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Finite Element and Finite Difference Methods for Elliptic and Parabolic Differential Equations

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1. Introduction

With the availability of powerful computers, the application of numerical methods to solve scientific and engineering problems is becoming the normal practice in engineering and scientific communities. Well-formed scientific theory with numerical methods may be used to study scientific and engineering problems. The numerical methods flourish where an experimental work is limited, but it may be imprudent to view a numerical method as a substitute for experimental work.

The growth in computer technology has made it possible to consider the application of partial deferential equations in science and engineering on a larger scale than ever. When experimental work is cost prohibitive, well-formed theory with numerical methods may be used to obtain very valuable information. In engineering, experimental and numerical solutions are viewed as complimentary to one another in solving problems. It is common to use the experimental work to verify the numerical method and then extend the numerical method to solve new design and system. The fast growing computational capacity also make it practical to use numerical methods to solve problems even for nontechnical people.

It is a common encounter that finite difference (FD) or finite element (FE) numerical methods-based applications are used to solve or simulate complex scientific and engineering problems. Furthermore, advances in mathematical models, methods, and computational capacity have made it possible to solve problems not only in science and engineering but also in social science, medicine, and economics. Finite elements and finite difference methods are the most frequently applied numerical approximations, although several numerical methods are available.

Finite element method (FEM) utilizes discrete elements to obtain the approximate solution of the governing differential equation. The final FEM system equation is constructed from the discrete element equations. However, the finite difference method (FDM) uses direct discrete points system interpretation to define the equation and uses the combination of all the points to produce the system equation. Both systems generate large linear and/or nonlinear system equations that can be solved by the computer.

Finite element and finite difference methods are widely used in numerical procedures to solve differential equations in science and engineering. They are also the basis for countless engineering computing and computational software. As the boundaries of numerical method applications expand to non-traditional fields, there is a greater need for basic understanding of numerical simulation.

This chapter is intended to give basic insight into FEM and FDM by demonstrating simple examples and working through the solution process. Simple one- and two-dimensional elliptic and parabolic equations are used to illustrate both FEM and FDM. All the basic mathematics is presented by considering a simplistic element type to define a system equation. The next section is devoted to the finite element method. It begins by discussing one- and two-dimensional linear elements. Then, a detailed element equation, and the forming of a final system equation are illustrated by considering simple elliptic and parabolic equations. In addition, a small number of approximations and methods used to simplify the system equation are, presented. The third section presents the finite difference method. It starts by illustrating how finite difference equations are defined for one- and two-dimensional fields. Then, it is followed by illustrative elliptic and parabolic equations.

2. Finite element method

Of all numerical methods available for solving engineering and scientific problems, finite element method (FEM) and finite difference methods (FDM) are the two widely used due to their application universality. FEM is based on the idea that dividing the system equation into finite elements and using element equations in such a way that the assembled elements represent the original system. However, FDM is based on the derivative that at a point is replaced by a difference quotient over a small interval (Smith, 1985).

It is impossible to document the basic concept of the finite element method since it evolves with time (Comini et al. 1994, Yue et al. 2010). However, the history and motivation of the finite element method as the basis for current numerical analysis is well documented (Clough, 2004; Zienkiewicz, 2004).

Finite element starts by discretizing the region of interest into a finite number of elements. The nodal points of the elements allow for writing a shape or distribution function. Polynomials are the most applied interpolation functions in finite element approximation. The element equations are defined using the distribution function, and when the element equations are combined, they yield a continuous equation that can approximate the system solution. The nodal points and corresponding functional values with shape function are used to write the finite element approximation (Segerlind, 1984):

$$\psi = N_1\psi_1 + N_2\psi_2 + \cdots + N_m\psi_m \tag{1}$$

where $\psi_1, \psi_2, \dots \psi_m$ are the functional values at the nodal points, and $N_1, N_2, \dots N_m$ are the shape functions. Thus, the system equation can be expressed by nodal values and element shape function.

2.1 One-dimensional linear element

Before we discuss the finite element application, we present the simple characteristic of a linear element. For simplicity, we will discuss only two nodes-based linear elements. But, depending on the number of nodes, any polynomial can be used to define the element characteristics. For two nodes element, the shape functions are defined using linear equations. Fig.1 shows one-dimensional linear element.

The one-dimensional linear element (Fig. 1) is defined as a line segment with a length (l) between two nodes at x_i and x_j . The node functional value can be denoted by ψ_i and ψ_j. When using the linear interpolation (shape), the value ψ varies linearly between x_i and x_j as

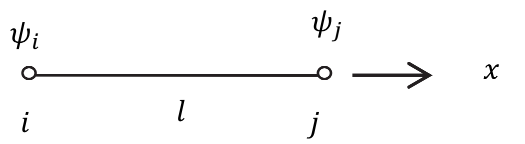

Fig. 1. One-dimensional linear element

$$\psi = mx + k \tag{2}$$

The functional value $\psi = \psi_i$ at node $i = x_i$ and $\psi = \psi_j$ at $j = x_j$. Using the functional and nodal values with the linear equation Eq. 2., the slope and the intercept are estimated as

$$m = \frac{\psi_j - \psi_i}{x_j - x_i} \text{ and } k = \frac{\psi_i x_j - \psi_j x_i}{x_j - x_i} \tag{3}$$

Substituting m and k in Eq. 2 gives

$$\psi = \left(\frac{\psi_i x_j - \psi_j x_i}{x_j - x_i}\right) + \left(\frac{\psi_j - \psi_i}{x_j - x_i}\right) x \tag{4}$$

Rearranging Eq. 4 and substituting l for the element size $(x_j - x_i)$ yields

$$\psi = \left(\frac{x_j - x}{l}\right)\psi_i + \left(\frac{x - x_i}{l}\right)\psi_j \tag{5}$$

By defining the shape functions as

$$\begin{aligned} N_i &= \left(\frac{x_j - x}{l}\right) \qquad \text{(a)} \\ N_j &= \left(\frac{x - x_i}{l}\right) \qquad \text{(b)} \end{aligned} \tag{6}$$

By introducing the shape function N_i and N_j in Eq. 5, the finite element equation can be rewritten as

$$\psi = N_i \psi_i + N_j \psi_j \tag{7}$$

The above equation is a one-dimensional linear standard finite element equation. It is represented by the shape functions $N_i$and N_j nodal values ψ_i and ψ_j.
The two shape functions profiles for a unit element are shown in Fig. 2. The main characters of the shape functions are depicted. These shape functions have a value of 1 at its own node and 0 at the opposing end. The two shape functions also sum up to one throughout.

2.2 Two-dimensional rectangular element

With the current computational methods and resources available, it is not clear whether or not using the FEM or modified FDM will provide an advantage over the other. However, in the early days of numerical analysis, one of the major advantages of using the finite element method was the simplicity and ease that FEM allows to solve complex and irregular two-dimensional problems (Clough 2004, Zienkiewicz, 2004, Dahlquist and Bjorck, 1974). Although several element shapes with various nodal points are used in many numerical simulations, our discussion is limited to simple rectangular elements. Our objective is to simply exhibit how two- dimensional elements are applied to define the elements and final system equation.

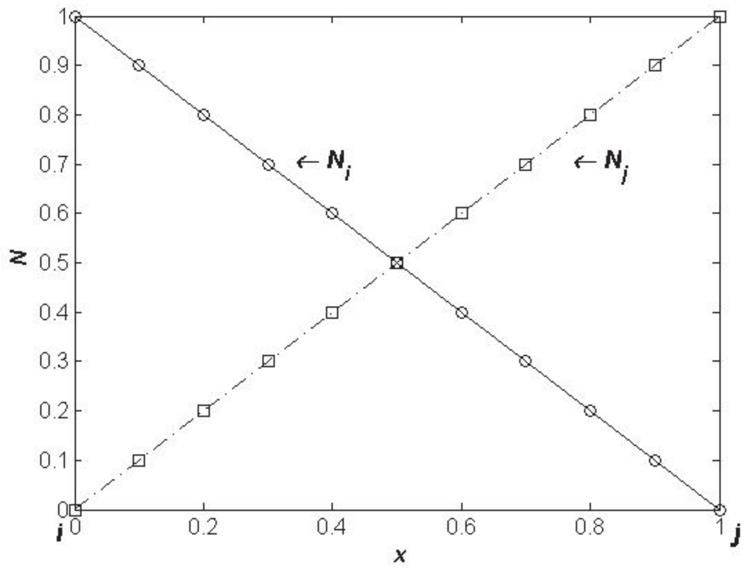

Fig. 2. Linear shape functions

Fig. 3 illustrates a linear rectangular element with four nodes. The nodes $i, j, k,$ and l have corresponding nodal values ψ_i , ψ_j, $\psi_k$and ψ_l at $(x_i, y_{i,}), (x_{j,}, y_{j,}), (x_k , y_k), and\ (x_l, y_l)$.

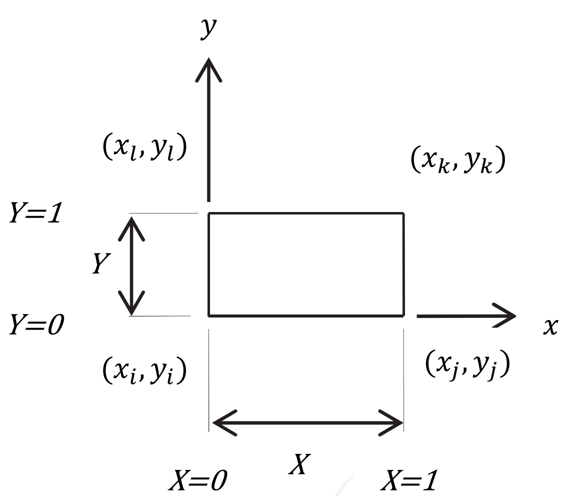

Fig. 3. Two-dimensional linear rectangular element.

The linear rectangular interpolation equation is defined as

$$\psi = k_1 + k_2x + k_3y + k_4xy \tag{8}$$

Applying the nodal and functional values$(x_i, y_{i,}) = (0,0), \psi = \psi_i$,$(x_j, y_{j,}) = (X, 0), \psi = \psi_j$, $(x_k, y_k) = (X, Y)$,$\psi = \psi_k$, and $(x_l, y_l) = (0, Y), \psi = \psi_l$ in Eq. 8 yields four equations and four unknowns as

$$\begin{aligned}
\psi_i &= k_1 && (a)\\
\psi_j &= k_1 + k_2\mathrm{X} && \text{(b)}\\
\psi_k &= k_1 + k_2\mathrm{X} + k_3\mathrm{Y} + k_4\mathrm{XY} && \text{(c)}\\
\psi_l &= k_1 + k_3\mathrm{Y} && \text{(d)}
\end{aligned} \tag{9}$$

Solving the unknown constants k_1 , k_2, $k_3$and k_4 in terms of the nodal values give

$$\begin{aligned} k_1 &= \psi_i && (a) \\ k_2 &= \frac{1}{\mathrm{X}}\left(\psi_j - \psi_i\right) && \text{(b)} \\ k_4 &= \frac{1}{\mathrm{XY}}\left(\psi_i - \psi_j + \psi_k - \psi_l\right) && (c) \\ k_3 &= \frac{1}{\mathrm{Y}}\left(\psi_l - \psi_i\right) && \text{(d)} \end{aligned} \tag{10}$$

Substituting the above values equations (Eq. 10.) into Eq. 8 and reorganizing in terms of nodal values give to finite element equation as

$$\psi = N_i\psi_i + N_j\psi_j + N_k\psi_k + N_l\psi_l \tag{11}$$

where

$$\begin{aligned} N_i &= \left(1 - \frac{x}{X}\right)\left(1 - \frac{y}{Y}\right) && \text{(a)} \\ N_j &= \left(\frac{x}{X}\right)\left(1 - \frac{y}{Y}\right) && \text{(b)} \\ N_k &= \left(\frac{x}{X}\right)\left(\frac{y}{Y}\right) && \text{(c)} \\ N_l &= \left(1 - \frac{x}{X}\right)\left(\frac{y}{Y}\right) && \text{(d)} \end{aligned} \tag{12}$$

Eq. 12 is two-dimensional rectangular shape functions based on element that is plotted in Fig. 3. The shape functions (Eq. 12) are plotted in Fig. 4. The shape functions satisfy the conditions: 1. the functions have a value of 1 at their own node and 0 at the other ends, 2. they vary linearly along the two adjacent edges, and 3. the shape functions sum up to one throughout.

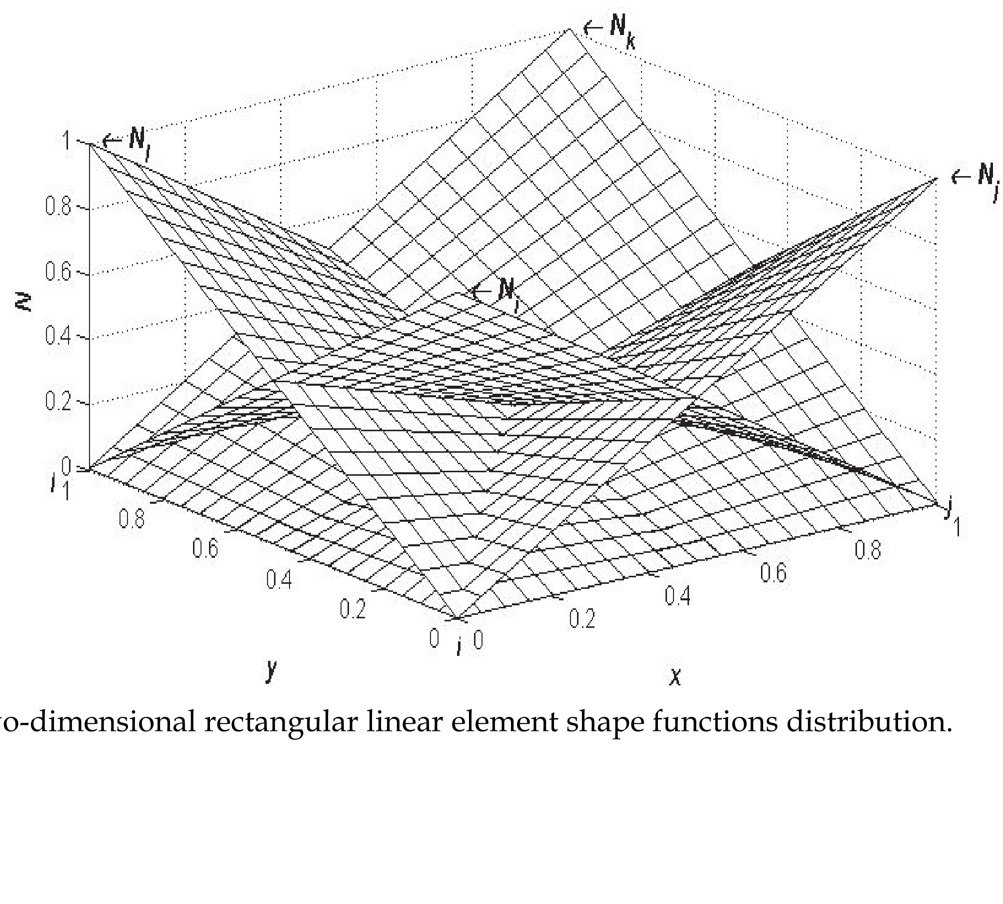

Fig. 4. Two-dimensional rectangular linear element shape functions distribution.

Finite element equation uses the element shape function to define the relationship between the nodal points. Once the element equation is defined, by assembling the element systematically the final system equation is structured. Next, we illustrate the application of the finite element method in a one-dimensional elliptic equation.

2.3 Elliptic equation in finite element method

In order to discuss the basic concept of finite element application in an elliptic equation, we start by illustrating a one-dimensional equation. A one-dimensional elliptic equation of function T can be written as:

$$D\frac{d^2T}{dx^2} + Q = 0 \tag{13}$$

This elliptic equation is used to describe the steady state heat conduction with heat generation where $D, T,$ and Q represent thermal diffusion, temperature, and heat generation. The distinctiveness of the solution of an elliptic equation is dependent on the boundary condition. Thus, it is sometimes called boundary value problem. Providing the appropriate boundary condition at the two ends, the unique solution exists for temperature distribution. The boundary condition could be a prescribed value (Dirichlet), the flux (Neumann), or a combination of both (Vichnevetsky, 1981). In order to demonstrate how the finite element method is used to solve an elliptic equation, we simplify by assuming that material has constant and uniform diffusion with heat generation. Building the finite element elliptic equation involves discretization, forming the element equation, assembling the element equation systematically, and forming the final algebraic equation of the system. Moreover, the uniqueness and the stability of the system equation depend on the specified boundary conditions, thus solving the algebraic equation requires the boundary condition to be introduced before the final equation is solved.

Before we start by forming the finite element equation of steady state heat conduction with heat generation, we have to address how the linear finite element equation is formed. One of the mathematical concepts used to generate the final system equation is called weighting residual method. In short, the weighting residual method is based on the fact that when an approximate solution is substituted in the differential equation, the error term resulted since the approximate solution does not completely satisfy the equation. Thus, the method of weighting residual is to force the product of residual and the weighting function to go to zero. In the finite element method, the weighting residual for each element nodal value is defined and the integral is evaluated using the interpolation function as

$$-\int w(x)R(x)dx = -\int W(x)\left(D\frac{d^2T}{dx^2} + Q\right)dx = 0 \tag{14}$$

where W is the weighting function and R is residual.

The major requirement to evaluate the above integral equation is that the functions that belong to the trail and weighting functions must be continuous. However, when the trail function is linear, the second derivative is not continuous and the integral cannot be evaluated as it is. Thus, in order to evaluate the integral with a lower degree of continuity by replacing the second derivative term with equivalent expression using the differentiation product rules, hence

$$\int W\frac{d^2T}{dx^2}dx = \int \frac{d}{dx}\left(W\frac{dT}{dx}\right)dx - \int \frac{dT}{dx}\frac{dW}{dx}dx \tag{15}$$

Eq. 15 shows that the degree of minimum continuity required to evaluate the integral for trail function is reduced while the continuity for weighting function is increased. The minimum continuity requirement for both weighting and trail can be fulfill with linear function and the integral can be evaluated as long as the functions are continuous within the integral interval. The finite element method is evolved from this need of finding appropriate sets of functions. The finite element method uses a systematic way of using polynomial approximate function that permits the evaluation of the integral equation. Introducing Eq. 15 in Eq. 14 gives the residual for the elliptic integral as

$$R = -\int D\frac{d}{dx}\left(W\frac{dT}{dx}\right)dx + \int D\frac{dT}{dx}\frac{dW}{dx}dx - \int QW\,dx \tag{16}$$

The finite element method uses the interpolation function as a weighting and trail functions. Even a linear element can satisfy the continuity requirement to evaluate the integral. Once we define the integral, in this case function, the next step is to evaluate the residual integral. By evaluating the integral for each element, the element contribution to the final system equation can be determined.

In order to determine the element contribution to the final system equation, we will consider linear element (e) with node i and j (Fig. 1) and evaluating the residual integral (Eq. 16) using the elements interpolation function (Eq. 12). Thus, the residual equation becomes

$$\begin{aligned} R_i^e &= -\int_i^j D\frac{d}{dx}\left(N_i\frac{dT}{dx}\right)dx + \int_i^j D\frac{dT}{dx}\frac{dN_i}{dx}dx - \int_i^j N_iQdx \quad \text{(a)} \\ R_j^e &= -\int_i^j D\frac{d}{dx}\left(N_j\frac{dT}{dx}\right)dx + \int_i^j D\frac{dT}{dx}\frac{dN_j}{dx}dx - \int_i^j N_jQdx \quad \text{(b)} \end{aligned} \tag{17}$$

The integral splits into two parts since the weighting functions are defined by two functions N_i and N_j. Consequently, (R_i^e) and (R_j^e) represent the two weighting functions contributions to the element nodal value residual (i) and (j), respectively. Fig. 5 shows that a system of linear interpolation functions. If we take the arbitrary element e that located anywhere in the field, except the two weighting functions N_i and N_j, all of the other weighting equations are zero contribution.

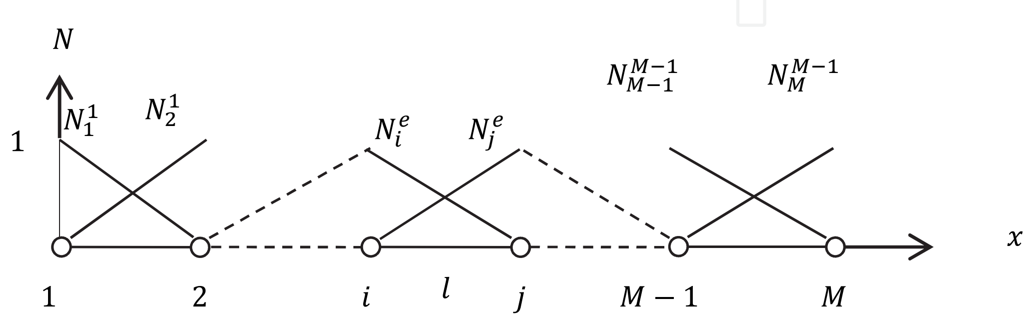

Fig. 5. System of elements shape function and nodes

The integrals on the right can be evaluated using the linear interpolation functions (Eq. 6) characteristics and the finite element equation (Eq. 7). The interpolation function characteristics are $N_i = 1$, at i and $N_i = 0$, at j. Similarly, $N_j = 1$, at j and $N_j = 0$, at i. Evaluating the first terms on the right (denote by q_i^e and q_j^e) gives

$$q_i^e = -\int_i^j D\frac{d}{dx}\left(N_i\frac{dT}{dx}\right)dx = -D\,N_i\frac{dT}{dx}\bigg|_i^j = \mathrm{D}\frac{dT}{dx} \qquad \text{(a)}$$
$$q_j^e = -\int_i^j D\frac{d}{dx}\left(N_j\frac{dT}{dx}\right)dx = -D\,N_j\frac{dT}{dx}\bigg|_i^j = -\mathrm{D}\frac{dT}{dx} \qquad \text{(b)} \qquad (18)$$

These terms (Eq. 18) are the inter element contribution and vanish since the derivative terms vanished between the neighboring elements. Thus, the finite element system equation formed without these inter elements except when the flux (derivative) boundary condition is specified. When the flux boundary specified, they used to apply the flux condition at the boundaries. Furthermore, they are used to compute the flux term once the system equation is solved.

We need the first derivative of the finite element equation (Eq. 7) and the interpolation functions (Eq. 6) in order to evaluate the second integrals on the right in Eq. 17. The first derivatives of the element equation using element length (l) is

$$\frac{dT^e}{dx} = \frac{1}{l}\left(-T_i + T_j\right) \qquad (19)$$

Furthermore, the derivatives of the weighting functions are

$$\frac{dN_i}{dx} = \frac{-1}{l} \qquad \text{(a)}$$
$$\frac{dN_j}{dx} = \frac{1}{l} \qquad \text{(b)} \qquad (20)$$

Thus, the residual from the second terms (denote by k_i^e and k_j^e) in Eq. 17 become

$$k_i^e = D\int_i^j\left(\frac{dN_i}{dx}\frac{dT}{dx}\right)dx = D\int_i^j\left(\frac{-1}{l}\frac{\left(T_j - T_i\right)}{l}\right)dx = \frac{D}{l}\left(T_i - T_j\right) \qquad \text{(a)}$$
$$k_j^e = D\int_i^j\left(\frac{dN_j}{dx}\frac{dT}{dx}\right)dx = D\int_i^j\left(\frac{1}{l}\frac{\left(T_j - T_i\right)}{l}\right)dx = \frac{D}{l}\left(T_j - T_i\right) \qquad \text{(b)} \qquad (21)$$

The last integrals in Eq. 17 are constant and evaluated using the linear weighting functions. Their contributions to the element residual are

$$f_i^e = Q\int_i^j N_i dx = \frac{Q}{l}\left(x_j x - \frac{x^2}{2}\right)\bigg|_i^j = \frac{Ql}{2} \qquad \text{(a)}$$
$$f_j^e = Q\int_i^j N_j dx = \frac{Q}{l}\left(\frac{x^2}{2} - x_i x\right)\bigg|_i^j = \frac{Ql}{2} \qquad \text{(b)} \qquad (22)$$

Substituting Eqs. 18, 21, and 22 in Eq. 17 yields

$$\begin{aligned} R_i^e &= D\left.\frac{dT}{dx}\right|_i + \frac{D}{l}\left(T_i - T_j\right) - \frac{Ql}{2} && \text{(a)} \\ R_j^e &= -D\left.\frac{dT}{dx}\right|_j + \frac{D}{l}\left(-T_i + T_j\right) - \frac{Ql}{2} && \text{(b)} \end{aligned} \tag{23}$$

For simplicity, we will introduce the matrix notation and rewrite the above terms of element contribution to residual equation in matrix form as

$$\begin{Bmatrix} R_i^e \\ R_j^e \end{Bmatrix} = \begin{Bmatrix} D\left.\dfrac{dT}{dx}\right|_i \\ -D\left.\dfrac{dT}{dx}\right|_j \end{Bmatrix} + \frac{D}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix} - \frac{Ql}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{24}$$

The matrix representation becomes very important particularly when illustrating more than one component is contributing to the element residual. Furthermore, it also comes helpful in two- and three-three dimensional spaces.
Thus, the residuals of element can represent as

$$\{R^e\} = \{q^e\} + [K^e]\{T^e\} - \{f^e\} \tag{25}$$

Eqs. 24 and 25 are representing the contribution of the element to the final system equation. The contribution are from the first right terms in Eq. 24 (denoted by $\{q\}$ at Eq.25) that are the inter element contribution and vanishes between the neighboring elements in the final system equation. The second terms, the element contribution to the final system equation, is referred as the stiffness matrix and is denoted by$[K]$. It can be easily determined from the interpolation function for each element as illustrated above and included in the final system equation. The final terms are referred as force vector and denoted by $\{f\}$. The final system equation is built by assembling the element matrices step by step or systematically. Thus the system equation becomes

$$\{R\} = \{q\} + [K]\{T\} - \{f\} = 0 \tag{26}$$

The final system equation is formed by assembling each element’s contribution and adding the contribution of each element’s based on the nodal points. When the system residual becomes zero, the approximate solution can be used to estimate the system. The number of elements used to define the final system equation has significant effect on the element residual. Thus, increasing the number of elements decreases the element residual and improves the approximate (FEM) solution.

2.3.1 Application of finite element for one-dimensional elliptic equation

To illustrate the application of FEM in one-dimensional elliptic equation, we will consider the temperature distribution of an insulated rod length $L = 1$ and thermal diffusivity $D = 10$. A constant heat is also being generated at the rate of $Q = 10$. The boundary conditions are specified as one end where $x = 0, T = 5$ and the opposite end where $x = 1, q = -10$. To illustrate the FEM solution this system, we uses four elements, the nodes of the elements are numbered from 1 to 5 and the element length is assumed to be uniform. Thus, the elliptic equation (Eq. 13) becomes

$$10\frac{d^2T}{dx^2} + 10 = 0 \tag{27}$$

The residual of element (Eq. 25) becomes

$$\begin{Bmatrix} R_i^e \\ R_j^e \end{Bmatrix} = \begin{Bmatrix} q_i \\ q_j \end{Bmatrix} + \frac{10}{0.25}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix} - \frac{10(0.25)}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{28}$$

The prescribe boundary condition at the end where the temperature is fixed $T_1 = 5$ and at the opposite end where the flux boundary applied $q_5 = 10$. The assembled final system equation for four elements becomes

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 5.03125 \\ 0.0625 \\ 0.0625 \\ -0.1875 \end{Bmatrix} \tag{29}$$

The finite element solution for the temperature profile produces the values $T_2 = 4.97, T_3 = 4.91, T_4 = 4.78$, and $T_5 = 4.60$. Furthermore; using the same principle shown above in detail a computer program is developed and the computer solution for 10 and 20 elements is shown in Fig. 6. To show that the number of elements has effect in quality of the FEM solution.

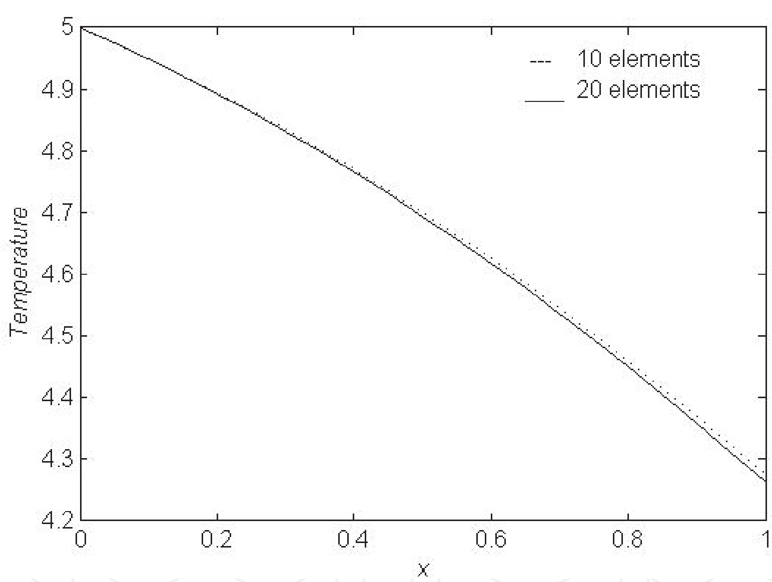

Fig. 6. Finite element approximation for steady state temperature profile for insulated rod

2.3.2 Two dimensional elliptic equation in finite element method

A two-dimensional elliptic equation is used to describe the steady state heat conduction with heat generation similar to the previous section but in a two-dimensional space. A two-dimensional steady state flow of heat in isometric material is expressed by an elliptic equation as

$$D\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + Q = 0 \tag{30}$$

It represents a bounded area. The solution uniqueness is dependent on the boundary condition. Like one-dimensional cases, the boundary condition can be specified as either the functional value or flux. However, in two-dimensional cases, the boundary values are specified at the edges while the region is an area.
In order to illustrate the finite solution of an elliptic equation, we will consider the temperature distribution in two-dimensional spaces that satisfies Eq. 30. The finite element solution satisfies the weighting integral function in two-dimensional space. For simplicity, we will use a linear rectangular element discussed in Sec. 2.2 to evaluate the integral for each elements and determine the elements contribution to the final system equation. Parallel to the one-dimensional finite element method, two-dimensional equations can be modeled by indentifying the implication of increasing dimensionality at the element integral. The interpolation functions for a linear rectangular element with four nodes are defined in (Eq. 12). For simplicity, we use a linear rectangular element with four nodes and also we use a matrix notation ($[N]^T$) to represent all nodal points of the elements instead of writing each node point contribution. Thus, the residual integral for a two-dimensional elliptic equation (Eq. 30) becomes

$$\{R_i^e\} = -\int [N]^T \left(D\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + Q \right) dA \tag{31}$$

The major difference from the one-dimensional case is that the residual integral is area integral and the boundary is line integral. Reducing the degree of continuity for the second derivative term by differentiation product rule (Eq. 15) further simplifies the element residual integral as

$$\{R_i^e\} = -D\left(\int \frac{\partial}{\partial x}\left([N]^T\frac{\partial T}{\partial x}\right) - \frac{\partial [N]^T}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}\left([N]^T\frac{\partial T}{\partial y}\right) - \frac{\partial [N]^T}{\partial y}\frac{\partial T}{\partial y}\right) dA - \int [N]^T Q\, dA \tag{32}$$

Substituting the element equation (quadratic linear element) $T^e = [N]\{T\}$ and rearranging the terms

$$\begin{aligned}\{R_i^e\} = -D\int\left(\frac{\partial}{\partial x}\left([N]^T\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left([N]^T\frac{\partial T}{\partial y}\right)\right) dA \\ + D\left(\int \frac{\partial [N]^T}{\partial x}\frac{\partial N}{\partial x}\{T^e\} + \frac{\partial [N]^T}{\partial y}\frac{\partial N}{\partial y}\{T^e\}\right) dA - \int [N]^T\{Q^e\}\, dA\end{aligned} \tag{33}$$

When the derivative boundary condition is applied, the first two terms are reduced to surface integral by using Green's theorem ($\int \frac{\partial}{\partial x}\left([N]^T\frac{\partial T}{\partial x}\right) dA = \int \left([N]^T\frac{\partial T}{\partial x}\right) d\Gamma\cos\theta$). These two terms on the right can be replaced by an integral around the boundary using the outward normal. Thus,

$$-D\int \frac{\partial}{\partial x}\left([N]^T\frac{\partial T}{\partial x}\right) dA + \frac{\partial}{\partial y}\left([N]^T\frac{\partial T}{\partial y}\right) dA = -D\int [N]^T\left(\frac{\partial T}{\partial x}cos\theta + \frac{\partial T}{\partial y}sin\theta\right) d\Gamma \tag{34}$$

The integral around the boundary of the element is done in a counterclockwise direction. For the rectangular element we considered here, it is the sum of four integrals. It includes the side where the boundary condition is specified and the inter-element side. The inter-element integral vanishes due to the element continuity requirements. However, when the flux boundaries are specified, the surface integrals need to be evaluated where applicable. The general derivative boundary condition can be given as a function of the surface temperature, constant, or zero as

$$-D\frac{\partial T}{\partial n} = C_1 T + C_2 \tag{35}$$

where $\partial T/\partial n$ is the normal gradient at the surface. When the boundary condition is insulated, $\partial T/\partial n = 0$, thus $C_1 = C_2 = 0$. When the derivative is the function of the surface temperature and constant, the boundary surface integral can be evaluated along the specified surface. Therefore, introducing a relationship given by the element equation $T^e = [N]\{T\}$ where $[N]$ represent the rectangular element interpolation functions (Eq. 12) and Eq. 35 is introduced in Eq. 34 gives

$$\{q_{bc}^e\} = \int C_1([N]^T[N])\{T^e\}d\Gamma + \int [N]^T C_2\, d\Gamma \tag{36}$$

Using Eq. 12, linear quadratic element, the above integral can be evaluated. The first integral has following terms

$$[K_q] = C_1 \int \begin{bmatrix} N_i^2 & N_iN_j & N_iN_k & N_iN_l \\ N_iN_j & {N_j}^2 & N_jN_k & N_jN_l \\ N_iN_k & N_jN_k & {N_k}^2 & N_kN_l \\ N_iN_l & N_jN_l & N_kN_l & {N_l}^2 \end{bmatrix} d\Gamma \tag{37}$$

and evaluated for arbitrary side l_{ij} where N_i and N_j are the only contributing functions gives

$$[K_{bc}^e] = \frac{C_1 l_{ij}}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{38}$$

The second term in Eq. 36 for arbitrary side l_{ij} becomes

$$\{f_{bc}^e\} = C_2 \int \begin{Bmatrix} N_i \\ N_j \\ N_k \\ N_l \end{Bmatrix} dx = \frac{C_2 l_{ij}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \tag{39}$$

Furthermore, the middle terms integral in Eq. 33 can be evaluated using the first derivatives of rectangular shape function Eq. 12. as

$$\frac{\partial[N]^T}{\partial x}\frac{\partial N}{\partial x} = \begin{bmatrix} \frac{\partial N_i}{\partial x}^2 & \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} & \frac{\partial N_i}{\partial x}\frac{\partial N_k}{\partial x} & \frac{\partial N_i}{\partial x}\frac{\partial N_l}{\partial x} \\ \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} & \frac{\partial N_j}{\partial x}^2 & \frac{\partial N_j}{\partial x}\frac{\partial N_k}{\partial x} & \frac{\partial N_j}{\partial x}\frac{\partial N_l}{\partial x} \\ \frac{\partial N_i}{\partial x}\frac{\partial N_k}{\partial x} & \frac{\partial N_j}{\partial x}\frac{\partial N_k}{\partial x} & \frac{\partial N_k}{\partial x}^2 & \frac{\partial N_k}{\partial x}\frac{\partial N_l}{\partial x} \\ \frac{\partial N_i}{\partial x}\frac{\partial N_l}{\partial x} & \frac{\partial N_j}{\partial x}\frac{\partial N_l}{\partial x} & \frac{\partial N_k}{\partial x}\frac{\partial N_l}{\partial x} & \frac{\partial N_l}{\partial x}^2 \end{bmatrix} \quad \text{(a)} \tag{40}$$

$$\frac{\partial[N]^T}{\partial y}\frac{\partial N}{\partial y} = \begin{bmatrix} \frac{\partial N_i}{\partial y}^2 & \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} & \frac{\partial N_i}{\partial y}\frac{\partial N_k}{\partial y} & \frac{\partial N_i}{\partial y}\frac{\partial N_l}{\partial y} \\ \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial y}^2 & \frac{\partial N_j}{\partial y}\frac{\partial N_k}{\partial y} & \frac{\partial N_j}{\partial y}\frac{\partial N_l}{\partial y} \\ \frac{\partial N_i}{\partial y}\frac{\partial N_k}{\partial y} & \frac{\partial N_j}{\partial y}\frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial y}^2 & \frac{\partial N_k}{\partial y}\frac{\partial N_l}{\partial y} \\ \frac{\partial N_i}{\partial y}\frac{\partial N_l}{\partial y} & \frac{\partial N_j}{\partial y}\frac{\partial N_l}{\partial y} & \frac{\partial N_k}{\partial y}\frac{\partial N_l}{\partial y} & \frac{\partial N_l}{\partial y}^2 \end{bmatrix} \quad \text{(b)} \tag{40}$$

Using Fig. 3 and Eq. 12, the integral of the terms above yields

$$[K^e] = \frac{DY}{6X}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{DX}{6Y}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \\ T_k \\ T_l \end{Bmatrix} \tag{41}$$

Using the rectangular interpolation functions, the last integral is evaluated and gives the residual as

$$\{f^e\} = \int [N]^T Q\, dA = Q\int \begin{Bmatrix} N_i \\ N_j \\ N_k \\ N_l \end{Bmatrix} dA = \frac{QXY}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{42}$$

Combining Eqs. 38, 39, 41, and 42 give all of the components contributing to the element residual integral (Eq. 33) in matrix form as

$$\{R^e\} = [K^e] + [K^e_{bc}]\{T^e\} + \{f^e_{bc}\} - \{f^e\} \tag{43}$$

2.3.3 Application of finite element for two-dimensional elliptic equation

To illustrate a two-dimensional elliptic equation, we will consider the temperature distribution of a two-dimensional rectangular region (Fig. 7) with a thermal diffusivity $D = 10$. A constant heat is being generated at the rate of $Q = 10$. Using four elements in each direction, the boundary conditions are specified where $y = 0,\ T = 5$ and at the opposite end where $y = 1,\ q = 3T - 6$ while the other regions are kept insulated. Assuming the material is isotropic and the elements are square. Thus, the elliptic two-dimensional equation (Eq. 30) becomes

$$10\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+10=0 \tag{44}$$

For simplicity, we use 4 elements to describe a unit square region. The numbering and the boundary conditions are show in Fig. 7. For illustrative purposes, we select element 3 and show all contribution for the system residual matrix mainly the flux boundary that is applied (l_{8-7}) top end. The element contribution becomes

$$\begin{aligned}\{R^3\} = \frac{10}{6}\begin{bmatrix}4 & -1 & -2 & -1\\ -1 & 4 & -1 & -2\\ -2 & -1 & 4 & -1\\ -1 & -2 & -1 & 4\end{bmatrix} + \frac{(3)(0.5)}{6}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}\begin{Bmatrix}T_4\\ T_5\\ T_8\\ T_7\end{Bmatrix} - \frac{6(0.5)}{2}\begin{Bmatrix}0\\ 0\\ 1\\ 1\end{Bmatrix}\\ - \frac{10(0.5)^2}{4}\begin{Bmatrix}1\\ 1\\ 1\\ 1\end{Bmatrix}\end{aligned} \tag{45}$$

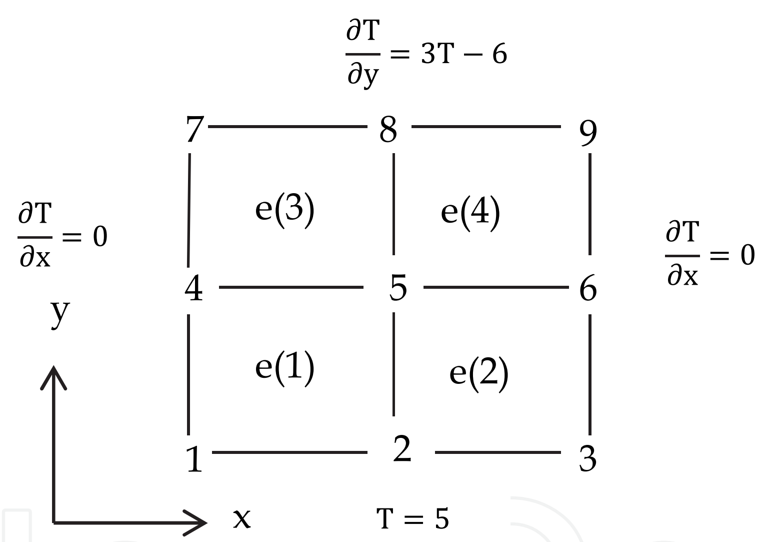

Fig. 7. Two-dimensional region divided into four square elements with boundary conditions

When combines and applied the specified boundary condition, the final system equation becomes 6 by 6 matrix as

$$\begin{bmatrix}8 & -2 & 0 & -1 & -2 & 0\\ -2 & 16 & -2 & -2 & -2 & -2\\ 0 & -2 & 8 & 0 & -2 & -1\\ 1 & -2 & 0 & 4.3 & -0.85 & 0\\ -2 & -2 & -2 & -0.85 & 8.6 & -0.85\\ 0 & -2 & -1 & 0 & -0.85 & 4.3\end{bmatrix}\begin{Bmatrix}T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9\end{Bmatrix} = \begin{Bmatrix}15.75\\ 31.50\\ 15.75\\ 1.275\\ 2.550\\ 1.275\end{Bmatrix} \tag{46}$$

As expected, the temperature profile is decreasing and symmetric as $T_4 = 4.97, T_5 = 4.97, T_6 = 4.97, T_7 = 4.69, T_8 = 4.69$, and $T_9 = 4.69$.

2.4 Parabolic equation in finite element method

The major characteristics of parabolic equations are that they require boundary and initial conditions (Awrejcewicz & Krysko, 2010). The general procedure for solving parabolic equations in finite element is by evaluating the residual integral with respect to space coordinates for fixed time. Using the initial value for the new value prediction, the time history is generated. In order to illustrate the fundamental procedure in solving a parabolic equation in the finite element method, we start by discussing a one-dimensional parabolic equation followed by two-dimensional equation. The one-dimensional scheme can be modified to include a two-dimensional equation with simple two-dimensional elements substitution.

2.4.1 One-dimensional parabolic equation

The cooling and heat process of material is considered parabolic in nature. The temperature change is expressed in terms of the rate of change in time and space. The heating and/or the cooling process of an insulated bar that is subjected to the different temperature can be considered a one-dimensional parabolic equation. In order to find the temperature in time, we need to solve the governing parabolic equation

$$D\frac{\partial^2 T}{\partial x^2} + Q - \lambda\frac{\partial T}{\partial t} = 0 \tag{47}$$

where λ is a rate constant. The finite element equation that gives the element contribution to the system residual is

$$\{R^e\} = -\int [N]^T\left(D\frac{\partial^2 T}{\partial x^2} + Q - \lambda\frac{\partial T}{\partial t}\right)dx \tag{48}$$

The first integral from the above equation is similar to Eq. 14 that yields the element contribution toward the residual integral as Eq. 25. What remain is solving the time-dependent integral, we use the average value assumption that the time derivatives ($\partial T/\partial t = \dot{T}$) varies linearly between the time interval. Using the shape function relationship that

$$\{\dot{T}^e\} = [N]\{\dot{T}^e\} \tag{49}$$

Then, the second term residual integral becomes

$$\{R_C^e\} = \lambda\int [N]^T[N]\{\dot{T}^e\}\,dx \tag{50}$$

The integral above is defined as capacitance matrix ($[C^e]$) and can be evaluated using the linear element interpolation function for one-dimensional element (Eq. 6). The integral result for the linear element is

$$\{R_C^e\} = \frac{\lambda l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \dot{T}_i \\ \dot{T}_j \end{Bmatrix} \tag{51}$$

The element contribution for final system equation becomes

$$\{R^e\} = \{q^e\} + [K^e]\{T^e\} - \{f^e\} - [C^e]\{\dot{T}^e\} \tag{52}$$

When the element equation is assembled, the sums of the residual vanish. As a result, the final system equation becomes

$$-[C]\{\dot{T}\} + [K]\{T\} + \{q\} - \{f\} = 0 \tag{53}$$

A time-dependent finite element equation requires solving the equation with time first. In order to approximate the time-dependent equation, the mean value-based equation is used. The mean (Sahoo & Riedel, 1998, Segerlind, 1984) rule is based on the hypothesis that the change in function (df/dt) at a location between two points (a, b) is proportional to the average change between two values of the function (f).

$$\frac{df}{dt} = \frac{f(b) - f(a)}{\Delta t} \tag{54}$$

The value at an arbitrary point c that is between a and b can be approximated as

$$f(c) = f(a) + (c - a)\frac{f(b) - f(a)}{\Delta t} \tag{55}$$

Let $(c - a)/\Delta t$ replaced by α

$$f = (1 - \alpha)f(a) + \alpha f(b) \tag{56}$$

Parallel to the mean value approximation, the time-dependent finite element solution (Eq. 53) can be approximated by introducing the vector containing the nodal values

$$[C]\{\dot{T}\} = \frac{[K]\{T\} + \{f\}_b - [K]\{T\} + \{f\}_a}{\Delta t} \tag{57}$$

The functional value between

$$([C] + \alpha\Delta t[K])\{T\}_b = ([C] - (1 - \alpha)\Delta t[K])\{T\}_a + \Delta t((1 - \alpha)\{f\}_a + \alpha\,\{f\}_b) \tag{58}$$

Thus, the nodal value can be predicted based on the known initial value and the time scale. When $\alpha = 1/2$, it is called the center difference method and the time-dependent finite element equation becomes

$$\left([C] + \frac{\Delta t}{2}[K]\right)\{T\}_b = \left([C] - \frac{\Delta t}{2}[K]\right)\{T\}_a + \frac{\Delta t}{2}\{f\}_a + \frac{\Delta t}{2}\,\{f\}_b \tag{59}$$

The above system equation has an equal number of unknown value and equation and can be solved by linear solvers.

2.4.2 Application of FEM in one-dimensional parabolic equation

To illustrate the application of the FEM in solving a one-dimensional parabolic equation, we will consider a finite element solution of an insulated shaft that is initially at known temperature (1) and places in the environment where the ends are subjected to 0 temperatures. The material diffusivity is $D = 10$ and heat capacity $\lambda = 1$. It is assumed to be one-dimensional since the lateral temperature change is insignificant to compare with the horizontal (x) direction. The length of a bar is 1 unit and for simplicity, we use four uniform elements (0.25) be used show the temperature distribution with time. Once the boundaries conditions are applied, the stiffness and capacitance matrix become

$$[K] = \frac{10}{0.25}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad \text{(a)}$$
$$[C] = \frac{0.25}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \qquad \text{(b)} \tag{60}$$
$$[f]^T = [0\ 0\ 0]$$

Unless the material property and time step change with time, the coefficient matrix is only evaluated once. Using the center difference method, the system matrix becomes similar to Eq. 59. Thus, using the previous temperature values to estimate the new value recursively, the time cooling process may be predicted by the finite element method. Using the time step of 0.001 s, the temperature profile of $T_2 = 0.30, T_3 = 0.43, T_4 = 0.30$ estimated after 0.01 s. In addition, a computer program is written by extending the above principle for 20 elements. The cooling process is solved using 0.001 s time step. The temperature profile with several times is shown in Fig. 8. As expected the rate cooling process with time is predicted using the finite element method and the solution also improves with elements number increases.

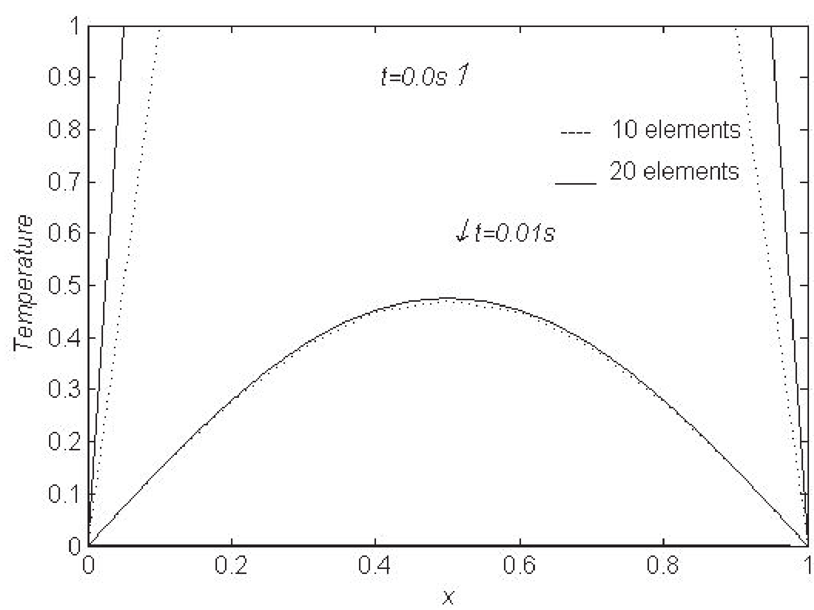

Fig. 8. The rate of cooling predicted with 10 and 20 linear elements using the finite element method.

2.4.3 Two-dimensional parabolic equation

A two-dimensional parabolic equation is represented by

$$D\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + Q - \lambda\frac{\partial T}{\partial t} \tag{61}$$

The element contribution to the residual is

$$\{R^e\} = -\int [N]^T \left(D\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + Q - \lambda\frac{\partial T}{\partial t}\right) dA \tag{62}$$

Parallel to the one-dimensional parabolic equation, the two-dimensional parabolic equation solution can be simply introduced by replacing the one-dimensional element integral for stiffness and capacitance matrix by the two-dimensional. In section 2.3.2, we showed that the first integral term using the linear rectangular element (Eq. 12 and the values in Fig 3.) yields the stiffness matrix and the derivative boundary conditions contribution ($[K_{bc}^{e}] + [K^{e}]$). Parallel to the one-dimensional parabolic case (Sec. 2.4.1), the time-dependent integral can be evaluated using the rectangular element (Eq. 12). This new capacitance matrix for two-dimensional element becomes

$$[C^e] = \int [N]^T \lambda [N] dA = \lambda \frac{XY}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \qquad (63)$$

Thus, the two-dimensional parabolic equation is similar to Eq. 53, but the vector and the matrix are going to be larger since the four nodal values are involved per element. The vector element is 1 by 4, while the matrix is 4 by 4 except for the boundary vectors.

2.4.4 Application of FEM in two-dimensional parabolic equation

To illustrate a two-dimensional parabolic equation application, we will consider the temperature history of the two-dimensional rectangular region shown in Fig. 9. We selected this problem for simplicity and illustrative purposes. Thermal diffusivity $D = 10$ and constant heat is being generated at the rate of $Q = 10$. The boundary conditions are specified as one end where $y = 0$, $T = 5$ and $y = 1.5$, $T = 1$ while the other regions are kept insulated. Initially, the surface temperature is kept at 5 degree before it is introduced into the environment. The objective of this to show that how FEM is applied to solve this parabolic equation. The region is discretized using six square elements size of 0.5 units (2 in horizontal and 3 in vertical direction).
The terms for two-dimensional, stiffness (Eq. 41), capacitance (Eq. 63), and the applied heat (Eq. 42) and boundary conditions (Eqs. 38 and 39) become

$$[K] = \frac{10}{6}\begin{bmatrix} 8 & -2 & 0 & -1 & -2 & 0 \\ -2 & 16 & -2 & -2 & -2 & -2 \\ 0 & -2 & 8 & 0 & -2 & -1 \\ -1 & -2 & 0 & 8 & -2 & 0 \\ -2 & -2 & -2 & -2 & 16 & -2 \\ 0 & -2 & -1 & 0 & -2 & 8 \end{bmatrix} \qquad \text{(a)} \qquad (64)$$

$$[C] = \frac{0.25}{36}\begin{bmatrix} 8 & 4 & 0 & 2 & 1 & 0 \\ 4 & 16 & 4 & 1 & 4 & 1 \\ 0 & 4 & 8 & 0 & 1 & 2 \\ 2 & 1 & 0 & 8 & 4 & 0 \\ 1 & 4 & 1 & 4 & 16 & 4 \\ 0 & 1 & 2 & 0 & 4 & 8 \end{bmatrix} \qquad (b) \qquad (64)$$

$$f = [-15.75 - 31.75 - 15.75 - 3.75 - 7.25 - 3.75]^T \quad (c)$$

Using the initial condition

$$\{T_0\} = \{5\} \qquad (65)$$

with center difference (Eq. 59) and the 0.001s time step, the temperature distribution of ($T_4 = 3.92, T_5 = 3.91, T_6 = 3.92, T_7 = 2.58, T_8 = 2.57,$ and $T_9 = 2.58$) is predicted after 0.1 seconds.

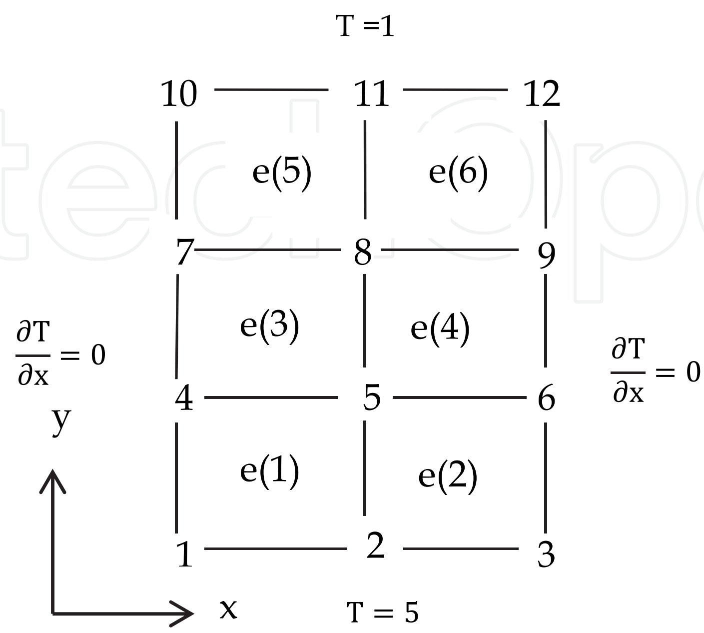

Fig. 9. Two dimensional square region using elements.

3. Finite difference method

The finite difference method is a direct interpretation of the differential equation into a discrete domain so that it can be solved using a numerical method. It is a direct representation of the governing equation ($\delta f/\delta x = (f_{i+1} - f_i)/(x_{i+1} - x_i)$. Using the discontinuous but connected regions, the governing equation is defined within the interval. In addition to direct interpretation, the deferential equation, the basic finite difference form, also can be derived from the Taylor-series expansion. Next, we will discuss the definition of one- and two-dimensional finite difference equations.

3.1 One-dimensional finite difference formulation

In order to define a finite difference representation in a one-dimensional space, we define a line space along the x-axis. The Taylor-series expansion for function f_{i+1} about point (i) is,

$$f_{i+1} = f_i + f'(x_{i+1} - x_i) + \tfrac{1}{2}f''(x_{i+1} - x_i)^2 + \tfrac{1}{6}f'''(x_{i+1} - x_i)^3 + \cdots \tag{66}$$

Let $(x_{i+1} - x_i) = \Delta x$

$$f_{i+1} = f_i + f'\Delta x + \frac{1}{2}f''\Delta x^2 + \frac{1}{6}f'''\Delta x^3 + \cdots \tag{67}$$

By rearranging

$$f' = \frac{(f_{i+1} - f_i)}{\Delta x} - \frac{1}{2}f''\Delta x - \frac{1}{6}f'''\Delta x^2 + \cdots \tag{68}$$

The above expression may be referred to as a forward difference. Furthermore, a similar expression may also be obtained by the backward difference

$$f' = \frac{(f_i - f_{i-1})}{\Delta x} \tag{69}$$

And center difference

$$f' = \frac{(f_{i+1} - f_{i-1})}{2\Delta x} \tag{70}$$

The higher order derivative of finite difference also may be derived from the Taylor-series expansion as

$$f_{i+1} = f_i + f'\Delta x + \frac{1}{2}f''\Delta x^2 + \frac{1}{6}f'''\Delta x^3 + \cdots \tag{71}$$

Using recursively and eliminate the first derivative term with previous approximation

$$\frac{1}{2}f''\Delta x^2 = f_{i+1} - f_i - \frac{(f_{i+1} - f_{i-1})}{2} - \frac{1}{6}f'''\Delta x^3 + \cdots \tag{72}$$

The final second-order derivative can be expressed as

$$f'' = \frac{1}{\Delta x^2}(f_{i+1} - 2f_i + f_{i-1}) - o(\Delta x) + \cdots \tag{73}$$

Similarly, the third-order derivative may also be defined as

$$f''' = \frac{1}{\Delta x^3}(-f_i + 3f_{i+1} - 3f_{i+2} + f_{i+3}) - o(\Delta x) \tag{74}$$

It also important to recognize that by using more points to form a discrete derivative, the error term may be minimized. For example, using five points instead of four in the equation above, the third-order derivate may be expressed as

$$f''' = \frac{1}{2\Delta x^3}(-f_{i-2} + 2f_{i-1} - 2f_{i+1} + f_{i+2}) - o(\Delta x^2) \tag{75}$$

So that, the error term is no longer a linear function but quadratic.

3.2 Two-dimensional finite difference expression

Parallel to one-dimensional forward, backward, and center difference expressions, the finite difference representation for a two-dimensional expression also can be defined. Similar to partial derivative, first, take the derivative two-dimensional space with one of the variables followed by the other as required. The first order partial derivatives of a function $f = f(x,y)$ in two- dimensional space (x,y) are expressed as

$$\begin{aligned} f'_x &= \frac{1}{2\Delta x}\left(f_{i+1,j} - f_{i-1,j}\right) \qquad (a) \\ f'_y &= \frac{1}{2\Delta y}\left(f_{i,j+1} - f_{i,j-1}\right) \qquad (b) \end{aligned} \tag{76}$$

Similarly, the second-order derivatives in x and y direction are

$$f_x'' = \frac{1}{\Delta x^2}\,(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}) \qquad (a)$$
$$f_y'' = \frac{1}{\Delta y^2}\,\left(f_{i,j+1} - 2f_{i,j} + f_{i,j-1}\right) \qquad (b) \tag{77}$$

Analogous to the partial derivate in the x and y direction, the function derivative with x and y, or vice versa, can be obtained by taking the first partial in one of the directions followed by the other. The final expression of the partial derivative is the same whether the x or y direction is used first or second.

$$f_{yx}'' = \frac{1}{4\Delta y \Delta x}\,\left(f_{i+1,j+1} - f_{i-1,j+1}\right) - \left(f_{i+1,j-1} - f_{i-1,j-1}\right) \tag{78}$$

3.3 Finite difference approximation of elliptic equation

Parallel to the previous sections, we start by considering the one-dimensional elliptic equation and how the FDM equation is formed. The major difference between the finite difference and the finite element method is that the finite difference method is based on the functional value at the nodal points, while the finite element is based on using the weighting function of the element to estimate the nodal values. We start by replacing the elliptic equation (Eq. 13) with the finite difference equation as

$$\frac{D}{\Delta x^2}(T_{i+1} - 2T_i + T_{i-1}) + Q = 0 \tag{79}$$

For simplicity, $\Delta x = h$ and be uniform, and $i = 1, 2 \ldots N$, thus

$$\frac{D}{h^2}(T_{i-1} - 2T_i + T_{i+1}) + Q = 0 \tag{80}$$

The solution of the elliptic equation is required the boundary condition. Thus, the boundary condition must be specified at, $i = 1$ and $i = N$. The number of equations and the unknown depends on the boundary condition whether or not the particular value or the derivative values are specified. When the derivative boundary condition is specified, the general flux equation (Eq. 35) with all discrete finite derivative methods (Sec. 3.1) can be used to replace the derivative boundary condition. The flux at the boundary can be estimated using the forward, backward, and center difference. Forward or backward difference can be used to define the flux using the boundary point and an ideal point next to it ($dT/dn = (T_{b+1} - T_b)/h$. Moreover, for a more accurate estimate, the center difference may be used ($dT/dn = (T_{b-1} - T_{b+1})/2h$). Both methods require the introduction of a new ideal point outside the region. The ideal point is eventually eliminated by combining the equations that include the specified boundary condition and the boundary node equation.

3.3.1 Application of FDM in one-dimensional elliptic equation

To illustrate a one-dimensional elliptic equation, we will consider the temperature distribution of a one-dimensional rod that is discussed in Sec. 2.3.1. Thus, when the material properties with all the assumption applied to the finite difference elliptic equation (Eq. 72) becomes

$$\frac{10}{0.25^2}(T_{i-1} - 2T_i + T_{i+1}) + 10 = 0 \tag{81}$$

When the prescribed boundary condition at the end where the temperature is fixed applied, T_1 becomes 5. Using the forward difference, the prescribed flux boundary at the opposite end gives the equation

$$\frac{\partial T}{\partial n} = D\frac{T_6 - T_5}{h} = -10 \tag{82}$$

where $T_6 = T_5 - 10h/D$. T_6 can be illuminated between the boundary condition and the last equation. Using all the element equation, the finite difference expression of the system becomes

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} -5.0625 \\ -0.0625 \\ -0.0625 \\ 0.1875 \end{Bmatrix} \tag{83}$$

The finite difference solution for the temperature profile produces the values $T_2 = 5.00$, $T_3 = 4.94$, $T_4 = 4.81$, and $T_5 = 4.62$. Indeed, the matrixes in Eq. 29 and Eq. 83 have some striking similarity considering they are formed from two different methods. As expected, the profiles in both FE and FD methods are the same. It is not also expected that the nodal values have differences. Furthermore, the computer solution for 10 and 20 elements is shown in Fig. 10. The profile indicates that with more nodal values, a better result can be estimated.

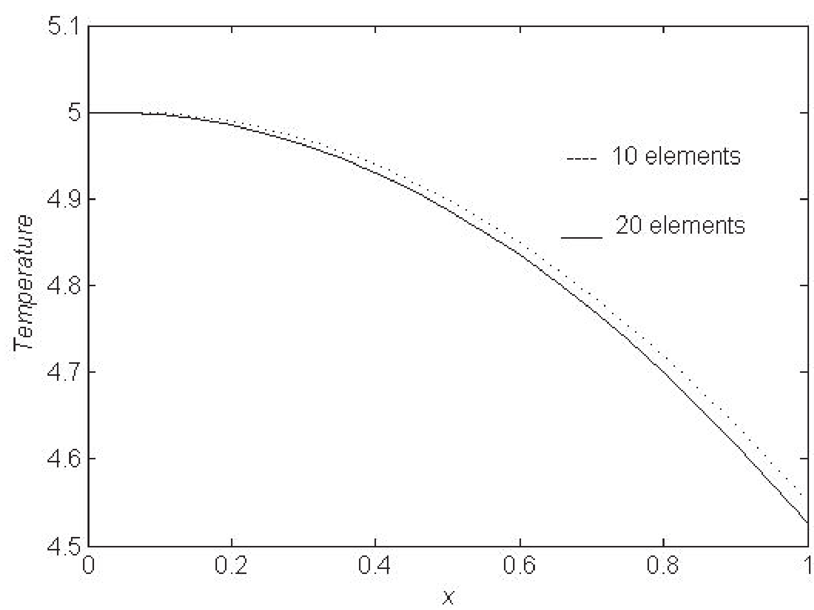

Fig. 10. Finite difference approximated temperature profile for one-dimensional elliptic equation rod.

3.3.2 Tow-dimensional elliptic finite difference solution

As previously discussed, elliptic equations are generally associated with steady state problems. The finite difference representation of two-dimensional elliptic equation (Eq. 30) for steady state temperature distribution is

$$\frac{D}{\Delta x^2}\left(T_{i+1,j} - 2T_{i,j} + T_{i-1,j}\right) + \frac{D}{\Delta y^2}\left(T_{i,j+1} - 2T_{i,j} + T_{i,j-1}\right) + Q = 0 \tag{84}$$

For simplicity, we let $\Delta x = \Delta y = h$, and assuming the diffusion is isotropic, we get

$$\frac{D}{h^2}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j}\right) + Q = 0 \tag{85}$$

As discussed earlier, the boundary condition may be a functional value and/or the flux at the edge, and the uniqueness of the solution depends on the boundary condition. There are vast methods developed over the years for direct and iterative solution methods in the form of explicit and implicit forms. As the name implies, the direct method involves a fixed number of operations to find a solution. In contrast, the iterative method starts with an approximation that successively improves (Dahlquist and Bjorck, 1974). Eq. 85 may be arranged in terms i or j so that the known value be solved using all the other known values. Furthermore by recognizing the relationship pattern among the neighboring points, the system equation can be generated as

$$T_{i+1,j} = \left(4T_{i,j} - T_{i-1,j} - T_{i,j+1} - T_{i,j-1}\right) - \frac{h^2}{D}Q \tag{86}$$

To illustrate the FD solution for two-dimensional elliptic equation, we will consider the problem discoursed before in Sec. 2.3.3 and Fig. 7. Thus, the FD system equation becomes

$$\begin{bmatrix} -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -3.15 & 2 & 0 \\ 0 & 1 & 0 & 1 & -3.15 & 1 \\ 0 & 0 & 1 & 0 & 2 & -3.15 \end{bmatrix} \begin{Bmatrix} T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \begin{Bmatrix} -5.25 \\ -5.25 \\ -5.25 \\ -0.55 \\ -0.55 \\ -0.55 \end{Bmatrix} \tag{87}$$

Solving Eq. 87 gives the temperature value for all nodal points as $T_4 = 5.0, T_5 = 5.0, T_6 = 5.0, T_7 = 4.7, T_8 = 4.7, \text{and } T_9 = 4.7$. The estimated temperature profile is similar to FEM solution in Sec. 2.3.3 with some nodal value variation.

3.4 Finite difference approximation of parabolic equation

The parabolic equation is a function of space and time. Thus, it involves at least two variables, time and space. It is always expressed in partial form since at least two variables are involved. The solution requires the boundary condition and the initial condition. The finite difference parabolic equation is different from an elliptic equation since the solution starts from the known time and propagates with increases in time.

3.4.1 One-dimensional parabolic finite difference equation

To illustrate the application of the finite difference method in solving a parabolic equation, we will consider the cooling process with time as previously noted in (Sec. 2.4.1). The one-dimensional finite difference time dependent equation can be generated by replacing one of the dimensions with time in Sec. 3.1. Furthermore, for parabolic equation the first and second derivative with time and space need to be defined. Thus, one-dimensional finite element parabolic equation becomes

$$\lambda\frac{(T_{i,\tau+1}-T_{i,\tau})}{\Delta t}=D\frac{1}{\Delta x^2}\left(T_{i+1,\tau}-2T_{i,\tau}+T_{i-1,\tau}\right)+Q \tag{88}$$

The subscript i and τ are used to reprecent the space dimension (x) and time (t), respectively. One can see that with the known boundary condition and initial condition and using the appropriate time step, the new value of T may be predicted. A very simplistic explanation of the finite difference expression above can be given by letting the boundary condition $(i+1)$ and $(i-1)$ and the initial condition $(T_{i,\tau})$ be known. Thus, the only unknown value is $T_{i,\tau+1}$ that is a function of the time step (Δt) the element size (Δx). The time step and the stability of the system are major parts of the parabolic equation. For example, for an explicit FD equation, the value for $\Delta t/\Delta x^2 \leq 0.5$ for the system to be stable and stability is a major part of numerical solutions (Smith, 1985; Dahlquist &Bjorck, 1974).

3.4.2 Application of FDM in one-dimensional parabolic equation

In order to demonstrate the application of FDM in parabolic equation, we will consider previously discussed cooling process of a thin insulated bar that initially at some temperature is placed in the environment where the heat allows to flow from the ends (Sec. 2.4.1). Similarly, , we will use four elements where the element size becomes 0.25. And let the time step be 0.001. Thus, the finite difference expression becomes

$$\frac{(T_{i,\tau+1}-T_{i,\tau})}{0.001}=\frac{10}{0.25^2}\left(T_{i+1,\tau}-2T_{i,\tau}+T_{i-1,\tau}\right) \tag{89}$$

The boundary conditions are $T_1=T_5=0$ and the initial condition is $t=0,$ all $T=1$ .Thus, the system equations becomes

$$\begin{Bmatrix} T_{2,\tau+1} \\ T_{3,\tau+1} \\ T_{4,\tau+1} \end{Bmatrix}=\begin{bmatrix} 0.68 & 0.16 & 0 \\ 0.16 & 0.68 & 0.16 \\ 0 & 0.16 & 0.68 \end{bmatrix}\begin{Bmatrix} T_{2,\tau} \\ T_{3,\tau} \\ T_{4,\tau} \end{Bmatrix} \tag{90}$$

Using the time step of 0.001 s, the temperature profile of $T_2=0.32, T_3=0.45, T_4=0.32$ estimated after 0.01 s. Furthermore, using the same process above, we wrote the computer program using 10 and 20 elements and solved the elliptic equation using 0.001s time step. The temperature profile with time is shown in Fig. 10. As expected, the rate cooling process with time is predicted using the finite element method.

3.4.3 Two-dimensional finite difference parabolic equation

Parallel to the above one-dimensional finite difference parabolic equation, the two-dimensional equation can be simply introduced by modifying the two-dimensional notation. The notation is modified to accommodate the time variable by using the superscripts instead of the subscripts for time. Thus, the finite difference two-dimensional parabolic (Eq. 61) becomes

$$\lambda\frac{(T_{i,j}^{\tau+1}-T_{i,j}^{\tau})}{\Delta t}=\frac{D}{\Delta x^2}\left(T_{i+1,j}^{\tau}-2T_{i,j}^{\tau}+T_{i-1,j}^{\tau}\right)+\frac{D}{\Delta y^2}\left(T_{i,j+1}^{\tau}-2T_{i,\tau}^{\tau}+T_{i,j+1}^{\tau}\right)+Q \tag{91}$$

Parallel to the previous one-dimensional case (Sec. 3.4.1), the boundary condition may be prescribed in several ways as a functional value and/or a flux depending on the situation. Since it is a time function, it also requires the initial condition.

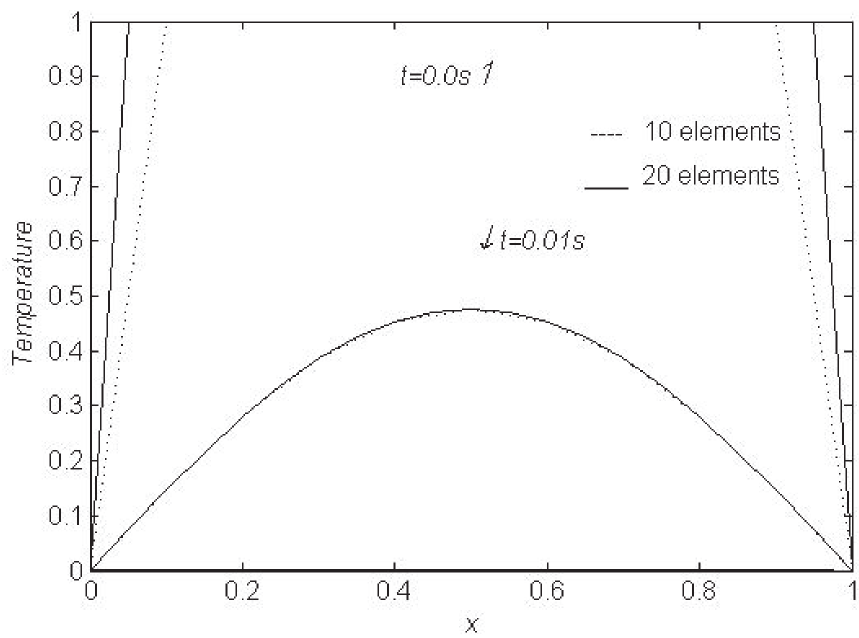

Fig. 11. The rate of cooling predicted with 10 and 20 linear elements using the finite difference method.

3.4.4 Application of FDM in two-dimensional finite difference parabolic equation

To illustrate the finite difference method application in solving two-dimensional parabolic equation, we will consider the temperature distribution of the two-dimensional rectangular region previously discussed in Sec. 2.4.4 and shown in Fig. 9. Using six elements, 2 in horizontal and 3 in vertical direction with specified boundary conditions, the two-dimensional FD parabolic equation becomes

$$\frac{(T_{i,j}^{\tau+1} - T_{i,j}^{\tau})}{\Delta t} = \frac{10}{0.5^2}\left(T_{i+1,j}^{\tau} - 2T_{i,j}^{\tau} + T_{i-1,j}^{\tau}\right) + \frac{10}{0.5^2}\left(T_{i,j+1}^{\tau} - 2T_{i,\tau}^{\tau} + T_{i,j+1}^{\tau}\right) + \frac{(10)0.5^2}{10} \quad (92)$$

Rearranging and forming in matrix form

$$\begin{Bmatrix} T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}^{t+\Delta t} = \begin{Bmatrix} T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}^{t} + \frac{10\Delta t}{0.5^2}\left(\begin{bmatrix} -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 2 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 2 & -4 \end{bmatrix}\begin{Bmatrix} T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}^{t} + \begin{Bmatrix} 5.25 \\ 5.25 \\ 5.25 \\ 1.25 \\ 1.25 \\ 1.25 \end{Bmatrix}\right) \quad (93)$$

Thus, finite difference two-dimensional elliptic equation (Eq. 93) solution can be generated by solving the space equation for a fixed time first (solving the second right term in the bracket first) and using that to get the new estimate. By using the time estimate recursively, the time process may be generated. Similar to Sec. 2.4.4, the 0.001s time step, the temperature distribution of ($T_4 = 3.95, T_5 = 3.95, T_6 = 3.95, T_7 = 2.62, T_8 = 2.62,$ and $T_9 = 2.62$) is predicted after 0.1 seconds. The predicted values have similar profile and values with FEM solution in Sec. 2.4.4.

4. Concluding remarks

In this chapter, we illustrated numerical solutions of elliptic and parabolic equations using both finite element and finite difference methods. Elliptic and parabolic equations are encountered in numerous areas of engineering and science. Finite element and finite difference methods are the two most frequently applied numerical approximations, although several numerical methods are available. We illustrated how finite element method utilizes discrete elements to obtain the approximate solution of the governing differential equation. In addition, we showed how the final system equation is constructed from the discrete element equations. In addition, we also showed how finite difference method uses points over intervals to define the equation and the combination of all the points to produce the system equation. Both systems generate large linear and/or nonlinear system equations that can be solved by computer.

FEM and FDM are evolving with technology. The growth in computer technology has made it even more possible to consider using them in many science and engineering applications. In addition, more people without science and engineering backgrounds are becoming numerical simulation users. Consequently, the fundamental understanding of numerical simulation is becoming increasingly very important. Thus, this chapter intended to give some fundamental introduction into FEM and FDM by considering simple and familiar examples. We illustrated the similarity and the differences in finite difference and finite element methods by considering the simple elliptic and parabolic equations. Indeed, for the problems considered, one can see that the similarity and the difference from the final system equations and approximate solution. We designed the chapter to be introductory. By considering simple examples, we have illustrated FEM and FDM are reasonable ways of estimating solutions.

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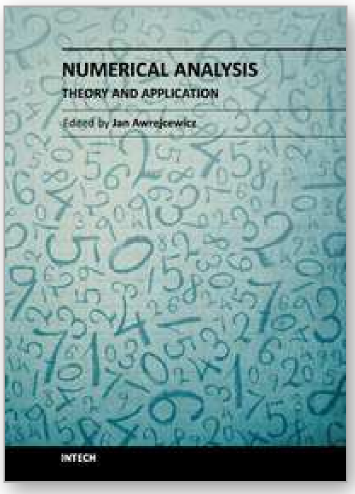

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Numerical Analysis â€“ Theory and Application is an edited book divided into two parts: Part I devoted to Theory, and Part II dealing with Application. The presented book is focused on introducing theoretical approaches of numerical analysis as well as applications of various numerical methods to either study or solving numerous theoretical and engineering problems. Since a large number of pure theoretical research is proposed as well as a large amount of applications oriented numerical simulation results are given, the book can be useful for both theoretical and applied research aimed on numerical simulations. In addition, in many cases the presented approaches can be applied directly either by theoreticians or engineers.