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Steady-State Performance Analyses of Adaptive Filters

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1. Introduction

Adaptive filters have become a vital part of many modern communication and control systems, which can be used in system identification, adaptive equalization, echo cancellation, beamforming, and so on [l]. The least mean squares (LMS) algorithm, which is the most popular adaptive filtering algorithm, has enjoyed enormous popularity due to its simplicity and robustness [2] [3]. Over the years several variants of LMS have been proposed to overcome some limitations of LMS algorithm by modifying the error estimation function from linearity to nonlinearity. Sign-error LMS algorithm is presented by its computational simplicity [4], least-mean fourth (LMF) algorithm is proposed for applications in which the plant noise has a probability density function with short tail [5], and the LMMN algorithm achieves a better steady state performance than the LMS algorithm and better stability properties than the LMF algorithm by adjusting its mixing parameter [6], [7].

The performance of an adaptive filter is generally measured in terms of its transient behavior and its steady-state behavior. There have been numerous works in the literature on the performance of adaptive filters with many creationary results and approaches [3]-[20]. In most of these literatures, the steady-state performance is often obtained as a limiting case of the transient behavior [13]-[16]. However, most adaptive filters are inherently nonlinear and time-variant systems. The nonlinearities in the update equations tend to lead to difficulties in the study of their steady-state performance as a limiting case of their transient performance [12]. In addition, transient analyses tend to require some more simplifying assumptions, which at times can be restrictive. Using the energy conservation relation during two successive iteration update , N. R. Yousef and A. H. Sayed re-derived the steady-state performance for a large class of adaptive filters [11],[12], such as sign-error LMS algorithm, LMS algorithm, LMMN algorithm, and so on, which bypassed the difficulties encountered in obtaining steady-state results as the limiting case of a transient analysis. However, it is generally observed that most works for analyzing the steady-state performance study individual algorithms separately. This is because different adaptive schemes have different nonlinear update equations, and the particularities of each case tend to require different arguments and assumptions. Some authors try to investigate the steady-state performance from a general view to fit more adaptive filtering algorithms, although that is a challenge task. Based on Taylor series expansion (TSE), S. C. Douglas and T. H. Meng obtained a general expression for the steady-state MSE for adaptive filters with error

nonlinearities [10]. However, this expression is only applicable for the cases with the real-valued data and small step-size. Also using TSE, our previous works have obtained some analytical expressions of the steady-state performance for some adaptive algorithms [8], [17], [19], [28]. Using the Price's theory, T. Y. Al-Naffouri and A. H. Sayed obtained the steady-state performance as the fixed-point of a nonlinear function in EMSE [11], [18]. For a lot of adaptive filters with error nonlinearities, their closed-form analytical expressions can not be obtained directly, and the Gaussion assumption condition of Price's theory is not adaptable for other noise. Recently, as a limiting case of the transient behavior, a general expression of the steady state EMSE was obtained by H. Husøy and M. S. E. Abadi [13]. Observing from the Table 1 in [13], we can see that this expression holds true only for the adaptive filters with most kinds of the preconditioning input data, and can not be used to analyze the adaptive filters with error nonlinearities.

These points motivate the development in this paper of a unified approach to get their general expressions for the steady-state performance of adaptive filters. In our analyses, second-order TSE will be used to analyze the performance for adaptive algorithms for real-valued cases. But for complex-valued cases, a so-called *complex Brandwood-form series expansion* (BSE), derived by G. Yan in [22], will be utilized. This series expansion is based on Brandwood's derivation operators [21] with respect to the complex-valued variable and its conjugate, and was used to analyze the MSE for Bussgang algorithm (BA) in noiseless environments [19], [20]. Here, the method is extended to analyze other adaptive filters in complex-valued cases.

1.1 Notation

Throughout the paper, the small boldface letters are used to denote vectors, and capital boldface letters are used to denote matrices, e.g., $\mathbf{w}_i$ and $\mathbf{R}_u$. All vectors are *column vectors*, except for the input vector $\mathbf{u}_i$, which is taken to be a row vector for convenience of notation. In addition, the following notations are adopted:

$\|\cdot\|$ Euclidean norm of a vector;	$\mathrm{Tr}(\cdot)$ Trace of a matrix;
$\mathrm{E}\{\cdot\}$ Expectation operator;	$\mathrm{Re}(\cdot)$ The real part of a complex-valued data;
$\mathbf{I}_{M\times M}$ $M\times M$ Identity matrix;	$(\cdot)^*$ Complex conjugation for scalars;
! Factorial;	!! Double factorial;

$f_x^{(1)}(a)$ 1th derivative of the function $f(x)$ with respect to x at the value a;
$f_{x,y}^{(2)}(a,b)$ 2th partial derivative of the function $f(x,y)$ with respect to x and y at the value (a,b) [1];
$C^i(D)$ The set of all functions for which $f^{(i)}(x)$ is continuous in definition domain D for each natural number i.

1.2 System model

Consider the following stochastic gradient approach for adaptive filters function [10]-[12][2]

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \mu g(\mathbf{u}_i)\mathbf{u}_i^{\mathrm{H}} f\left(e_i, e_i^*\right), \tag{1}$$

[1] Similar notations can be used for $f_{x,x}^{(2)}(a,b)$ and $f_{y,y}^{(2)}(a,b)$.

[2] If e is complex-valued, the estimation error function $f(e,e^*)$ has two independent variables: e and e^*. In addition, due to $e = e^*$, $f(e,e^*)$ can be replaced by $f(e)$ if e is real-valued. Here, we use the general form $f(e,e^*)$.

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_i \,, \tag{2}$$

$$d_i = \mathbf{u}_i \mathbf{w}_{o,i} + v_i \,, \tag{3}$$

where

μ step-size; $\mathbf{u}_i$ $(1 \times M)$ row input (regressor) vector;
H conjugate and transpose; $\mathbf{w}_i$ $(M \times 1)$ weight vector;
e_i scalar-valued error signal; d_i scalar-valued noisy measurement;
$g(\mathbf{u}_i)$ scalar variable factor for step-size.
$\mathbf{w}_{o,i}$ $(M \times 1)$ unknown column vector at constant i that we wish to estimate;
v_i accounts for both measurement noise and modeling errors, whose support region is D_v.
$f(e_i, e_i^*)$ memoryless nonlinearity function acting upon the error e_i and its complex conjugate e_i^*. Different choices for $f(e_i, e_i^*)$ result in different adaptive algorithms. For example, Table 1 defines $f(e_i, e_i^*)$ for many well-known special cases of (1) [10]-[12].

The rest of the paper is organized as follows. In the next section, the steady-state performances for complex and real adaptive filters are derived, which are summarized in Theorem 1 based on separation principle and Theorem 2 for white Gaussian regressor, respectively. In section 3, based on Theorem 1 and Theorem 2, the steady-state performances for the real and complex least-mean p-power norm (LMP) algorithm, LMMN algorithm and their normalized algorithms, are investigated, respectively. Simulation results are given in Section 4, and conclusions are drawn in Section 5.

Algorithms	**Estimation errors**
LMP	$\lvert e_i \rvert^{p-2} e_i$
LMMN	$e_i \left(\delta + (1-\delta) \lvert e_i \rvert^2 \right)$
ε - NLMP	$\lvert e_i \rvert^{p-2} e_i \Big/ \left(\varepsilon + \lVert \mathbf{u} \rVert^2 \right)$
ε - LMMN	$e_i \left(\delta + (1-\delta) \lvert e_i \rvert^2 \right) \Big/ \left(\varepsilon + \lVert \mathbf{u} \rVert^2 \right)$

Notes:

1. The parameter p is the order of the cost function of LMP algorithm, which includes LMS ($p = 2$), LMF ($p = 4$) algorithms.
2. The parameter δ, such that $0 \le \delta \le 1$, is the mixing paramter of LMMN algorithms. $\delta = 1$ results in the LMS algorithm and $\delta = 0$ results in the LMF algorithm.
3. The parameter ε of ε - NLMP algorithm or ε - LMMN algorithm is a small positive real value.

Table 1. Examples for the estimation error

2. Steady-state performance analyses

Define so-call *a priori* estimation error $e_a(i) = \mathbf{u}_i \tilde{\mathbf{w}}_i$, where $\tilde{\mathbf{w}}_i = \mathbf{w}_{o,i} - \mathbf{w}_i$ is the weight-error vector. Then, under (2) and (3), the relation between e_i and $e_a(i)$ can be expressed as

$$e_i = e_a(i) + v_i \,. \tag{4}$$

The steady-state MSE for an adaptive filter can be written as $\zeta_{MSE} = \lim_{i\to\infty} \mathrm{E}|e_i|^2$. To get ζ_{MSE}, we restrict the development of statistical adaptive algorithm to a small step-size, long filter length, an appropriate initial conditions of the weights and finite input power and noise variance in much of what follows[3], which is embodied in the following two assumptions:

A.1: The noise sequence $\{v_i\}$ with zero-mean and variance σ_v^2 is independent identically distributed (i.i.d.) and statistically independent of the regressor sequence $\{\mathbf{u}_i\}$.

A.2: The a priori estimation error $e_a(i)$ with zero-mean is independent of v_i. And for complex-valued cases, it satisfies the circularity condition, namely, $\mathrm{E}e_a^2(i) = 0$.

The above assumptions are popular, which are commonly used in the steady-state performance analyses for most of adaptive algorithms [11]-[14]. Then, under A.1 and A.2, the steady-state MSE can be written as $\zeta_{MSE} = \sigma_v^2 + \zeta$, where ζ is the steady-state EMSE, defined by

$$\zeta = \lim_{i\to\infty} \mathrm{E}\left|e_a(i)\right|^2 . \tag{5}$$

That is to say, getting ζ is equivalent to getting the MSE.

A first-order random-walk model is widely used to get the tracking performance in nonstationary environments [11], [12], which assumes that $\mathbf{w}_{o,i}$ appearing in (3) undergoes random variations of the form

$$\mathbf{w}_{o,i+1} = \mathbf{w}_{o,i} + \mathbf{q}_i , \tag{6}$$

where $\mathbf{q}_i$ is $(M\times 1)$ column vector and denotes some random perturbation.

A.3: The stationary sequence $\{\mathbf{q}_i\}$ is i.i.d., zero-mean, with $(M\times M)$ covariance matrix $\mathrm{E}\left(\mathbf{q}_i\mathbf{q}_i^{\mathrm{H}}\right) = \mathbf{Q}$, which is independent of the regressor sequences $\{\mathbf{u}_i\}$ and weight-error vector $\tilde{\mathbf{w}}_i$.

In stationary environments, the iteration equation of (6) becomes $\mathbf{w}_{o,i+1} = \mathbf{w}_{o,i}$, i.e., $\mathbf{w}_{o,i}$ does not change during the iteration because of $\{\mathbf{q}_i\}$ being a zero sequence. Here, the covariance matrix of $\{\mathbf{q}_i\}$ becomes $\mathrm{E}\left(\mathbf{q}_i\mathbf{q}_i^{\mathrm{H}}\right) = \mathbf{0}$, where $\mathbf{0}$ is a $(M\times M)$ zero matrix.

Substituting (6) and the definition of $\tilde{\mathbf{w}}_i$ into (1), we get the following update

$$\tilde{\mathbf{w}}_{i+1} = \tilde{\mathbf{w}}_i - \mu g(\mathbf{u}_i)\mathbf{u}_i^{\mathrm{H}} f\left(e_i, e_i^*\right) + \mathbf{q}_i \tag{7}$$

By evaluating the energies of both sides of the above equation, we obtain

$$\begin{aligned} \left\|\tilde{\mathbf{w}}_{i+1}\right\|^2 &= \left\|\tilde{\mathbf{w}}_i\right\|^2 - \mu\tilde{\mathbf{w}}_i^{\mathrm{H}}\mathbf{u}_i^{\mathrm{H}} g(\mathbf{u}_i) f\left(e_i, e_i^*\right) - \mu\mathbf{u}_i\tilde{\mathbf{w}}_i g^*(\mathbf{u}_i) f^*\left(e_i, e_i^*\right) \\ &+ \mu^2\left\|\mathbf{u}_i\right\|^2\left|g(\mathbf{u}_i)\right|^2\left|f\left(e_i, e_i^*\right)\right|^2 + \tilde{\mathbf{w}}_i^{\mathrm{H}}\mathbf{q}_i + \mathbf{q}_i^{\mathrm{H}}\tilde{\mathbf{w}}_i - \mu\mathbf{q}_i^{\mathrm{H}}\mathbf{u}_i^{\mathrm{H}} g(\mathbf{u}_i) f\left(e_i, e_i^*\right) \\ &- \mu\mathbf{u}_i\mathbf{q}_i g^*(\mathbf{u}_i) f^*\left(e_i, e_i^*\right) + \left\|\mathbf{q}_i\right\|^2 \end{aligned} \tag{8}$$

[3] As described in [25] and [26], the convergence or stability condition of an adaptive filter with error nonlinearity is related to the initial conditions of the weights, the step size, filter length, input power and noise variance. Since our works mainly focus on the steady-state performances of adaptive filters, the above conditions are assumed to be satisfied.

Taking expectations on both sides of the above equation and using A.3 and $e_a(i) = \mathbf{u}_i\tilde{\mathbf{w}}_i$, we get

$$\begin{aligned}\mathrm{E}\|\tilde{\mathbf{w}}_{i+1}\|^2 = \mathrm{E}\|\tilde{\mathbf{w}}_i\|^2 - \mu\mathrm{E}\left[e_a^*(i)g(\mathbf{u}_i)f\left(e_i,e_i^*\right)\right] - \mu\mathrm{E}\left[e_a(i)g^*(\mathbf{u}_i)f^*\left(e_i,e_i^*\right)\right] \\ + \mu^2\mathrm{E}\left[\|\mathbf{u}_i\|^2\left|g(\mathbf{u}_i)\right|^2 f\left(e_i,e_i^*\right)\right|^2\Big] + \mathrm{E}\|\mathbf{q}_i\|^2\end{aligned}. \tag{9}$$

At steady state, the adaptive filters hold $\lim_{i\to\infty}\mathrm{E}\|\tilde{\mathbf{w}}_{i+1}\|^2 = \lim_{i\to\infty}\mathrm{E}\|\tilde{\mathbf{w}}_i\|^2$ [11], [12]. Then, the variance relation equation (9) can be rewritten compactly as

$$2\mathrm{Re}\,\mathrm{E}\left[e_a^* g(\mathbf{u})f\left(e,e^*\right)\right] = \mu\mathrm{E}\left[\|\mathbf{u}\|^2\left|g(\mathbf{u})\right|^2\left|f\left(e,e^*\right)\right|^2\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}). \tag{10}$$

where $\mathrm{Tr}(\mathbf{Q}) = \mathrm{E}\|\mathbf{q}_i\|^2$, and the time index '*i*' has been omitted for the easy of reading. Specially, in stationary environments, the second term in the light-hand side of (10) will be removed since $\{\mathbf{q}_i\}$ is a zero sequence (i.e., $\mathrm{Tr}(\mathbf{Q}) = 0$).

2.1 Separation principle

At steady-state, since the behavior of e_a in the limit is likely to be less sensitive to the input data when the adaptive filter is long enough, the following assumption can be used to obtain the steady-state EMSE for adaptive filters, i.e.,

A.4: $\|\mathbf{u}\|^2$ and $g(\mathbf{u})$ are independent of e_a.

This assumption is referred to as the *separation principle* in [11]. Under the assumptions A.2 and A.4, and using (4), we can rewrite (10) as

$$\alpha_u\mathrm{E}\varphi\left(e,e^*\right) = \mu\beta_u\mathrm{E}q\left(e,e^*\right) + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) \tag{11}$$

where

$$\begin{aligned}&\alpha_u = \mathrm{E}g(\mathbf{u}), \quad \beta_u = \mathrm{E}\left[\|\mathbf{u}\|^2\left|g(\mathbf{u})\right|^2\right] \\ &\varphi\left(e,e^*\right) = 2\mathrm{Re}\left[e_a^* f\left(e,e^*\right)\right], \quad q\left(e,e^*\right) = \left|f\left(e,e^*\right)\right|^2\end{aligned}. \tag{12}$$

Lemma 1 If e is complex-valued, and $\phi\left(e,e^*\right)$ and $q\left(e,e^*\right)$ are defined by (12), then[4]

$$\begin{aligned}&\varphi\left(v,v^*\right) = 0, \quad \varphi_{e,e^*}^{(2)}\left(v,v^*\right) = 2\mathrm{Re}\,f_e^{(1)}\left(v,v^*\right), \quad q_{e,e^*}^{(2)}\left(v,v^*\right) = \left|f_e^{(1)}\left(v,v^*\right)\right|^2 \\ &+\left|f_{e^*}^{(1)}\left(v,v^*\right)\right|^2 + 2\mathrm{Re}\left[f^*\left(v,v^*\right)f_{e,e^*}^{(2)}\left(v,v^*\right)\right]\end{aligned}. \tag{13}$$

The proofs of Lemma 1 and all subsequent lemmas in this paper are given in the APPENDIXS.

[4] Since e and e^* are assumed to be two independent variables, all $f\left(e,e^*\right)$ in Table 1 can be considered as a 'real' function with respect to e and e^*, although $f\left(e,e^*\right)$ may be complex-valued. Then, the accustomed rules of derivative with respect to two variables e and e^* can be used directly.

Lemma 2 If e is real-valued, and $\varphi(e)$ and $q(e)$ are defined by (12)[5], then

$$\varphi(v)=0,\quad \varphi^{(2)}_{e,e}(v)=4f^{(1)}_{e}(v),\quad q^{(2)}_{e,e}(v)=2\left|f^{(1)}_{e}(v)\right|^2+2f(v)f^{(2)}_{e,e}(v). \tag{14}$$

Theorem 1—Steady-state performance for adaptive filters by separation principle: Consider adaptive filters of the form (1) – (3), and suppose the assumptions A.1-A.4 are satisfied and $f\left(e,e^*\right)\in C^2(D_v)$. Then, if the following condition is satisfied, i.e.,

C.1 $A\alpha_u > \mu B\beta_u$,

the steady-state EMSE (ζ_{EMSE}), tracking EMSE (ζ_{TEMSE}), and the optimal step-size (μ_{opt}) for adaptive filters can be approximated by

$$\zeta_{EMSE}=\frac{\mu C\beta_u}{A\alpha_u-\mu B\beta_u} \tag{15}$$

$$\zeta_{TEMSE}=\frac{\mu^{-1}\mathrm{Tr}(\mathbf{Q})+\mu C\beta_u}{A\alpha_u-\mu B\beta_u} \tag{16}$$

$$\mu_{opt}=\sqrt{\left[\frac{B\mathrm{Tr}(\mathbf{Q})}{AC\alpha_u}\right]^2+\frac{\mathrm{Tr}(\mathbf{Q})}{C\beta_u}}-\frac{B\mathrm{Tr}(\mathbf{Q})}{AC\alpha_u} \tag{17}$$

where

$$A=2\mathrm{Re}\,\mathrm{E}f^{(1)}_{e}\left(v,v^*\right),\quad B=\mathrm{E}\left\{\left|f^{(1)}_{e}\left(v,v^*\right)\right|^2+\left|f^{(1)}_{e^*}\left(v,v^*\right)\right|^2+2\mathrm{Re}\left[f^*\left(v,v^*\right)f^{(2)}_{e,e^*}\left(v,v^*\right)\right]\right\},$$
$$C=\mathrm{E}\left|f\left(v,v^*\right)\right|^2 \tag{18a}$$

for complex-valued data cases, and

$$A=2\mathrm{E}f^{(1)}_{e}(v),\quad B=\mathrm{E}\left|f^{(1)}_{e}(v)\right|^2+\mathrm{E}\left[f(v)f^{(2)}_{e,e}(v)\right],\quad C=\mathrm{E}\left|f(v)\right|^2 \tag{18b}$$

for real-valued data cases, respectively.

Proof:

First, we consider the complex-valued cases. The complex BSE of the function $\varphi\left(e,e^*\right)$ with respect to $\left(e,e^*\right)$ around $\left(v,v^*\right)$ can be written as [19]-[22]

$$\begin{aligned}\varphi\left(e,e^*\right)&=\varphi\left(v,v^*\right)+\varphi^{(1)}_{e}\left(v,v^*\right)e_a+\varphi^{(1)}_{e^*}\left(v,v^*\right)e^*_a\\&+\frac{1}{2}\left[\varphi^{(2)}_{e,e}\left(v,v^*\right)e_a^2+\varphi^{(2)}_{e^*,e^*}\left(v,v^*\right)\left(e^*_a\right)^2+2\varphi^{(2)}_{e,e^*}\left(v,v^*\right)\left|e_a\right|^2\right]+\mathbf{O}\left(e_a,e^*_a\right)\end{aligned} \tag{19}$$

[5] In real-valued cases, $f\left(e,e^*\right)$ can be simplified to $f(e)$ since $e=e^*$, and $\varphi\left(e,e^*\right)$ and $q\left(e,e^*\right)$ can also be replaced by their simplified forms $\varphi(e)$ and $q(e)$, respectively.

where $\mathbf{O}\left(e_a, e_a^*\right)$ denotes third and higher-power terms of e_a or e_a^*. Ignoring $\mathbf{O}\left(e_a, e_a^*\right)$[6], and taking expectations of both sides of the above equation, we get

$$\begin{aligned}\mathrm{E}\varphi\left(e, e^*\right) = \mathrm{E}\varphi\left(v, v^*\right) + \mathrm{E}\left[\varphi_e^{(1)}\left(v, v^*\right) e_a\right] + \mathrm{E}\left[\varphi_{e^*}^{(1)}\left(v, v^*\right) e_a^*\right] \\ + \frac{1}{2}\mathrm{E}\left[\varphi_{e,e}^{(2)}\left(v, v^*\right) e_a^2\right] + \frac{1}{2}\mathrm{E}\left[\varphi_{e^*,e^*}^{(2)}\left(v, v^*\right)\left(e_a^*\right)^2\right] + \mathrm{E}\left[\varphi_{e,e^*}^{(2)}\left(v, v^*\right)\left|e_a\right|^2\right].\end{aligned} \tag{20}$$

Under A.2, (i.e. $\{v, e_a\}$ are mutually independent, and $\mathrm{E}e_a = \mathrm{E}e_a^2 = 0$), we obtain

$$\mathrm{E}\varphi\left(e, e^*\right) = \mathrm{E}\varphi\left(v, v^*\right) + \mathrm{E}\varphi_{e,e^*}^{(2)}\left(v, v^*\right)\zeta_{TEMSE} \tag{21}$$

where ζ_{TEMSE} is defined by (5). Here, to distinguish two kinds of steady-state EMSE, we use different subscripts for ζ, i.e., ζ_{EMSE} for steady-state MSE and ζ_{TEMSE} for tracking performance. Similarly, replacing $\varphi\left(e, e^*\right)$ in (20) by $q\left(e, e^*\right)$ and using A.2, we get

$$\mathrm{E}q\left(e, e^*\right) = \mathrm{E}q\left(v, v^*\right) + \mathrm{E}q_{e,e^*}^{(2)}\left(v, v^*\right)\zeta_{TEMSE}. \tag{22}$$

Substituting (21) and (22) into (11) yields

$$\left[\alpha_u \mathrm{E}\varphi_{e,e^*}^{(2)}\left(v, v^*\right) - \mu\beta_u \mathrm{E}q_{e,e^*}^{(2)}\left(v, v^*\right)\right]\zeta_{TEMSE} = \alpha_u \mathrm{E}\varphi\left(v, v^*\right) + \mu\beta_u \mathrm{E}q\left(v, v^*\right) + \mu^{-1}\mathrm{Tr}\left(\mathbf{Q}\right). \tag{23}$$

Under Lemma 1, the above equation can be rewritten as

$$\left[A\alpha_u - \mu B\beta_u\right]\zeta_{TEMSE} = \mu\beta_u C + \mu^{-1}\mathrm{Tr}\left(\mathbf{Q}\right). \tag{24}$$

where parameters A, B, C are defined by (18a). Since $\mu C\mathrm{Tr}\left(\mathbf{R}_u\right) \geq 0$, $\mu^{-1}\mathrm{Tr}\left(\mathbf{Q}\right) \geq 0$, and $\zeta_{TEMSE} \geq 0$, if the condition C.1 is satisfied[7], i.e., $A\alpha_u > \mu B\beta_u$, removing the coefficient of ζ_{TEMSE} in (24) to the right-hand side, we can obtain (16) for the tracking EMSE in nonstationary environments in complex-valued cases.

Next, we consider the real-valued cases. The TSE of $\varphi(e)$ with respect to e around v can be written as

$$\varphi(e) = \varphi(v) + \varphi_e^{(1)}(v) e_a + \frac{1}{2}\varphi_{e,e}^{(2)}(v) e_a^2 + \mathbf{O}\left(e_a\right) \tag{25}$$

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[6] At steady-state, since the *a priori* estimation error e_a becomes small if step size is small enough, ignoring $\mathbf{O}\left(e_a, e_a^*\right)$ is reasonable, which has been used in to analyze the steady-state performance for adaptive filters [11], [12], [19], [20].

[7] The restrictive condition C.1 can be used to check whether the expressions (15) - (17) are able to be used for a special case of adaptive filters. In the latter analyses, we will show that C.1 is not always satisfied for all kinds of adaptive filters. In addition, due to the influences of the initial conditions of the weights, step size, filter length, input power, noise variance and the residual terms $\mathrm{O}\left(e_a, e_a^*\right)$ having been ignored during the previous processes, C.1 can not be a strict mean square stability condition for an adaptive filter with error nonlinearity.

where $\mathrm{O}(e_a)$ denotes third and higher-power terms of e_a. Neglecting $\mathrm{O}(e_a)$ and taking expectations of both sides of (25) yields

$$\mathrm{E}\varphi(e) = \mathrm{E}\varphi(v) + \mathrm{E}\left[\varphi_e^{(1)}(v)e_a\right] + \frac{1}{2}\mathrm{E}\left[\varphi_{e,e}^{(2)}(v)e_a^2\right] \tag{26}$$

Under A.2, we get

$$\mathrm{E}\varphi(e) = \mathrm{E}\varphi(v) + \frac{1}{2}\mathrm{E}\varphi_{e,e}^{(2)}(v)\zeta_{TEMSE} \tag{27}$$

where ζ_{TEMSE} is defined by (5). Similarly, replacing $\varphi(e)$ in (26) by $q(e)$ and using A.2, we get

$$\mathrm{E}q(e) = \mathrm{E}q(v) + \frac{1}{2}\mathrm{E}q_{e,e}^{(2)}(v)\zeta_{TEMSE} \tag{28}$$

Substituting (27) and (28) into (11), and using Lemma 2, we can obtain (24), where parameters A, B, C are defined by (18b). Then, if the condition C.1 is satisfied, we can obtain (16) for real-valued cases.

In stationary environments, letting $\mathrm{Tr}(\mathbf{Q}) = 0$ in (16), we can obtain (15) for the steady-state EMSE, i.e., ζ_{EMSE}.

Finally, Differentiating both-hand sides of (16) with respect to μ, and letting it be zero, we get

$$\frac{\partial}{\partial\mu}\zeta_{TEMSE}\bigg|_{\mu=\mu_{opt}} = \frac{\partial}{\partial\mu}\left[\frac{\mu^{-1}\mathrm{Tr}(\mathbf{Q}) + \mu C\beta_u}{A\alpha_u - \mu B\beta_u}\right]_{\mu=\mu_{opt}} = 0\,. \tag{29}$$

Simplifying the above equation, we get

$$\mu_{opt}^2 + \frac{2B\mathrm{Tr}(\mathbf{Q})}{A\alpha_u C}\mu_{opt} - \frac{\mathrm{Tr}(\mathbf{Q})}{C\beta_u} = 0\,. \tag{30}$$

Solving the above equality, we can obtain the optimum step-size expressed by (17). Here, we use the fact $\mu > 0$. This ends the proof of Theorem 1.

Remarks:

1. Substituting (17) into (16) yields the minimum steady-state TEMSE.
2. Observing from (18), we can find that the steady-state expressions of (15) ~ (17) are all second-order approximate.
3. In view of the step-size μ being very small, $\mu B\beta_u << A\alpha_u$, and the expressions (15) ~ (17) can be simplified to

$$\zeta_{EMSE} = \frac{\mu\beta_u C}{A\alpha_u}, \tag{31}$$

$$\zeta_{TEMSE} = \frac{\mu^{-1}\mathrm{Tr}(\mathbf{Q}) + \mu C\beta_u}{A\alpha_u}, \tag{32}$$

$$\mu_{opt} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{C\beta_u}} \tag{33}$$

Substituting (33) into (32) yields the minimum steady-state TEMSE

$$\zeta_{\min} = \frac{2}{A\alpha_u}\sqrt{C\beta_u \mathrm{Tr}(\mathbf{Q})}\,. \tag{34}$$

In addition, since $\mu B\beta_u$ in the denominator of (15) has been ignored, C.1 can be simplified to $A > 0$, namely $\mathrm{Re}\mathrm{E}f_e^{(1)}(v, v^*) > 0$ for complex-valued data cases, and $\mathrm{E}f_e^{(1)}(v) > 0$ for real-valued data cases, respectively. Here, the existing condition of the second-order partial derivative of $f(e, e^*)$ can be weakened, i.e., $f(e, e^*) \in C^1(D_v)$.

4. For fixed step-size cases, substituting $g(\mathbf{u}) = 1$ into (12), we get

$$\alpha_u = 1, \beta_u = \mathrm{E}\|\mathbf{u}\|^2 = \mathrm{Tr}(\mathbf{R}_u)\,. \tag{35}$$

Substituting (35) into (31) yields $\zeta_{EMSE} = \mu C\mathrm{Tr}(\mathbf{R}_u)/A$. For the real-valued cases, this expression is the same as the one derived by S. C. Douglas and T. H.-Y. Meng in [10] (see e.g. Eq. 35). That is to say, Eq. 35 in [10] is a special case of (15) with small step-size, $g(\mathbf{u}) = 1$, and real-valued data.

2.2 White Gaussian regressor

Consider $g(\mathbf{u}_i) = 1$, and let *M*-dimensions regressor vector $\mathbf{u}$ have a circular Gaussian distribution with a diagonal covariance matrix, namely,

$$\mathbf{R}_u = \sigma_u^2 \mathbf{I}_{M\times M}\,. \tag{36}$$

Under the following assumption (see e.g. 6.5.13) in [11] at steady state, i.e.,

A.5 $\tilde{\mathbf{w}}$ *is independence of* $\mathbf{u}$,

the term $\mathrm{E}\left[\|\mathbf{u}\|^2 q(e, e^*)\right]$ that appears in the right-hand side of (10) can be evaluated explicitly without appealing to the separation assumption (e.g. A.4), and its steady-state EMSE for adaptive filters can be obtained by the following theorem.

Theorem 2 —Steady-state performance for adaptive filters with white Gaussian regressor: Consider adaptive filters of the form (1) – (3) with white Gaussian regressor and $g(\mathbf{u}_i) = 1$, and suppose the assumptions A.1 – A.3, and A.5 are satisfied. In addition, $f(e, e^*) \in C^2(D_v)$. Then, if the following condition is satisfied, i.e.,

C.2 $A > \mu B(M + \gamma)\sigma_u^2$,

the steady-state EMSE, TEMSE and the optimal step-size for adaptive filters can be approximated by

$$\zeta_{EMSE} = \frac{\mu CM\sigma_u^2}{A - \mu B(M+\gamma)\sigma_u^2}\,, \tag{37}$$

$$\zeta_{TEMSE} = \frac{\mu^{-1}\mathrm{Tr}(\mathbf{Q}) + \mu MC\sigma_u^2}{A - \mu B(M+\gamma)\sigma_u^2}\,, \tag{38}$$

$$\mu_{opt} = \sqrt{\left[\frac{B(M+\gamma)\mathrm{Tr}(Q)}{AMC}\right]^2 + \frac{\mathrm{Tr}(\mathbf{Q})}{MC\sigma_u^2}} - \frac{B(M+\gamma)\mathrm{Tr}(\mathbf{Q})}{AMC}, \tag{39}$$

where $\gamma = 1$, A, B and C are defined by (18a) for complex-valued data, and $\gamma = 2$, A, B and C are defined by (18b) for real-valued data, respectively.
The proofs of Theorem 2 is given in the APPENDIX D.
For the case of μ being small enough, the steady-state EMSE, TEMSE, the optimal step-size, and the minimum TEMSE can be expressed by (31) ~ (33), respectively, if we replace $\mathrm{Tr}(\mathbf{R}_u)$ by $M\sigma_u^2$ and $g(\mathbf{u}_i) = 1$. That is to say, when the input vector $\mathbf{u}$ is Gaussian with a diagonal covariance matrix (36), the steady-state performance result obtained by separation principle coincides with that under A.5 for the case of μ being small enough.

3. Steady-state performance for some special cases of adaptive filters

In this section, based on Theorem 1 and Theorem 2 in Section Ⅱ, we will investigate the steady-state performances for LMP algorithm with different choices of parameter p, LMMN algorithm, and their normalized algorithms, respectively. To begin our analysis, we first introduce a lemma for the derivative operation about a complex variable.
Lemma 3: Let z be a complex variable and p be an arbitrary real constant number except zero, then

$$\frac{\partial}{\partial z}|z|^p = \frac{p}{2}|z|^{p-2}z^*,$$
$$\frac{\partial}{\partial z^*}|z|^p = \frac{p}{2}|z|^{p-2}z$$

3.1 LMP algorithm

The estimation error signal of LMP algorithm can be expressed as [23]

$$f(e,e^*) = |e|^{p-2}e = \left(ee^*\right)^{(p-2)/2}e \tag{40}$$

where $p > 0$ is a positive integral. $p = 2$ results in well-known LMS algorithm, and $p = 4$ results in LMF algorithm. Here, we only consider $p \geq 2$.
Using (40) and Lemma 3, we can obtain the first-order and second-order partial derivatives of $f(e,e^*)$, expressed by

$$\begin{aligned} f_e^{(1)}(e) &= (p-1)|e|^{p-2} \\ f_{e,e}^{(2)}(e) &= (p-1)(p-2)|e|^{p-4}e \end{aligned} \tag{41a}$$

in real-valued cases, and

$$\begin{aligned} f_e^{(1)}(e,e^*) &= \frac{p}{2}|e|^{p-2} \\ f_{e^*}^{(1)}(e,e^*) &= \frac{p-2}{2}|e|^{p-4}e^2 \\ f_{e,e^*}^{(2)}(e,e^*) &= \frac{p(p-2)}{4}|e|^{p-4}e \end{aligned} \quad . \tag{41b}$$

in complex-valued cases, respectively. Substituting (41) into (18a) and (42) into (18b), respectively, we get

$$\begin{aligned} A &= a\xi_v^{p-2} \\ B &= b\xi_v^{2p-4} \\ C &= \xi_v^{2p-2} \end{aligned} \tag{42}$$

where $\xi_v^k = \mathrm{E}|v|^k$ denote the *k*-order absolute moment of v, and

$$\begin{aligned} &a = 2(p-1),\ b = (p-1)(2p-3) && \text{real-valued cases} \\ &a = p,\ b = (p-1)^2 && \text{complex-valued cases} \end{aligned} \tag{43}$$

Then, under Theorem 1, the condition C.1 becomes

$$\xi_v^{p-2} > \frac{\mu b \beta_u}{a\alpha_u}\xi_v^{2(p-2)}, \tag{44}$$

and the steady-state performance for real LMP algorithms can be written as

$$\zeta_{EMSE} = \frac{\mu\beta_u\xi_v^{2p-2}}{a\alpha_u\xi_v^{p-2} - \mu b\beta_u\xi_v^{2(p-2)}}. \tag{45a}$$

$$\zeta_{TEMSE} = \frac{\mu\beta_u\xi_v^{2p-2} + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{a\alpha_u\xi_v^{p-2} - \mu b\beta_u\xi_v^{2(p-2)}} \tag{45b}$$

$$\mu_{opt} = -\frac{b\xi_v^{2p-4}\mathrm{Tr}(\mathbf{Q})}{a\xi_v^{p-2}\xi_v^{2p-2}\alpha_u} + \sqrt{\left[\frac{b\xi_v^{2p-4}\mathrm{Tr}(\mathbf{Q})}{a\xi_v^{p-2}\xi_v^{2p-2}\alpha_u}\right]^2 + \frac{\mathrm{Tr}(\mathbf{Q})}{\xi_v^{2p-2}\beta_u}}. \tag{45c}$$

Similarly, substituting (42) into Theorem 2, we can also obtain the corresponding expressions for the steady-state performance of LMP algorithms with white Gaussian regressor.

Example 1: For LMS algorithm, substituting $p = 2$ and (35) into (45a) ~ (45c), and substituting (42) and $p = 2$ into Theorem 2, yield the same steady-state performance results (see e.g. Lemma 6.5.1 and Lemma 7.5.1) in [11]. For LMF algorithm, substituting $p = 4$ and (34) into (45a) ~ (45c), and substituting (42) and $p = 4$ into Theorem 2, yield the same steady-state performance results (see e.g. Lemma 6.8.1 and Lemma 7.8.1 with $\delta = 0$ [8]) in [11]. That is to say, the results of Lemma 6.5.1, Lemma 7.5.1, Lemma 6.8.1 and Lemma 7.8.1 in [11] are all second-order approximate.

Example 2: Consider the real-valued data in Gaussian noise environments. Based on the following formula, described in [23]

[8] The parameters a, b, c in (44)-(45) are different from those in Lemma 6.8.1 and Lemma 7.8.1 in [11].

$$\xi_v^k = \begin{cases} (k-1)!!\sigma_v^k & k\text{:even} \\ \sqrt{\dfrac{2^k}{\pi}}\left(\dfrac{k-1}{2}\right)!\sigma_v^k & k\text{:odd} \end{cases}, \tag{46}$$

where $(k-1)!! = 1\cdot 3\cdot 5\cdots(k-1)$, (42) becomes

$$\left.\begin{aligned} A &= 2(p-1)(p-3)!!\sigma_v^{p-2} \\ B &= (p-1)(2p-3)!!\sigma_v^{2p-4} \\ C &= (2p-3)!!\sigma_v^{2p-2} \end{aligned}\right\} \quad p\text{: even}$$

$$\left.\begin{aligned} A &= \sqrt{2^p/\pi}\,(p-1)\left[(p-3)/2\right]!\sigma_v^{p-2} \\ B &= (p-1)(2p-3)!!\sigma_v^{2p-4} \\ C &= (2p-3)!!\sigma_v^{2p-2} \end{aligned}\right\} \quad p\text{: odd} \tag{47}$$

Then, substituting (47) into Theorem 1 and Theorem 2 or substituting (46) into (45a) - (45c), yield the steady-state performance results for real LMP algorithm in Gaussian noise environments. Here, we only give the expression for EMSE

$$\zeta_{EMSE} = \begin{cases} \dfrac{\mu(2p-3)!!\sigma_v^p\beta_u}{2(p-1)(p-3)!!\alpha_u - \mu(p-1)(2p-3)!!\sigma_v^{p-2}\beta_u}, & p\text{:even} \\ \dfrac{\mu(2p-3)!!\sigma_v^p\beta_u}{\sqrt{\dfrac{2^p}{\pi}}(p-1)\left(\dfrac{p-3}{2}\right)!\alpha_u - \mu(p-1)(2p-3)!!\sigma_v^{p-2}\beta_u}, & p\text{:odd} \end{cases}. \tag{48}$$

The above expression is also applicable for LMS algorithm by means of $(-1)!! = 1$.

Example 3: Consider the real-valued data in uniformly distributed noise environments, whose interval is $[-\Delta, \Delta]$ and *k*-order absolute moment can be written as

$$\xi_v^k = \frac{\Delta^k}{k+1}. \tag{49}$$

Substituting the above equation into (42), we get

$$\begin{aligned} A &= 2\Delta^{p-2} \\ B &= (p-1)\Delta^{2p-4} \\ C &= \frac{\Delta^{2p-2}}{2p-1} \end{aligned}. \tag{50}$$

Then, substituting (50) into Theorem 1 and Theorem 2 yields the steady-state performance for real LMP algorithm in uniformly distributed noise environments. Here, we also only give the EMSE expression, expressed by

$$\zeta_{EMSE} = \frac{\mu\Delta^p\beta_u}{2(2p-1)\alpha_u - \mu(2p-1)(p-1)\Delta^{p-2}\beta_u}. \tag{51}$$

Example 4: In complex Gaussian noise environments, using the following formula, summarized from [24]

$$\xi_v^k = \mathrm{E}|v|^k = \begin{cases} \frac{k}{2}!\sigma_v^k & k\text{:even} \\ 0 & k\text{:odd} \end{cases}, \tag{52}$$

(42) becomes

$$\begin{aligned} A &= p\left(\frac{p-2}{2}\right)!\sigma_v^{p-2} \\ B &= (p-1)(p-1)!\sigma_v^{2p-4} \\ C &= (p-1)!\sigma_v^{2p-2}, \end{aligned} \tag{53}$$

for even p. Then, substituting (53) into Theorem 1 and Theorem 2 or substituting (52) into (45a) ~ (45c), we can obtain the steady-state performances for complex LMP algorithms with even p in Gaussian noise environments. For instance, the EMSE expression can be written as

$$\zeta_{EMSE} = \frac{\mu(p-1)!\sigma_v^p\beta_u}{p\left(\frac{p-2}{2}\right)!\alpha_u - \mu(p-1)(p-1)!\sigma_v^{p-2}\beta_u}. \tag{54}$$

But for odd p, substituting (40) and (52) into (18a) yields $A = 0$, which leads to the conditions C.1 and C.2 being not satisfied again. That is to say, the proposed theorems are unsuitable to analyze the steady-state performances in this case.

Example 5: Tracking performance comparison with LMS

We now compare the ability of the LMP algorithm with $p > 2$ to track variations in nonstationary environments with that of the LMS algorithm. The ratio of the minimum achievable steady-state EMSE of each of the LMS algorithm is used as a performance measure. In addition, the step-size of this minimum value is often sufficient small, which leads to that (34) can be used directly. Substituting (42) into (34), we obtain the minimum TEMSE for LMP algorithm, expressed as

$$\zeta_{\min}^{LMP} = \gamma\frac{\sqrt{\xi_v^{2p-2}\beta_u\mathrm{Tr}(\mathbf{Q})}}{\alpha_u\xi_v^{p-2}}. \tag{55}$$

where $\gamma = 2/p$ for complex-valued cases, and $\gamma = 1/(p-1)$ for real-valued cases. Then the ratio between $\zeta_{\min}^{LMS} = \sigma_v\sqrt{\mathrm{Tr}(\mathbf{R}_u)\mathrm{Tr}(\mathbf{Q})}$ (which can be obtained by substituting $p = 2$ and (35) into (55)) and $\zeta_{\min}^{LMP}$ can be written as

$$\frac{\zeta_{\min}^{LMS}}{\zeta_{\min}^{LMP}} = \frac{\alpha_u\xi_v^{p-2}\sigma_v\sqrt{\mathrm{Tr}(\mathbf{R}_u)}}{\gamma\sqrt{\beta_u\xi_v^{2p-2}}}. \tag{56}$$

For the case of LMF algorithm, substituting $p = 4$ and (35) into (56), we can obtain the same result (see e.g. Eq.7.9.1) in [11].

3.2 LMMN algorithm

The estimation error of the LMMN algorithm is [6], [7], [11], [12]

$$f\left(e, e^*\right) = e\left(\delta + \bar{\delta}\left|e\right|^2\right), \tag{57}$$

where $0 \le \delta \le 1$ and $\bar{\delta} = 1 - \delta$. $\delta = 1$ results in the LMS algorithm and $\delta = 0$ results in the LMF algorithm.
Using (57) and Lemma 3, we get

$$\begin{aligned} f_e^{(1)}\left(e, e^*\right) &= \delta + 2\bar{\delta}\left|e\right|^2 \\ f_{e^*}^{(1)}\left(e, e^*\right) &= \bar{\delta}\left|e\right|^2 \\ f_{e,e^*}^{(2)}\left(e, e^*\right) &= 2\bar{\delta}e \end{aligned} \tag{58a}$$

for complex-valued cases, and

$$\begin{aligned} f_e^{(1)}\left(e\right) &= \delta + 3\bar{\delta}e^2 \\ f_{e,e}^{(2)}\left(e\right) &= 6\bar{\delta}e \end{aligned} . \tag{58b}$$

for real-valued cases, respectively. Substituting (58a) into (18a), or substituting (58b) into (18b), respectively, we get

$$\begin{aligned} A &= 2\left(\delta + k_0\bar{\delta}\xi_v^2\right) \\ B &= \delta^2 + k_1\delta\bar{\delta}\xi_v^2 + k_2\bar{\delta}^2\xi_v^4 \\ C &= \delta^2\xi_v^2 + 2\delta\bar{\delta}\xi_v^4 + \bar{\delta}^2\xi_v^6 \end{aligned} . \tag{59}$$

where $k_0 = 3, k_1 = 12, k_2 = 15$ for real-valued cases $k_0 = 2, k_1 = 8, k_2 = 9$ for complex-valued cases. Then, under Theorem 1, the condition C.1 becomes

$$\alpha_u > \frac{\mu\beta_u\left(\delta^2 + k_1\delta\bar{\delta}\xi_v^2 + k_2\bar{\delta}^2\xi_v^4\right)}{2\left(\delta + k_0\bar{\delta}\xi_v^2\right)}, \tag{60}$$

and the steady-state performance for LMMN algorithms (here, we only give the expression for EMSE) can be written as

$$\zeta_{EMSE} = \frac{\mu\beta_u\left(\delta^2\xi_v^2 + 2\delta\bar{\delta}\xi_v^4 + \bar{\delta}^2\xi_v^6\right)}{2\left(\delta + k_0\bar{\delta}\xi_v^2\right)\alpha_u - \mu\beta_u\left(\delta^2 + k_1\delta\bar{\delta}\xi_v^2 + k_2\bar{\delta}^2\xi_v^4\right)} . \tag{61}$$

Example 6: Consider the cases with $g(\mathbf{u}) = 1$. Substituting (35) and $A = 2b`, C = a`, B = c`$ or $A = 2b, C = a, B = c$ into (15) - (17) yields the steady-state performances for real and complex

LMMN algorithms, which coincide with the results (see e.g. Lemma 6.8.1 and Lemma 7.8.1) in [11].

Example 7: In Gaussian noise environments, based on (46) and (52), we can obtain

$$\zeta_{EMSE} = \frac{\mu\beta_u\left(\delta^2\sigma_v^2 + k_3\delta\bar{\delta}\sigma_v^4 + k_4\bar{\delta}^2\sigma_v^6\right)}{2\left(\delta + k_0\bar{\delta}\sigma_v^2\right)\alpha_u - \mu\beta_u\left(\delta^2 + k_1\delta\bar{\delta}\sigma_v^2 + k_2\bar{\delta}^2\sigma_v^4\right)}. \tag{62}$$

where $k_0 = 3, k_1 = 12, k_2 = 45, k_3 = 6, k_4 = 15$ for real-valued cases, $k_0 = 2, k_1 = 8, k_2 = 18, k_3 = 4, k_4 = 6$ for complex-valued cases.

3.3 Normalized type algorithms

Being similar with LMF algorithm [25]-[27], there are the stability and convergence problems in the LMP algorithm with $p > 2$, LMMN algorithm, and other adaptive filters with error nonlinearities. In this subsection, ε-normalized method, extended from ε-normalized LMS (ε-NLMS) algorithm [11], will be introduced for the LMP algorithm and LMMN algorithm, which are so-called ε-NLMP algorithm and ε-NLMMN algorithm.

The estimation errors for ε-NLMP algorithm and ε-NLMMN algorithm are expressed by (40) and (57), respectively, and its variable factor for step-size can be written as

$$g(\mathbf{u}) = \frac{1}{\varepsilon + \|\mathbf{u}\|^2}. \tag{63}$$

Substituting (63) into (12), we get

$$\alpha_u = \mathrm{E}\left(\frac{1}{\varepsilon + \|\mathbf{u}\|^2}\right), \quad \beta_u = \mathrm{E}\left[\frac{\|\mathbf{u}\|^2}{\left(\varepsilon + \|\mathbf{u}\|^2\right)^2}\right]. \tag{64}$$

In general, $\varepsilon << \|\mathbf{u}\|^2$, so α_u equals to β_u, and can be expressed as

$$\alpha_u = \beta_u = \mathrm{E}\left(\frac{1}{\|\mathbf{u}\|^2}\right). \tag{65}$$

Substituting (65) into (15) yields a simplified expression for steady-state EMSE

$$\zeta_{EMSE} = \frac{\mu C}{A - \mu B} \tag{66}$$

Observing from the above equation, we can find that ζ_{EMSE} is no longer related to the regressor.

4. Simulation results

In section Ⅲ, some well-known real and complex adaptive algorithms, such as LMS algorithm, LMF algorithm and LMMN algorithm have shown the accuracy of the

corresponding analysis results. In this section, we will give the computer simulation for the steady-state performance of real LMP algorithm with odd parameter $p > 2$ (here $p = 3$), ε-NLMP and ε-NLMMN algorithms (here $\delta = 0.5$), which have not been involved in the previous literatures.

4.1 Simulation model

In all the cases, a 11-tap adaptive filter with tap-centered initialization is used. The data are generated according to model (3), the experimental value for different step-size is obtained by running adaptive algorithm for different iteration number and averaging the squares-error curve over 60 experiments in order to generate the ensemble-average curve. The average of the last 4×10^4 entries of the ensemble-average curve is then used as the experimental value for the MSE. The noise with variance $\sigma_v^2 = 0.001$ is used, which is generated as the following two models:

N.1 $\sigma_v \,\mathrm{randn}(\,)$ is used in Gaussian noise environments;

N.2 $\sigma_v\left[-1+2\,\mathrm{rand}(\,)\right]$ is used in uniformly distributed noise environments, whose distributed interval is $[-1,1]$, i.e. $\Delta = 1$.

Here, the function $\mathrm{randn}(\,)$ is used to generate the normally distributed (Gaussian) sequence with zero mean and unit covariance in Matlab software, and $\mathrm{rand}(\,)$ is used to generate the uniformly distributed sequence.

The regressors $\{\mathbf{u}_i\}$ are generated as the following two models.

M.1 The regressors $\{\mathbf{u}_i\}$ are generated as independent realizations of a Gaussian distribution with a covariance matrix $\mathbf{R}_u$ (a diagonal unit matrix).

M.2 The regressors $\{\mathbf{u}_i\}$ have shift structure, and are generated by feeding correlated data into a tapped delay time, which are expressed as [11]

$$u(i+1) = au(i) + \sqrt{1-a^2}\,s(i), \tag{67}$$

where $\mathbf{u}_i = \left[u(i), u(i-1), \cdots, u(i-L+1)\right]$, and $s(i)$ is a unit-variance i.i.d. Gaussian random process. Here, we set $a = 0.8$.

4.2 MSE and tracking performance simulation

Fig. 1 - Fig. 4 show the theoretical and simulated MSE curves for real LMP algorithm, Fig. 5 for real ε-NLMP algorithm, Fig. 6 for real ε-NLMMN algorithm, and Fig. 7 for complex ε-NLMMN algorithm. Fig. 8 ~ Fig. 11 show the theoretical and simulated tracking MSE curves for real LMP algorithm. The range of step-size are all set from 0.001 to 0.1 except for ε-NLMP algorithm, which is from 0.001 to 0.6. Other conditions (including the regressor model and the noise model) for these figures are shown in Table 2. In addition, Fig. 1, Fig. 2, Fig. 8 and Fig. 9 show two theoretical curves, one curve is plotted under Theorem 1, another under Theorem 2.

From these figures (Fig. 1 ~ Fig. 11), we can see that the simulation and theoretical results are matched reasonable well. Specially, as shown in Fig. 1, Fig. 2, Fig. 8, and Fig. 9, there is a marginal difference between the MSE values based on Theorem 1 and the values based on Theorem 2 for white Gaussian regressors. For the tracking performance, Fig. 8 ~ Fig.

11, whose corresponding σ_q are $\left(5\times10^{-5},1\times10^{-5},1\times10^{-5},1\times10^{-5}\right)$, where $Q=\sigma_q^2\mathbf{I}$, also show the optimal step-sizes, at which the steady-state MSE possess minimum values. These tracking figures show that these minimum values are in good agreement with the corresponding theoretical values, written by $\left(0.0283,\ 0.0074,\ 0.0058,\ 0.0074\right)$, respectively.

Figures	Regressor models	Noise models
Fig. 1	M.1	N.1
Fig. 2	M.1	N.2
Fig. 3	M.2	N.1
Fig. 4	M.2	N.2
Fig. 5	M.2	N.1
Fig. 6	M.2	N.1
Fig. 7	M.2	N.1
Fig. 8	M.1	N.1
Fig. 9	M.1	N.2
Fig. 10	M.2	N.1
Fig. 11	M.2	N.2

Table 2. Conditions for simulation examples

4.3 Tracking ability comparison with LMS algorithm

Consider the real-valued cases with $g(\mathbf{u}_i)=1$. Substituting (35), (46) and (49) into (56), we can obtain

$$\frac{\zeta_{\min}^{LMS}}{\zeta_{\min}^{LMP}}=\begin{cases}(p-3)!!(p-1)\Big/\sqrt{(2p-3)!!}\ , & p\text{:even}\\ \sqrt{\dfrac{2^{(p-2)}}{\pi}}\left(\dfrac{p-3}{2}\right)!(p-1)\Big/\sqrt{(2p-3)!!}\ , & p\text{:odd}\end{cases}. \tag{68}$$

in Gaussian noise environments, and

$$\frac{\zeta_{\min}^{LMS}}{\zeta_{\min}^{LMP}}=\frac{\sqrt{2p-1}}{\sqrt{3}}, \tag{69}$$

in uniformly distributed noise environments, respectively. Under different parameter p (from 2 to 6), Fig. 12 shows two curves for the ratio of $\zeta_{\min}^{LMS}/\zeta_{\min}^{LMP}$ (dB) in Gaussian noise environments and uniformly distributed noise environments. From the figure, we can observe that the superiority of the LMS algorithm over LMP with $p>2$ for tracking nonstationary systems in Gaussian noise environments ($\zeta_{\min}^{LMS}/\zeta_{\min}^{LMP}(\text{dB})<0$), and inferiority in uniformly distributed noise environments ($\zeta_{\min}^{LMS}/\zeta_{\min}^{LMP}(\text{dB})>0$). Similar analyses can be done for the complex-valued cases.

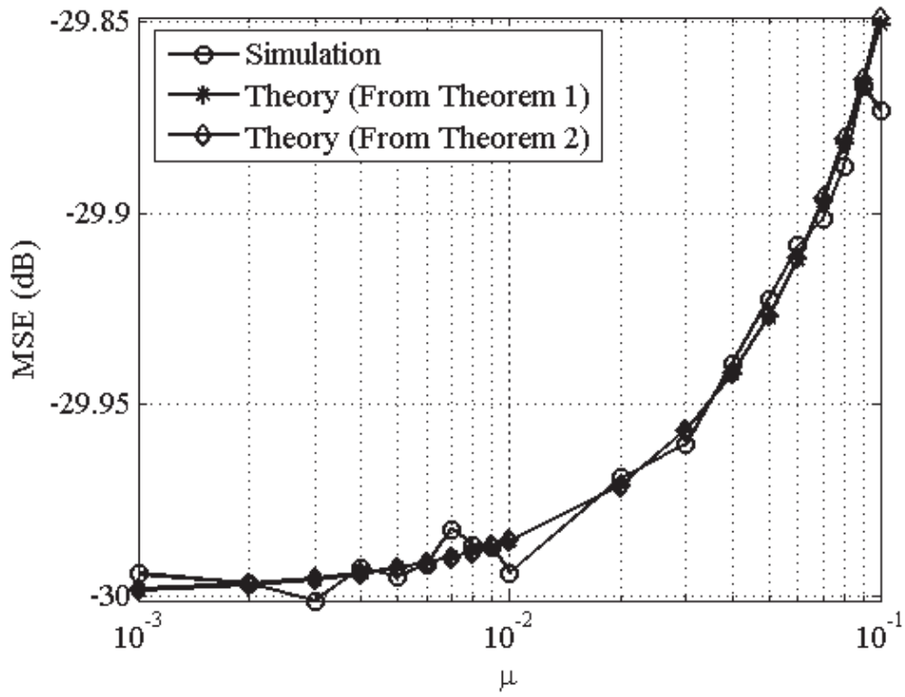

Fig. 1. Two theoretical (Theorem 1 and Theorem 2) and simulated MSEs for real LMP algorithm under the regressor model M.1 and the noise model N.1.

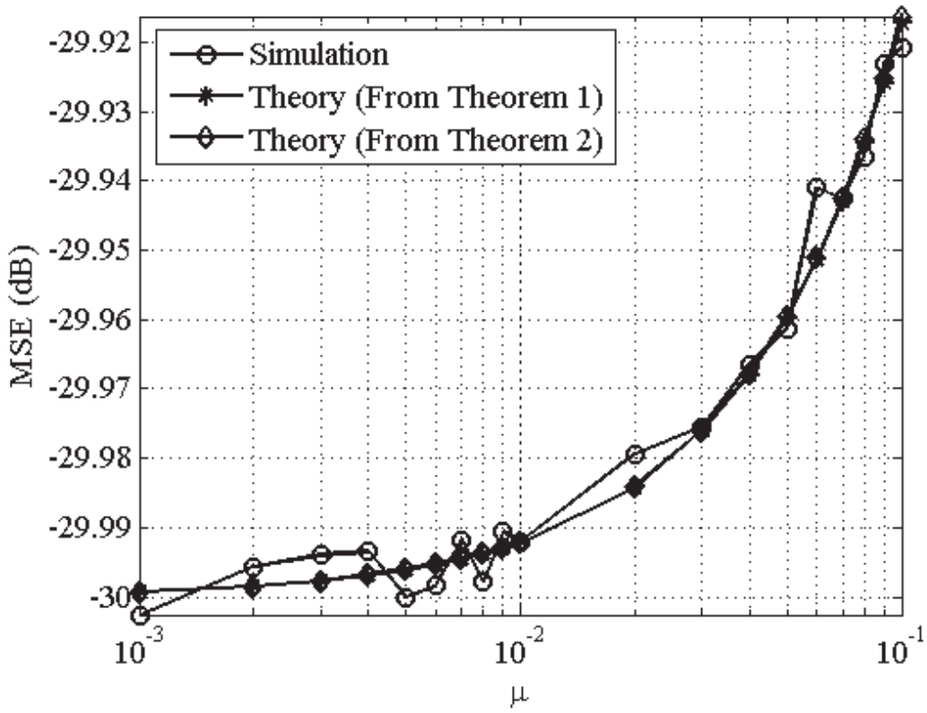

Fig. 2. Two theoretical (Theorem 1 and Theorem 2) and simulated MSEs for real LMP algorithm under the regressor model M.1 and the noise model N.2.

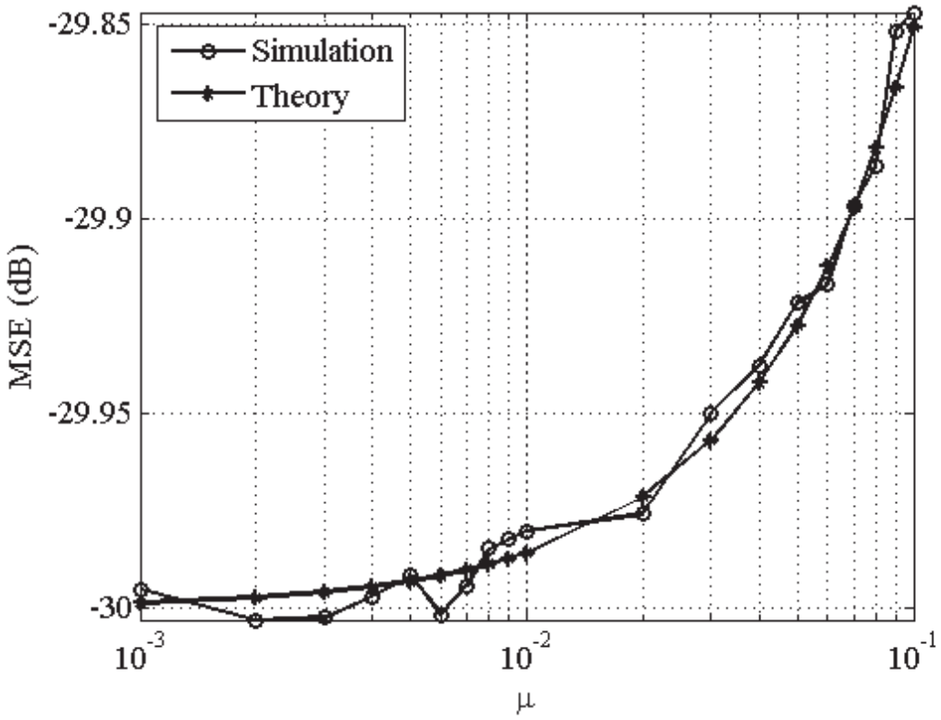

Fig. 3. Theoretical and simulated MSEs for real LMP algorithm under the regressor model M.2 and the noise model N.1.

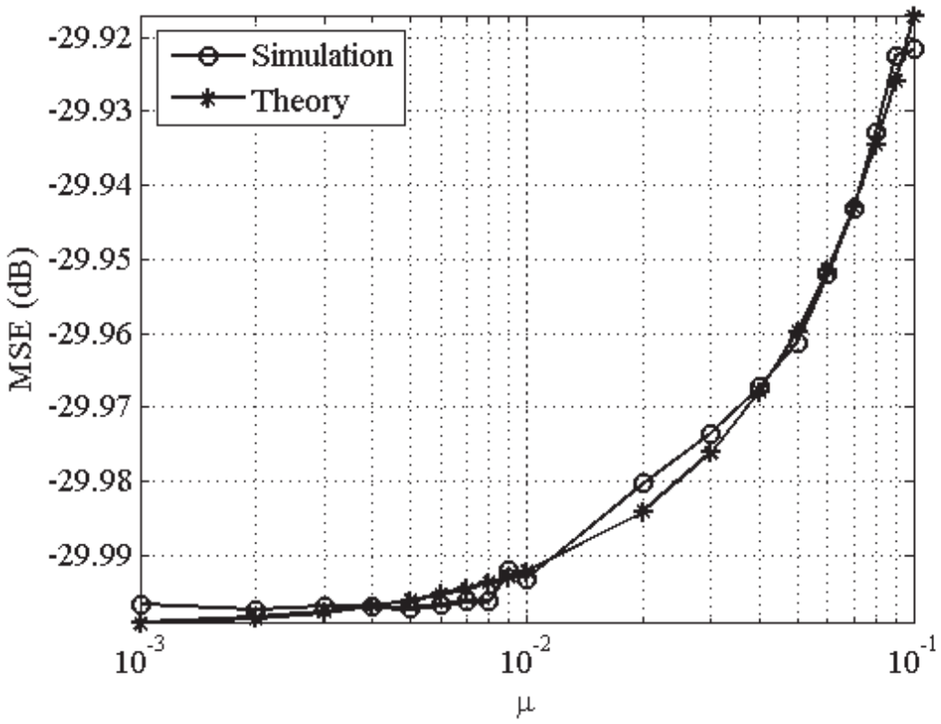

Fig. 4. Theoretical and simulated MSEs for real LMP algorithm under the regressor model M.2 and the noise model N.2.

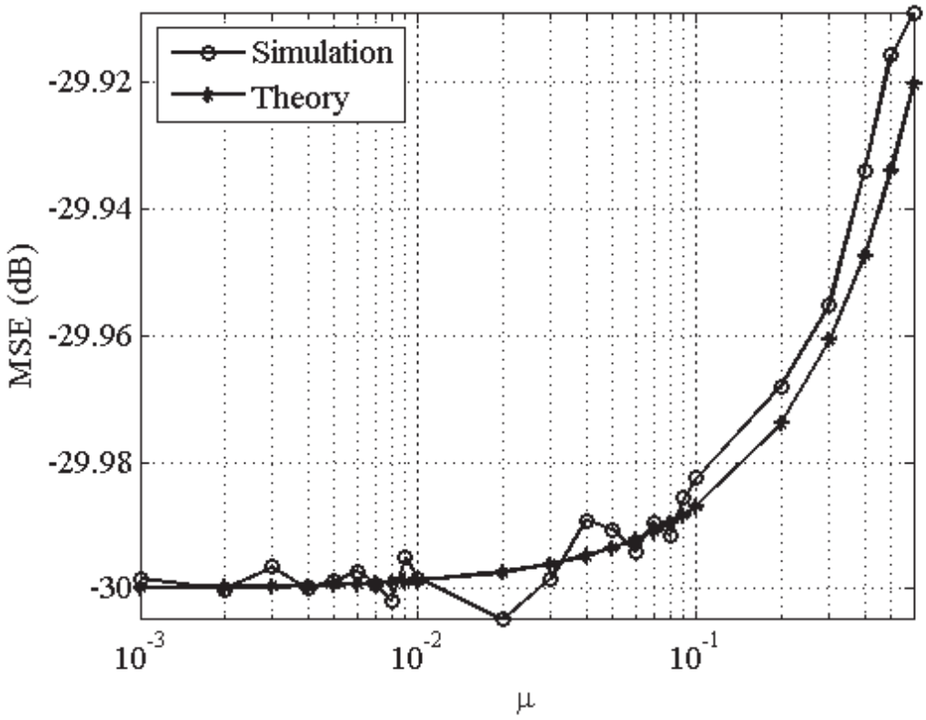

Fig. 5. Theoretical and simulated MSEs for real NLMP algorithm under the regressor model M.2 and the noise model N.1.

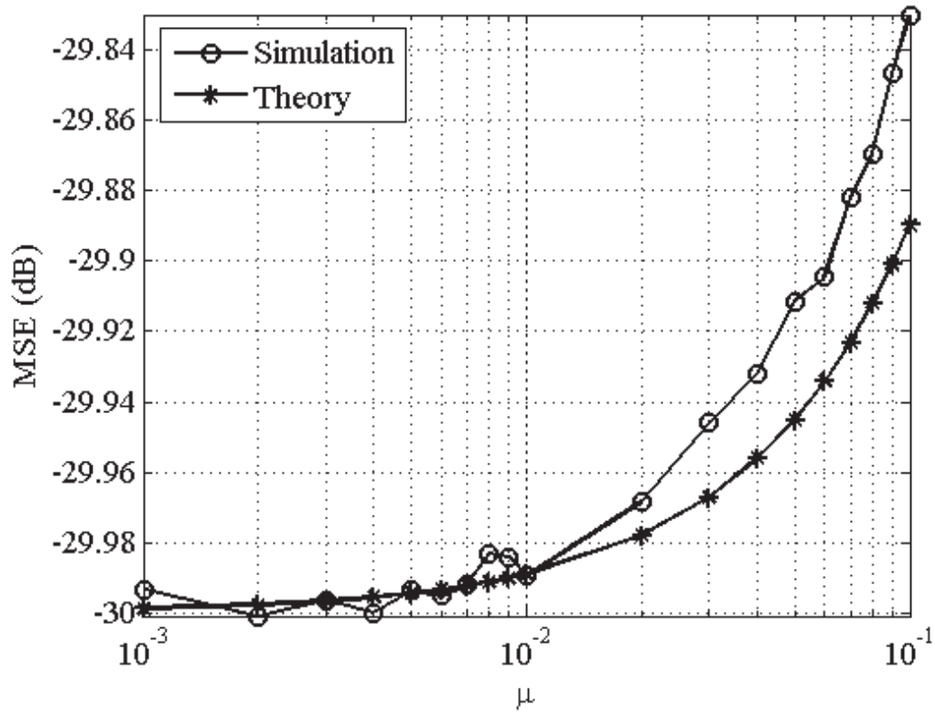

Fig. 6. Theoretical and simulated MSEs for real LMMN algorithm under the regressor model M.2 and the noise model N.1

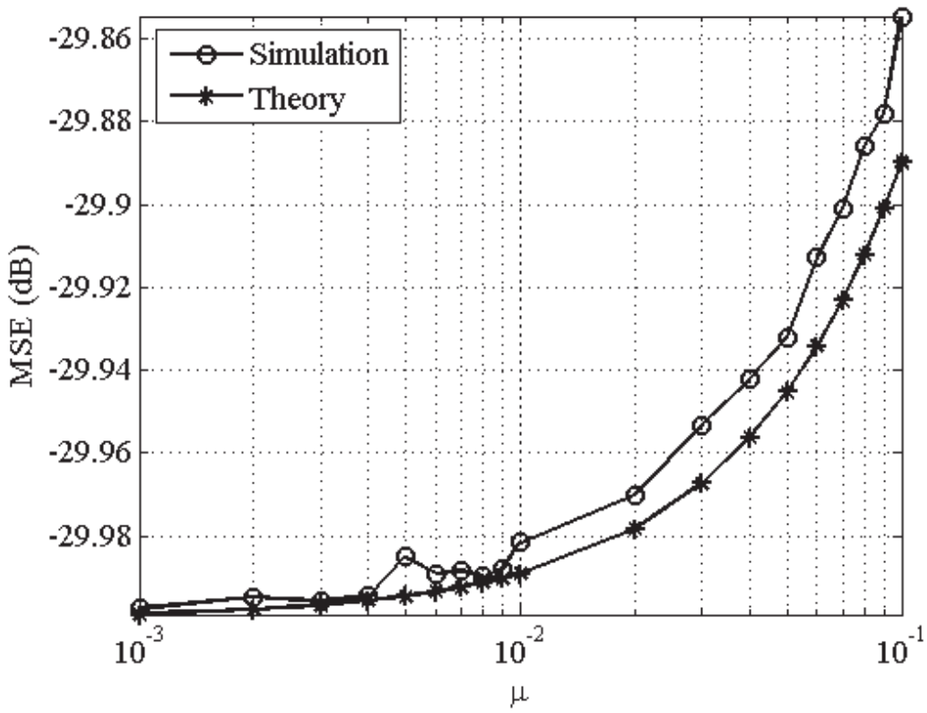

Fig. 7. Theoretical and simulated MSEs for complex LMMN algorithm under the regressor model M.2 and the noise model N.1.

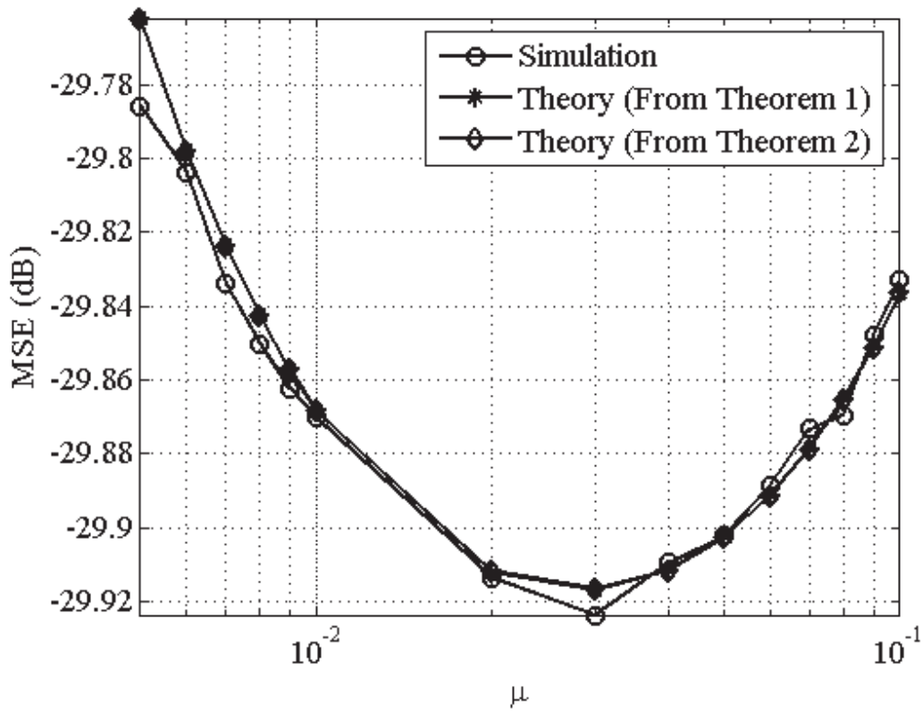

Fig. 8. Two theoretical (Theorem 1 and Theorem 2) and simulated tracking MSEs for real LMP algorithm under the regressor model M.1 and the noise model N.1.

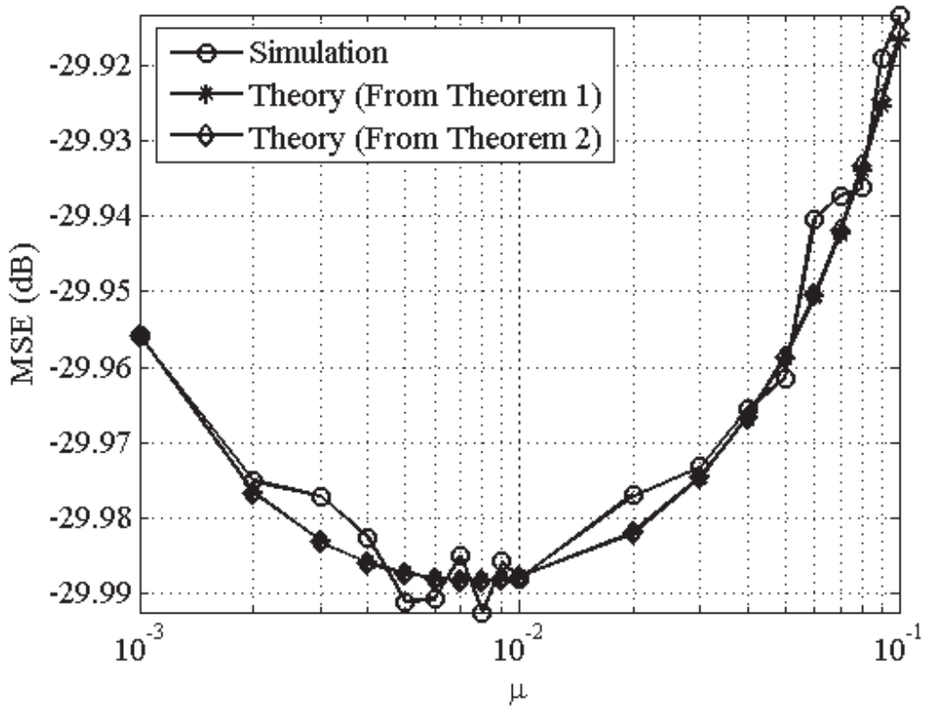

Fig. 9. Two theoretical (Theorem 1 and Theorem 2) and simulated tracking MSEs for real LMP algorithm under the regressor model M.1 and the noise model N.2.

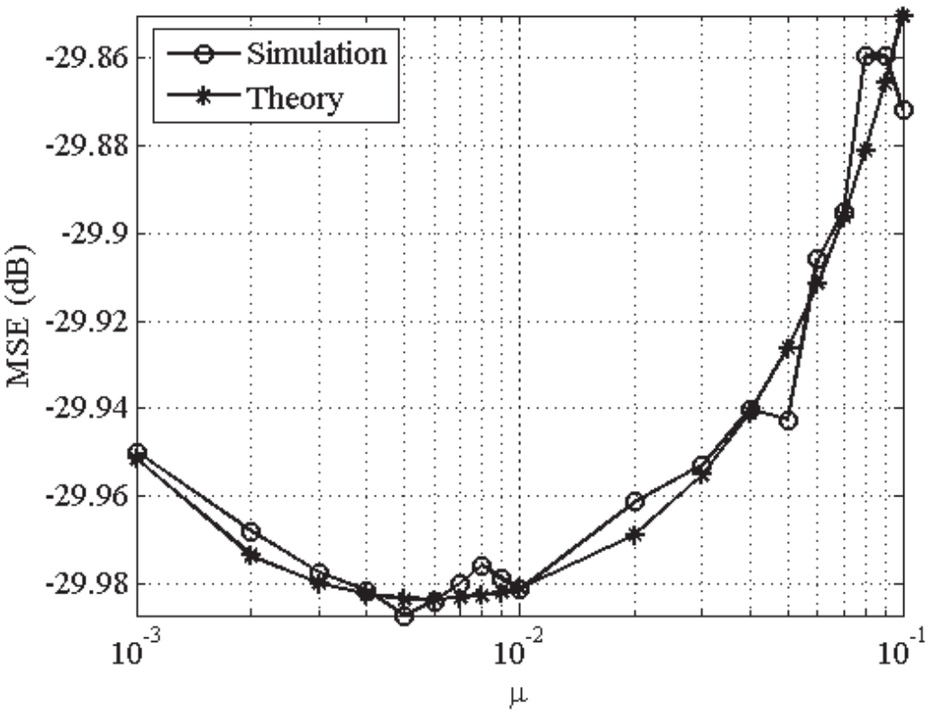

Fig. 10. Theoretical and simulated tracking MSEs for real LMP algorithm under the regressor model M.2 and the noise model N.1.

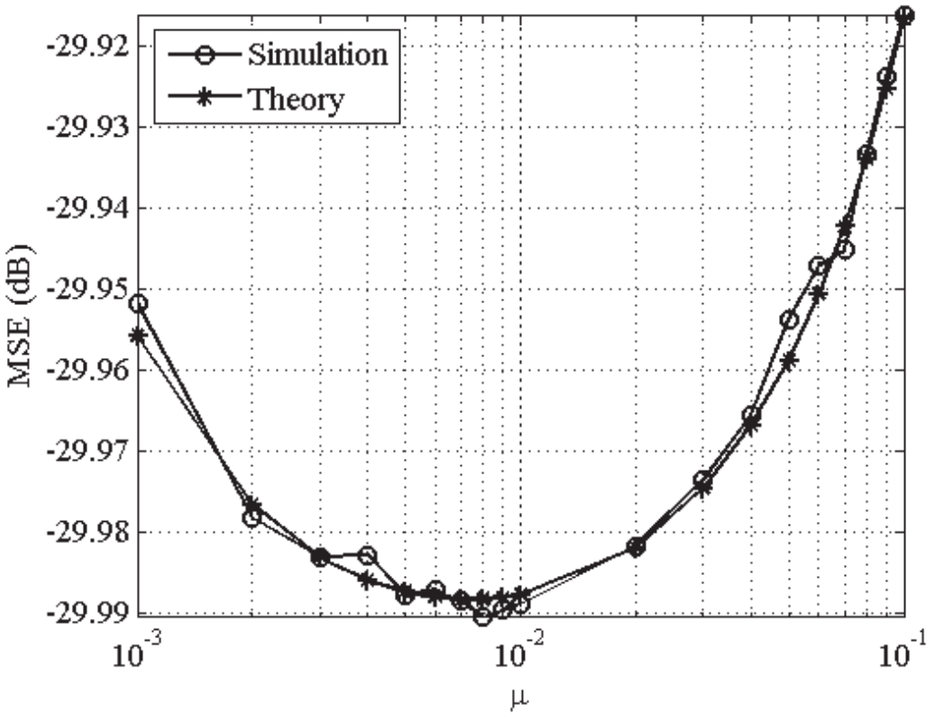

Fig. 11. Theoretical and simulated tracking MSEs for real LMP algorithm under the regressor model M.2 and the noise model N.2.

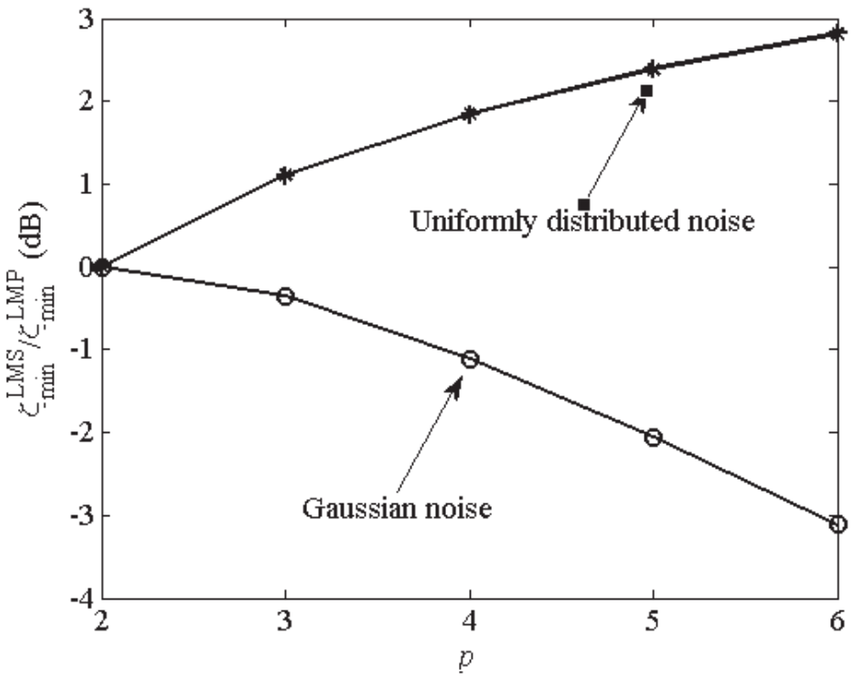

Fig. 12. Comparisons of the tracking performance between LMS algorithm and LMP algorithm in Gaussian noise environments and uniformly distributed noise environments.

5. Conclusions

Based on the Taylor series expansion (TSE) and so-called *complex Brandwood-form series expansion* (BSE), this paper develops a unified approach for the steady-state mean-square-error (MSE) and tracking performance analyses of adaptive filters. The general closed-form analytical expressions for the steady-state mean square error (MSE), tracking performance, optimal step-size, and minimum MSE are derived. These expressions are all second-order approximate. For some well-known adaptive algorithms, such as least-mean-square (LMS) algorithm, least-mean-forth (LMF) algorithm and least-mean-mixed norm (LMMN) algorithm, the proposed results are all the same as those summarized by A. H. Sayed in [11]. For least-mean p-power (LMP) algorithm, the normalized type LMMN algorithm and LMP algorithm (i.e., ε-NLMMN and ε-NLMP), their steady-state performances are also investigated. In addition, comparisons with tracking ability between LMP algorithm with $p > 2$ and LMS algorithm, show that the superiority of the LMS algorithm over LMP algorithm in Gaussian noise environments, and inferiority in uniformly distributed noise environments. Extensive computational simulations show the accuracy of our analyses.

APPENDIX A. Proof of lemma 1

Under (4), we have $e_a = e - v$. Then, substituting this result into (12), we get

$$\varphi\left(v,v^*\right) = 2\mathrm{Re}\left[e_a^* f\left(e,e^*\right)\right]\Big|_{e=v,e^*=v^*} = 2\mathrm{Re}\left[\left(e^*-v^*\right) f\left(e,e^*\right)\right]\Big|_{e=v,e^*=v^*} = 0. \tag{A.1}$$

$$\begin{aligned}
\varphi_{e,e^*}^{(2)}\left(v,v^*\right) &= \frac{\partial^2}{\partial e\partial e^*} 2\mathrm{Re}\left[e_a^* f\left(e,e^*\right)\right]\Big|_{\substack{e=v,\\ e^*=v^*}} = \frac{\partial^2}{\partial e\partial e^*}\left[\left(e-v\right)^* f\left(e,e^*\right) + \left(e-v\right) f\left(e,e^*\right)^*\right]\Big|_{\substack{e=v,\\ e^*=v^*}} \\
&= \frac{\partial}{\partial e^*}\left[\left(e-v\right)^* f_e^{(1)}\left(e,e^*\right) + f\left(e,e^*\right)^* + \left(e-v\right) f_e^{(1)}\left(e,e^*\right)^*\right]\Big|_{\substack{e=v,\\ e^*=v^*}} \\
&= f_e^{(1)}\left(e,e^*\right) + f_{e^*}^{(1)}\left(e,e^*\right)^* + \left(e-v\right)^* f_{e,e^*}^{(2)}\left(e,e^*\right) + \left(e-v\right) f_{e,e^*}^{(2)}\left(e,e^*\right)^*\Big|_{\substack{e=v,\\ e^*=v^*}} \\
&= 2\mathrm{Re}\, f_e^{(1)}\left(v,v^*\right).
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
q_{e,e^*}^{(2)}\left(v,v^*\right) &= \frac{\partial}{\partial e^*\partial e}\left|f\left(e,e^*\right)\right|^2\Big|_{e=v,e^*=v^*} = \frac{\partial}{\partial e^*\partial e}\left[f\left(e,e^*\right) f\left(e,e^*\right)^*\right]\Big|_{e=v,e^*=v^*} \\
&= \frac{\partial}{\partial e^*}\left[f\left(e,e^*\right) f_e^{(1)}\left(e,e^*\right)^* + f\left(e,e^*\right)^* f_e^{(1)}\left(e,e^*\right)\right]\Big|_{e=v,e^*=v^*} \\
&= f_{e^*}^{(1)}\left(e,e^*\right) f_e^{(1)}\left(e,e^*\right)^* + f_e^{(1)}\left(e,e^*\right) f_{e^*}^{(1)}\left(e,e^*\right)^* \\
&\quad + f\left(e,e^*\right) f_{e,e^*}^{(2)}\left(e,e^*\right)^* + f\left(e,e^*\right)^* f_{e,e^*}^{(2)}\left(e,e^*\right)\Big|_{e=v,e^*=v^*} \\
&= \left|f_e^{(1)}\left(v,v^*\right)\right|^2 + \left|f_{e^*}^{(1)}\left(v,v^*\right)\right|^2 + 2\mathrm{Re}\left[f^*\left(v,v^*\right) f_{e,e^*}^{(2)}\left(v,v^*\right)\right]
\end{aligned} \tag{A.3}$$

Here, we use two equalities: $\left[f_e^{(1)}\left(e,e^*\right)\right]^* = f_{e^*}^{(1)}\left(e,e^*\right)^*$ and $\left[f_{e,e^*}^{(2)}\left(e,e^*\right)\right]^* = f_{e,e^*}^{(2)}\left(e,e^*\right)^*$.

APPENDIX B. Proof of lemma 2

Consider the real-valued cases. Substituting $e_a = e - v$ into (12), we get

$$\varphi(v) = 2\left[e_a f(e)\right]\Big|_{e=v} = 2\left[(e-v) f(e)\right]\Big|_{e=v} = 0. \tag{B.1}$$

$$\begin{aligned}\varphi_{e,e}^{(2)}(v) &= \frac{\partial^2}{\partial e \partial e} 2\left[e_a f(e)\right]\Big|_{e=v} = 2\frac{\partial}{\partial e}\left[(e-v) f_e^{(1)}(e) + f(e)\right]\Big|_{e=v} \\ &= 2\left[f_e^{(1)}(e) + (e-v) f_{e,e}^{(2)}(e) + f_e^{(1)}(e)\right]\Big|_{e=v} = 4 f_e^{(1)}(v)\end{aligned} \tag{B.2}$$

$$\begin{aligned}q_{e,e}^{(2)}(v) &= \frac{\partial^2}{\partial e \partial e} f^2(e)\Big|_{e=v} = \frac{\partial}{\partial e}\left[2 f(e) f_e^{(1)}(e)\right]\Big|_{e=v} \\ &= 2\left|f_e^{(1)}(e)\right|^2 + 2 f(e) f_{e,e}^{(2)}(e)\Big|_{e=v} = 2\left[\left|f_e^{(1)}(v)\right|^2 + f(v) f_{e,e}^{(2)}(v)\right]\end{aligned} \tag{B.3}$$

APPENDIX C. Proof of lemma 3

Let $z = x + jy$, where x and $y \neq 0$ are all real variables, then

$$\begin{aligned}\frac{\partial}{\partial z}|z|^p &= \frac{\partial}{\partial z}\left(|z|^2\right)^{p/2} = \frac{1}{2}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right)\left(x^2+y^2\right)^{p/2} \\ &= \frac{p}{4}\left(x^2+y^2\right)^{p/2-1} 2x - j\frac{p}{4}\left(x^2+y^2\right)^{p/2-1} 2y \\ &= \frac{p}{2}|z|^{p-2}(x - jy) = \frac{p}{2}|z|^{p-2} z^*\end{aligned} \tag{C.1}$$

Likewise, we can obtain $\frac{\partial}{\partial z^*}|z|^p = \frac{p}{2}|z|^{p-2} z$. Here, we use the Brandword's derivation operators [21].

APPENDIX D. Proof of theorem 2

First, we consider the complex-valued cases. Substituting the complex BSE of $q\left(e,e^*\right)$ (i.e., replacing $\varphi\left(e,e^*\right)$ in (19) by $q\left(e,e^*\right)$) into $\mathrm{E}\left[\|\mathbf{u}\|^2 q\left(e,e^*\right)\right]$, and neglecting $\mathrm{O}\left(e_a, e_a^*\right)$, we obtain

$$\begin{aligned}\mathrm{E}\left[\|\mathbf{u}\|^2 q\left(e,e^*\right)\right] &= \mathrm{E}\left[\|\mathbf{u}\|^2 q\left(v,v^*\right)\right] + \mathrm{E}\left[\|\mathbf{u}\|^2 q_e^{(1)}\left(v,v^*\right) e_a\right] + \mathrm{E}\left[\|\mathbf{u}\|^2 q_{e^*}^{(1)}\left(v,v^*\right) e_a^*\right] \\ &+ \frac{1}{2}\mathrm{E}\left[\|\mathbf{u}\|^2 q_{e,e}^{(2)}\left(v,v^*\right) e_a^2\right] + \frac{1}{2}\mathrm{E}\left[\|\mathbf{u}\|^2 q_{e^*,e^*}^{(2)}\left(v,v^*\right)\left(e_a^*\right)^2\right] + \mathrm{E}\left[\|\mathbf{u}\|^2 q_{e,e^*}^{(2)}\left(v,v^*\right)|e_a|^2\right]\end{aligned} . \tag{D.1}$$

Due to v being independent of e_a and $\mathbf{u}$ (i.e., A.1 and A.2), the above equation can be rewritten as

$$\begin{aligned}\mathrm{E}\left[\|\mathbf{u}\|^2 q\left(e,e^*\right)\right] &= \mathrm{E}\|\mathbf{u}\|^2 \mathrm{E}q\left(v,v^*\right)+\mathrm{E}\left(\|\mathbf{u}\|^2 e_a\right)\mathrm{E}q_e^{(1)}\left(v,v^*\right)+\mathrm{E}\left(\|\mathbf{u}\|^2 e_a^*\right)\mathrm{E}q_{e^*}^{(1)}\left(v,v^*\right)\\ &+\frac{1}{2}\mathrm{E}\left(\|\mathbf{u}\|^2 e_a^2\right)\mathrm{E}q_{e,e}^{(2)}\left(v,v^*\right)+\frac{1}{2}\mathrm{E}\left[\|\mathbf{u}\|^2\left(e_a^*\right)^2\right]\mathrm{E}q_{e^*,e^*}^{(2)}\left(v,v^*\right)\\ &+\mathrm{E}\left(\|\mathbf{u}\|^2|e_a|^2\right)\mathrm{E}q_{e,e^*}^{(2)}\left(v,v^*\right)\end{aligned} . \quad \text{(D.2)}$$

Using $e_a = \mathbf{u}\tilde{\mathbf{w}}$ and A.5, we get

$$\mathrm{E}\left(\|\mathbf{u}\|^2 e_a\right) = \mathrm{E}\left(\|\mathbf{u}\|^2 e_a^*\right)=\mathrm{E}\left(\|\mathbf{u}\|^2 e_a^2\right) = \mathrm{E}\left[\|\mathbf{u}\|^2\left(e_a^*\right)^2\right] = 0 . \quad \text{(D.3)}$$

Hence, substituting (D.3) into (D.2) and using Lemma 1, we can obtain

$$\mathrm{E}\left[\|\mathbf{u}\|^2 q\left(e,e^*\right)\right] = C\mathrm{E}\|\mathbf{u}\|^2 + B\mathrm{E}\left(\|\mathbf{u}\|^2|e_a|^2\right) \quad \text{(D.4)}$$

where B and C are defined by (18a). Substituting the following formula [see e.g. 6.5.18] in [11], i.e.,

$$\mathrm{E}\left(\|\mathbf{u}\|^2|e_a|^2\right) = (M+1)\sigma_u^2\mathrm{E}|e_a|^2 \quad \text{(D.5)}$$

into (D.4), and using (5) and $\mathrm{E}\|\mathbf{u}\|^2 = M\sigma_u^2$, we have

$$\mathrm{E}\left[\|\mathbf{u}\|^2 q\left(e,e^*\right)\right] = CM\sigma_u^2 + B(M+1)\sigma_u^2\zeta_{TEMSE} . \quad \text{(D.6)}$$

Then, substituting (21) and (D.6) into (10) yields

$$\left[A-\mu B(M+1)\sigma_u^2\right]\zeta_{TEMSE} = \mu CM\sigma_u^2 + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) . \quad \text{(D.7)}$$

where A is defined by (18a). Obviously, if the condition C.2 is satisfied, the tracking EMSE expression of (38) can be obtained while in nonstationary environments.

Next, we consider the real-valued cases. Similarly, substituting the TSE of $q(e)$ (i.e., replacing $\varphi(e)$ in (25) by $q(e)$) into $\mathrm{E}\left[\|\mathbf{u}\|^2 q(e)\right]$, neglecting $\mathrm{O}(e_a)$, and using A.1, A.2 and A.5, we get

$$\mathrm{E}\left[\|\mathbf{u}\|^2 q(e)\right] = C\mathrm{E}\|\mathbf{u}\|^2 + B\mathrm{E}\left(\|\mathbf{u}\|^2|e_a|^2\right) . \quad \text{(D.8)}$$

where B and C are defined by (18b). Using $\mathrm{E}\|\mathbf{u}\|^2 = M\sigma_u^2$ and substituting the following formula [see e.g. 6.5.20] in [11], i.e.,

$$\mathrm{E}\left(\|\mathbf{u}\|^2 |e_a|^2\right) = (M+2)\sigma_u^2 \mathrm{E}|e_a|^2 \tag{D.9}$$

into (D.8), we obtain

$$\mathrm{E}\left[\|\mathbf{u}\|^2 q(e)\right] = CM\sigma_u^2 + B(M+2)\sigma_u^2 \zeta_{TEMSE}\,. \tag{D.10}$$

Hence, substituting (27) and (D.10) into the real-form equation of (10) yields

$$\left[A - \mu B(M+2)\sigma_u^2\right] \zeta_{TEMSE} = \mu CM\sigma_u^2 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\,. \tag{D.11}$$

where A is defined by (18b). Then, if the condition C.2 is satisfied, we can obtain (37) and (38) while $\gamma = 2$, respectively.

Next, letting $\mathrm{Tr}(\mathbf{Q}) = 0$, we can obtain the EMSE expression of (37) in stationary environments.

Finally, differentiating both-hand sides of (38) with respect to μ, and letting it be zero, we get

$$\mu_{opt}^2 + \frac{2B(M+\gamma)\mathrm{Tr}(\mathbf{Q})}{AMC}\mu_{opt} - \frac{\mathrm{Tr}(\mathbf{Q})}{MC\sigma_u^2} = 0 \tag{D.12}$$

Then, we can obtain (39) by solving the above equation. This ends the proof of Theorem 2.

6. References

[1] S. Haykin, *Adaptive Filter Theory*. 4th edn. Englewood Cliffs, NJ: Prentice Hall 2002.

[2] B. Widrow and M. E. Hoff, Jr., "Adaptive switching circuits," *IRE WESCON Conv. Rec.*, Pt. 4, pp. 96-104, 1960.

[3] B. Windrow, J. M. Mccool, M. G. Larimore, and C. R. Hohnson, "Stationary and nonstationary learning characteristics of the LMS adaptive filter, " *Proc. IEEE*, vol.46, pp. 1151-1161, 1976.

[4] V. J. Mathews and S. Cho, "Improved convergence analysis of stochastic gradient adaptive filters using the sign algorithm," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 35, pp. 450-454, 1987.

[5] E. Walah and B. Widrow, "The least-mean fourth (LMF) adaptive algorithm and its family," *IEEE Trans. Information Theory*, vol. 30, pp. 275-283, 1984.

[6] J. A. Chambers, O. Tanrikulu and A. G. Constantinides, "Least mean mixed-norm adaptive filtering," *Electron. Lett.*, vol. 30, pp. 1574-1575, 1994.

[7] O. Tanrikulu and J. A. Chambers, "Convergence and steady-state properties of the least-mean mixed-norm (LMMN) adaptive algorithm," *IEE Proceedings- Vision, Image Signal Processing*, vol. 143, pp137-142, 1996.

[8] B. Lin, R. He, X. Wang and B. Wang, "Excess MSE analysis of the concurrent constant modulus algorithm and soft decision-directed scheme for blind equalization," *IET Signal Processing*, vol. 2, pp. 147-155, Jun., 2008.

[9] D. L. Duttweiler, "Adaptive filter performance with nonlinearities in the correlation multiplier," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp.578-586, Aug., 1982.

[10] S. C. Douglas and T. H.-Y. Meng, "Stochastic gradient adaptation under general error criteria," *IEEE Trans. Signal Processing*, vol.42, pp.1335-1351, Jun., 1994.
[11] A. H. Sayed, *Fundamentals of adaptive filtering*, New York:Wiley, 2003.
[12] N. R. Yousef and A. H. Sayed, "A unified approach to the steady-state and tracking analyses of adaptive filters", *IEEE Trans. Signal Processing*, vol. 49, pp. 314-324, Feb., 2001.
[13] J. H. Husøy and M. S. E. Abadi, "Unified approach to adaptive filters and their performance," *IET Signal Processing*, vol. 2, pp. 97-109, Jun., 2008.
[14] T. Y. Al-Naffouri and A. H. Sayed, "Transient analysis of adaptive filters with error nonlinearities," *IEEE Trans. Signal Processing*, vol. 51, pp. 653-663, Mar., 2003.
[15] O. Dabeer, and E. Masry, "Analysis of mean-square error and transient speed of the LMS adaptive algorithm," *IEEE Trans. Information Theory*, vol. 48, pp. 1873-1894, July, 2002.
[16] N. J. Bershad, J. C. M. Bermudez and J. Y. Tourneret, "An affine combination of two LMS adaptive filters-transient mean-square analysis," *IEEE Trans. Signal Processing*, vol. 56, pp. 1853-1864, May, 2008.
[17] B. Lin, R. He, L. Song and B. Wang, "Steady-state performance analysis for adaptive filters with error nonlinearities," *Proc. of ICASSP*, Taibei, Taiwan, pp. 3093-3096, Apr., 2009.
[18] N. R. Yousef and A. H. Sayed, "Fixed-point steady-state analysis of adaptive filters," *Int. J. Contr. Signal Processing*, vol. 17, pp. 237-258, 2003.
[19] B. Lin, R. He, X. Wang and B. Wang, "The excess mean square error analyses for Bussgang algorithm," *IEEE Signal Processing Letters*, vol. 15, pp. 793-796, 2008.
[20] A. Goupil and J. Palicot, "A geometrical derivation of the excess mean square error for Bussgang algorithms in a noiseless environment", *ELSEVIER Signal Processing*, vol. 84, pp. 311-315, May, 2004.
[21] D. H. Brandwood, "A complex gradient operator and its application in adaptive array theory," *Proc. Inst. Elect. Eng. F, H*, vol. 130, pp. 11-16, Feb., 1983.
[22] G. Yan and H. Fan, "A Newton-like algorithm for complex variables with applications in blind equalization," *IEEE Trans. Signal Processing*, vol. 48, pp. 553-556, Feb., 2000.
[23] S. C. Pei, and C. C. Tseng, "Least mean *p*-power error criterion for adaptive FIR filter," *IEEE Journal on Selected Areas in Communications*, vol. 12, pp. 1540-1547, Dec. 1994.
[24] I. S. Reed, "On a moment theorem for complex Gaussian process," *IRE Trans. Information Theory*, pp. 194-195, April, 1962.
[25] V. H. Nascimento and J. C. M. Bermudez, "Probability of divergence for the least-mean fourth (LMF) algorithm," *IEEE Trans. Signal Processing*, vol. 54, pp. 1376-1385, Apr., 2006.
[26] J. C. M. Bermudez and V. H. Nascimento, "A mean-square stability analysis of the least mean fourth adaptive algorithm," *IEEE Trans. Signal Processing*, vol. 55, pp. 4018-4028, Apr., 2007.
[27] A. Zerguine, "Convergence and steady-state analysis of the normalized least mean fourth algorithm," *Digital Signal Processing*, vol. 17 (1), pp. 17-31, Jan., 2007.
[28] B. Lin, R. He, X. Wang and B. Wang, "The steady-state mean square error analysis for least mean *p*-order algorithm," *IEEE Signal Processing Letter*, vol. 16, pp. 176-179, 2009.

Adaptive Filtering

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ISBN 978-953-307-158-9
Hard cover, 398 pages
Publisher InTech
Published online 06, September, 2011
Published in print edition September, 2011

Adaptive filtering is useful in any application where the signals or the modeled system vary over time. The configuration of the system and, in particular, the position where the adaptive processor is placed generate different areas or application fields such as prediction, system identification and modeling, equalization, cancellation of interference, etc., which are very important in many disciplines such as control systems, communications, signal processing, acoustics, voice, sound and image, etc. The book consists of noise and echo cancellation, medical applications, communications systems and others hardly joined by their heterogeneity. Each application is a case study with rigor that shows weakness/strength of the method used, assesses its suitability and suggests new forms and areas of use. The problems are becoming increasingly complex and applications must be adapted to solve them. The adaptive filters have proven to be useful in these environments of multiple input/output, variant-time behaviors, and long and complex transfer functions effectively, but fundamentally they still have to evolve. This book is a demonstration of this and a small illustration of everything that is to come.

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