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# Reynolds Stress Transport Modelling

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## 1. Introduction

The Reynolds-averaged Navier-Stokes (RANS) approach is the most commonly employed approach in CFD for industrial applications, and is likely to continue to be so for the foreseeable future. The need to handle complex wall-bounded flows, and the need to evaluate large numbers of design variations usually prohibits high-fidelity approaches such as direct numerical simulation (DNS), and large-eddy simulation (LES). The application of Reynolds-averaging to the equations of motion introduces a set of unclosed terms, the *Reynolds Stresses*, into the mean flow momentum equations, and turbulence models are needed to provide closure of these terms before the set of equations can be solved. Within the framework of RANS approaches, a hierarchy of modelling schemes exists based on the level of sophistication in which these unclosed terms are modelled. In Reynolds stress transport (RST) modelling, rather than assuming a direct (linear or non-linear) link between the Reynolds stresses and mean strain, a separate transport equation for each of the stress components is solved. This in principle provides a number of advantages over other RANS models, which will be reviewed here.

This chapter aims to provide a general introduction and overview of Reynolds Stress transport modelling. The first section will provide a brief historical background on the development of this class of models. Next, the theoretical background and rationale underlying the most common modelling practises within this framework are presented. This is followed by a discussion of some numerical implementation issues specific to RST modelling within the context of the finite volume method. Finally the chapter is closed with some concluding remarks.

## 2. Development of RST modelling

Early work leading to the development of Reynold-stress transport (RST) modelling was mainly theoretical, due to the relative complexity of this level of modelling compared to the available computational capabilities of the time. Chou (1945) constructed a formal solution to the fluctuating pressure Poisson equation that is the basis for current models of the pressure-strain-rate correlation. Later Rotta (1951), laid the foundation for Reynolds stress transport modelling by being the first to develop a closed model of all the terms in the exact equation (Speziale, 1991). Because of limited computational capability at the time, successful computations were not carried out until several decades later (Speziale, 1991). Another important development came when the continuum mechanics community speculated on the potential similarity between turbulent flow and the flow of non-Newtonian fluids (Gatski,

2004). This meant that tensor representation results from the continuum mechanics literature could be used to formulate expressions for the Reynolds-stress tensor, as first proposed by Rivlin (1957). These ideas were then expanded by Crow (1968; 1967), and Lumley (1967; 1970). Computational work accelerated in the 1970's with the works of Daly & Harlow (1970), Reynolds (1970), Donaldson (1971), Naot et al. (1972), Hanjalić & Launder (1972), and Lumley & Khajeh-Nouri (1974). In a landmark paper, Launder, Reece & Rodi (1975), developed a hierarchy of Reynolds-stress transport models by consolidating the work of various separate groups into a unified framework. They were able to successfully apply the models to a variety of free-shear and wall-bounded flows of practical interest (Launder et al., 1975). Their model, particularly the simple version (the 'Basic' model), has since been one of the most widely used RST models in engineering applications because of the combined advantage of being simple in form, yet retaining the ability to overcome many of the weaknesses of eddy-viscosity formulations (Hanjalić & Jakirlić, 2002).

Later Schumann (1977) introduced the concept of *realisability* as a constraint to guide model formulation. By this it is meant that models should be designed to prevent certain unphysical solutions, such as negative normal stress components, or a stress tensor that violates the Cauchy-Schwartz inequality. Lumley (1978) extensively discussed the significance and implementation of realisability requirements. He devised and used anisotropy invariant maps, or 'Lumley triangles', to illustrate the limiting states of turbulence with respect to values of the second and third invariants of the Reynolds-stress anisotropy tensor. Lumley pointed out that to prevent a negative normal stress component from arising during computations, the time derivative of the component must be made to vanish at the instant when the component itself vanishes, thereby preventing a negative value to arise as time progresses. Such a situation can arise near a wall or a free-surface, where the interface-normal component decays much faster than the other components as the interface is approached, thus approaching a *two-component* limit. Shih & Lumley (1985) later used these arguments to devise a realisable model for the pressure-strain-rate correlation. Their model, however, did not perform well in simple shear flows, and higher order corrections were later added to achieve better agreement with these flows (Craft & Launder, 2002). Speziale (1985; 1987) used arguments of material-frame indifference in the limit of two-dimensional turbulence to develop a model for the rapid pressure-strain-rate correlation. Speziale et al. (1991) later considered the simplest topologically equivalent form (returning the same equilibrium states) to that of the Speziale (1987) model, to arrive at a more simplified, similarly performing version (Speziale et al., 1991). This latter model is also in relatively common use in engineering RST computations.

The UMIST group, starting with the work of Fu et al. (1987), Fu (1988), Craft et al. (1989), and Craft (1991) developed a model also based on ensuring realizability in the two-component limit, but using an approach slightly different from that used by Shih & Lumley. This model (the 'TCL' model, in what follows) uses a cubic expansion of the rapid pressure-strain-rate correlation in  $k$  and  $a_{ij}$ . It was shown to achieve significant improvements over previous models in a wide range of flows.

### 3. Modelling practises

In this section the basic equations for the mean flow of incompressible fluids are presented, along with the equations for the relevant turbulence statistics. At the level of Reynolds stress transport modelling, the Reynolds averaged Navier-Stokes (RANS) equations are solved, along with separate equations for each independent component of the Reynolds stress tensor, as well as a transport equation for the scalar rate of dissipation of turbulent kinetic energy.

The modelling approach used for the various terms appearing in the exact Reynolds stress transport equation are briefly reviewed.

### 3.1 Basic equations of turbulent flow

Turbulent flows are characterised by highly fluctuating velocity, pressure, and other field variables. One approach for dealing with this fluctuating nature of the flow, the one most widely used by engineers, is to work with an averaged form of the basic equations. In *Reynolds averaging* the instantaneous flow variables are decomposed into an average quantity and a fluctuation. Thus,

$$\begin{aligned}\tilde{U}_i &= U_i + u_i \\ \tilde{P} &= P + p,\end{aligned}\tag{1}$$

where capital letters denote averaged quantities, and small letters denote purely fluctuating quantities. The averaging can be either over time or over a repeated realisation of an experiment with the same nominal conditions. The latter, *ensemble* averaging, will be implied in the following, to allow for temporal variations of mean quantities. When this decomposition is substituted into the Navier Stokes equations for incompressible flow, and the result is ensemble averaged, one obtains the Reynolds-averaged Navier-Stokes (RANS) equations

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} - \frac{\partial \overline{u_i u_j}}{\partial x_j}.\tag{2}$$

When the decomposition is substituted into the continuity equation for incompressible flow, and averaging is applied, one obtains for the mean flow

$$\frac{\partial U_i}{\partial x_i} = 0.\tag{3}$$

If this is subtracted from the instantaneous continuity equation, the continuity condition for the fluctuating velocity is obtained

$$\frac{\partial u_i}{\partial x_i} = 0,\tag{4}$$

meaning that both the mean and fluctuating velocity fields are individually divergence free. The last term in the RANS equation (2) contains the Reynolds stress tensor  $\overline{u_i u_j}$ . Thus the averaging process introduced a new unknown tensor term, and the set of equations is no longer closed. This is called the *closure* problem of averaging approaches. The task of turbulence modelling is to construct appropriate models for these stresses that relate them to the mean flow quantities, and thus to construct a closed set of equations allowing numerical solutions to be obtained. An additional implied objective in the engineering context is for the models to be as computationally inexpensive as possible while being able to reproduce the behaviour and phenomena of relevance to the problem in question, at the required level of accuracy.

A transport equation for the fluctuating velocity can be obtained by subtracting the RANS equation (2) from the Navier-Stokes equation. Using the divergence-free property of the fluctuating field, the result can be written as

$$\frac{\overline{D} u_i}{\overline{D} t} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial}{\partial x_j} [u_i u_j - \overline{u_i u_j} - \nu \frac{\partial u_i}{\partial x_j}]\tag{5}$$

The operator  $\overline{D}/\overline{Dt}$  is used to denote the material derivative following the mean flow

$$\frac{\overline{D}}{\overline{Dt}} = \frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x_j}. \quad (6)$$

Since this interpretation will be used exclusively here, the over-bar on this mean-flow material derivative will subsequently be dropped. An exact equation for the Reynolds stresses can be obtained by using (5) to construct

$$\frac{D\overline{u_i u_j}}{Dt} = \overline{u_j \frac{Du_i}{Dt}} + \overline{u_i \frac{Du_j}{Dt}},$$

where it has been assumed that averaging and taking the material derivative (6) commute. The result is

$$\begin{aligned} \frac{D\overline{u_i u_j}}{Dt} = & - \left( \overline{u_i u_k \frac{\partial U_j}{\partial x_k}} + \overline{u_j u_k \frac{\partial U_i}{\partial x_k}} \right) \\ & + \frac{p}{\rho} \left( \overline{\frac{\partial u_i}{\partial x_j}} + \overline{\frac{\partial u_j}{\partial x_i}} \right) \\ & + \frac{\partial}{\partial x_k} \left[ \overline{v \frac{\partial \overline{u_i u_j}}{\partial x_k}} - \overline{u_i u_j u_k} - \frac{p}{\rho} (u_i \delta_{jk} + u_j \delta_{ik}) \right] \\ & - 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}. \end{aligned} \quad (7)$$

The first term on the right hand side above is the production rate of Reynolds stresses by mean velocity gradients. This term is closed at the RST level since it is given in terms of quantities that are being solved for at this level. All the remaining terms in the equation, except for viscous diffusion, require modelling. The second term is a correlation between the fluctuating pressure and the fluctuating strain rate. From continuity this term is traceless, so it does not contribute directly to the kinetic energy of the turbulence. Its effect is to redistribute the energy between the stress components, so it plays a very important role in determining the degree of anisotropy of the stresses. Accordingly, it has received much attention from researchers, and continues to do so. The third term in (7) is a combination of several diffusion terms, all having the effect of spatial redistribution of the Reynolds stresses. Finally, the last term is the dissipation rate of Reynolds stresses by viscous action at the smallest scales of turbulence. Since the smallest scales of motion are assumed to be isotropic, the dissipation rate tensor is frequently modelled as  $\varepsilon_{ij} = \frac{2}{3} \varepsilon \delta_{ij}$ , where  $\varepsilon$  is the scalar dissipation rate of turbulent kinetic energy. This approximation is not applicable near walls or free surfaces, where the dissipation tensor becomes markedly anisotropic. Equation (7) can be written in short form as

$$\frac{D\overline{u_i u_j}}{Dt} = P_{ij} + \phi_{ij} + \mathcal{D}_{ij} - \varepsilon_{ij}, \quad (8)$$

where it is understood that each term above defines the notation for the corresponding term in (7).

An equation for the kinetic energy associated with the turbulent fluctuations,  $k = \frac{\overline{u_i u_i}}{2}$ , can be obtained by taking half the contraction of (7). The resulting equation is

$$\frac{Dk}{Dt} = -\overline{u_i u_k} \frac{\partial U_i}{\partial x_k} + \frac{\partial}{\partial x_k} \left[ \nu \frac{\partial k}{\partial x_k} - \overline{u_i u_i u_k} - \frac{1}{\rho} \overline{p u_i} \delta_{ik} \right] - \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k}}. \quad (9)$$

In short form, this can be written

$$\frac{Dk}{Dt} = P_\kappa + \mathcal{D} - \varepsilon \quad (10)$$

The first term on the right hand side of (9) is the production of turbulent kinetic energy by mean velocity gradients. The next term is the diffusion of turbulent kinetic energy by various mechanisms. Finally, the last term is the scalar dissipation rate of turbulent kinetic energy. The short form (10) defines the notation that will be used in the following for the respective terms in (9). It is often convenient to work with the deviatoric Reynolds stress anisotropy tensor  $a_{ij}$  defined as,

$$a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3} \delta_{ij}. \quad (11)$$

### 3.2 Pressure–strain rate correlation

Modelling of the pressure–strain rate correlation is to a large extent guided by consideration of the exact equation for it. An equation for the fluctuating pressure can be obtained by taking the divergence of (5), and invoking continuity (4). This gives

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_k \partial x_k} = -2 \frac{\partial U_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j - \overline{u_i u_j}) \quad (12)$$

A formal solution to this Poisson equation can be constructed using the method of Green's functions, as first demonstrated by Chou (1945). The Green's function of the Laplacian operator is

$$g(\mathbf{x}|\mathbf{x}') = \frac{-1}{4\pi|\mathbf{x} - \mathbf{x}'|}.$$

The fluctuating pressure is thus given by

$$\frac{p}{\rho} = \frac{1}{4\pi} \iiint_V \left[ -2 \frac{\partial U_i}{\partial x'_j} \frac{\partial u_j}{\partial x'_i} - \frac{\partial^2}{\partial x'_i \partial x'_j} (u_i u_j - \overline{u_i u_j}) \right]_{\mathbf{x}',t} \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} + \text{Surface integral}. \quad (13)$$

It can be seen from this equation that the fluctuating pressure can be decomposed into three components (Pope, 2000), corresponding to the three terms appearing on the right-hand side of (13). The first term is linear in the turbulent fluctuations, and responds directly to changes in mean velocity gradient. It is thus called the *rapid* pressure,  $p^r$ . The second is a turbulence-turbulence interaction term, that does not respond directly to changes in the mean flow, but through the turbulent cascade process, and is thus called the *slow* pressure,  $p^s$ . The last term is the solution to the homogeneous (Laplace) equation and satisfies appropriate boundary conditions that ensure the superposition of the three parts,  $p$ , satisfies its own boundary conditions (Pope, 2000). This final term is only significant close to a wall or a free surface, and, since the emphasis here is on modelling regions away from walls, it will be neglected. Wall effects on  $\phi_{ij}$  are considered in Section 3.3.



Based on the above decomposition, the pressure–strain rate correlation can similarly be decomposed into rapid, slow, and *wall influence* terms. The rapid part can be constructed as follows

$$\phi_{ij}^r = \overline{\frac{p^r}{\rho} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)} \quad (14)$$

$$\begin{aligned} \frac{p^r}{\rho} \left( \frac{\partial u_i}{\partial x_j} \right) &= \frac{1}{2\pi} \iiint_{-\infty}^{\infty} \left( \frac{\partial U_k}{\partial x'_l} \frac{\partial u_l}{\partial x'_k} \right)_{\mathbf{x}',t} \left( \frac{\partial u_i}{\partial x_j} \right)_{\mathbf{x},t} \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \\ &= \frac{1}{2\pi} \frac{\partial U_k}{\partial x_l} \iiint_{-\infty}^{\infty} \frac{\partial^2}{\partial x_j \partial x'_k} (u_i(\mathbf{x}) u_l(\mathbf{x}')) \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}. \end{aligned} \quad (15)$$

In taking  $\frac{\partial U_k}{\partial x'_l}$  outside the integral it is assumed that this term is reasonably constant over the volume integral. In homogeneous flows, that is of course exact, but is an approximation in inhomogeneous ones. One can thus write:

$$\phi^r = \frac{\partial U_k}{\partial x_l} (\mathcal{M}_{iljk} + \mathcal{M}_{jlik}), \quad (16)$$

where the fourth rank tensor  $\mathcal{M}_{iljk}$  is given by

$$\mathcal{M}_{iljk} = \frac{-1}{2\pi} \iiint_{-\infty}^{\infty} \frac{\partial^2 u_i(\mathbf{x}) u_l(\mathbf{x} + \mathbf{r})}{\partial r_j \partial r_k} \frac{d\mathbf{r}}{|\mathbf{r}|}, \quad (17)$$

using  $\mathbf{r} = \mathbf{x}' - \mathbf{x}$  for the separation distance. The  $\mathcal{M}_{iljk}$  tensor is symmetric in the first two indices, and in the last two

$$\mathcal{M}_{iljk} = \mathcal{M}_{lijk} = \mathcal{M}_{ilkj}. \quad (18)$$

The divergence-free velocity condition means that contraction over the middle indices results in the quantity vanishing:

$$\mathcal{M}_{ijjk} = 0, \quad (19)$$

and contraction over the last two indices can be shown to yield (twice) the Reynolds stress tensor

$$\mathcal{M}_{ilkk} = 2 \overline{u_i u_l}. \quad (20)$$

The last of these kinematic conditions (20) suggested to workers that the  $\mathcal{M}$  tensor could be modelled as a function of the Reynolds stresses (Launder et al., 1975). The approach taken was to model  $\mathcal{M}$  as a polynomial function in the stresses. The most general fourth-rank tensor linear in the Reynolds stresses satisfying the symmetry conditions (18) is

$$\begin{aligned} \mathcal{M}_{ijkl} &= \alpha \delta_{kl} \overline{u_i u_j} + \beta (\delta_{ik} \overline{u_j u_l} + \delta_{il} \overline{u_j u_k} + \delta_{jk} \overline{u_i u_l} + \delta_{jl} \overline{u_i u_k}) \\ &\quad + \gamma \delta_{ij} \overline{u_k u_l} + [\eta \delta_{ij} \delta_{kl} + v (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})] k, \end{aligned} \quad (21)$$

where the coefficients  $\alpha, \beta, \gamma, \eta, v$  are constants (or functions of the invariants of  $a_{ij}$ ). The continuity condition (19), and the normalisation condition (20) can be used to reduce the number of undetermined constants to one. When this is done, and the resulting modelled  $\mathcal{M}_{ijkl}$  is substituted into (16) the resulting linear rapid pressure–strain rate model is

$$\phi_{ij}^r = -\frac{\gamma + 8}{11} (P_{ij} - 2/3 P \delta_{ij}) - \frac{30\gamma - 2}{55} k \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{(8\gamma - 2)}{11} (D_{ij} - 2/3 D \delta_{ij}), \quad (22)$$

where  $D_{ij}$  is given by

$$D_{ij} = -\overline{u_i u_k} \frac{\partial U_k}{\partial x_j} - \overline{u_j u_k} \frac{\partial U_k}{\partial x_i}, \quad (23)$$

and  $D = D_{ii}/2$ . This is the first of the two Launder-Reece-Rodi (LRR) models in Launder et al. (1975), called the *Quasi-Isotropic* model (LRR-QI). A simplified version of (22) was also suggested in Launder et al. (1975) by observing that the dominant term in this equation is the first one appearing on the right hand side. The model thus obtained, first proposed by Naot et al. (1972), is sometimes termed the *isotropization of production* model (LRR-IP),

$$\phi_{ij}^r = -c_2(P_{ij} - \frac{2}{3}P\delta_{ij}). \quad (24)$$

Various other models have been proposed following similar lines of reasoning, in which  $\mathcal{M}$  is modelled as a tensor-polynomial function of the Reynolds stress tensor or, equivalently, expressed in terms of  $k$  and  $a_{ij}$

$$\mathcal{M} = \mathcal{M}(k, \mathbf{a}). \quad (25)$$

It is worth pointing out at this stage that there is an intrinsic weakness in all such models of the form (25). The tensor  $\mathcal{M}$ , as defined by (17), contains two kinds of directional information – the direction of the energetic velocity components, and the direction of variation or dependence of the two-point correlation (Pope, 2000). Only the former type of information is contained in the Reynolds stress tensor, so two fields having the same Reynolds stresses can have different  $\mathcal{M}$  tensors. More explicitly put, the evolution of the Reynolds stresses is not uniquely determined by the Reynolds stresses (Pope, 2000). This is an intrinsic limitation in RST modelling, that is difficult to overcome without significantly complicating the modelling approach and/or computational cost (Johansson & Hallbäck, 1994; Kassinos & Reynolds, 1994). This limitation is known to cause poor results in flows where the velocity gradient has a strong rotational component, such as in pure (or dominant) rotation, and in high shear rate flows (Johansson & Hallbäck, 1994). However, in many other flows, including ones with significant rotational effects, RST models have been shown to produce very good results.

As for the slow pressure-strain-rate term,  $\phi_{ij}^s$ , it is difficult to extract anything from the exact expression, pertaining to the non-linear turbulence-turbulence interaction part of (13). Most early models followed Rotta's (1951) linear return to isotropy model for the slow term

$$\phi_{ij}^s = -C_1 \varepsilon a_{ij}. \quad (26)$$

This model is motivated by the decay of homogeneous anisotropic turbulence in the absence of mean velocity gradients. It is generally observed that in such cases turbulence progressively tends towards an isotropic state, hence the negative sign in (26).

Experimental evidence shows that the return-to-isotropy process is in fact non-linear in  $a_{ij}$  (Chung & Kim, 1995). When plotted on anisotropy invariant maps, the paths taken during return-to-isotropy experiments are not straight lines, and have different behaviour depending on the sign of the third invariant (Pope, 2000). It is also found that the rate of return is highly dependent on the Reynolds number. A number of nonlinear models for the slow pressure strain term have been suggested in the literature.

### 3.3 Wall effects on $\phi_{ij}$

The presence of a wall alters pressure fluctuations by viscous effect through the no-slip condition, and by inviscid effect through the impermeability condition. DNS results show that



the viscous effect is confined to a region within  $y^+ \approx 15$  from the wall (Mansour et al., 1988). The inviscid wall-blocking effect on the other hand is significant where the distance from the wall is of the same order as the turbulent length scale. Wall blocking causes two opposing effects; *wall reflection* of the fluctuating pressure field increases the energy-redistributing pressure fluctuations, which pushes turbulence towards isotropy, while it also causes selective damping of the wall-normal fluctuating velocity component in turbulent eddies, thereby increasing anisotropy. The latter effect dominates, and turbulence anisotropy near a wall is higher than that in a free shear flow at a similar rate of shear. To account for this, Gibson & Launder (1978) proposed two additive corrections to  $\phi_{ij}$  using the unit normal vector to the wall,  $n_i$ . The first, based on the proposal of Shir (1973), is an additive correction to the slow part

$$\phi_{ij}^{s,w} = C_1^w \frac{\varepsilon}{k} \left( \overline{u_k u_m} n_k n_m \delta_{ij} - \frac{3}{2} \overline{u_i u_k} n_k n_j - \frac{3}{2} \overline{u_j u_k} n_k n_i \right) f_w \quad (27)$$

and the second, is a correction to the rapid part

$$\phi_{ij}^{r,w} = C_2^w \left( \phi_{km}^r n_k n_m \delta_{ij} - \frac{3}{2} \phi_{ik}^r n_k n_j - \frac{3}{2} \phi_{jk}^r n_k n_i \right) f_w \quad (28)$$

where  $C_1^w = 0.5$ ,  $C_2^w = 0.3$ , and  $f_w = 0.4k^{3/2}/(\varepsilon x_n)$  is a damping function based on the ratio of the turbulence length scale to the normal distance to the wall,  $x_n$ .

### 3.4 Modelling dissipation

While modelling of the turbulent kinetic energy, and of the pressure-strain rate correlation, has been to at least some degree guided by consideration of their exact equations, the same is not true for the standard dissipation rate model (Pope, 2000). Dissipation of turbulent kinetic energy is associated with the smallest scales of the fluctuating field, while the kinetic energy itself is mostly contained in the largest scales of fluctuations. The exact dissipation rate equation is comprised of a large number of terms that are all related to dissipative-scale processes, and all but one of the source-terms require modelling. It is thus not a useful starting point for modelling the dissipation rate. Instead the more empirical approach taken is motivated by the spectral energy transfer view of dissipation. The kinetic energy of the larger energy containing eddies is transferred by vortex-stretching in the presence of mean velocity gradients to smaller eddies, and the same process occurs at the 'next' smaller scales, and so on to the smallest dissipative scales, where kinetic energy is finally converted to heat by viscous (molecular) action. If the molecular viscosity is somehow changed, all that happens is that the size of the dissipative scales change to accommodate the rate of energy they receive, but the rate itself is not affected. Thus even though the *mechanism* of dissipation is governed by processes that occur at the smallest scales, dissipation can also be viewed as an energy-transfer rate that readjusts itself with the amount of energy it receives. In this sense, the amount (as opposed to the mechanism) of dissipation is in fact determined by the energy in larger scales. Under the assumption of spectral equilibrium, the transfer rate of energy across the spectrum of turbulence scales is constant and determined by the rate of energy input. Based on this assumption, and the preceding arguments, the conventional equation for dissipation is assumed to be of the form

$$\frac{D\varepsilon}{Dt} = C_{\varepsilon 1} \frac{\varepsilon}{k} P + \mathcal{D}_\varepsilon - C_{\varepsilon 2} \frac{\varepsilon^2}{k}, \quad (29)$$

where  $\mathcal{D}_\varepsilon$  is the diffusion of  $\varepsilon$ . The modelled production term above reflects the assumed direct link between a single rate of transfer of energy across the spectrum and production

of energy at the large scales. This assumption is an obvious weakness in the model when the turbulence is not in equilibrium, as when unsteady solutions are sought, or where the time-scale of the mean flow is of the same order or smaller than the characteristic time-scale of turbulence. In such cases the small-scale turbulence may not have enough time to adjust to the large-scale scale variations, and the instantaneous link implied by the production term in (29) is questionable.

The destruction term in (29) is motivated by consideration of the decay of homogeneous isotropic turbulence in the absence of production (Pope, 2000). In such a flow one expects that the turbulence will decay in a self-similar form in which the rates of decay of  $k$  and  $\varepsilon$  are proportional

$$\frac{k/\frac{dk}{dt}}{\varepsilon/\frac{d\varepsilon}{dt}} = \frac{-k/\varepsilon}{\varepsilon/\frac{d\varepsilon}{dt}} = C$$

If this proportionality constant is labelled  $C_{\varepsilon 2}$ , the following destruction term is implied

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2} \frac{\varepsilon^2}{k} \quad (30)$$

### 3.5 Diffusion modelling

There are three diffusive transport terms on the right hand side of (7). The first is the viscous diffusion term

$$V_{ij} = \nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} \quad (31)$$

which is closed and does not require modelling. The following two terms are the pressure diffusion and turbulent convection, respectively. Most commonly these are modelled together as a combined turbulent diffusion term,  $T_{ij}$ , using the generalised gradient diffusion hypothesis (GGDH) of Daly & Harlow (1970),

$$T_{ij} = \frac{\partial}{\partial x_l} \left( C_s \frac{\varepsilon}{k} \overline{u_l u_k} \frac{\partial \overline{u_i u_j}}{\partial x_k} \right) \quad (32)$$

where  $C_s$  is typically 0.22.

A deficiency of this model is that it does not preserve the symmetry under cyclic permutation of indices that is exhibited by the triple velocity moments  $\overline{u_i u_j u_k}$ . This is only significant when the triple moments and pressure diffusion are modelled separately. In such case an improved model that has been suggested by Hanjalić & Launder (1972) is often used,

$$\overline{u_i u_j u_k} = -C_s \frac{k}{\varepsilon} \left( \overline{u_i u_l} \frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l} \frac{\partial \overline{u_k u_i}}{\partial x_l} + \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} \right). \quad (33)$$

More elaborate models exist in the literature, as in (Craft, 1998) for example, but the models mentioned above are the ones more commonly used.

### 3.6 Accounting for low-Re effects

Viscous effects on turbulence properties and their implications on modelling are considered next. The absence of viscous terms in the equation for fluctuating pressure (12) suggests that viscous effects on the fluctuating pressure will be of secondary importance compared to the inviscid effects due to impermeability, considered in section 3.3. The focus of the discussion is thus directed to the dissipation rate tensor, and the transport equation for the scalar dissipation rate. When discussing low-Re effects, reference is frequently made to the turbulent Reynolds number,  $Re_t$ , defined as

$$Re_t = \frac{k^2}{\nu \varepsilon}. \quad (34)$$

As previously mentioned, at high Reynolds numbers the dissipation rate tensor is assumed to be isotropic,  $\varepsilon_{ij} = \frac{2}{3} \varepsilon \delta_{ij}$ . This, however, will cease to be true near a wall where the high anisotropy of the turbulence is expected to be increasingly felt at the smaller scales as the wall is approached. The simplest model accounting for this effect is that of Rotta (1951), which is based on the idea that the anisotropy of the dissipation rate tensor is similar to the stress anisotropy, thus

$$\varepsilon_{ij} = \frac{\overline{u_i u_j}}{k} \varepsilon. \quad (35)$$

This model was used by Hanjalić & Launder (1976) to give the following blending approximation for the dissipation rate tensor

$$\varepsilon_{ij} = \frac{2}{3} \varepsilon \left[ (1 - f_s) \delta_{ij} + f_s \frac{3}{2} \frac{\overline{u_i u_j}}{k} \right], \quad (36)$$

where  $f_s$  is a function of  $Re_t$  whose value ranges from 1 to 0 as  $Re_t$  ranges from 0 to  $\infty$ , ensuring the desired behaviour of  $\varepsilon_{ij}$  in these limits. The near-wall model (35) is the simplest form accounting for near-wall anisotropy of the dissipation tensor. Launder & Reynolds (1983) have shown that this form does not give the correct near-wall asymptotic behaviour of the individual tensor elements, which are rather given by

$$\begin{aligned} \frac{\varepsilon_{ij}}{\varepsilon} &= \frac{\overline{u_i u_j}}{k}, \quad i \neq 2, j \neq 2 \\ \frac{\varepsilon_{12}}{\varepsilon} &= 2 \frac{\overline{u_1 u_2}}{k}, \quad i \neq 2 \\ \frac{\varepsilon_{22}}{\varepsilon} &= 4 \frac{\overline{u_2 u_2}}{k}. \end{aligned} \quad (37)$$

What is needed then is a term to replace the Rotta model in (36) which yields the correct asymptotic behaviour described by (37), and which contracts to  $2\varepsilon$ . One possible form that satisfies these requirements is

$$\varepsilon_{ij}^* = \frac{\varepsilon/k \left( \overline{u_i u_j} + \overline{u_i u_k} n_j n_k + \overline{u_j u_k} n_i n_k + \overline{u_k u_l} n_k n_l \delta_{ij} \right)}{\left( 1 + \frac{5}{2} n_p n_q \overline{u_p u_q} / k \right)}, \quad (38)$$

where  $n_i$  represents a component of the wall-normal unit vector (Pope, 2000). The use of the wall vector in a model is undesirable because of the ambiguity it introduces in complex

geometries. One way to avoid it is based on the observation that the quantity  $\nabla k^{1/2}$ , evaluated near a wall, is a vector that points in the wall-normal direction. Thus

$$\vec{n} = \frac{\nabla k^{1/2}}{|\nabla k^{1/2}|}, \quad (39)$$

and using the value of the dissipation at the wall for a wall with  $\vec{n} = (0, 1, 0)$ ,

$$|\nabla k^{1/2}|_{x_2=0} = \left( \frac{\partial k^{1/2}}{\partial x_2} \frac{\partial k^{1/2}}{\partial x_2} \right)_{x_2=0}^{1/2} = \sqrt{\frac{\varepsilon}{2\nu}}. \quad (40)$$

Quantities of the form  $n_i n_j$  appearing in (38) can therefore be replaced by

$$n_i n_j = \frac{2\nu}{\varepsilon} \frac{\partial k^{1/2}}{\partial x_i} \frac{\partial k^{1/2}}{\partial x_j}. \quad (41)$$

Following Hanjalić & Launder (1976), when considering the implications of Low-Re effects on dissipation rate modelling, it is instructive to consider the exact transport equation for the energy dissipation rate. This is given by (Daly & Harlow, 1970)

$$\begin{aligned} \frac{D\varepsilon}{Dt} = & -2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_l} \frac{\partial u_k}{\partial x_l}} - 2 \overline{\left( \nu \frac{\partial^2 u_i}{\partial x_k \partial x_l} \right)^2} - \frac{\partial}{\partial x_k} \left[ \overline{u_k \varepsilon} + \frac{2\nu}{\rho} \overline{\frac{\partial u_k}{\partial x_l} \frac{\partial p}{\partial x}} - \nu \overline{\frac{\partial \varepsilon}{\partial x_k}} \right] \\ & - 2\nu \left( \overline{\frac{\partial u_i}{\partial x_l} \frac{\partial u_l}{\partial x_i}} + \overline{\frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_k}} \right) \frac{\partial U_i}{\partial x_k} - 2\nu \overline{u_k \frac{\partial u_i}{\partial x_l} \frac{\partial^2 U_i}{\partial x_k \partial x_l}}. \end{aligned} \quad (42)$$

All the terms on the right hand side above are unclosed, with the exception of viscous diffusion. The first two terms on the right hand side of (42) are the dominant ones in high Re flows. Respectively they represent generation and destruction of  $\varepsilon$ . The third term, which represents a combination of diffusive processes, can be of the same order as the difference of the first two, and must therefore be retained. These three terms are modelled by the three terms that typically appear in high-Re  $\varepsilon$  transport models, as in section 3.4. The fourth and fifth terms are respectively of order  $\text{Re}_t^{1/2}$  and  $\text{Re}_t$  smaller than the other terms (Hanjalić & Launder, 1976), and are thus neglected in high-Re model versions. In low-Re models these terms need to be reconsidered and accounted for if necessary. The last term is often modelled as

$$-2\nu \overline{u_k \frac{\partial u_i}{\partial x_l} \frac{\partial^2 U_i}{\partial x_k \partial x_l}} = C_{\varepsilon 3} \nu \frac{\overline{k u_j u_k}}{\varepsilon} \left( \frac{\partial^2 U_i}{\partial x_j \partial x_l} \right) \left( \frac{\partial^2 U_i}{\partial x_k \partial x_l} \right). \quad (43)$$

This term is present in several Low-Re models developed by the Manchester group. As for the fourth term, initial proposals meant to account for it by allowing the coefficient of the production and destruction terms,  $C_{\varepsilon 1}$  and  $C_{\varepsilon 2}$ , to be functions of  $\text{Re}_t$ . Similarly, possible viscous effects on the diffusion terms were to be accounted for by allowing the term  $C_{\varepsilon}$  to depend on  $\text{Re}_t$  (Hanjalić & Launder, 1976). However, computations revealed that adding the term in (43) alone was sufficient in producing good agreement between computed energy profiles and available data to within experimental accuracy. Thus dependence of the coefficients  $C_{\varepsilon 1}$ ,  $C_{\varepsilon 2}$ ,  $C_{\varepsilon}$  on the turbulence Reynolds number is often (not always) abandoned. Finally the viscous diffusion term, neglected in high-Re models, is retained in its exact form.

### 3.7 The Launder–Reece–Rodi models

In their seminal 1975 paper, Launder, Reece & Rodi laid out a hierarchy of RST models based on arguments presented in section 3.2. Two rapid pressure-strain rate models were proposed. The first is the quasi-isotropic model (LRR-QI), which has the most general linear tensorial form satisfying the required symmetry conditions, and is given by

$$\phi_{ij}^r = -C_2(P_{ij} - \frac{2}{3}\delta_{ij}P_\kappa) - C_3(D_{ij} - \frac{2}{3}\delta_{ij}P_\kappa) - 2C_4kS_{ij}, \quad (44)$$

where  $S_{ij}$  is the mean strain rate tensor, defined as:

$$S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad (45)$$

and the coefficients have the following values

$$C_2 = 0.764, \quad C_3 = 0.182, \quad C_4 = 0.109. \quad (46)$$

The second rapid pressure-strain rate model is the isotropization of production model (LRR-IP), which is also referred to as the ‘Basic’ model, and simply retains the first term of the QI model and neglects the other two. Thus,

$$\phi_{ij}^r = -C_2(P_{ij} - \frac{2}{3}\delta_{ij}P_\kappa), \quad (47)$$

where the coefficient  $C_2$  is now set at 0.6. Both models use the Rotta return-to-isotropy model for the slow pressure-strain rate term,

$$\phi_{ij}^s = -C_1\epsilon a_{ij}, \quad (48)$$

but the coefficient  $C_1$  is 1.5 for the QI model and 1.8 for the IP model.

In the original proposal turbulent diffusion  $T_{ij}$  is modelled using (33) for the triple velocity moments (pressure diffusion is usually neglected). In many later implementations this is replaced by the simpler GGDH. Thus the models can be written as

$$\frac{Du_i u_j}{Dt} = P_{ij} - C_1\epsilon a_{ij} + \phi_{ij}^r + \phi_{ij}^{s,w} + \phi_{ij}^{r,w} + \frac{\partial}{\partial x_l} \left( C_\epsilon \frac{\epsilon}{k} \overline{u_l u_k} \frac{\partial \overline{u_i u_j}}{\partial x_k} \right) - \frac{2}{3}\delta_{ij}\epsilon \quad (49)$$

where  $\phi_{ij}^r$  is replaced by either the QI or IP models, and the wall-reflection terms  $\phi_{ij}^{s,w}, \phi_{ij}^{r,w}$  are given by (27) and (28), respectively. Since these models are intended as high Re models, the viscous diffusion term is neglected and an isotropic dissipation rate tensor is assumed.

Finally, closure is completed with the standard high-Re dissipation rate equation, given by

$$\frac{D\epsilon}{Dt} = C_{\epsilon 1} \frac{\epsilon}{k} P_\kappa - C_{\epsilon 2} \frac{\epsilon^2}{k} + \frac{\partial}{\partial x_k} \left( C_\epsilon \frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial \epsilon}{\partial x_l} \right), \quad (50)$$

where

$$C_{\epsilon 1} = 1.44, \quad C_{\epsilon 2} = 1.92, \quad C_\epsilon = 0.15. \quad (51)$$

### The Shima low-Re model

In its original form, the Launder & Shima (1989) model is a low-Re version of the Basic model that uses wall reflection terms and includes  $Re_t$ -based damping coefficients to return the correct near-wall behaviour. Shima (1998) later proposed a low-Re model, based on the QI pressure-strain rate model, that does away with the wall reflection terms in the interest of more general applicability to complex geometries. The model admittedly gives stress anisotropy results in steady channel flow that are inferior to his previous low-Re formulation, but this is a compromise made in order to discard the wall reflection terms with their associated difficulties related to complex geometries. The pressure-strain rate coefficients are no longer constant, and are given by the following expressions:

$$C_1 = 1 + 2.45A_2^{0.25}A^{0.75}[1 - \exp(-49A^2)] \times \{1 - \exp[-(Re_t/60)^2]\} \quad (52a)$$

$$C_2 = 0.7A \quad (52b)$$

$$C_3 = 0.3A^{0.5} \quad (52c)$$

$$C_4 = 0.65A(0.23C_1 + C_2 - 1) + 1.3A_2^{0.25}C_3 \quad (52d)$$

where  $A_2, A_3$  are the second and third invariants of the stress anisotropy tensor:

$$A_2 = a_{ij}a_{ji} \quad A_3 = a_{ij}a_{jk}a_{ki} \quad (53)$$

and  $A$  is the 'flatness' parameter first defined by Lumley (1978),

$$A = 1 - \frac{9}{8}(A_2 - A_3). \quad (54)$$

Turbulent diffusion, comprising the triple velocity correlation and the pressure velocity correlation, is modelled using the simple gradient diffusion of Daly & Harlow (1970)

$$T_{ij} = \frac{\partial}{\partial x_k} \left( C_s \frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} \right) \quad (55)$$

where  $C_s = 0.22$ .

The dissipation equation is given by

$$\frac{D\varepsilon}{Dt} = C_{\varepsilon 1} \frac{\varepsilon}{k} P - C_{\varepsilon 2} \frac{\varepsilon \tilde{\varepsilon}}{k} + \frac{\partial}{\partial x_k} \left( C_{\varepsilon} \frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \varepsilon}{\partial x_l} + \nu \frac{\partial \varepsilon}{\partial x_k} \right). \quad (56)$$

where  $\tilde{\varepsilon}$  is the homogeneous dissipation rate, defined as:

$$\tilde{\varepsilon} = \varepsilon - 2\nu \left( \frac{\partial k^{1/2}}{\partial x_i} \right)^2. \quad (57)$$

The coefficients  $C_{\varepsilon 2}, C_{\varepsilon}$  retain their typical values 1.92, 0.15 respectively, but  $C_{\varepsilon 1}$  is prescribed as:

$$C_{\varepsilon 1} = 1.44 + \beta_1 + \beta_2, \quad (58a)$$

$$\beta_1 = 0.25A \min(\lambda/2.5 - 1, 0) - 1.4A \min(P/\varepsilon - 1, 0), \quad (58b)$$

$$\beta_2 = 1.0A\lambda^2 \max(\lambda/2.5 - 1, 0), \quad (58c)$$

$$\lambda = \min(\lambda^*, 4), \quad (58d)$$

$$\lambda^* = \left[ \frac{\partial}{\partial x_i} \left( \frac{k^{1.5}}{\varepsilon} \right) \frac{\partial}{\partial x_i} \left( \frac{k^{1.5}}{\varepsilon} \right) \right]. \quad (58e)$$



### 3.8 The Speziale–Sarkar–Gatski model

Speziale et al. (1991) developed a pressure-strain rate model that is quadratic in  $a_{ij}$  by first considering the most general form for  $\phi_{ij}$  (slow and rapid) that is linear in the mean strain and rotation tensors and quadratic in  $a_{ij}$ . Then they obtained their model by considering the simplest subset of that general form that has an equivalent structural equilibrium in plane homogeneous flows. The resulting model has a rapid part that is linear in  $a_{ij}$ , and a quadratic slow part, given by

$$\begin{aligned} \phi_{ij} = & - (2d_1\varepsilon + d_1^*P_\kappa) \frac{a_{ij}}{2} + \frac{d_2}{4}\varepsilon(a_{ik}a_{kj} - \frac{1}{3}a_{kl}a_{kl}\delta_{ij}) \\ & + \left(d_3 - d_3^* \frac{\sqrt{A_2}}{2}\right) kS_{ij} + \frac{d_4}{2}k(a_{ik}S_{jk} + a_{jk}S_{ik} - \frac{2}{3}a_{kl}S_{kl}\delta_{ij}) \\ & + \frac{d_5}{2}k(a_{ik}\Omega_{jk} + a_{jk}\Omega_{ik}), \end{aligned} \quad (59)$$

where  $\Omega_{ij}$  is the mean vorticity tensor defined as:

$$\Omega_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right), \quad (60)$$

and the coefficients have the following values

$$d_1 = 1.7, \quad d_1^* = 1.8, \quad d_2 = 4.2, \quad d_3 = \frac{4}{5}, \quad d_3^* = 1.3, \quad d_4 = 1.25, \quad d_5 = 0.4. \quad (61)$$

The rapid part of the SSG model, aside from the nonlinear dependence on  $A_2$  in third term of (59), is tensorially equivalent to the QI model.

Diffusion is modelled using the GGDH, and the standard high-Re version of the  $\varepsilon$  equation (50) is used, but the coefficient  $C_{\varepsilon 2}$  is assigned the slightly lower value of 1.83.

### 3.9 The Hanjalić–Jakirlić low-Re model

Jakirlić & Hanjalić (1995) developed a low-Re RSTM that is based on the LRR-IP model, and the Gibson & Launder (1978) wall corrections (27) and (28), making modifications to handle Low-Re and near-wall effects. The modifications are expressed in terms of  $Re_t$ , the stress anisotropy invariants,  $A_2, A_3$ , in addition to invariants of the stress dissipation rate anisotropy tensor,  $E_2, E_3$ , defined as:

$$E_2 = e_{ij}e_{ji} \quad E_3 = e_{ij}e_{jk}e_{ki}, \quad (62)$$

$$e_{ij} = \frac{\varepsilon_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij}. \quad (63)$$

A ‘flatness’ parameter based on the stress dissipation rate anisotropy invariants is also used:

$$E = 1 - \frac{9}{8}(E_2 - E_3). \quad (64)$$

The modelled RST equation is given by:

$$\begin{aligned} \frac{D\overline{u_i u_j}}{Dt} = & P_{ij} - C_1 \varepsilon a_{ij} - C_2 \left( P_{ij} - \frac{2}{3} \delta_{ij} P_\kappa \right) + \phi_{ij}^{s,w} + \phi_{ij}^{r,w} \\ & + \frac{\partial}{\partial x_l} \left( C_s \frac{\varepsilon}{k} \overline{u_l u_k} \frac{\partial \overline{u_i u_j}}{\partial x_k} \right) - \varepsilon_{ij}. \end{aligned} \quad (65)$$

The coefficients are specified by:

$$C_1 = C + \sqrt{AE^2}, \quad C = 2.5AF^{1/4}f, \quad F = \min(0.6, A_2), \quad (66)$$

$$C_2 = 0.8\sqrt{A}, \quad (67)$$

$$f = \min \left[ \left( \frac{\text{Re}_t}{150} \right)^{3/2}, 1 \right], \quad (68)$$

$$C_2^w = \max(1 - 0.7C, 0.3), \quad C_2^w = \min(A, 0.3). \quad (69)$$

The damping coefficient appearing in the wall correction terms (27) and (28) is given by:

$$f_w = \min \left[ \frac{k^{3/2}}{2.5\epsilon x_n}, 1.4 \right]. \quad (70)$$

The modelled dissipation rate transport equation is given by:

$$\begin{aligned} \frac{D\epsilon}{Dt} = & C_{\epsilon 1} \frac{\epsilon}{k} P_k - C_{\epsilon 2} f_{\epsilon} \frac{\epsilon \tilde{\epsilon}}{k} + \frac{\partial}{\partial x_k} \left( C_{\epsilon} \frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial \epsilon}{\partial x_l} + \nu \frac{\partial \tilde{\epsilon}}{\partial x_k} \right) \\ & + C_{\epsilon 3} \nu \frac{k}{\epsilon} \overline{u_i u_j} \frac{\partial^2 U_k}{\partial x_i \partial x_l} \frac{\partial^2 U_k}{\partial x_j \partial x_l} - C_{\epsilon 4} f_4 k \Omega_k \Omega_k + S_l. \end{aligned} \quad (71)$$

The coefficients have the following specified values:

$$C_{\epsilon 1} = 2.6 \quad C_{\epsilon 2} = 1.92 \quad C_{\epsilon 3} = 0.25 \quad C_{\epsilon 4} = 0.1 \quad C_{\epsilon} = 0.18, \quad (72)$$

and

$$f_{\epsilon} = 1 - \frac{C_{\epsilon 2} - 1.4}{C_{\epsilon 2}} \exp \left[ - \left( \frac{\text{Re}_t}{6} \right)^2 \right]. \quad (73)$$

The length-scale growth correction,  $S_l$ , is given by:

$$S_l = \max \left\{ \left[ \left( \frac{1}{C_l} \frac{\partial l}{\partial x_n} \right)^2 - 1 \right] \left( \frac{1}{C_l} \frac{\partial l}{\partial x_n} \right)^2, 0 \right\} \frac{\tilde{\epsilon} \epsilon}{k} A, \quad (74)$$

where  $l = k^{3/2}/\epsilon$ , and  $C_l = 2.5$ .

The anisotropic stress dissipation rate tensor is modelled as:

$$\epsilon_{ij} = f_s \epsilon_{ij}^* + (1 - f_s) \frac{2}{3} \delta_{ij} \epsilon, \quad (75)$$

where  $\epsilon_{ij}^*$  is given by:

$$\epsilon_{ij}^* = \frac{\epsilon \overline{u_i u_j} + (\overline{u_i u_k} n_j n_k + \overline{u_j u_k} n_i n_k + \overline{u_k u_l} n_k n_l n_i n_j) f_d}{1 + \frac{3}{2} n_p n_q \frac{\overline{u_p u_q}}{k} f_d}, \quad (76)$$

$$f_s = 1 - \sqrt{AE^2}, \quad f_d = (1 + 0.1 \text{Re}_t)^{-1}. \quad (77)$$

### 3.10 The Two-Component-Limit model

Researchers at UMIST, starting with the work of Fu et al. (1987), and Craft et al. (1989), developed a stress transport model that satisfies the constraint of realizability in the limit of two component turbulence. An outline of the derivation of the model is presented in Craft & Launder (2002). Using similar arguments as in (21), but retaining up to cubic terms in  $a_{ij}$ , and using the additional constraint of realizability, the following model for  $\phi_{ij}^r$  was obtained

$$\begin{aligned} \phi_{ij}^r = & -0.6 \left( P_{ij} - 2/3 \delta_{ij} P \right) + 0.6 a_{ij} P \\ & - 0.2 \left\{ \frac{\overline{u_k u_j} \overline{u_l u_i}}{k} \left[ \frac{\partial U_k}{\partial x_l} + \frac{\partial U_l}{\partial x_k} \right] - \frac{\overline{u_l u_k}}{k} \left[ \frac{\partial U_j}{\partial x_l} + \frac{\partial U_i}{\partial x_k} \right] \right\} \\ & - c_2 \left\{ A_2 (P_{ij} - D_{ij}) + 3 a_{mi} a_{nj} (P_{mn} - D_{mn}) \right\} \\ & + c'_2 \left\{ \left( \frac{7}{15} - \frac{A_2}{4} \right) (P_{ij} - 2/3 \delta_{ij} P) \right. \\ & + 0.2 [a_{ij} - 1/2 (a_{ik} a_{kj} - 1/3 \delta_{ij} A_2)] P - 0.05 a_{ij} a_{lk} P_{kl} \\ & + 0.1 \left[ \frac{\overline{u_i u_m}}{k} P_{mj} + \frac{\overline{u_j u_m}}{k} P_{mi} - 2/3 \delta_{ij} \frac{\overline{u_l u_m}}{k} P_{ml} \right] \\ & + 0.1 \left[ \frac{\overline{u_l u_i} \overline{u_k u_j}}{k^2} - 1/3 \delta_{ij} \frac{\overline{u_l u_m} \overline{u_k u_m}}{k^2} \right] \cdot \left[ 6 D_{lk} + 13 k \left( \frac{\partial U_l}{\partial x_k} + \frac{\partial U_k}{\partial x_l} \right) \right] \\ & \left. + 0.2 \frac{\overline{u_l u_i} \overline{u_k u_j}}{k^2} (D_{lk} - P_{lk}) \right\} \end{aligned} \quad (78)$$

where  $A_2$  is the second invariant of the stress anisotropy tensor defined in (53). In the earliest, high-Re, version of the model the recommended values of the coefficients,  $C_2$ ,  $C'_2$ , are

$$C_2 = 0.55, \quad C'_2 = 0.6.$$

As for the slow pressure-strain-rate term, a second-order expression in  $a_{ij}$  is used, where the coefficients are allowed to depend on the stress anisotropy invariants in such a way as to satisfy realizability (Craft & Launder, 2002). Dependency on the third invariant,  $A_3$ , is introduced through the flatness parameter,  $A$ , defined in (54). The flatness parameter becomes zero when one stress component vanishes; thus using the form

$$\phi_{ij}^s = -C_1 \varepsilon [a_{ij} + c'_1 (a_{ij} a_{jk} - \frac{1}{3} A_2 \delta_{ij})] - f'_A \varepsilon a_{ij}, \quad (79)$$

where the coefficients are given by

$$C_1 = 3.1 (A_2 A)^{1/2} \quad C'_1 = 1.1 \quad f'_A = A^{1/2},$$

ensures that  $\phi_{ij}^s$  drops to zero when the turbulence is two-component.

### Low-Re TCL model

A low-Re version of the TCL model was presented by Craft (1998). This version adopts a slightly different decomposition of the velocity-pressure gradient correlation  $\Pi_{ij}$  (which appears in the exact RST equation *before* it is decomposed, as in (7)). The alternate decomposition was found to be more appropriate when modelling inhomogeneous flows. Where this correlation is typically decomposed into the pressure-strain-rate correlation and pressure diffusion, an alternative decomposition is obtained by defining:

$$\phi_{ij}^* = \Pi_{ij} - \frac{1}{3}\delta_{ij}\Pi_{kk}. \quad (80)$$

Constructing  $\phi_{ij}^*$  in this way ensures that it is redistributive in nature, since it is traceless and thus cannot contribute to the level of kinetic energy. This redistributive quantity is modelled as

$$\phi_{ij}^* = \phi_{ij}^{*,s} + \phi_{ij}^{*,r} + \phi_{ij}^{\text{inh},s} + \phi_{ij}^{\text{inh},r}. \quad (81)$$

The quantities  $\phi_{ij}^{*,s}, \phi_{ij}^{*,r}$  have the same form as their homogeneous counterparts (79) and (78), respectively, but the coefficients  $C_1, C_2$  and  $C_2'$  are prescribed by

$$C_1 = 3.1f_A f_{\text{Re}_t} A_2^{1/2}, \quad (82a)$$

$$C_2 = \min \left\{ 0.55 \left[ 1 - \exp \left( \frac{-A^{1.5} \text{Re}_t}{100} \right) \right], \frac{3.2A}{1+S^*} \right\}, \quad (82b)$$

$$C_2' = \min(0.6, A) + \frac{3.5(S^* - \Omega^*)}{3 + S^* + \Omega^*} - 2S_I, \quad (82c)$$

where

$$f_A' = \sqrt{A} f_{\text{Re}_t} + A(1 - f_{\text{Re}_t}), \quad (83)$$

$$f_{\text{Re}_t} = \min[(\text{Re}_t/160)^2, 1], \quad (84)$$

$$f_A = \begin{cases} (A/14)^{1/2} & A < 0.05 \\ A/0.8367 & 0.05 < A < 0.7 \\ A^{1/2} & A > 0.7, \end{cases} \quad (85)$$

$$S^* = Sk/\varepsilon, \quad \Omega^* = \Omega k/\varepsilon, \quad (86)$$

$$S = (2S_{ij}S_{ji})^{1/2}, \quad \Omega = (2\Omega_{ij}\Omega_{ji})^{1/2}, \quad (87)$$

$$S_I = \frac{2\sqrt{2}S_{ij}S_{jk}S_{ki}}{(S_{lm}S_{ml})^{3/2}}, \quad (88)$$

and

$$S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad \Omega_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right). \quad (89)$$

The inhomogeneous corrections are independent of the wall-normal vector, and are given by

$$\begin{aligned} \phi_{ij}^{\text{inh},s} = & f_{w1} \frac{\varepsilon}{k} (\overline{u_l u_k} d_l^A \delta_{ij} - \frac{3}{2} \overline{u_i u_k} d_j^A - \frac{3}{2} \overline{u_j u_k} d_i^A) d_k^A \\ & + f_{w2} \frac{\varepsilon}{k^2} \overline{u_l u_n} (\overline{u_n u_k} d_k^A \delta_{ij} - \frac{3}{2} \overline{u_i u_n} d_j^A - \frac{3}{2} \overline{u_j u_n} d_i^A) d_l^A \\ & + f_{w3} \nu \left( a_{il} \frac{\partial \sqrt{k}}{\partial x_l} \frac{\partial \sqrt{k}}{\partial x_j} + a_{jl} \frac{\partial \sqrt{k}}{\partial x_l} \frac{\partial \sqrt{k}}{\partial x_i} - \frac{2}{3} a_{nl} \frac{\partial \sqrt{k}}{\partial x_l} \frac{\partial \sqrt{k}}{\partial x_n} \delta_{ij} - \frac{4}{3} a_{ij} \frac{\partial \sqrt{k}}{\partial x_l} \frac{\partial \sqrt{k}}{\partial x_l} \right) \\ & + f'_{w1} \frac{k^2}{\varepsilon} \left( \frac{\overline{u_k u_l}}{u_k u_l} \frac{\partial \sqrt{A}}{\partial x_k} \frac{\partial \sqrt{A}}{\partial x_l} \delta_{ij} - \frac{3}{2} \frac{\overline{u_i u_k}}{u_i u_k} \frac{\partial \sqrt{A}}{\partial x_k} \frac{\partial \sqrt{A}}{\partial x_j} - \frac{3}{2} \frac{\overline{u_j u_k}}{u_j u_k} \frac{\partial \sqrt{A}}{\partial x_k} \frac{\partial \sqrt{A}}{\partial x_i} \right), \end{aligned} \quad (90)$$

$$\phi_{ij}^{\text{inh},r} = f_I k \frac{\partial U_l}{\partial x_n} d_l d_n (d_i d_j - \frac{1}{3} d_k d_k \delta_{ij}), \quad (91)$$

where the 'normalised length-scale gradients',  $d_i, d_i^A$ , introduced by Craft & Launder (1996), are used indicate the direction of strong inhomogeneity, when present, without the use of a wall-normal vector. These are defined by

$$d_i = \frac{N_i}{0.5 + (N_k N_k)^{0.5}}, \quad \text{where} \quad N_i = \frac{\partial(k^{1.5}/\varepsilon)}{\partial x_i}, \quad (92a)$$

$$d_i^A = \frac{N_i^A}{0.5 + (N_k^A N_k^A)^{0.5}}, \quad \text{where} \quad N_i^A = \frac{\partial(k^{1.5} A^{0.5}/\varepsilon)}{\partial x_i}. \quad (92b)$$

The coefficients appearing in the inhomogeneous corrections are given by:

$$f_{w1} = 0.4 + 1.6 \min \left\{ 1, \max \left[ 0, 1 - \frac{\text{Re}_t - 55}{20} \right] \right\}, \quad (93)$$

$$f_{w2} = 0.1 + 0.8 A_2 \min \left\{ 1, \max \left[ 0, 1 - \frac{\text{Re}_t - 50}{85} \right] \right\}, \quad (94)$$

$$f_{w3} = 2.5 \sqrt{A}, \quad (95)$$

$$f'_{w1} = 0.22, \quad (96)$$

$$f_I = 2.5 f_A. \quad (97)$$

As discussed in Section 3.6, the dissipation tensor near a wall or free surface is anisotropic, and the low-Re TCL accordingly prescribes the following anisotropic model for the dissipation rate tensor,

$$\varepsilon_{ij} = (1 - f_\varepsilon) \frac{\varepsilon'_{ij} + \varepsilon''_{ij} + \varepsilon'''_{ij}}{D} + \frac{2}{3} f_\varepsilon \varepsilon \delta_{ij}, \quad (98)$$

where

$$\begin{aligned}
 \epsilon'_{ij} &= \epsilon \frac{\overline{u_i u_j}}{k} + 2\nu \frac{\overline{u_l u_n}}{k} \frac{\partial \sqrt{k}}{\partial x_l} \frac{\partial \sqrt{k}}{\partial x_n} \delta_{ij} \\
 &\quad + 2\nu \frac{\overline{u_l u_i}}{k} \frac{\partial \sqrt{k}}{\partial x_j} \frac{\partial \sqrt{k}}{\partial x_l} + 2\nu \frac{\overline{u_l u_j}}{k} \frac{\partial \sqrt{k}}{\partial x_i} \frac{\partial \sqrt{k}}{\partial x_l}, \\
 \epsilon''_{ij} &= \epsilon \left( 2 \frac{\overline{u_l u_k}}{k} d_l^A d_k^A \delta_{ij} - \frac{\overline{u_l u_i}}{k} d_l^A d_j^A - \frac{\overline{u_l u_j}}{k} d_l^A d_i^A \right), \\
 \epsilon'''_{ij} &= C_{\epsilon 3} \nu k \left( \frac{\partial \sqrt{A}}{\partial x_k} \frac{\partial \sqrt{A}}{\partial x_k} \delta_{ij} + 2 \frac{\partial \sqrt{A}}{\partial x_i} \frac{\partial \sqrt{A}}{\partial x_j} \right), \\
 D &= \frac{\epsilon'_{kk} + \epsilon''_{kk} + \epsilon'''_{kk}}{2\epsilon},
 \end{aligned} \tag{99}$$

and the coefficients are taken as  $f_\epsilon = A^{3/2}$ ,  $C_{\epsilon 3} = 0.2$ . The term  $\epsilon'_{ij}$  is similar in nature to the model in (38), and its purpose is to ensure the correct wall-limiting behaviour of  $\epsilon_{ij}$ , as discussed in Section 3.6. The term  $\epsilon''_{ij}$  serves the specific purpose of producing the dip in  $\epsilon_{12}$  near  $y/\delta = 0.1$  observed in DNS studies of plane channel flow, and finally the term  $\epsilon'''_{ij}$  improves the behaviour of  $\epsilon_{ij}$  at a free surface where there is strong inhomogeneity even without significant viscous effects (Craft & Launder, 1996).

### Dissipation rate equation

Early high-Re implementations of the TCL model used the same transport equation for the scalar dissipation rate (50) as in the LRR models. In later versions of the TCL model (Batten et al., 1999; Craft, 1998), an equation for the homogeneous dissipation rate,

$$\tilde{\epsilon} = \epsilon - 2\nu \left( \frac{\partial k^{1/2}}{\partial x_i} \right)^2, \tag{100}$$

is solved, which takes the form

$$\begin{aligned}
 \frac{D\tilde{\epsilon}}{Dt} &= C_{\epsilon 1} \frac{\tilde{\epsilon}}{k} P_K - C_{\epsilon 2} \frac{\tilde{\epsilon}^2}{k} - C'_{\epsilon 2} \frac{(\epsilon - \tilde{\epsilon})\tilde{\epsilon}}{k} + \frac{\partial}{\partial x_k} \left( C_\epsilon \frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial \tilde{\epsilon}}{\partial x_l} + \nu \frac{\partial \tilde{\epsilon}}{\partial x_k} \right) \\
 &\quad + C_{\epsilon 3} \nu \frac{k}{\epsilon} \overline{u_i u_j} \frac{\partial^2 U_k}{\partial x_i \partial x_l} \frac{\partial^2 U_k}{\partial x_j \partial x_l} + Y_E.
 \end{aligned} \tag{101}$$

The term  $Y_E$  is a length-scale correction based on the proposal of Iacovides & Raisee (1997), and is given by

$$Y_E = C_{\epsilon l} \frac{\tilde{\epsilon}^2}{k} \max[F(F+1)^2, 0], \tag{102}$$

and  $F$  in turn is given by

$$F = \left( \frac{\partial l}{\partial x_j} \frac{\partial l}{\partial x_j} \right) - C_l \{ [1 - \exp(-B_\epsilon \text{Re}_t)] + B_\epsilon C_l \text{Re}_t \exp(-B_\epsilon \text{Re}_t) \}, \tag{103}$$

$$l = k^{3/2}/\epsilon, \quad B_\epsilon = 0.1069, \quad C_l = 2.55. \tag{104}$$



The remaining coefficients are given by

$$\begin{aligned} C_{\varepsilon 1} &= 1.0, \quad C_{\varepsilon 2} = \frac{1.92}{1 + 0.7 A_d \sqrt{A_2}}, \quad A_d = \max(A, 0.25), \\ C'_{\varepsilon 2} &= 1.0, \quad C_{\varepsilon 3} = 0.875, \\ C_{\varepsilon l} &= 0.5, \quad C_{\varepsilon} = 0.15. \end{aligned} \quad (105)$$

#### 4. Numerical issues specific to RST modelling

There are a number of numerical difficulties associated with the use of RST models that are not present when using eddy viscosity formulations. In particular, the use of RST models results in relatively large source terms that increase the stiffness of the algebraic equation system, in addition to the fact that the equation set becomes highly non-linear and strongly coupled (Leschziner & Lien, 2002; Lien & Leschziner, 1994). When using a collocated grid, there is also the issue of odd-even decoupling of the velocities and the Reynolds stresses.

The use of an eddy-viscosity approach adds to the momentum equations a momentum diffusion term that can be treated implicitly, thus enhancing stability. Since no such term is present in RST model equations, one approach to improve stability when applying RST models is to add and subtract a gradient-diffusion term based on an effective viscosity,  $\nu_{\text{eff}}$ . Considering the stress term  $\overline{u^2}$ , for example, one may write

$$\overline{u^2} = \left( \overline{u^2} + \nu_{\text{eff}} \frac{\partial U}{\partial x} \right) - \nu_{\text{eff}} \frac{\partial U}{\partial x},$$

allowing the unbracketed term to be treated implicitly in the  $U$ -momentum equation.

Since the effective viscosity does not affect the final converged solution, it is not uniquely specified. One would, in general, simply be trying to significantly reduce the residual stress term that must be treated explicitly in the source term. One way to specify the effective viscosity is by reference to a simplified form of the Basic Reynolds stress model equations. What is needed is to construct a relation between  $\overline{u^2}$  and  $\frac{\partial U}{\partial x}$ , between  $\overline{v^2}$  and  $\frac{\partial V}{\partial x}$ , and so on. Take  $\overline{u^2}$  for example, and start by assuming its transport equation is source dominated:

$$P_{11} + \phi_{11} - \frac{2}{3} \varepsilon \delta_{ij} = 0. \quad (106)$$

Substituting for  $\phi_{11}$  from the Basic model,

$$P_{11} - C_1 \varepsilon \left( \frac{u^2}{k} - \frac{2}{3} \right) - C_2 \left[ P_{11} - \frac{1}{3} (P_{11} + P_{22} + P_{33}) \right] - \frac{2}{3} \varepsilon \delta_{ij} = 0. \quad (107)$$

This leads to

$$\begin{aligned} -2\overline{u^2} \frac{\partial U}{\partial x} \left( 1 - \frac{2}{3} C_2 \right) - C_1 \frac{\varepsilon}{k} \overline{u^2} + \\ \left( \text{other terms not containing } \overline{u^2} \text{ or } \frac{\partial U}{\partial x} \right) = 0, \end{aligned} \quad (108)$$

or

$$\overline{u^2} = \frac{(2 - \frac{4}{3} C_2) \overline{u^2}}{C_1} \frac{k}{\varepsilon} \frac{\partial U}{\partial x} + \text{O.T.} \quad (109)$$

Thus a suitable choice for  $\nu_{11}$  is

$$\nu_{11} = \frac{2 - \frac{4}{3}C_2}{C_1} \frac{k}{\varepsilon} \overline{u^2}. \quad (110)$$

Similar consideration of the  $\overline{v^2}$  transport equation leads to the specification

$$\nu_{22} = \frac{2 - \frac{4}{3}C_2}{C_1} \frac{k}{\varepsilon} \overline{v^2}, \quad (111)$$

and relating the shear stress  $\overline{uv}$  to  $\frac{\partial U}{\partial y}$  leads to the following specification for  $\nu_{12}$

$$\nu_{12} = \frac{1 - C_2}{C_1} \frac{k}{\varepsilon} \overline{v^2}. \quad (112)$$

Maintaining the required coupling between the velocity and Reynolds stress components can be accomplished through a Rhie-Chow-type interpolation (Leschziner & Lien, 2002):

$$\overline{u_P^2} = \frac{1}{a_P} \left( \underbrace{\sum_i a_i \overline{u_i^2} + S_u}_{H_P/a_P} \right) + \frac{S_u}{a_P} + \nu_{11}^P \frac{(U_w - U_e)_P}{\Delta x}. \quad (113)$$

Similarly,

$$\overline{u_E^2} = \frac{H_E}{a_E} + \nu_{11}^E \frac{(U_w - U_e)_E}{\Delta x}, \quad \overline{u_e^2} = \frac{H_e}{a_e} + \nu_{11}^e \frac{(U_P - U_E)}{\Delta x}.$$

Using linear interpolation for  $\nu_{11}^e$  and  $H_e/a_e$ , one obtains for the value at face  $e$ :

$$\begin{aligned} \overline{u_e^2} = & \underbrace{\frac{1}{2}(\overline{u_P^2} + \overline{u_E^2})}_{\text{linear interpolation}} \\ & + \underbrace{\frac{1}{2\Delta x} \left\{ [\nu_{11}^P + \nu_{11}^E](U_P - U_E) - \nu_{11}^P(U_w - U_e)_P - \nu_{11}^E(U_w - U_e)_E \right\}}_{\text{velocity smoothing}}. \end{aligned} \quad (114)$$

Similar expressions can be constructed for the remaining faces, and for the remaining stress terms.

## 5. Concluding remarks

This chapter has provided an introduction to the subject of Reynolds stress transport modelling. A brief historical account of the development of this class of RANS models was presented. This was followed by an account of the theoretical background, assumptions, approximations, as well as the rationale behind the most commonly adopted RST modelling practises. Finally, some numerical implementation issues specific to RST models were briefly discussed.

The account served to illustrate areas of strength of this class of RANS models, such as the exact form of the stress production terms, and the abandoning of the incorrectly assumed direct link between stress and strain that characterises eddy-viscosity formulations. The presentation also serves to illustrate some inherent weaknesses of present RST models,

which might also be thought of as areas for potential improvement. These weaknesses are a natural result of the complexity of turbulent phenomena, and of the persistent *closure* problem—transport equations for any level of statistical moments will always contain unclosed higher moment terms.

The realizability constraint is ultimately a kinematic constraint that serves to prevent certain un-physical results. Aside from that, it does not prescribe any particular dynamic *stimulus–response* type of link between the strain field and inter-component redistribution processes. Therefore there is no reason to expect that redistributive models, in the form of tensor polynomial expansions in stress anisotropy and velocity gradient, satisfying such constraints should return the correct response to all possible strain fields and histories, particularly ones far removed from those for which the models were calibrated.

This does not diminish the value of RST models, but rather serves to emphasise the importance of testing and validation in order to understand the limits of validity and accuracy for intended applications. As discussed earlier, there is always a trade-off between accuracy and computational cost, and the need for reliable RANS models for many types engineering simulations is not likely to be replaced by LES or DNS in the near future. More importantly, these arguments emphasise the need to strive for a deeper and more general understanding of the complex turbulent phenomena described by the unclosed terms in the transport equations, with the aim of building better models.

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