_2Q_2^{-1}R_2+d)+\alpha_{22}(b_1Q_1^{-1}R_1+C_1)+\underbrace{\alpha_{23}(b_2Q_2^{-1}R_2+C_2)}_{0} \\ \alpha_{31}(a_1Q_1^{-1}R_1+a_2Q_2^{-1}R_2+d)+\underbrace{\alpha_{32}(b_1Q_1^{-1}R_1+C_1)}_{0}+\alpha_{33}(b_2Q_2^{-1}R_2+C_2) \end{vmatrix}_{\substack{m_0x1 \\ m_1x1 \\ m_2x1}} \tag{13}$$

*Determination of* $A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}R+D)$

After a substitution of (2) in (13) it follows

$$\begin{vmatrix} a_{1\,n_1 x m_0}^T & b_{1\,m_1 x m_0}^T & 0_{n_1 x m_2} \\ a_{2\,n_2 x m_0}^T & 0_{n_2 x m_1} & b_{2\,n_2 x m_2}^T \end{vmatrix} (AQ^{-1}A^T)^{-1}(AQ^{-1}R + D) =$$

$$= \begin{vmatrix} \left|a_1^T \quad b_1^T\right| \begin{matrix} \underbrace{\alpha_{11}}_{m_0 x m_0} \\ \underbrace{\alpha_{21}}_{m_1 x m_0} \end{matrix} \underbrace{(a_1 Q_1^{-1} R_1 + a_2 Q_2^{-1} R_2 + d)}_{m_0 x 1} + \left|a_1^T \quad b_1^T\right| \begin{matrix} \underbrace{\alpha_{12}}_{m_0 x m_1} \\ \underbrace{\alpha_{22}}_{m_1 x m_1} \end{matrix} \underbrace{(b_1 Q_1^{-1} R_1 + C_1)}_{m_1 x 1} + \left|a_1^T \quad b_1^T\right| \begin{matrix} \underbrace{\alpha_{13}}_{m_0 x m_2} \\ \underbrace{0}_{m_1 x m_2} \end{matrix} \underbrace{(b_2 Q_2^{-1} R_2 + C_2)}_{m_2 x 1} \\ \left|a_2^T \quad b_2^T\right| \begin{matrix} \underbrace{\alpha_{11}}_{m_0 x m_0} \\ \underbrace{\alpha_{31}}_{m_2 x m_0} \end{matrix} \underbrace{(a_1 Q_1^{-1} R_1 + a_2 Q_2^{-1} R_2 + d)}_{m_0 x 1} + \left|a_2^T \quad b_2^T\right| \begin{matrix} \underbrace{\alpha_{12}}_{m_0 x m_1} \\ \underbrace{0}_{m_1 x m_2} \end{matrix} \underbrace{(b_1 Q_1^{-1} R_1 + C_1)}_{m_1 x 1} + \left|a_2^T \quad b_2^T\right| \begin{matrix} \underbrace{\alpha_{13}}_{m_0 x m_2} \\ \underbrace{\alpha_{33}}_{m_2 x m_2} \end{matrix} \underbrace{(b_2 Q_2^{-1} R_2 + C_2)}_{m_2 x 1} \end{vmatrix}$$

Consequently, the term $A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}R + D)$ becomes

$$A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}R + D) = \begin{vmatrix} \left[ \left|a_1^T \quad b_1^T\right| \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & 0 \end{matrix} \left| \begin{matrix} a_1 Q_1^{-1} R_1 + a_2 Q_2^{-1} R_2 + d \\ b_1 Q_1^{-1} R_1 + C_1 \\ b_2 Q_2^{-1} R_2 + C_2 \end{matrix} \right| \right] \\ \left[ \left|a_2^T \quad b_2^T\right| \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{31} & 0 & \alpha_{33} \end{matrix} \left| \begin{matrix} a_1 Q_1^{-1} R_1 + a_2 Q_2^{-1} R_2 + d \\ b_1 Q_1^{-1} R_1 + C_1 \\ b_2 Q_2^{-1} R_2 + C_2 \end{matrix} \right| \right] \end{vmatrix} \tag{14}$$

After putting (3) and (14) in (5) the analytical solutions of the initial problem (1) are

$$x_1^{opt} = -Q_1^{-1}R_1 + Q_1^{-1} \left[ \left|a_1^T \quad b_1^T\right| \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & 0 \end{matrix} \left| \begin{matrix} a_1 Q_1^{-1} R_1 + a_2 Q_2^{-1} R_2 + d \\ b_1 Q_1^{-1} R_1 + C_1 \\ b_2 Q_2^{-1} R_2 + C_2 \end{matrix} \right| \right] \tag{15}$$

$$x_2^{opt} = -Q_2^{-1}R_2 + Q_2^{-1} \left[ \left|a_2^T \quad b_2^T\right| \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{31} & 0 & \alpha_{33} \end{matrix} \left| \begin{matrix} a_1 Q_1^{-1} R_1 + a_2 Q_2^{-1} R_2 + d \\ b_1 Q_1^{-1} R_1 + C_1 \\ b_2 Q_2^{-1} R_2 + C_2 \end{matrix} \right| \right]$$

Analytical relations (15) are a result of applying a goal coordination for solving the initial problem (1). They are useful only if the components $\alpha_{ij}$ of the inverse matrix (12) are known. However, if $\alpha_{ij}$ are not known (the usual case) relations (15) can not be applied.

### 2.2 Decomposition of the initial problem by predictive coordination

According to the hierarchical approach, the subsystems work independently. The idea of the predictive coordination is that the coordinator influences to each subsystem by independent impacts instead of common impact in goal coordination. For the initial problem (1)

decomposition by goal coordination can not be fully accomplished because of the connected relation $a_1x_1 + a_2x_2 = d$ . Applying predictive coordination, the connected restriction can be decomposed to:

$$a_1x_1 = y_1 \; ; \; a_2x_2 = y_2 \tag{16}$$

observing the condition for resource limitation

$$y_1 + y_2 = d \tag{17}$$

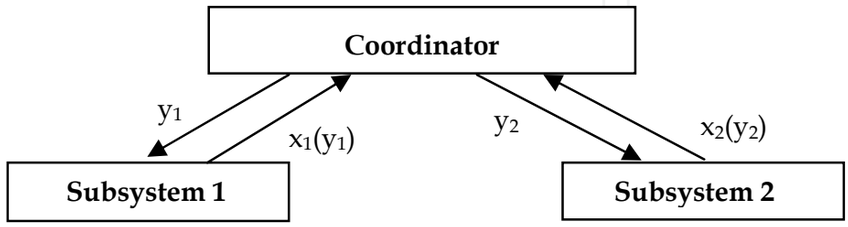



Fig. 1. Hierarchical approach for solving (1)

Applying (16), the initial optimization problem (1) is decomposed to two optimization subproblems with lower dimensions than the initial one:

$$\min\left\{\frac{1}{2}x_1^TQ_1x_1 + R_1^Tx_1\right\} \; \min\left\{\frac{1}{2}x_2^TQ_2x_2 + R_2^Tx_2\right\} \tag{18}$$

$$\underbrace{a_1}_{m_0xn_1} \; x_1 = \underbrace{y_1}_{m_0x1} \quad \underbrace{a_2}_{m_0xn_2} \; x_2 = \underbrace{y_2}_{m_0x1}$$

$$\underbrace{b_1}_{m_1xn_1} \; x_1 = \underbrace{C_1}_{m_1x1} \; ; \; \underbrace{b_2}_{m_2xn_2} \; x_2 = \underbrace{C_2}_{m_2x1}$$

where

$$A_1\big|_{(m_0+m_1+m_2)xn_1} = \begin{vmatrix} a_1 \\ b_1 \\ 0 \end{vmatrix}\begin{matrix} m_0xn_1 \\ m_1xn_1 \\ m_2xn_1 \end{matrix} ; A_2\big|_{(m_0+m_1+m_2)xn_2} = \begin{vmatrix} a_2 \\ 0 \\ b_2 \end{vmatrix}\begin{matrix} m_0xn_2 \\ m_1xn_2 \\ m_2xn_2 \end{matrix} ; \overset{=}{y}_1 = \begin{vmatrix} d_1 \\ C_1 \\ 0 \end{vmatrix}\begin{matrix} m_0x1 \\ m_1x1 \\ m_2x1 \end{matrix} ; \overset{=}{y}_2 = \begin{vmatrix} d_2 \\ 0 \\ C_2 \end{vmatrix}\begin{matrix} m_0x1 \\ m_1x1 \\ m_2x1 \end{matrix}$$

and it can be realized

$$\overset{=}{y}_1 + \overset{=}{y}_2 = D \text{ or } y_1 + y_2 = d \; .$$

The analytical solution of the first subproblem (18), according to (4), is

$$x_1^{opt} = -Q_1^{-1}\left[R_1 - A_1^T(A_1Q_1^{-1}A_1^T)^{-1}(A_1Q_1^{-1}R_1 + D)\right].$$

The analysis of matrix $A_1$ shows that it has zero rows. Respectively, the square matrix

$$A_1 Q_1^{-1} A_1^T = \left| \begin{matrix} a_1 \\ b_1 \\ 0 \end{matrix} \right|_{(m+l_1+l_2)xn_1} Q_1 \Big|_{n_1 x n_1} \left| a_1^T \quad b_1^T \quad 0 \right|_{n_1 x (m+l_1+l_2)} = \left| \begin{matrix} a_1 Q_1^{-1} a_1^T & a_1 Q_1^{-1} b_1^T & 0 \\ b_1 Q_1^{-1} a_1^T & b_1 Q_1^{-1} b_1^T & 0 \\ 0 & 0 & 0 \end{matrix} \right|$$

has zero rows and columns, which means that the inverse matrix $(AQ^{-1}A^T)^{-1}$ does not exist. However, in the solution of problem (1) takes part a sum of the matrices $(A_i Q_i^{-1} A_i^T)^{-1}$, so that the sum matrix $(AQ^{-1}A^T)^{-1}$ has a full rank. This matrix has a high dimension and for it can not be used the specific structure $A_i Q_i^{-1} A_i^T$. To use the less rank of matrices $A_i Q_i^{-1} A_i^T$, the definition of subproblems (18) has to be done by rejecting the zero rows in matrices $A_1$ and $A_2$.

Respectively, the subproblems are obtained of the initial problem (1) by additional modification of the admissible areas, determined by the matrices $A_1$ and $A_2$ instead of direct decomposition. In that manner the modified subproblems will present only the corresponding meaning components as follows:

$$A_1 = \left| \begin{matrix} a_1 \\ b_1 \\ 0 \end{matrix} \right| \Rightarrow \bar{A}_1 = \left| \begin{matrix} a_1 \\ b_1 \end{matrix} \right| ; A_2 = \left| \begin{matrix} a_2 \\ 0 \\ b_2 \end{matrix} \right| \Rightarrow \bar{A}_2 = \left| \begin{matrix} a_2 \\ b_2 \end{matrix} \right| ; \overset{=}{y}_1 = \left| \begin{matrix} y_1 \\ C_1 \\ 0 \end{matrix} \right| \Rightarrow \bar{y}_1 = \left| \begin{matrix} y_1 \\ C_1 \end{matrix} \right| ; \overset{=}{y}_2 = \left| \begin{matrix} y_2 \\ 0 \\ C_2 \end{matrix} \right| \Rightarrow \bar{y}_2 = \left| \begin{matrix} y_2 \\ C_2 \end{matrix} \right| .$$

The modified subproblems (19) have lower dimension in comparison with (18), obtained by direct decomposition

$$\min \left\{ \frac{1}{2} x_1^T Q_1 x_1 + R_1^T x_1 \right\} ; \min \left\{ \frac{1}{2} x_2^T Q_2 x_2 + R_2^T x_2 \right\} \tag{19}$$

$$\bar{A}_1 x_1 = \bar{y}_1 \; ; \bar{A}_2 x_2 = \bar{y}_2$$

$$\bar{A}_1 = \left| \begin{matrix} a_1 \\ b_1 \end{matrix} \right|_{(m_0+m_1)xn_1} ; \bar{A}_2 = \left| \begin{matrix} a_2 \\ b_2 \end{matrix} \right|_{(m_0+m_2)xn_2}$$

$$\bar{y}_1 = \left| \begin{matrix} y_1 \\ C_1 \end{matrix} \right|_{(m_0+m_1)x1} ; \bar{y}_2 = \left| \begin{matrix} y_2 \\ C_2 \end{matrix} \right|_{(m_0+m_2)x1} .$$

The solutions of (18), obtained in analytical forms using (4), are

$$x_i^{opt} = -Q_i^{-1} \left[ R_i - \bar{A}_i^T (\bar{A}_i Q_i^{-1} \bar{A}_i^T)^{-1} (\bar{A}_i Q_i^{-1} R_i + \bar{y}_i) \right] \quad i = 1,2 \cdot$$

After substitution of matrices $\bar{A}_i$ with the corresponding matrix components, it follows

$$x_1(y_1) = -Q_1^{-1} \left[ R_1 - \left| a_1^T \quad b_1^T \right| \left| \begin{matrix} a_1 Q_1^{-1} a_1^T & a_1 Q_1^{-1} b_1^T \\ b_1 Q_1^{-1} b_1^T & b_1 Q_1^{-1} a_1^T \end{matrix} \right|^{-1} \left| \begin{matrix} a_1 Q_1^{-1} R_1 + y_1 \\ b_1 Q_1^{-1} R_1 + C_1 \end{matrix} \right| \right] .$$

It is put

$$\begin{vmatrix} \underbrace{a_1 Q_1^{-1} a_1^T}_{m_0 x m_0} & \underbrace{a_1 Q_1^{-1} b_1^T}_{m_0 x m_1} \\ \underbrace{b_1 Q_1^{-1} a_1^T}_{m_1 x m_0} & \underbrace{b_1 Q_1^{-1} b_1^T}_{m_1 x m_1} \end{vmatrix}^{-1} = \beta = \begin{vmatrix} \underbrace{\beta_{11}}_{m_0 x m_0} & \underbrace{\beta_{12}}_{m_0 x m_1} \\ \underbrace{\beta_{21}}_{m_1 x m_0} & \underbrace{\beta_{22}}_{m_1 x m_1} \end{vmatrix}. \tag{20}$$

where the matrix $\beta$ is a symmetric one by definition. Consequently, $x_1(y_1)$ can be developed to:

$$x_1(y_1) = -Q_1^{-1} R_1 + Q_1^{-1} \left[ \begin{vmatrix} a_1^T & b_1^T \end{vmatrix} \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix} \begin{vmatrix} a_1 Q_1^{-1} R_1 + y_1 \\ b_1 Q_1^{-1} R_1 + C_1 \end{vmatrix} \right]. \tag{21}$$

Analogically, $x_2(y_2)$ is

$$x_2(y_2) = -Q_2^{-1} R_2 + Q_2^{-1} \left[ \begin{vmatrix} a_2^T & b_2^T \end{vmatrix} \begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} \begin{vmatrix} a_2 Q_2^{-1} R_2 + y_2 \\ a_2 Q_2^{-1} R_2 + C_2 \end{vmatrix} \right]. \tag{22}$$

where matrix $\gamma$ is a symmetric one by definition, $\gamma_{12} = \gamma_{21}^T$

$$\gamma = \begin{vmatrix} \underbrace{\gamma_{11}}_{m_0 x m_0} & \underbrace{\gamma_{12}}_{m_0 x m_2} \\ \underbrace{\gamma_{21}}_{m_2 x m_0} & \underbrace{\gamma_{22}}_{m_2 x m_2} \end{vmatrix} = \begin{vmatrix} \underbrace{a_2 Q_2^{-1} a_2^T}_{m_0 x m_0} & \underbrace{a_2 Q_2^{-1} b_2^T}_{m_0 x m_2} \\ \underbrace{b_2 Q_1^{-1} a_2^T}_{m_2 x m_0} & \underbrace{b_2 Q_2^{-1} b_2^T}_{m_2 x m_2} \end{vmatrix}^{-1}.$$

If the optimal resources $y_1^{opt}$, $y_2^{opt}$ are known, after their substitution in (21)-(22), the solution of the initial problem (1) can be obtained

$$x_1^{opt} = x_1(y_1^{opt}), \qquad x_2^{opt} = x_2(y_2^{opt}) \ .$$

The determination of the optimal resources $y_1^{opt}$, $y_2^{opt}$ is done by solution of the coordination problem.

### 2.2.1 Determination of the coordination problem

After substitution of relations $x_1(y_1)$ and $x_2(y_2)$ in the initial problem (1) and taking into account the resource constraint (17), the coordination problem becomes

$$\min_{y \in S_y} w(y) = \min \left\{ \frac{1}{2} x_1^T(y_1) Q_1 x_1(y_1) + R_1^T x_1(y_1) + \frac{1}{2} x_2^T(y_2) Q_2 x_2(y_2) + R_2^T x_2(y_2) \right\}$$

$$S_y \equiv y_1 + y_2 = d \ ,$$

or

$$\min\{w(y) = w_1(y_1) + w_2(y_2)\} \tag{23}$$

$$y_1 + y_2 = d$$

where

$$w_i(y_i) = \frac{1}{2} x_i^T(y_i) Q_i x_i(y_i) + R_i^T x_i(y_i), \quad i = 1,2 .$$

As $x_i(y_i)$ is inexplicit function, it can be approximated in Mac-Laurin series at point $y_i$=0

$$x_1(y_1)_{n_1 x 1} = x_{10 n_1 x 1} + X_{1 n_1 x m_0} y_{1 m_0 x 1} \tag{24}$$

where

$$x_{i0}^1 = -Q_1^{-1} R_1 + Q_1^{-1} \begin{vmatrix} a_1^T & b_1^T \end{vmatrix} \begin{Vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{Vmatrix} \begin{vmatrix} a_1 Q_1^{-1} R_1 \\ b_1 Q_1^{-1} R_1 + C_1 \end{vmatrix} \tag{25}$$

$$X_{1 n_1 x m_0} = Q_1^{-1}{}_{n_1 x n_1} \begin{vmatrix} \underbrace{a_1^T}_{n_1 x m_0} & \underbrace{b_1^T}_{n_1 x m_1} \end{vmatrix} \begin{Vmatrix} \beta_{11 m_0 x m_0} \\ \beta_{21 m_1 x m_0} \end{Vmatrix} \tag{26}$$

where $x_{10}$ is solution of subproblem (19) having zero resource, $y_1$=0 .
Analogically, for the second subproblem is valid:

$$x_2(y_2)_{n_2 x 1} = x_{20 n_2 x 1} + X_{2 n_2 x m_0} y_{2 m_0 x 1} \tag{27}$$

where

$$x_{20} = -Q_2^{-1} R_2 + Q_2^{-1} \begin{vmatrix} a_2^T & b_2^T \end{vmatrix} \begin{Vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{Vmatrix} \begin{vmatrix} a_2 Q_2^{-1} R_2 \\ b_2 Q_2^{-1} R_2 + C_2 \end{vmatrix} \tag{28}$$

$$X_{2 n_2 x m_0} = Q_2^{-1}{}_{n_2 x n_2} \begin{vmatrix} \underbrace{a_2^T}_{n_2 x m_0} & \underbrace{b_2^T}_{n_2 x m_2} \end{vmatrix} \begin{Vmatrix} \gamma_{11 m_0 x m_0} \\ \gamma_{21 m_2 x m_0} \end{Vmatrix} \tag{29}$$

After substitution of (24) in $w_1(y_1)$ of (23), it follows:

$$w_1(y_1) = \frac{1}{2}(x_{10}^T + y_1^T X_1^T) Q_1 (x_{10} + X_1 y_1) + R_1^T (x_{10} + X_1 y_1) =$$

$$= \frac{1}{2} x_{10}^T Q_1 x_{10} + \frac{1}{2} x_{10}^T Q_1 X_1 y_1 + \frac{1}{2} y_1^T X_1^T Q_1 x_{10} + \frac{1}{2} y_1^T X_1^T Q_1 X_1 y_1 + R_1^T x_{10} + R_1^T X_1 y_1$$

The components $x_{10}^T Q_1 X_1 y_1$ and $y_1^T X_1^T Q_1 x_{10}$ are equal, as they are transposed of corresponding equal relations. Consequently, the coordination problem becomes

$$w_1(y_1) \equiv \frac{1}{2} y_1^T X_1^T Q_1 X_1 y_1 + y_1^T X_1^T Q_1 x_{10} + y_1^T X_1^T R_1 \tag{30}$$

or

$$w_1(y_1) \equiv \frac{1}{2} y_1^T q_1 y_1 + y_1^T r_1$$

where

$$q_1 = X_1^T Q_1 X_1 \, ; \, r_1 = X_1^T Q_1 x_{10} + X_1^T R_1$$

Analogically, for the second subproblem, it follows:

$$w_2(y_2) \equiv \frac{1}{2} y_2^T X_2^T Q_2 X_2 y_2 + y_2^T X_2^T Q_2 x_{20} + y_2^T X_2^T R_2 \tag{31}$$

or

$$w_2(y_2) \equiv \frac{1}{2} y_2^T q_2 y_2 + y_2^T r_2$$

where

$$q_2 = X_2^T Q_2 X_2 \, ; \, r_2 = X_2^T Q_2 x_{20} + X_2^T R_2 \, .$$

Functions $w_i(y_i)$ has to be presented in terms of the initial problem (1) by the following transformations.

*Development of $q_1$*

Relation $q_1$ is presented like

$$q_1 = X_1^T Q_1 X_1 = \left| \beta_{11}^T \quad \beta_{21}^T \right| \left| \begin{matrix} a_1 \\ b_1 \end{matrix} \right| Q_1^{-1} \underbrace{Q_1 Q_1^{-1}}_{I} \left| a_1^T \quad b_1^T \right| \left| \begin{matrix} \beta_{11} \\ \beta_{21} \end{matrix} \right| = \left| \beta_{11}^T \quad \beta_{21}^T \right| \left| \begin{matrix} a_1 \\ b_1 \end{matrix} \right| Q_1^{-1} \left| a_1^T \quad b_1^T \right| \left| \begin{matrix} \beta_{11} \\ \beta_{21} \end{matrix} \right|$$

or

$$q_1 = \left| \beta_{11}^T \quad \beta_{21}^T \right| \left| \begin{matrix} a_1 Q_1^{-1} a_1^T & a_1 Q_1^{-1} b_1^T \\ b_1 Q_1^{-1} a_1^T & b_1 Q_1^{-1} b_1^T \end{matrix} \right| \left| \begin{matrix} \beta_{11} \\ \beta_{21} \end{matrix} \right| = \left| \beta_{11}^T \quad \beta_{21}^T \right| \left| \begin{matrix} a_1 Q_1^{-1} a_1^T \beta_{11} + a_1 Q_1^{-1} b_1^T \beta_{21} \\ b_1 Q_1^{-1} a_1^T \beta_{11} + b_1 Q_1^{-1} b_1^T \beta_{21} \end{matrix} \right| . \tag{32}$$

According to the manner of definition of matrix $\beta$ from (20) the following matrix equality is performed

$$\left| \begin{matrix} \underbrace{a_1 Q_1^{-1} a_1^T}_{m_0 x m_0} & \underbrace{a_1 Q_1^{-1} b_1^T}_{m_0 x m_1} \\ \underbrace{b_1 Q_1^{-1} a_1^T}_{m_1 x m_0} & \underbrace{b_1 Q_1^{-1} b_1^T}_{m_1 x m_1} \end{matrix} \right| \left| \begin{matrix} \underbrace{\beta_{11}}_{m_0 x m_0} & \underbrace{\beta_{12}}_{m_0 x m_1} \\ \underbrace{\beta_{21}}_{m_1 x m_0} & \underbrace{\beta_{22}}_{m_1 x m_1} \end{matrix} \right|^{-1} = \left| \begin{matrix} \underbrace{I}_{m_0 x m_0} & \underbrace{0}_{m_0 x m_1} \\ \underbrace{0}_{m_1 x m_0} & \underbrace{I}_{m_1 x m_1} \end{matrix} \right| \tag{33}$$

i.e. an unit matrix is obtained. Consequently, the following equations are performed

$$a_1 Q_1^{-1} a_1^T \beta_{11} + a_1 Q_1^{-1} b_1^T \beta_{21} = I_{m_0 x m_0} \tag{34}$$

$$b_1 Q_1^{-1} a_1^T \beta_{11} + b_1 Q_1^{-1} b_1^T \beta_{21} = 0_{m_1 x m_0} .$$

After substitution of (34) in (32) it is obtained

$$q_1 = \left| \underbrace{\beta_{11}^T}_{m_0 x m_0} \quad \underbrace{\beta_{21}^T}_{m_0 x m_1} \right| \left| \begin{matrix} \underbrace{I}_{m_0 x m_0} \\ \underbrace{0}_{m_1 x m_0} \end{matrix} \right| = \underbrace{\beta_{11}}_{m_0 x m_0}$$

as $\beta_{11}$ is a symmetric and square or

$$q_1 = \beta_{11} . \tag{35}$$

Analogically,

$$q_2 = \gamma_{11} \tag{36}$$

*Development of $r_i$.*
In a similar way the relations of $r_i$ are developed to the expressions

$$r_1 = X_1^T (Q_1 x_{10} + R_1) = \left| \beta_{11}^T \quad \beta_{21}^T \right| \left| \begin{matrix} a_1 Q_1^{-1} R_1 \\ b_1 Q_1^{-1} R_1 + C_1 \end{matrix} \right| \tag{37}$$

$$r_2 = X_2^T (Q_2 x_{20} + R_2) = \left| \gamma_{11}^T \quad \gamma_{21}^T \right| \left| \begin{matrix} a_2 Q_2^{-1} R_2 \\ b_2 Q_2^{-1} R_2 + C_2 \end{matrix} \right| .$$

The coordination problem becomes

$$\min \left\{ w_1(y_1) + w_2(y_2) = \frac{1}{2} y_1^T q_1 y_1 + r_1^T y_1 + \frac{1}{2} y_2^T q_2 y_2 + r_2^T y_2 \right\} \tag{38}$$

$$y_1 + y_2 = d \quad \Rightarrow \quad \left| I_{m_0 x m_0} \quad I_{m_0 x m_0} \right|_{m_0 x 2 m_0} ; y = \left| \begin{matrix} y_1 \\ y_2 \end{matrix} \right| ; \; q = \left| \begin{matrix} q_1 & 0 \\ 0 & q_2 \end{matrix} \right| \; r = \left| \begin{matrix} r_1 \\ r_2 \end{matrix} \right| ;$$

The coordination problem (38) is a linear-quadratic one and its solution can be found in an analytical form according to (4) or

$$y^{opt} = -q^{-1} \left[ r - A_I^T (A_I q^{-1} A_I^T)^{-1} (A_I q^{-1} r + d) \right] . \tag{39}$$

Relation (39) is developed additionally to the form

$$y_1^{opt} = -q_1^{-1} r_1 + q_1^{-1} (q_1^{-1} + q_2^{-1})^{-1} (q_1^{-1} r_1 + q_2^{-1} r_2 + d) \tag{40}$$

$$y_2^{opt} = -q_2^{-1} r_2 + q_2^{-1} (q_1^{-1} + q_2^{-1})^{-1} (q_1^{-1} r_1 + q_2^{-1} r_2 + d) .$$

### 2.2.2 Presenting the resources $y_i^{opt}$ in terms of the initial problem

It is necessary the values of $y_i^{opt}$ to be presented by the matrices and vectors of the initial problem $a_i$, $b_i$, $Q_i$, $R_i$, $C_i$, $\beta$, $\gamma$, $i$=1,2. According to (35) and (36), it is performed

$$q_1 = \beta_{11} \quad \Rightarrow \quad q_1^{-1} = \beta_{11}^{-1} \tag{41}$$

$$q_2 = \gamma_{11} \quad \Rightarrow \quad q_2^{-1} = \gamma_{11}^{-1}.$$

After additional transformations it follows

$$y_1^{opt} = \beta_{11}^{-1}\left|\beta_{11}^T \quad \beta_{21}^T\right|\left|\begin{matrix} a_1Q_1^{-1}R_1 \\ b_1Q_1^{-1}R_1 + C_1 \end{matrix}\right| + \beta_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\left[\beta_{11}^{-1}\left|\beta_{11}^T \quad \beta_{21}^T\right|\left|\begin{matrix} a_1Q_1^{-1}R_1 \\ b_1Q_1^{-1}R_1 + C_1 \end{matrix}\right| + \gamma_{11}^{-1}\left|\gamma_{11}^T \quad \gamma_{21}^T\right|\left|\begin{matrix} a_2Q_2^{-1}R_2 \\ b_2Q_2^{-1}R_2 + C_2 \end{matrix}\right| + d\right] \tag{42}$$

$$y_2^{opt} = \gamma_{11}^{-1}\left|\gamma_{11}^T \quad \gamma_{21}^T\right|\left|\begin{matrix} a_2Q_2^{-1}R_2 \\ b_2Q_2^{-1}R_2 + C_2 \end{matrix}\right| + \gamma_{11}^{-1}(\gamma_{11}^{-1}+\beta_{11}^{-1})^{-1}\left[\beta_{11}^{-1}\left|\beta_{11}^T \quad \beta_{21}^T\right|\left|\begin{matrix} a_1Q_1^{-1}R_1 \\ b_1Q_1^{-1}R_1 + C_1 \end{matrix}\right| + \gamma_{11}^{-1}\left|\gamma_{11}^T \quad \gamma_{21}^T\right|\left|\begin{matrix} a_2Q_2^{-1}R_2 \\ b_2Q_2^{-1}R_2 + C_2 \end{matrix}\right| + d\right].$$

After substitution of optimal resources $y_i^{opt}$, $i$=1,2 from (42) in the expressions of $x_1(y_1)$ from (21) and $x_2(y_2)$ from (22) the analytical relations $x_1(y_1^{opt})$ and $x_2(y_2^{opt})$, which are solutions of the initial problem (1) are obtained. To get the explicit analytical form of relations $x_i(y_i^{opt})$, (42) is substituted in (21) and (22) and after transformations follows

$$x_1(y_1^{opt}) = -Q_1^{-1}R_1 + Q_1^{-1}\left|a_1^T \quad b_1^T\right| \begin{vmatrix} (\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1} & (\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^T & (\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^T \\ \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1} & \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^T + & \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^T \\ & +\beta_{22} - \beta_{21}\beta_{11}^{-1}\beta_{21}^T & \end{vmatrix} \tag{43}$$

$$* \begin{vmatrix} a_1Q_1^{-1}R_1 + a_2Q_2^{-1}R_2 + d \\ b_1Q_1^{-1}R_1 + C_1 \\ b_2Q_2^{-1}R_2 + C_2 \end{vmatrix}$$

$$x_2(y_2^{opt}) = -Q_2^{-1}R_2 + Q_2^{-1}\left|a_2^T \quad b_2^T\right| \begin{vmatrix} (\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1} & (\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^T & (\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^T \\ \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1} & \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^T & \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^T + \\ & & +\gamma_{22} - \gamma_{21}\gamma_{11}^{-1}\gamma_{21}^T \end{vmatrix} * \tag{44}$$

$$\begin{vmatrix} a_1Q_1^{-1}R_1 + a_2Q_2^{-1}R_2 + d \\ b_1Q_1^{-1}R_1 + C_1 \\ b_2Q_2^{-1}R_2 + C_2 \end{vmatrix}.$$

The obtained results in (43) and (44) $x_i(y_i^{opt})$, $i = 1,2$ are after applying the predictive coordination for solving the initial problem (1). The solutions $x_i^{opt}$, $i = 1,2$ from (15) are obtained by applying goal coordination to the same initial problem. As the solutions $x_i(y_i^{opt})$, $i = 1,2$ and $x_i^{opt}$, $i = 1,2$ are equal, after equalization of (15) with (43) and (44)

relations among the components of the inverse matrix $\alpha$ and the components of the inverse $\beta$ and $\gamma$ are obtained. According to (15) and (43), it follows:

$$\begin{gathered}\alpha_{11}=(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1};\ \alpha_{12}=(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^{T};\ \alpha_{13}=(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^{T}\\ \alpha_{21}=\beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1};\ \alpha_{22}=\beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^{T}+\beta_{22}-\beta_{21}\beta_{11}^{-1}\beta_{21}^{T};\\ \alpha_{23}=\beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^{T}\\ \alpha_{31}=\gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1};\ \alpha_{32}=\gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^{T};\\ \alpha_{33}=\gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1}+\gamma_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^{T}+\gamma_{22}-\gamma_{21}\gamma_{11}^{-1}\gamma_{21}^{T}.\end{gathered} \tag{45}$$

Consequently, after applying the both coordination strategies towards the same initial problem (1) analytical relations (15) and respectively (43) and (44) are obtained. This allows to be received analytical relations among the components of the inverse matrices $\alpha$, $\beta$ and $\gamma$, which were not able to be determined directly because by definition:

$$\alpha=\begin{vmatrix} a_1Q_1^{-1}a_1^T+a_2Q_2^{-1}a_2^T & a_1Q_1^{-1}b_1^T & a_2Q_2^{-1}b_2^T\\ b_1Q_1^{-1}a_1^T & b_1Q_1^{-1}b_1^T & 0\\ b_2Q_2^{-1}a_2^T & 0 & b_2Q_2^{-1}b_2^T\end{vmatrix}^{-1};$$

$$\beta=\begin{vmatrix} a_1Q_1^{-1}a_1^T & a_1Q_1^{-1}b_1^T\\ b_1Q_1^{-1}a_1^T & b_1Q_1^{-1}b_1^T\end{vmatrix}^{-1}\quad \gamma=\begin{vmatrix} a_2Q_2^{-1}a_2^T & a_2Q_2^{-1}b_2^T\\ b_2Q_2^{-1}a_2^T & b_2Q_2^{-1}b_2^T\end{vmatrix}^{-1}.$$

Consequently, using (45) the components of the inverse matrix $\alpha$ can be determined when $\beta$ and $\gamma$ are given. This allows the matrix $\alpha$ to be determined by fewer calculations in comparison with its direct inverse transformation because the inverse matrices $\beta$ and $\gamma$ have less dimensions. Relations (45) can be applied for calculation of the components $\alpha_{ij}$ of the inverse matrix $\alpha$ (with large dimension) by finding the inverse matrices $\beta$ and $\gamma$ (with fewer dimensions). The computational efficiency for evaluating the inverse matrix with high dimension using relations (45) is preferable in comparison with its direct calculation (Stoilova & Stoilov, 2007).

## 3. Predictive coordination for block-diagonal problem of quadratic programming with three and more subsystems

Analytical relations for predictive coordination strategy for the case when the subsystems in the bi-level hierarchy are more than two are developed. The case of bi-level hierarchical system with three subsystems is considered, figure 2

The initial optimization problem, solved by the hierarchical system is stated as

$$\min\frac{1}{2}\begin{vmatrix} x_1^T & x_2^T & x_3^T\end{vmatrix}\begin{vmatrix} Q_1 & 0 & 0\\ 0 & Q_2 & 0\\ 0 & 0 & Q_3\end{vmatrix}\begin{vmatrix} x_1\\ x_2\\ x_3\end{vmatrix}+\begin{vmatrix} R_1^T & R_2^T & R_3^T\end{vmatrix}\begin{vmatrix} x_1\\ x_2\\ x_3\end{vmatrix} \tag{46}$$

$$a_1 x_{1m_0xn_1} + a_2 x_{2m_0xn_2} + a_3 x_{3m_0xn_3} = d \tag{47}$$

$$\begin{aligned} b_1 x_{1m_1xn_1} &= C_1 \\ b_2 x_{2m_2xn_2} &= C_2 \\ b_3 x_{3m_3xn_3} &= C_3 \end{aligned}$$

where the dimensions of the vectors and matrices are appropriately defined

$$x_{1n_1x1}\,;\, x_{2n_2x1}\,;\, x_{3n_3x1}\,;\, R_{1n_1x1}\,;\, R_{2n_2x1}\,;\, R_{3n_3x1}\,;$$

$$a_{1m_0xn_1}\,;\, a_{2m_0xn_2}\,;\, a_{3m_0xn_3}\,;\, d_{\ m_0x1}\,;$$

$$b_{1m_1xn_1}\,;\, b_{2m_2xn_2}\,;\, b_{3m_3xn_3}\,;\, C_{1m_1x1}\,;\, C_{2m_2x1}\,;\, C_{3m_3x1}\,;$$

$$C = \begin{vmatrix} d \\ C_1 \\ C_2 \\ C_3 \end{vmatrix}_{(m_0+m_1+m_2+m_3)x(n_1+n_2+n_3)}\!\!\!

$$A = \left| A_1 \quad A_2 \quad A_3 \right|_{(m_0+m_1+m_2+m_3)x(n_1+n_2+n_3)}.$$

Analogically to the previous case with two subsystems, analytical relations for determining the inverse matrix components by matrices with fewer dimensions are obtained

$$\alpha_{11} = (\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1};\ \alpha_{12} = (\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^{T} \tag{49}$$

$$\alpha_{13} = (\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^{T};\ \alpha_{14} = (\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\delta_{11}^{-1}\delta_{21}^{T}$$

$$\alpha_{21} = \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1};\ \alpha_{22} = \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^{T} + \beta_{22} - \beta_{21}\beta_{11}^{-1}\beta_{21}^{T}$$

$$\alpha_{23} = \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^{T};\ \alpha_{24} = \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\delta_{11}^{-1}\delta_{21}^{T}$$

$$\alpha_{31} = \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1};\ \alpha_{32} = \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^{T}$$

$$\alpha_{33} = \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^{T} + \gamma_{22} - \gamma_{21}\gamma_{11}^{-1}\gamma_{21}^{T};\ \alpha_{34} = \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\delta_{11}^{-1}\delta_{21}^{T}$$

$$\alpha_{41} = \delta_{21}\delta_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1};\ \alpha_{42} = \delta_{21}\delta_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^{T}$$

$$\alpha_{43} = \delta_{21}\delta_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^{T};\ \alpha_{44} = \delta_{21}\delta_{11}^{-1}(\beta_{11}^{-1} + \gamma_{11}^{-1} + \delta_{11}^{-1})^{-1}\delta_{11}^{-1}\delta_{21}^{T} + \delta_{22} - \delta_{21}\delta_{11}^{-1}\delta_{21}^{T}.$$

The initial problem can be solved by four or more subsystems. The relations between the components of the matrix $\alpha = (\alpha_{ij})$ and the matrices with lower sizes $\beta, \gamma, \ldots \omega$ are given below

$$\begin{vmatrix}
\alpha_{11} = & \alpha_{12} = & \alpha_{13} = & & \alpha_{1j} = \\
(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} & (\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1}\beta_{11}^{-1}\beta_{21}^{T} & (\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1}\gamma_{11}^{-1}\gamma_{21}^{T} & \cdots & (\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * \\
 & & & & *\omega_{11}^{-1}\omega_{21}^{T} \\
\alpha_{21} = & \alpha_{22} = & \alpha_{23} = & & \alpha_{2j} = \\
\beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} & \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * & \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * & \cdots & \beta_{21}\beta_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * \\
 & *\beta_{11}^{-1}\beta_{21}^{T} + \beta_{22} - \beta_{21}\beta_{11}^{-1}\beta_{21}^{T} & *\gamma_{11}^{-1}\gamma_{21}^{T} & & *\omega_{11}^{-1}\omega_{21}^{T} \\
\alpha_{31} = & \alpha_{32} = & \alpha_{33} = & & \alpha_{3j} = \\
\gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} & \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * & \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * & \cdots & \gamma_{21}\gamma_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * \\
 & *\beta_{11}^{-1}\beta_{21}^{T} & *\gamma_{11}^{-1}\gamma_{21}^{T} + \gamma_{22} - \gamma_{21}\gamma_{11}^{-1}\gamma_{21}^{T} & & *\omega_{11}^{-1}\omega_{21}^{T} \\
\vdots & & & & \\
\alpha_{i1} = & \alpha_{i2} = & \alpha_{i3} = & & \alpha_{ij} = \\
\omega_{21}\omega_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} & \omega_{21}\omega_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * & \omega_{21}\omega_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * & \cdots & \omega_{21}\omega_{11}^{-1}(\beta_{11}^{-1} + \cdots + \omega_{11}^{-1})^{-1} * \\
 & *\beta_{11}^{-1}\beta_{21}^{T} & *\gamma_{11}^{-1}\gamma_{21}^{T} & & \omega_{11}^{-1}\omega_{21}^{T} + \omega_{22} - \omega_{21}\omega_{11}^{-1}\omega_{21}^{T}
\end{vmatrix} \tag{50}$$

## 4. Assessment of the calculation efficiency of the analytical results for determination of inverse matrix components

For simplicity of working the notations for right matrices with lower dimensions are introduced. In the case of 2 subsystems, the matrices *c* and *d* are the corresponding right

matrices of the inverse matrices $\beta$ and $\gamma$. By definition having in mind (46) $c$ and $d$ are symmetric ones

$$c=\begin{vmatrix} \underbrace{c_{11}}_{m_0xm_0} & \underbrace{c_{12}}_{m_0xm_1} \\ \underbrace{c_{21}}_{m_1xm_0} & \underbrace{c_{22}}_{m_1xm_1} \end{vmatrix}=\begin{vmatrix} \underbrace{a_1Q_1^{-1}a_1^T}_{m_0xm_0} & \underbrace{a_1Q_1^{-1}b_1^T}_{m_0xm_1} \\ \underbrace{b_1Q_1^{-1}a_1^T}_{m_1xm_0} & \underbrace{b_1Q_1^{-1}b_1^T}_{m_1xm_1} \end{vmatrix};\ \beta=\begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix}=c^{-1}$$

$$d=\begin{vmatrix} \underbrace{d_{11}}_{m_0xm_0} & \underbrace{d_{12}}_{m_0xm_2} \\ \underbrace{d_{21}}_{m_2xm_0} & \underbrace{d_{22}}_{m_2xm_2} \end{vmatrix}=\begin{vmatrix} \underbrace{a_2Q_2^{-1}a_2^T}_{m_0xm_0} & \underbrace{a_2Q_2^{-1}b_2^T}_{m_0xm_2} \\ \underbrace{b_2Q_1^{-1}a_2^T}_{m_2xm_0} & \underbrace{b_2Q_2^{-1}b_2^T}_{m_2xm_2} \end{vmatrix}\quad \gamma=\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix}=d^{-1}.$$

Analogically, for 3 subsystems the right matrix is $e$ and the corresponding inverse matrix is $\delta$ ($e-\delta$); for 4 subsystems - ($f-\varphi$).

An *example* for computational efficiency of the proposed relations (45) /2 subsystems is given below with a symmetric matrix $AL$ with dimension from 17×17 to 26×26 which varies according to variation of dimension $m_1$ from 2 to 11, while $m_0$ is 4 and $m_2$ is 11. Matrix $AL$ is in the form

$AL=$

| $m_0$=4 | $m_1$=2÷11 | $m_2$=11 |
|---|---|---|
| $c_{11}+d_{11}$ | $c_{12}$ | $d_{12}$ |
| $c_{21}$ | $c_{22}$ | 0 |
| $d_{21}$ | 0 | $d_{22}$ |

The MATLAB's codes are given below:

```
%example of inversion of AL  (dimension 26x26) when m1=11

c11=[1 2 3 1;  2 3 2 2;  3 2 4 2;   1 2 2 1];

c12=[ -2  1  0  -1  2  4   0  -1  -3  2  2;   -5  0  2  1  4  6  -5
7  1  9  1;   0  7  2  1  3  -8  9  1  0  2  3;   4  2 -1  0  2  -1
2  1  3  1  1];

c21=c12';

c22=[ 1   2   1  -3   1   0   1   5   2   8   2;
    2   3  -4   1   0   2   9   8   5   1  -3;
    1  -4   1  -1   1   5  -6   2  1    3   1;
    -3  1  -1   4   3   2   1   0  -1  -2   2;
          1   0   1   3   2   1   1   5  -3   3  -1;
    0   2   5   2   1   4   2   1   0   1   1;
    1   9  -6   1   1   2   6   3   1   0   2;
    5   8   2   0   5   1   3   5  -2   7   0;
          2   5   1  -1  -3  0   1  -2    3   1   1;
```

```
    8   1   3  -2   3  1   0    7   1   4   2;
    2  -3  1    2  -1  1   2   0    1   2  4];

c=[c11 c12; c21 c22];
d11=[3  0  0 -2;      0 -6  2  0;     0  2 -2 -1;      -2  0 -1  2];

d12=[1  0  2  1  3  -4  2  0  1  -3  1 ;
    3 -1  0  2 -2  -1  0  1  1   0  2 ;
    3  0  1  0  5   3  7  1  2  -2  0;
    2  2  0  1  3   2  1  0  1  -1  1];

d21=d12';

d22=[ 1   3  7  2  2  1  1  0  -2  -1  1;
          3   2  1  1  1 -1 -2 -3   0   1  0 ;
          7   1  5  2  4  2  3 -1  -3  -2  1 ;
          2   1  2  1  2  3  0  1  -2  -4  2;
          2   1  4  2  1 -1  3  7  -1   2  0;
          1  -1  2  3 -1  0  2  1   4   1  2;
          1  -2  3  0  3  2  2  1  -3  -1  1;
          0  -3 -1  1  7  1  1  0   1   2  1;
         -2  0 -3 -2 -1  4 -3  1  -2   0  2;
         -1  1 -2 -4  2  1 -1  2   0   1  1;
          1   0  1  2  0  2  1  1   2   1  5];

d=[d11 d12; d21 d22];

%definition of AL

mm=size(c12);
m0=mm(1);            %m0=4
m1=mm(2);            %m1=11
mm=size(d12);
m2=mm(2);            %m2=11
m10=zeros(m1,m2);
m20=m10';

al=[c11+d11   c12   d12;  c21   c22   m10;   d21   m20   d22];

flops(0);
alpha1=inv(al);         %direct inversion  of AL
fl_al=flops             %flops for direct matrix inversion

%matrix inversion by hierarchical approach
flops(0);

beta=inv(c);
gama=inv(d);
invbeta11=inv(beta(1:m0,1:m0));
invgama11=inv(gama(1:m0,1:m0));
invbeta11beta21T=invbeta11*beta(m0+1:m0+m1,1:m0)';
invgama11gama21T=invgama11*gama(m0+1:m0+m2,1:m0)';
```

```
ff=flops
flops(0);
alpha11=inv(invbeta11+invgama11);
alpha12=alpha11*invbeta11beta21T;
alpha13=alpha11*invgama11*gama(m0+1:m0+m2,1:m0)';

alpha21=alpha12';
alpha22=invbeta11beta21T'*alpha12+beta(m0+1:m0+m1,m0+1:m0+m1)-
beta(m0+1:m0+m1,1:m0)*invbeta11beta21T;
alpha23=invbeta11beta21T'*alpha13;

alpha31=alpha13';
alpha32=alpha23';
alpha33=invgama11gama21T'*alpha13+gama(m0+1:m0+m2,m0+1:m0+m2)-
gama(m0+1:m0+m2,1:m0)*invgama11gama21T;

alpha=[alpha11 alpha12 alpha13; alpha21 alpha22 alpha23; alpha31
alpha32 alpha33];

fl_nic=flops
fl_full=ff+fl_nic              %flops using noniterative coordination


al2=inv(alpha);          %verification
```

This code has been used for two types of calculations:

1. Direct calculation of $\alpha$ - inversion of matrix $AL$ by built-in MATLAB function *INV*. The amount of calculations is presented as a dashed red line in Figure 3.
2. Evaluation of $\alpha$ applying relations (45). The amount of calculation is presented as a solid blue line in Figure 3.

The comparison of the both manners of calculations shows that the analytical relations are preferable when the matrix dimension increases. From experimental considerations it is preferable to hold the relation $3m_0 < m_1 + m_2$, which gives boundaries for the decomposition of the initial matrix $AL$. For the initial case of $m_0$=4 $m_1$=2 $m_2$=11 these values are near to equality of the above relation and that is why the decomposition approach does not lead to satisfactory result.

*Second example* A 29×29 symmetric block-diagonal matrix denoted by $AL$ is considered. It has to be inversed to the matrix $\alpha$ by two manners: direct MATLAB's inversion and using relations (45) and (50). This matrix will be calculated by hierarchical approach and decomposition with 2, 3, and 4 subsystems.

*Case 1*. The right matrix $AL$ can be inversed to $\alpha$ by the above analytical relations applying 4 subsystems where $AL$ is presented by the matrices $c$, $d$, $e$ and $f$ in the manner:

$$AL = \begin{array}{|c|c|c|c|c|} \hline m_0 & m_1 & m_2 & m_3 & m_4 \\ \hline c_{11}+d_{11}+e_{11}+f_{11} & c_{12} & d_{12} & e_{12} & f_{12} \\ \hline c_{21} & c_{22} & 0 & 0 & 0 \\ \hline d_{21} & 0 & d_{22} & 0 & 0 \\ \hline e_{21} & 0 & 0 & e_{22} & 0 \\ \hline f_{21} & 0 & 0 & 0 & f_{22} \\ \hline \end{array}$$

$$AL=$$

| $m_0$=3 | | | $m_1$=6 | | | | | | $m_2$=6 | | | | | | $m_3$=7 | | | | | | | $m_4$=7 | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 1 | -3 | -1 | 2 | 4 | -1 | 2 | 1 | -4 | 1 | 0 | 1 | -3 | 2 | 0 | 2 | 1 | 1 | -2 | 0 | -3 | 1 | 0 | 3 | 6 | -1 | 1 | 4 |
| 1 | 4 | 5 | -3 | 1 | 0 | 5 | 4 | -1 | -2 | -1 | 0 | -2 | 1 | 4 | 2 | 4 | 0 | -5 | 7 | 1 | -4 | 3 | 2 | 1 | -1 | 4 | -1 | -2 |
| -3 | 5 | -4 | 3 | 2 | 1 | 0 | 3 | 2 | -2 | 1 | 0 | 1 | -1 | -2 | 1 | -2 | 1 | 3 | 1 | 0 | 1 | 4 | 1 | 3 | 2 | 4 | 1 | -2 |
| -1 | -3 | 3 | 3 | -4 | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 2 | -4 | 1 | -1 | 1 | 5 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 1 | 1 | -1 | 7 | 3 | 2 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -1 | 5 | 0 | 0 | 1 | 3 | 9 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 4 | 3 | 2 | 5 | 2 | 1 | 8 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | -1 | 2 | 1 | -1 | 3 | 2 | -1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -4 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | -2 | -1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -2 | 3 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -2 | 3 | -1 | 1 | -2 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | -2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 1 | -2 | -4 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -3 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | -2 | -4 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 4 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 4 | 2 | 3 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 5 | 1 | 1 | -2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 4 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 | 2 | 4 | 1 | 3 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 2 | 3 | 2 | 1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | -5 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 4 | 2 | 5 | -1 | 3 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -2 | 7 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | -1 | 6 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -2 | 3 | -1 | 3 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -3 | -4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 1 | -2 | -1 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 3 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 3 | 1 | 2 | 1 | 4 |
| 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 2 | 1 | 3 | -2 | 1 | 0 |
| 3 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 1 | 1 | -2 | -1 | -2 | 1 |
| 6 | -1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 | -2 | 1 | -3 | 1 | -2 |
| -1 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | -2 | -1 | -3 | 2 | 1 | -1 |
| 1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -2 | 1 | 1 | 3 | 1 |
| 4 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 1 | -2 | -1 | 1 | 1 |

$$c=$$

| $m_0$=3 | | | $m_1$=6 | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | -1 | 2 | -1 | 2 | 4 | -1 | 2 | 1 |
| -1 | 1 | 2 | -3 | 1 | 0 | 5 | 4 | -1 |
| 2 | 2 | 3 | 3 | 2 | 1 | 0 | 3 | 2 |
| -1 | -3 | 3 | 3 | -4 | 1 | 0 | 2 | 1 |
| 2 | 1 | 2 | -4 | 1 | -1 | 1 | 5 | -1 |
| 4 | 0 | 1 | 1 | -1 | 7 | 3 | 2 | 3 |
| -1 | 5 | 0 | 0 | 1 | 3 | 9 | 1 | 2 |
| 2 | 4 | 3 | 2 | 5 | 2 | 1 | 8 | -1 |
| 1 | -1 | 2 | 1 | -1 | 3 | 2 | -1 | 2 |

$$d=$$

| $m_0$=3 | | | $m_2$=6 | | | | | |
|---|---|---|---|---|---|---|---|---|
| -1 | -1 | -1 | -4 | 1 | 0 | 1 | -3 | 2 |
| -1 | 2 | 0 | -2 | -1 | 0 | -2 | 1 | 4 |
| -1 | 0 | -2 | -2 | 1 | 0 | 1 | -1 | -2 |
| -4 | -2 | -2 | 1 | -1 | -2 | -1 | -1 | 1 |
| 1 | -1 | 1 | -1 | -2 | 3 | 0 | 1 | 2 |
| 0 | 0 | 0 | -2 | 3 | -1 | 1 | -2 | 4 |
| 1 | -2 | 1 | -1 | 0 | 1 | -2 | -4 | 2 |
| -3 | 1 | 1 | -1 | 1 | -2 | -4 | 1 | 3 |
| 2 | 4 | -2 | 1 | 2 | 4 | 2 | 3 | 2 |

(51)

$e =$

| $m_0$=3 | | | $m_3$=7 | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | -2 | 0 | 2 | 1 | 1 | -2 | 0 | -3 |
| 0 | -4 | 2 | 2 | 4 | 0 | -5 | 7 | 1 | -4 |
| -2 | 2 | 1 | 1 | -2 | 1 | 3 | 1 | 0 | 1 |
| 0 | 2 | 1 | 2 | 1 | 5 | 1 | 1 | -2 | 1 |
| 2 | 4 | -2 | 1 | 3 | 2 | 4 | 1 | 3 | 2 |
| 1 | 0 | 1 | 5 | 2 | 3 | 2 | 1 | -1 | 1 |
| 1 | -5 | 3 | 1 | 4 | 2 | 5 | -1 | 3 | -2 |
| -2 | 7 | 1 | 1 | 1 | 1 | -1 | 6 | 1 | -1 |
| 0 | 1 | 0 | -2 | 3 | -1 | 3 | 1 | 2 | 1 |
| -3 | -4 | 1 | 1 | 2 | 1 | -2 | -1 | 1 | 3 |

$f =$

| $m_0$=3 | | | $m_4$=7 | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | -2 | 1 | 0 | 3 | 6 | -1 | 1 | 4 |
| 3 | 5 | 1 | 3 | 2 | 1 | -1 | 4 | -1 | -2 |
| -2 | 1 | -4 | 4 | 1 | 3 | 2 | 4 | 1 | -2 |
| 1 | 3 | 4 | 1 | -1 | 3 | 1 | 2 | 1 | 4 |
| 0 | 2 | 1 | -1 | 2 | 1 | 3 | -2 | 1 | 0 |
| 3 | 1 | 3 | 3 | 1 | 1 | -2 | -1 | -2 | 1 |
| 6 | -1 | 2 | 1 | 3 | -2 | 1 | -3 | 1 | -2 |
| -1 | 4 | 4 | 2 | -2 | -1 | -3 | 2 | 1 | -1 |
| 1 | -1 | 1 | 1 | 1 | -2 | 1 | 1 | 3 | 1 |
| 4 | -2 | -2 | 4 | 0 | 1 | -2 | -1 | 1 | 1 |

Here is assessed the efficiency of usage of relations (50) for finding the inverse matrix $\alpha$ when the matrices with fewer dimensions *c, d, e* and *f* are given. The assessment is done by measurement of "flops" in MATLAB environment. A part of the MATLAB's codes which performs relations (50) for inverse matrix calculations and assess the computational performance are given below

```
al=[c11+d11+e11   c12   d12   e12 ;   c21   c22   m10   m30;   d21
m20   d22   m40;   e21   m50   m60   e22];
flops(0);
alpha1=inv(al);
fl_al=flops
flops(0);
beta=inv(c);
gama=inv(d);
delta=inv(e);
invbeta11=inv(beta(1:m0,1:m0));
invgama11=inv(gama(1:m0,1:m0));
invdelta11=inv(delta(1:m0,1:m0));
invbeta11beta21T=invbeta11*beta(m0+1:m0+m1,1:m0)';
invgama11gama21T=invgama11*gama(m0+1:m0+m2,1:m0)';
invdelta11delta21T=invdelta11*delta(m0+1:m0+m3,1:m0)';
ff=flops
flops(0);
alpha11=inv(invbeta11+invgama11+invdelta11);
alpha12=alpha11*invbeta11beta21T;
alpha13=alpha11*invgama11*gama(m0+1:m0+m2,1:m0)';
alpha14=alpha11*invdelta11delta21T;
alpha21=alpha12';
alpha22=invbeta11beta21T'*alpha12+beta(m0+1:m0+m1,m0+1:m0+m1)-
beta(m0+1:m0+m1,1:m0)*invbeta11beta21T;
alpha23=invbeta11beta21T'*alpha13;
alpha24=alpha21*invdelta11delta21T;
alpha31=gama(m0+1:m0+m2,1:m0)*invgama11*alpha11;
alpha32=alpha31*invbeta11beta21T;
alpha33=invgama11gama21T'*alpha13+gama(m0+1:m0+m2,m0+1:m0+m2)-
gama(m0+1:m0+m2,1:m0)*invgama11gama21T;
alpha34=alpha31*invdelta11delta21T;
```

```
alpha41=delta(m0+1:m0+m3,1:m0)*invdelta11*alpha11;
alpha42=alpha41*invbeta11beta21T;
alpha43=alpha41*invgama11gama21T;
alpha44=alpha41*invdelta11delta21T+delta(m0+1:m0+m3,m0+1:m0+m3)-
delta(m0+1:m0+m3,1:m0)*invdelta11delta21T;
alpha=[alpha11   alpha12   alpha13   alpha14;
alpha21    alpha22   alpha23   alpha24;
alpha31    alpha32   alpha33   alpha34;
alpha41    alpha42   alpha43   alpha44];
fl_nic=flops;
fl_full=ff+fl_nic
al2=inv(alpha);%verification
```

For direct inversion of $AL$ the flops are 50220 and for using (50) - 16329, figure 4.

*Case 2.* The same matrix $AL$ is given however $\alpha$ is determined by a different manner - by 3 subsystems:

$$AL = \begin{array}{|c|c|c|c|} \hline c_{11}+d_{11}+e_{11} & c_{12} & d_{12} & e_{12} \\ \hline c_{21} & c_{22} & 0 & 0 \\ \hline d_{21} & 0 & d_{22} & 0 \\ \hline e_{21} & 0 & 0 & e_{22} \\ \hline \end{array}$$

where $c$ and $d$ are the same as in (51) , however the right matrix $e$ is different. It utilizes the previous matrices $e$ and $f$:

$e =$

| $m_0$=3 | | | $m_3$=14 | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | -4 | 0 | 2 | 1 | 1 | -2 | 0 | -3 | 1 | 0 | 3 | 6 | -1 | 1 | 4 |
| 3 | 1 | 3 | 2 | 4 | 0 | -5 | 7 | 1 | -4 | 3 | 2 | 1 | -1 | 4 | -1 | -2 |
| -4 | 3 | -5 | 1 | -2 | 1 | 3 | 1 | 0 | 1 | 4 | 1 | 3 | 2 | 4 | 1 | -2 |
| 0 | 2 | 1 | 2 | 1 | 5 | 1 | 1 | -2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 4 | -2 | 1 | 3 | 2 | 4 | 1 | 3 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 5 | 2 | 3 | 2 | 1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | -5 | 3 | 1 | 4 | 2 | 5 | -1 | 3 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -2 | 7 | 1 | 1 | 1 | 1 | -1 | 6 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | -2 | 3 | -1 | 3 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -3 | -4 | 1 | 1 | 2 | 1 | -2 | -1 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 3 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 3 | 1 | 2 | 1 | 4 |
| 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 2 | 1 | 3 | -2 | 1 | 0 |
| 3 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 1 | 1 | -2 | -1 | -2 | 1 |
| 6 | -1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 | -2 | 1 | -3 | 1 | -2 |
| -1 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | -2 | -1 | -3 | 2 | 1 | -1 |
| 1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -2 | 1 | 1 | 3 | 1 |
| 4 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 1 | -2 | -1 | 1 | 1 |

The calculations in flops for direct inversing of $AL$ are 50220 and using (49) - 23082, figure 4

*Case 3.* The inverse matrix $\alpha$ is determined by 2 subsystems and $AL$ is in the form

$$AL \quad \begin{array}{|c|c|c|} \hline c_{11}+d_{11} & c_{12} & d_{12} \\ \hline c_{21} & c_{22} & 0 \\ \hline d_{21} & 0 & d_{22} \\ \hline \end{array}$$

where $c$ is the same as in (51) but the right matrix $d$ covers $d$, $e$ and $f$ from *Case* 1 or $d$ and $e$ from *Case* 2.

The calculations for direct inversing $AL$ are 50232 flops and for using (45) are 37271, figure 4

The results of the experiments of the second example show that if the number of the subsystems increases, the computational efficiency increases because the matrices' dimensions decrease. This is in harmony with the multilevel hierarchical idea for decomposition of the initial problem leading to better efficiency of the system's functionality.

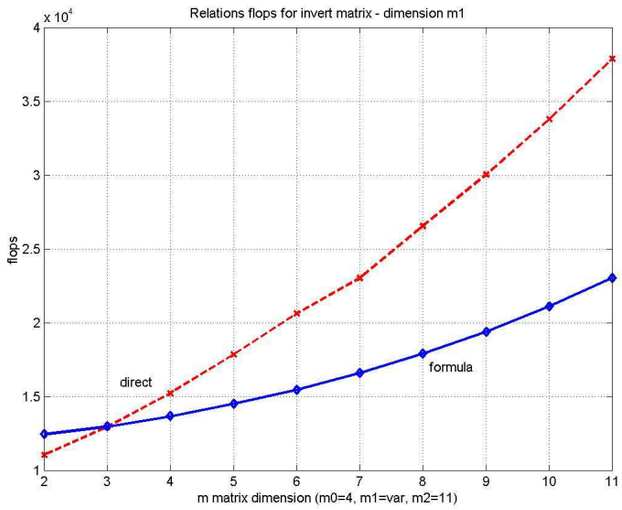



Fig. 3. Relation flops- $m_1$ matrix dimension

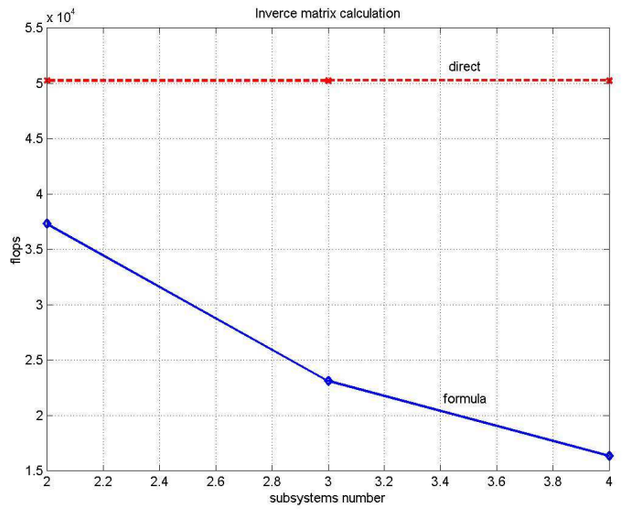



Fig. 4. Relation flops-subsystems number

## 5. Conclusion

The inverse matrix evaluations are decomposed to a set of operations, which does not consist of calculations of inverse high order matrix. Such decomposition benefits the inverse calculations when the initial large scale matrix is composed of low order matrices, which inverses are calculated with less computational efforts. The decomposition approach benefits the case when an initial matrix is known with its inverse, but few modifications of its components change it and new inverse has to be calculated. The decomposition approach for the inverse calculations is assessed and increase of the computational efficiency is illustrated. The MATLAB implementation of the presented sequence of calculations (49)−(50) is easy to perform because it results in consequent matrix sum and multiplications, and low order inverse matrix evaluations.

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A well-known statement says that the PID controller is the â€œbread and butterâ€ of the control engineer. This is indeed true, from a scientific standpoint. However, nowadays, in the era of computer science, when the paper and pencil have been replaced by the keyboard and the display of computers, one may equally say that MATLAB is the â€œbreadâ€ in the above statement. MATLAB has became a de facto tool for the modern system engineer. This book is written for both engineering students, as well as for practicing engineers. The wide range of applications in which MATLAB is the working framework, shows that it is a powerful, comprehensive and easy-to-use environment for performing technical computations. The book includes various excellent applications in which MATLAB is employed: from pure algebraic computations to data acquisition in real-life experiments, from control strategies to image processing algorithms, from graphical user interface design for educational purposes to Simulink embedded systems.

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