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Methods for Structural and Parametric Synthesis of Bio-Economic Models

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1. Introduction

The communities of animals and plants are some examples of biological systems where population dynamics $\mathbf{X}(t) \in \mathsf{R}^d$, $t \in [t_0, t_1]$, depends not only on environmental variability and internal transformations, but also on human control factor $\mathbf{u}(t) \in \mathsf{R}^r$, $t \in [t_0, t_1]$. (We will call also $\mathbf{X}(t)$ a phase variable and $\mathbf{u}(t)$ a control variable.)

Since many biological systems have to stay in some dynamic equilibrium, the control representing the action, decision, or policy of the decision-makers can only correct the system development with moderate effects on its natural behavior. In this case controllers are interested in the selection of a nonaticipative decision among the ones satisfying all the changes of existence conditions and inter-connections among different communities.

Suppose that the controller is interested in renewable resource management (fishery or forestry) over the planning horizon $[t_0, t_1]$.

The goal of this task is maximization with respect to $\mathbf{u}(t) \in U(t)$ ($U(t)$ is a given metric space, $t \in [t_0, t_1]$) of the functional

$$\mathfrak{J}\big(\mathbf{X}(\cdot),\mathbf{u}(\cdot)\big) = \sup_{u \in U}\left\{ \int_{t_0}^{t_1} f\big(t,\mathbf{X}(t),\mathbf{u}(t)\big)dt + \psi_0\big(\mathbf{X}(t_0),\mathbf{X}(t_1)\big)\right\} \tag{1.1}$$

subjected to the constraints

$$d\mathbf{X}(t) = \tilde{\mathsf{T}}\big(t,\mathbf{X}(t),\mathbf{u}(t)\big)dt, \tag{1.2}$$

$$\psi_1\big(\mathbf{X}(t_0),\mathbf{X}(t_1)\big) \le 0, \tag{1.3}$$

$$\psi_2\big(\mathbf{X}(t_0),\mathbf{X}(t_1)\big) = 0; \tag{1.4}$$

$$g_i\big(t,\mathbf{X}(t)\big) \le 0,\ \forall t \in [t_0, t_1], i = 1,\ldots,\ell_1, \tag{1.5}$$

$$\phi_j\big(t,\mathbf{u}(t)\big) \le 0, j = 1,\ldots,\ell_2, \tag{1.6}$$

where $\tilde{T}(t, \mathbf{X}(t), \mathbf{u}(t))$ is an operator of a mathematical model of the resource $\mathbf{X}(t)$, $f: \mathsf{R} \times \mathsf{R}^d \times \mathsf{R}^r \to \mathsf{R}$, $\varphi: \mathsf{R} \times \mathsf{R}^d \times \mathsf{R}^r \to \mathsf{R}^d$, $\psi_0: \mathsf{R}^d \times \mathsf{R}^d \to \mathsf{R}$, $\psi_1: \mathsf{R}^d \times \mathsf{R}^d \to \mathsf{R}^{k_1}$, $\psi_2: \mathsf{R}^d \times \mathsf{R}^d \to \mathsf{R}^{k_2}$, $g_i: \mathsf{R} \times \mathsf{R}^d \to \mathsf{R}$ ($1 \le i \le \ell_1$), $\phi_j: \mathsf{R} \times \mathsf{R}^r \to \mathsf{R}$ ($1 \le j \le \ell_2$); d, r, k_1, k_2, ℓ_1, $\ell_2 \in \mathsf{N}$, and $\sup(\cdot)$ is the least upper bound.

The task (1.1) - (1.6) is formulated as an optimal control task [Milyutin & Osmolovski, 1998]. Its solution and accuracy of this solution depend on many different factors, mainly on the successful selection of the object equation. The wrong specification of a model and as a consequence wrong parametric identification can lead to erroneous solution of (1.1) - (1.6). Since the factor of uncertainty is always present at the resource description, it is reasonable to take (1.2) as an stochastic differential equation (SDE).

Let the renewable resource describes a certain population of fish, which natural growth rate depends on different biological parameters. Very often these parameters are evaluating over the time because environmental conditions are not constant. In this case it is reasonable to treat the parameters of SDE as the bivariate functions (or the SDE with time-varying parameters). Since the structure of the SDE is selected, the next step is the construction of the estimation procedure.

It is not easy to describe the bivariate functions by means of certain functional forms. Flexible model does not assume any specific form of the functions. This data-analytic approach called nonparametric regression can be found in statistical literature. However the direct application of the ideas does not bring desired results. The improvements of the identification procedures were presented in [Fan at al. 2003]. The main idea of this work was based on the discretization of the SDE and further approximation of parameter functions by constants at the discretization points. It is clear that the accuracy of the estimates depends on the accuracy of the discretization method. To overcome this problem we propose to consider bivariate functions as an control functions and solve the identification task as an optimal control problem using the maximum principle.

The rest of the paper is organized as follows. The second section is dedicated to the system analysis of a bio-economic models. The third section presents the identification methods based on the ideas [Bastogne at al., 2007], [Hansen & Penland, 2007], [Hurn at al., 2003], [Jang at al., 2003], [McDonald & Sandal, 1998], [Shoji & Ozaki, 1998]. The fourth section shows the solution for the bivariate functions. The paper is ended by the conclusions.

2. System analysis of a bio-economic model

2.1 Problems of structural and parametric synthesis

The mathematical description of any bio-economic model requires taking into account all the elements of the system and all the interrelations among them. Detailed analysis of the "complete" model allows to forecast its behavior and introduce optimal management strategies. Unfortunately, this analysis is mathematically difficult. From one side it is impossible to detect all the elements and nature of their interconnections. From other side there are several sources of uncertainty: the growth, mortality, reproduction rate vary in random manner causing random effects on genetics and age structure of exploited population; the price of the resource depends on economical situation on stock markets, political situation, climate and etc.

Distinctive property of each stochastic object is contained with ambiguous respond on the same input signals. Even for simplest one-dimensional object and for non-stochastic input signal, output signal of stochastic object can't be considered as deterministic one. For output

variable of this objects class scattering growths with increasing of objects "noise properties". For that reason uncertainty in behavior can be explained by noise influence that in addition brings deficiency of a priori information about system.

Therefore, it is impossible to detect a model, whose properties and mathematical description would correspond to the "exact" behavior of the system. Every mathematical model will be only similar with the system. In this case model selection has to be done under two groups of disjoint requirements:

- the main features of the system have to be reflected as precise as possible (the degree of similarity);
- existence of the theoretical methods, which allow to use the model for forecast, optimization, control, etc.

The dynamics of bio-economic system (1.2) can be described by means of its states. Let a union of the values of the phase variable $\mathbf{X}(t) \in \mathsf{R}^d$, $t \in [t_0, t_1]$, denote the state $\mathsf{S}_t \in \mathsf{S}^*$ (S^* is some space) of the system in time t, $\mathsf{S}_{t+\Delta} \in \mathsf{S}^*$ denote the state in time $t+\Delta$ (where Δ is time increment, $\Delta \in \mathsf{R}$, $0 < \Delta < t_1$). The dynamics of the system (1.2) can be written as

$$\mathsf{S}(t+\Delta) - \mathsf{S}(t) = \mathsf{T}\left(\mathsf{S}^*, \left\langle \mathsf{C}(t), \mathsf{G}(t) \right\rangle\right), \tag{2.1}$$

where $\mathsf{T}(t)$ is an operator of a mathematical model, $\mathsf{C}(t) = \{\mathsf{c}_0(t), \mathsf{c}_1(t), ..., \mathsf{c}_k(t)\}$ is a set of k parameters, $\mathsf{G}(t) = \{\mathsf{g}_0(t), \mathsf{g}_1(t), ..., \mathsf{g}_k(t)\}$ is the set of the structural relations of the phase variables of the system.

The formalization of the model (2.1) depends on the selection of:

- the set of phase variables $\mathbf{X}(t) \in \mathsf{R}^{d-n}$, $0 \le n < d$;
- the theoretical method to define S^*;
- the structural relations $\mathsf{G}(t)$ among the selected phase variables;
- the parametric identification method for the estimation of the parameters $\mathsf{C}(t)$.

Even though it is well known task, the problem of the object (2.1) formalization can be solved by a few mathematical tools. Among basic groups of exact and approximated methods, which are used to solve the problem, we can name: exact methods, methods of task simplification, methods of the task linearization, numerical methods, methods of integral transformation, method of infinite series, variation methods, and methods of reduction to the systems of the ordinary differential equations.

2.2 Stochastic differential equation as a bio-economic model

We consider a certain population of fish, whose size at time t is denoted by $X(t) \in \mathsf{R}$, $t \in [t_0, t_1]$. This population has a natural growth rate $\varphi^*(t, X(t))$:

$$\frac{dX(t)}{dt} = \varphi^*(t, X(t)), \ X(t_0) = X_0, \tag{2.2}$$

where $\varphi^* : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$ is assumed to be a concave function with given properties.

Fish stock $X(t)$ has natural fluctuations and is subjected to many stochastic effects. To take them into account we have to improve the model (2.2) adding some stochastic terms. In fact

the increment $\dfrac{dX(t)}{dt}$ is not a deterministic, thus the structure of the model (2.2) has to be reorganized.

Let $\left(\Omega, \mathsf{F}, \{\mathsf{F}_t\}_{t\geq 0}, \mathsf{P}\right)$ be a stochastic basis satisfying the usual conditions. Let the phase variable be $X : \Omega \to \mathsf{R}$, $\omega \in \Omega$, $X(\omega) = x$ and $\{v(t), t \in [t_0, t_1]\}$ be a continuous stochastic process, defined on $\left(\Omega, \mathsf{F}, \{\mathsf{F}_t\}_{t\geq 0}, \mathsf{P}\right)$, such that its mean value function $\mathsf{E}[v(t)] = 0$ for every $t \in [t_0, t_1]$ and $v(t_0) = 0$. By the intuition we add the term $v(t)$ to (2.2)

$$\frac{dX(t)}{dt} = \varphi^{**}\left(t, X(t), v(t)\right), \ X(t_0) = X_0, \ t \in [t_0, t_1], \tag{2.3}$$

where $\varphi^{**} :: \mathsf{R} \times \mathsf{R} \times \mathsf{R} \to \mathsf{R}$.

The model (2.3) presents a stochastic differential equation, whose solution depends on the selection of the stochastic process $v(t)$ and its properties. Let the process be the Brownian motion process $\{B(t), t \in [t_0, t_1]\}$. This process has many useful theoretical properties, namely

1. the independence and stationarity of the increments $B(t+h) - B(\tau+h)$ for every $t > \tau \in [t_0, t_1]$ and every $h > 0$;

2. the mean square continuity of $B(t)$ for every $t \in [t_0, t_1]$;

3. the regularity conditions, i.e. $\mathsf{E}\left[\left|B(t_0)\right|^2\right] < \infty$ and $\mathrm{var}\left[B(t+h) - B(\tau+h)\right] < \infty$ (where $\mathsf{E}[\cdot]$ denotes the expectation operator and $\mathrm{var}[\cdot]$ denotes the variance operator).

We replace the stochastic term $v(t)$ by $B(t)$ and rewrite the model (2.3) as a stochastic differential equation

$$dX(t) = a\left(t, X(t)\right)dt + b\left(t, X(t)\right)dB(t), \ X(t_0) = X_0, \tag{2.4}$$

or as an integral equation

$$X(t) - X(t_0) = \int_{t_0}^{t} a\left(\tau, X(\tau)\right)dt + \int_{t_0}^{t} b\left(\tau, X(\tau)\right)dB(\tau), \tag{2.5}$$

where $a : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$ and $b : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$ are Baire functions, $dB(t)$ is an increment of the Brownian motion process. The first integral in (2.5) will be a mean square Riemann integral, whereas the second one will be the Ito stochastic integral. Equation (2.4) is therefore called Ito stochastic differential equation.

Equation (2.4) can be used for the description of the object (1.2). In this case, the optimal control solution of the task (1.1) – (1.6) requires the existence and uniqueness of the solution of the equation (2.4). This solution can be treated in strong or weak sense. The SDE (2.4) has

•   a unique strong solution, if any two solutions $X(t)$ and $\tilde{X}(t)$, $t \in [t_0, t_1]$, coincide by all trajectories of the process so that

$$\mathsf{p}\left(\sup_{t \in [t_0,t_1]} \left| X_t - \tilde{X}_t \right| > 0 \right) = 0 \text{,}$$

- a unique weak solution, if its solutions coincide by probability, but not obligatory by trajectories.

Let us consider the solution of SDE (2.3) only in the strong sense. The following theorem guaranties the existence and uniqueness of the strong solution for (2.4).

Theorem 2.1.   *The solution of equation (2.3)* $X(t)$ *(* $t \in [t_0,t_1]$ *) exists and is unique in strong sense, if the following conditions are held.*

A1 *(Measurability):* $a(t,x)$ *and* $b(t,x)$ *are jointly* $L^2 - measurable$ *in* $(t,x) \in [t_0,t_1] \times \mathsf{R}$.

A2 *(Lipschitz condition): There exists a constant* $K > 0$ *such that*

$$\left| a(t,x) - a(t,y) \right| \leq K \left| x - y \right|$$

and

$$\left| b(t,x) - b(t,y) \right| \leq K \left| x - y \right|$$

for all $t \in [t_0,t_1]$ *and* $x,y \in \mathsf{R}$.

A3 *(Linear growth bound): There exists a constant* $K > 0$ *such that*

$$\left| a(t,x) \right|^2 \leq K^2 \left(1 + \left| x \right|^2 \right)$$

and

$$\left| b(t,x) \right|^2 \leq K^2 \left(1 + \left| x \right|^2 \right)$$

for all $t \in [t_0,t_1]$ *and* $x \in \mathsf{R}$.

A4 *(Initial value):* $X(t_0)$ *is* A_{t_0} *-measurable with* $\mathsf{E}\left(\left| X(t_0) \right|^2 \right) < \infty$.

                                                                                                                ∎

In particular, assumptions (A2) and (A3) ensure that a solution of an SDE does not explode. This is very important to insure the stability of a numerical approximation of $X(t)$, $t \in [t_0,t_1]$, and avoid the "stiffness" and "ill-posed" problems. Among other properties are boundedness and continuity on $t \in [t_0,t_1]$, which guarantee the existence of an adjoint system and partly the optimality of the solution of the task (1.1) – (1.6).

The description of a bio-economic model is not completed since we have not introduce the set of parameters $\mathsf{C}(t)$, $t \in [t_0,t_1]$, in the structure (2.1). Denoting the parameters as $\boldsymbol{\theta}(t) \in \mathsf{R}^q$ we rewrite (2.4)

$$dX(t) = a\big(t, X(t); \boldsymbol{\theta}(t)\big) dt + b\big(t, X(t); \boldsymbol{\theta}(t)\big) dB(t) \text{,}\ X(t_0) = X_0 \text{.} \qquad (2.6)$$

It is clear that the values of the parameters $\boldsymbol{\theta}(t)$ are unknown and have to be evaluate by the proper estimation procedure. If one uses the maximum likelihood estimation method the following can be helpful.

The process $X(t)$, $t \in [t_0, t_1]$, generated by the Ito stochastic differential equation (2.4) is a Markov process and is therefore characterized by the density function $\mathsf{p}(X(t)) = \mathsf{p}(t,x)$ for all $t \in [t_0, t_1]$ and the transition probability density function $\mathsf{p}(X(t), X(\tau)) = \mathsf{p}_{X(t)|, X(\tau)}(x|y) = \mathsf{p}(t,x;\tau,y)$ for every $t > \tau \in [t_0, t_1]$. Assume the existence of the continuous partial derivatives $\frac{\partial \mathsf{p}}{\partial t}$, $\frac{\partial[\mathsf{p}a(t,x)]}{\partial x}$, $\frac{\partial^2[\mathsf{p}b^2(t,x)]}{\partial x^2}$, where p stands for $\mathsf{p}(t,x)$ and $\mathsf{p}(t,x;\tau,y)$. In this case we can write two equations, namely

a. Kolmogorov's forward equation

$$\frac{\partial \mathsf{p}(t,x;\tau,y)}{\partial t} = -\frac{\partial\big[\mathsf{p}(t,x;\tau,y)a(t,x)\big]}{\partial x} + \frac{1}{2}\frac{\partial^2\big[\mathsf{p}(t,x;\tau,y)b^2(t,x)\big]}{\partial x^2}, \tag{2.7}$$

b. Kolmogorov's backward equation

$$-\frac{\partial \mathsf{p}(t,x;\tau,y)}{\partial \tau} = a(\tau,y)\frac{\partial \mathsf{p}(t,x;\tau,y)}{\partial y} + \frac{1}{2}b^2(\tau,y)\frac{\partial^2 \mathsf{p}(t,x;\tau,y)}{\partial y^2} \tag{2.8}$$

with initial conditions selected in proper manner.

3. Estimation methods for time independent parameters of SDE

3.1 Maximum likelihood method

Consider the stochastic process $X = \{X_t, t \in [t_0, t_1]\}$, which is assumed to be the unique strong solution of the SDE (2.6) with $\boldsymbol{\theta}(t) = \boldsymbol{\theta}$

$$dX(t) = a(t, X(t); \boldsymbol{\theta})dt + b(t, X(t); \boldsymbol{\theta})dB(t), \ X(t_0) = X_0. \tag{3.1}$$

Assume that there are $n \in \mathsf{N}$ real valued observations $Y_0, Y_1, ..., Y_n \in \mathsf{R}$ of the process X, which were made at the discretization times $t_0 \le \tau_0, \tau_1, ..., \tau_n \le t_1$. (We consider, for simplicity, equidistant time discretization with $\tau_n = n\Delta$, where $\Delta = \frac{t_1}{n} \in (0,1)$ and some $n \in \mathsf{N}$). The observations contain information about the parameters $\theta_1, \theta_2, ..., \theta_q \in \Theta \subseteq \mathsf{R}$ (where Θ specifies the set of allowable values for the parameters, $q \in \mathsf{N}$) that we wish to estimate. The maximum likelihood estimator $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_q)^{\mathrm{T}}$ is the best estimate of $\boldsymbol{\theta} \in \Theta$, where

$$\mathsf{L}^* = \mathsf{L}(\hat{\boldsymbol{\theta}}) = \sup_{\boldsymbol{\theta} \in \Theta} \mathsf{L}(\boldsymbol{\theta}), \tag{3.2}$$

here $\sup(\cdot)$ is the least upper bound of $\mathsf{L}(\boldsymbol{\theta})$ over all $\boldsymbol{\theta} \in \Theta$.
This means, $\hat{\boldsymbol{\theta}}$ is the parameter that maximizes the likelihood function with respect to the set of permitted parameter values $\boldsymbol{\theta} \in \Theta$. The hypothesized density in the given parametrized family of probability densities $\mathsf{f}_Y(\cdot, \hat{\boldsymbol{\theta}})$ represents the most probable density from the given class of densities having observed $Y_0, Y_1, ..., Y_n \in \mathsf{R}$.

Let $p(t_k, y_k | t_{k-1}, y_{k-1}; \boldsymbol{\theta})$ be the transition probability density of (t_k, y_k) starting from (t_{k-1}, y_{k-1}) given the vector $\boldsymbol{\theta}$, $k \in \{1, 2, ..., n\}$. The density of the initial state is $p_0(y_0 | \boldsymbol{\theta})$. The joint density corresponds to the likelihood function

$$L(\boldsymbol{\theta}) = p_0(y_0; \boldsymbol{\theta}) \prod_{k=1}^{n} p(t_k, y_k | t_{k-1}, y_{k-1}; \boldsymbol{\theta}) \tag{3.3}$$

or in more convenient for the numerical simulation form

$$\mathsf{L}(\boldsymbol{\theta}) = -\ln\big(L(\boldsymbol{\theta})\big). \tag{3.4}$$

The task (3.2) for (3.4) can be solved as initial value problem for forward Kolmogorov's equation (2.7) with the initial condition given as $\lim_{t \to \tau} p_{Y_k | Y_{k-1}}(y_k | y_{k-1}) = \delta(y_k - y_{k-1})$, where $\delta(\cdot)$ stands for a forward difference [Jazwinski, 2007].

Let the SDE (3.1) be the liner SDE

$$dX(t) = \theta_1 X(t) dt + \theta_2 X(t) dB(t), \ X(t_0) = X_0, \tag{3.5}$$

where θ_1 and θ_2 are unknown parameters, which have to be estimated using the observations $Y_0, Y_1, ..., Y_n$.

Taking $Y_0 = X_0$, we rewrite (3.4) as

$$\mathsf{L}(\theta_1, \theta_2) = \sum_{k=1}^{n} \ln \left\{ \frac{\exp\left\{ -\left[\ln\left(\frac{Y_k}{Y_{k-1}}\right) - \left(\theta_1 - \frac{\theta_2^2}{2}\right)\Delta \right]^2 \bigg/ 2\theta_2^2 \Delta \right\}}{Y_k \sqrt{2\pi\theta_2^2 \Delta}} \right\}. \tag{3.6}$$

Under suitable conditions, when the true parameter is an interior point of Θ, the maximum likelihood estimate $\hat{\boldsymbol{\theta}}$ can be obtained as a root of the first order conditions

$$\frac{\partial \mathsf{L}(\boldsymbol{\theta})}{\partial \theta_i} = 0, \tag{3.7}$$

for all $i \in \{1, 2, ..., q\}$, where $\frac{\partial}{\partial \theta_i}$ denotes the partial derivative with respect to θ_i.

Using the conditions (3.7) for (3.6) we get the estimates of the parameters

$$\hat{\theta}_1 = \frac{1}{t_1} \ln\left(\frac{Y_n}{Y_0}\right) + \frac{\hat{\theta}_2^2}{2} \tag{3.8}$$

and

$$\hat{\theta}_2 = \sqrt{\frac{1}{n-1} \sum_{k=1}^{n} \Delta \left[\frac{1}{\Delta} \ln\left(\frac{Y_k}{Y_{k-1}}\right) - \frac{1}{t_1} \ln\left(\frac{Y_n}{Y_0}\right) \right]^2}. \tag{3.9}$$

Unfortunately, Kolmogorov's forward equation (2.7) has been solved only in a few simple cases. So, the alternative approach is required to solve the problem (3.4).

3.2 Monte Carlo methods

One difficulty in finding the optimal values $\boldsymbol{\theta}^{opt}$ is that the transition densities are not known. Recall that $\mathsf{p}(t,x;\tau,x^*)$ is a random variable since it depends on values x^* taken on by x_τ. We can take the expectation of (2.7) and, interchanging expectation with differentiation, obtain

$$\frac{\partial \mathsf{p}(t,x)}{\partial t} = -\frac{\partial\big[\mathsf{p}(t,x)a(t,x)\big]}{\partial x} + \frac{1}{2}\frac{\partial^2\big[\mathsf{p}(t,x)b^2(t,x)\big]}{\partial x^2}. \tag{3.10}$$

By the same reasoning as before we rewrite the likelihood function (3.3) as

$$\mathsf{L}(\boldsymbol{\theta}) = -\ln \mathsf{p}_0(y_0;\boldsymbol{\theta}) - \ln \sum_{k=1}^{n} \mathsf{p}(t_k,y_k;\boldsymbol{\theta}). \tag{3.11}$$

The main problem here is how to find the estimates $\hat{\mathsf{p}}(t_k,y_k;\boldsymbol{\theta})$, $k=1,2,..,n$, since we have only the sequence $(Y_n)_{n\in\{0,1,...,n_{t_1}\}}$. One of the ways to solve this task is to use Monte Carlo simulation method. We will consider to two possibilities, namely for one sample data and for the panel data.

3.2.1 One sample data set

Let for the random variable Y_k ($Y_k:\Omega \to \mathsf{R}$, $\omega\in\Omega$, $Y_k(\omega)=y_k$, $k\in\{0,1,...,n_{t_1}\}$) the distribution function be denoted as $\mathsf{F}_{Y_k}(y_k)$ and the density function as $\mathsf{f}_{Y_k}(y_k)$. For the simplicity of the reasoning we omit the index n. Both functions are connected as follows

$$\mathsf{f}_Y(y) = \frac{d}{dy}\mathsf{F}_Y(y). \tag{3.12}$$

The empirical estimate of the distribution function is

$$\hat{\mathsf{F}}_Y(y) \equiv \frac{1}{M}\sum_{j=1}^{M} 1\{\tilde{y}_j \le y\}, \tag{3.13}$$

where $1\{\cdot\}$ is set belonging indicator, M is a number of the realizations $\tilde{y}_j$.

Taking into account (3.12) and (3.13) the estimate for the density function is

$$\hat{\mathsf{f}}_Y(y) = \frac{\hat{\mathsf{F}}_Y(y+h) - \hat{\mathsf{F}}_Y(y)}{h} = \frac{1}{Mh}\sum_{j=1}^{M} 1\{y < \tilde{y}_j \le y+h\}, \tag{3.14}$$

where h is the bandwidth, $h>0$.

Since $\hat{\mathsf{F}}_Y(y)$ is an unbiased estimate of $\mathsf{F}_Y(y)$, we have the unbiased estimate for the density function so that

$$\begin{aligned}
\mathsf{E}\big[\hat{\mathsf{f}}_Y(y)\big] = \mathsf{f}_Y(y) &= \mathsf{E}\left[\frac{\hat{\mathsf{F}}_Y(y+h) - \hat{\mathsf{F}}_Y(y)}{h}\right] - \mathsf{f}_Y(y) \\
&= \frac{1}{Mh}\sum_{j=1}^{M} 1\{y < \tilde{y}_j \le y+h\} - \mathsf{f}_Y(y)
\end{aligned} \tag{3.15}$$

for $h \to 0$ and $M \to \infty$ and the variance

$$
\begin{aligned}
\mathrm{var}\left[\hat{f}_Y(y)\right] &= \mathrm{var}\left[\frac{1}{Mh}\sum_{j=1}^{M}\mathbb{1}\{y < \tilde{y}_j \le y+h\}\right] \\
&= \frac{1}{Mh^2}\mathrm{var}\left[\mathbb{1}\{y < \tilde{y}_j \le y+h\}\right] \\
&= \frac{1}{Mh}\left[\frac{F_Y(y+h)-F_Y(y)}{h}\left(1-\left(F_Y(y+h)-F_Y(y)\right)\right)\right] \\
&= \frac{f_Y(y)}{Mh}+\mathrm{O}\left(\frac{1}{M}\right)
\end{aligned}
\tag{3.16}
$$

is striving to zero for $Mh \to \infty$ and $M \to \infty$. It is clear that quality of the estimate $f_Y(y)$ depends on values M and h selection. In order $M \to \infty$, it is enough to increase the number of sample paths Y^j, thus we will consider problem of h parameter selection.

Let $h \to 0$ and $Mh \to \infty$ by means of the kernel function the approximation of the mean squared error for the estimate $\hat{f}_Y(y)$ can be written as

$$
MSE\left(\hat{f}_Y(y)\right) = \frac{1}{Mh}\left\|K(\rho)\right\|_2^2 + \frac{h^4}{4}\left(\mu\left(K(\rho)\right)\right)^2\left\|f_Y''(y)\right\|_2^2,
\tag{3.17}
$$

where $\left\|K(\rho)\right\|_2^2$ and $\mu\left(K(\rho)\right)$ are some constants, depending on kernel function ρ; f_Y'' is the second derivative of the function f_Y.

The minimization of (3.17) with respect to h gives following results

$$
h^{opt} = \left(\frac{1}{M}\frac{\left\|K(\rho)\right\|_2^2}{\left\|f_Y''(y)\right\|_2^2\left(\mu\left(K(\rho)\right)\right)^2}\right)^{1/5},
\tag{3.18}
$$

where $f_Y''(y)$ is the only unknown term.

Now the solution of (3.18) depends only on the kernel function selection. For the simplicity we use the parametric identification Epanichnikov kernel function

$$
K(\rho) = \frac{3}{4}\left(1-\rho^2\right)\mathbb{1}\left(|\rho| \le 1\right),
\tag{3.19}
$$

where $\rho = \left\{\rho_j = \frac{1}{h}\left(\tilde{y}_j - y\right), j = 1,2,\ldots,M\right\}$.

Substituting (3.19) into (3.18) we get

$$
h^{opt} \approx 0.9\sigma_Y M^{-1/5},
\tag{3.20}
$$

where the standard deviation of the sample is given as

$$
\sigma_Y = \sqrt{\frac{1}{M-1}\left[\sum_{j=1}^{M}\tilde{y}_j^2 - \frac{1}{M}\left(\sum_{j=1}^{M}\tilde{y}_j\right)^2\right]}
$$

for $j = 1, 2, ..., M$.

Next, let $\tilde{Y}$ be a discrete time strong approximation of $X = \{X_t, t \in [t_0, t_1]\}$ and the sequences $(\tilde{y}_{kj})_{k \in \{0,1,...,n_{t_1}\}, j \in \{1,2,...,M\}}$ of values of the numerical approximation at the discretization times $\tau_0, \tau_1, ..., \tau_{n_{t_1}}$ be computed in an iterative manner. Now for the observations Y any value of $\mathsf{f}_{Y_k}(y_k)$ can be estimated as follows

$$\hat{\mathsf{f}}_{Y_k}(y_k) = \frac{3}{4Mh} \sum_{j=1}^{M} \left(1 - \left(\frac{y_k - \tilde{y}_{kj}}{h}\right)^2\right) 1\{y_k - h \le \tilde{y}_{kj} \le y_k + h\}. \tag{3.21}$$

As far as $h \to 0$, but $h \ne 0$, then estimates (3.21) are biased $\mathsf{E}\left[\ln \hat{\mathsf{f}}_{Y_k}(y_k)\right] - \ln \mathsf{f}_{Y_k}(y_k) \ne 0$. The value of $\ln \hat{\mathsf{f}}_{Y_k}(y_k)$ has to be corrected for every $k \in \{0, 1, ..., n_{t_1}\}$, namely by repeating the estimation procedure S times for each $\hat{\mathsf{f}}_{Y_k}(y_k)$

$$\ln \mathsf{f}_{Y_k}(y_k) = \ln\left(\mathsf{E}\left[\hat{\mathsf{f}}_{Y_k}(y_k)\right]\right) - \frac{1}{2} \frac{\mathrm{var}\left[\hat{\mathsf{f}}_{Y_k}(y_k)\right]}{\left(\mathsf{E}\left[\hat{\mathsf{f}}_{Y_k}(y_k)\right]\right)^2}, \tag{3.22}$$

where

$$\mathsf{E}\left[\hat{\mathsf{f}}_{Y_k}(y_k)\right] = \frac{1}{S} \sum_{s=1}^{S} \hat{\mathsf{f}}_{Y_k}^{(S)}(y_k)$$

and

$$\mathrm{var}\left[\hat{\mathsf{f}}_{Y_k}(y_k)\right] = \frac{1}{S-1} \sum_{s=1}^{S} \left(\hat{\mathsf{f}}_{Y_k}^{(S)}(y_k) - \mathsf{E}\left[\hat{\mathsf{f}}_{Y_k}(y_k)\right]\right)^2.$$

Since the estimate (3.20) is found on the basis of values of the numerical approximation, the accuracy of the estimation procedure for (3.11) and in consequence of the solution of (3.2). Therefore, the optimization of (3.2) has to be done under constrains. Let us recall the definition.

Definition 3.1. A discrete time approximation Y^Δ converges strongly with order $\gamma > 0$ at time t_1 if there exists a positive constant C, which does not depend on Δ, and a $\delta_0 > 0$, such that

$$\varepsilon(\Delta) = \mathsf{E}\left(\left|X_{t_1} - Y_{t_1}^\Delta\right|\right) \le C\Delta^\gamma, \tag{3.23}$$

for each $\Delta \in (0, \delta_0)$.   ∎

The order of strong convergence γ is usually know for selected numerical scheme and cant be improved. The only possibility to get better accuracy for the parameters of SDE (3.1) is the chose of the discretization time $\Delta \in (0, \delta_0)$. As it was set before $\Delta = \frac{t_1}{n}$, so this means that $\varepsilon(\Delta) \to 0$ if $n \to \infty$. Taking into account that the generated values $\tilde{y}_{kj}$ depend also on the

unknown parameters $\boldsymbol{\theta}$, we set $\hat{f}_{Y_k}(y_k) := \hat{f}_{Y_k}(y_k;\boldsymbol{\theta})$, $k=0,1,...,n$, and formulate the identification task (3.1) as minimization of the functional

$$J(\boldsymbol{\theta}) = \inf_{\boldsymbol{\theta} \in \Theta} \left\{ -\ln \hat{f}_{Y_0}(y_0;\boldsymbol{\theta}) - \sum_{k=1}^{n} \hat{f}_{Y_k}(y_k;\boldsymbol{\theta}) \right\} \tag{3.24}$$

subjected to the constraints (3.1) and (3.23).

3.2.2 Panel data sample set

Let us consider a very typical situation for the bio-economic modeling, when the data on the population dynamics $X = \{X_t, t \in [t_0,t_1]\}$ are obtained by different observers, say there are M observers. In this case the parameters of the stochastic differential equation (3.1) can be estimated on the basis of the panel Y_k^j, where $j=1,2,..,M$ stands for the observer, $k=0,1,...,n$ refers to the discretization times $t_0 \le \tau_0, \tau_1,...,\tau_n \le t_1$ and $Y_0 = \mathsf{E}\left[Y_0^j\right]$. It is not difficult to conclude that the hypothesized distribution in the given parametrized family of probability distributions $\mathsf{F}_X(\cdot,\hat{\boldsymbol{\theta}})$ represents the most probable distribution from the given class of distributions having observed Y_k^j, $j=1,2,..,M$, $k=0,1,...,n$.

We suppose, that for the stochastic process $X = \{X_t, t \in [t_0,t_1]\}$ there exists the equivalent stochastic process $\tilde{X} = \{\tilde{X}_t, t \in [t_0,t_1]\}$, which sample paths w.p.1 are continuous on the interval $[t_0,t_1]$, so that both processes have equivalent distributions, i.e. $\mathsf{F}_X(x) = \mathsf{F}_{\tilde{X}}(\tilde{x})$. The empirical estimate of $\mathsf{F}_X(x)$ can be found on the basis of Y_k^j as

$$\mathsf{F}_{Y_k}(y_k) = \frac{1}{M} \sum_{j=1}^{M} \mathbf{1}_{(-\infty,y_j]}\left(Y_k^j\right), \tag{3.25}$$

where $j=1,2,..,M$, $k=0,1,...,n$.

For the same estimate of $\mathsf{F}_{\tilde{X}}(\tilde{x})$ the generated sample paths are required

$$\hat{\mathsf{F}}_{\tilde{Y}_k}(\tilde{y}_k;\boldsymbol{\theta}) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{(-\infty,y_i]}\left(\tilde{Y}_k^i\right), \tag{3.26}$$

where N is the number of simulated sample paths of the equivalent stochastic process given by (3.1) with the set of the parameters $\boldsymbol{\theta}$.

Now, the identification task can be solved by means of the testing the hypothesis about the equivalence of the distributions (3.25) and (3.26), using, for example, Kolmogorov-Smirnov's goodness-of-fit test

$$D_{N,M}(\tau_k;\boldsymbol{\theta}) = \sup_{y,\tilde{y} \in \mathsf{R}} \left| \hat{\mathsf{F}}_{Y_k}(y_k) - \hat{\mathsf{F}}_{\tilde{Y}_k}(\tilde{y}_k;\boldsymbol{\theta}) \right| \tag{3.27}$$

for all $\tau_k \in [t_0,t_1]$.

The statistic (3.27) has asymptotic null distribution

$$KS\big(D(\tau_k;\boldsymbol{\theta})\big) = \lim_{N,M\to\infty} \mathsf{P}\left(\sqrt{\frac{NM}{N+M}} D_{N,M}(\tau_k;\boldsymbol{\theta}) \le D^*\right), \tag{3.28}$$

where D^* is critical value of Kolmogorov's distribution.
The expression (3.28) can be presented also by

$$KS\big(D(\tau_k;\boldsymbol{\theta})\big) = 1 - 2\sum_{\ell=1}^{+\infty}(-1)^{\ell-1}\exp\left[-2\ell^2 D(\tau_k;\boldsymbol{\theta})^2\right]. \tag{3.29}$$

A large value of $D(\tau_k;\boldsymbol{\theta})$, and therefore a small value $KS\big(D(\tau_k;\boldsymbol{\theta})\big)$, indicates that the distributions are not equivalent, whereas small values of $D(\tau_k;\boldsymbol{\theta})$ support that the distributions are equivalent. This fact can be used for the formulation of the identification task for (3.1), that is to say one has to maximize the functional

$$J(\boldsymbol{\theta}) = \sup_{\boldsymbol{\theta}\in\Theta}\left\{\sum_{k=0}^{n} KS\big(D(\tau_k;\boldsymbol{\theta})\big)\right\} \tag{3.30}$$

subjected to (3.1) and (3.23).

4. Identification method for the time-varying parameters

4.1 Basic assumptions

Let $\big(\Omega, \mathsf{F}, \{\mathsf{F}_t\}_{t\ge 0}, \mathsf{P}\big)$ be a complete probability space on which some m-dimensional stochastic process $\mathbf{B} = \left\{\mathbf{B}_t = \big(B_t^1, B_t^2, ..., B_t^m,\big)^{\mathrm{T}}, t\in[t_0,t_1]\right\}$ is defined such that $\{\mathsf{F}_t\}_{t\ge 0}$ is the natural filtration generated by $\mathbf{B}(\cdot)$, augmented by all the P-null sets in F. We suppose that these stochastic processes are independent and replace (2.6) by the following SDE

$$d\mathbf{X}(t) = \mathbf{a}\big(t,\mathbf{X}(t),\boldsymbol{\theta}(t)\big)dt + \mathbf{b}\big(t,\mathbf{X}(t),\boldsymbol{\theta}(t)\big)d\mathbf{B}, \ \mathbf{X}(t_0) = \mathbf{X}_0 \in \mathsf{R}^d, \ t\in[t_0,t_1], \tag{4.1}$$

where $\mathbf{a}:[t_0,t_1]\times\mathsf{R}^d\times\Theta\to\mathsf{R}^d$ and $\mathbf{b}:[t_0,t_1]\times\mathsf{R}^d\times\Theta\to\mathsf{R}^{d\times m}$ with Θ being a given metric space, which specifies the set of allowable values for the parameters $\boldsymbol{\theta}$, $\boldsymbol{\theta}(\cdot)$ is the unknown non-random vector of parameters.

The goal is to present the estimation method for the parameters $\boldsymbol{\theta}(\cdot)$ taking into account some properties of the stochastic process, which is assumed to be the unique strong solution of (4.1). For the simplicity in further reasoning we will consider one-dimensional case ($d=m=1$) of the SDE (4.1) and limit the family of stochastic processes $\mathbf{B}(\cdot)$ to one-dimensional ordinary Brownian motion (fBm). This gives the possibility to rewrite (4.1) as

$$dX(t) = a\big(t,X(t),\boldsymbol{\theta}(t)\big)dt + b\big(t,X(t),\boldsymbol{\theta}(t)\big)dB(t), \ X(t_0) = X_0 \in \mathsf{R}, \ t\in[t_0,t_1], \tag{4.2}$$

We point out that although the SDE (4.2) is now assumed to be one-dimensional, results can be extended to m-dimensional case of the SDE (4.1) with the same ideas.

4.2 Estimation principle

There are many possibilities to solve the general optimal control problem (1.1), (1.3) - (1.6), (4.2) with respect to the identification problem of the parameters $\boldsymbol{\theta}$. Since the solution of the object equation (4.2) is a stochastic process, it is reasonable to use stochastic principles as it was done in [Hu at al., 2003]. However, in our case we are not going to solve "pure" optimal control task, because we consider a non-random vector of parameters and thus SDE (4.2) can be converted to an ordinary differential equation (ODE) by means of moment equations.

Let $m_1(t) = \mathsf{E}\big[X(t)\big]$ and $m_2(t) = \mathsf{E}\big[X^2(t)\big]$ be the first and second moments of stochastic process $X(t)$, $t \in [t_0, t_1]$, generated by the SDE (4.2). Denote a new state variable

$$\mathbf{y}(t) = \big[m_1(t), m_2(t)\big] \in \mathsf{R}^2,$$

where $\mathbf{y}(t_0) = \big[m_1(t_0), m_2(t_0)\big]$ $(m_1(t_0) = \mathsf{E}\big[X_{t_0}\big]$, $m_2(t_0) = \mathsf{E}\big[X_{t_0}^2\big])$, and describe object dynamics using a system of the ODEs

$$d\mathbf{y}(t) = \varphi(t, \mathbf{y}(t), \boldsymbol{\theta}(t)) dt \quad \text{a.e.} \quad t \in [t_0, t_1] . \tag{4.3}$$

In this manner we have the possibility to use the principle maximum in a form, described in [Milyutin & Osmolovskii, 1998] or [Milyutin at al., 2004], to solve the parameter estimation problem. Now we introduce several definitions, which help to construct the estimation method.

Definition 4.1. Any $\boldsymbol{\theta}(\cdot)$ is called a feasible parameters vector $\boldsymbol{\theta}_f(\cdot)$, if

- $\boldsymbol{\theta}(\cdot) \in \mathbf{v}[t_0, t_1]$, where $\mathbf{v}[t_0, t_1] \triangleq \big\{\boldsymbol{\theta} : [t_0, t_1] \to \Theta \big| \boldsymbol{\theta}(\cdot) \text{ is measurable}\big\}$;

- $\mathbf{y}(\cdot)$ is the unique solution of the system of the ODEs (4.3) under $\boldsymbol{\theta}(\cdot)$;

- the state constraints (1.3) and (1.4) are satisfied;

- $f(t, \mathbf{y}(t), \boldsymbol{\theta}(t))$ belongs to the set of Lebesgue measurable functions such that

$$\int_{t_0}^{t_1} \big|f(t, \mathbf{y}(t), \boldsymbol{\theta}(t))\big| dt < \infty . \qquad \blacksquare$$

Definition 4.2. $\hat{\boldsymbol{\theta}}(\cdot)$ is called an optimal estimate of $\boldsymbol{\theta}(\cdot)$, if $\mathsf{J}\big(\hat{\mathbf{y}}(\cdot), \hat{\boldsymbol{\theta}}(\cdot)\big)$ is measurable and there exists $\varepsilon > 0$ such that for any $\mathbf{u}_f(\cdot)$ the following inequalities are fulfilled

$$\big\|\mathbf{y}(\cdot) - \hat{\mathbf{y}}(\cdot)\big\|_{\mathsf{C}\big([t_0, t_1], \mathsf{R}^2\big)} < \varepsilon ,$$

$$\mathsf{J}\big(\mathbf{y}(\cdot), \boldsymbol{\theta}(\cdot)\big) \geq \mathsf{J}\big(\hat{\mathbf{y}}(\cdot), \hat{\boldsymbol{\theta}}(\cdot)\big),$$

where $\mathtt{C}\big([t_0,t_1],\mathsf{R}^2\big)$ is the set of all continuous functions, $\hat{\ }$ denotes the estimate.

The definitions (4.1) and (4.2) allow us to propose the goal function (goal) as follows

$$\mathtt{J}\big(\mathbf{y}(\cdot),\boldsymbol{\theta}(\cdot)\big)=\inf_{\boldsymbol{\theta}\in\Theta}\int_{t_0}^{t_1}f\big(t,\mathbf{y}(t),\boldsymbol{\theta}(t)\big)dt$$

where

$$f\big(t,\mathbf{y}(t),\boldsymbol{\theta}(t)\big)=\big\|\varphi(t,\mathbf{y},\boldsymbol{\theta})-\varphi\big(t,\hat{\mathbf{y}},\hat{\boldsymbol{\theta}}\big)\big\|_2^2$$

and $\inf(\cdot)$ is the greatest lower bound.   ■

The phase constraints (1.3) and state constraints (1.4) can be defined on the basis of the properties of the stochastic process [Shyryaev, 1998]. As it was said before the Pontryagin's type maximum principle will be used to find the solution to the estimation problem. In this case we introduce the Pontryagin's function

$$\mathtt{H}\big(t,\mathbf{y}(t),\boldsymbol{\theta}(t),\psi(t)\big)=\psi(t)\varphi\big(t,\mathbf{y}(t),\boldsymbol{\theta}(t)\big)-\alpha_0 f\big(t,\mathbf{y}(t),\boldsymbol{\theta}(t)\big),$$

where $\psi(t)\in\big(\mathsf{R}^2\big)'$ is an adjoint function of bounded variation ($\psi:[t_0,t_1]\to\mathsf{R}^2$ is an absolutely continues function), α_0 is a number.

The theorem below, based on Dubovitski-Milyutin method [Milyutin at al., 2004], gives the possibility to find an optimal estimate $\hat{\boldsymbol{\theta}}(\cdot)$ of $\boldsymbol{\theta}(\cdot)$ for SDE (4.2).

Theorem 4.1. Let $\hat{\boldsymbol{\theta}}(\cdot)$ be an optimal estimate of $\boldsymbol{\theta}(\cdot)$ and $\big(\hat{\mathbf{y}}(\cdot),\hat{\boldsymbol{\theta}}(\cdot)\big)$ be an optimal pair ($\boldsymbol{\theta}(\cdot)\in\mathtt{L}^\infty\big([t_0,t_1],\mathsf{R}^2\big)$, $\mathbf{y}(\cdot)\in\mathtt{C}\big([t_0,t_1],\mathsf{R}^2\big)$). Then there exist a number α_0, a function of bounded variation $\psi(t)$ (which defines the measure $d\psi$), a function of bounded variation $\lambda(t)$ (which defines the measure $d\lambda$) such that the following conditions hold:

- nontriviality $|\alpha_0|+\|d\lambda\|>0$,
- nonnegativity $\alpha_0\geq 0$, $d\lambda\geq 0$,
- complementary slackness $d\lambda(t)g(t,\mathbf{y}(t))=0$,
- adjoint equation

$$-d\psi(t)=\psi(t)\varphi_\mathbf{y}\big(t,\hat{\mathbf{y}}(t),\hat{\boldsymbol{\theta}}(t)\big)-\alpha_0 f_\mathbf{y}\big(t,\hat{\mathbf{y}}(t),\hat{\boldsymbol{\theta}}(t)\big)-g_\mathbf{y}\big(t,\hat{\mathbf{y}}(t)\big)d\lambda,$$

- transversality condition $\psi(t_1)=0$,
- the local maximum condition

$$\psi(t)\varphi_\boldsymbol{\theta}\big(t,\hat{\mathbf{y}}(t),\hat{\boldsymbol{\theta}}(t)\big)-f_\boldsymbol{\theta}\big(t,\hat{\mathbf{y}}(t),\hat{\boldsymbol{\theta}}(t)\big)=0.$$

                                                                                      ■

The proof of the theorem 4.1 is not complicated and can be found in [Milyutin & Osmolovskii, 1998].

5. Conclusions

The stochastic differential equation was considered as the bio-economic model in the task of optimal control of the resource management. Several groups of the parameter estimation methods for the different types of the stochastic differential equation were proposed. First group of the estimation procedures is based on the maximum likelihood method, second one uses principles of Monte Carlo simulations and the last one employs the Pontryagin's type maximum principle. First and second group are very sensitive to the structural selection of the stochastic differential equation, not useful in the case of time-varying parameters or system of stochastic differential equations. However, they can be used for the "first iteration" in the time-varying case. The last method can be easily applied for the mentioned problems. Its scheme, formulated as the theorem, can be used if one is interested in the parametric identification of a system of the ordinary differential equations. In future, the numerical experiments are intended to take place in order to investigate the accuracy of the method.

6. References

Bastogne T.; Thomassin M. & Masse J. 2007. Selection and identification of physical parameters from passive observation. Application to a winding process. *Control Engineering Practice*, 15: 1051 - 1061.

Fan J.; Jiang J.; Zhang Ch. & Zhou Z. 2003. Time-dependent diffusion models for term struture dynamics. *Statistica Sinica*, 13: 965 - 992.

Hansen J.A. & Penland C. 2007. On stochastic parameter estimation using data assimilation. *Physica D*, 230: 88 - 98.

Hu Y.; Øksendal B. & Sulem A. 2003. Optimal consumption and portfolio in a Back Scholes market driven by fractional Brownian motion: Infinite dimensional analysis, *Quantum Probability and Related Topics*, 6 (4): 519 - 536.

Hurn A.S., Lindsay K.A., Martin V.L. 2003. On the efficacy of simulated maximum likelihood for estimating the parameters of stochastic differential equations. Journal of Time Series Analysis, 24: 45 - 63.

Jang M.-J.; Wang J.-S. & Liu Y.-C. 2003. Applying differential transformation method to parameter identification problems. *Applied Mathematics and Computation*, 139: 491 - 502.

Jazwinski A.H. (2007). Stochastic Processes and Filtering Theory. Dover Publiscations Inc.

McDonald A.D. & Sandal L.K. 1999. Estimating the parameters of stochastic differential equations using a criterion function based on the Kolmogorov-Smirnov statistic. *J. Statist. Comput. Simul.*, 64: 235 - 250.

Milyutin A.A. & Osmolovskii N.P. 1998. Calculus of Variations and Optimal Control, *Translations of Mathematical Monographs*, volume 180, American Mathematical Society, Providence.

Milyutin A.A.; Dmitruk A.V. & Osmolovskii N.P. 2004. Maximum Principle in Optimal Control, *Moscow State University, Moscow* (in Russian).

Shoji I. & T. Ozaki 1998. Estimation for the nonlinear stochastic differential equations by a local linearization method, Stochastic Analysis and Applications, 16: 733 - 752, 1998.

Shyryaev A.N. (1998). The Basis of Stochastic Financial Mathematics: Facts, Models (in Russian).
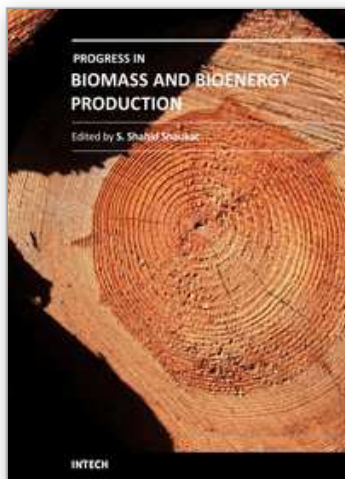
Progress in Biomass and Bioenergy Production
Edited by Dr. Shahid Shaukat

Alternative energy sources have become a hot topic in recent years. The supply of fossil fuel, which provides about 95 percent of total energy demand today, will eventually run out in a few decades. By contrast, biomass and biofuel have the potential to become one of the major global primary energy source along with other alternate energy sources in the years to come. A wide variety of biomass conversion options with different performance characteristics exists. The goal of this book is to provide the readers with current state of art about biomass and bioenergy production and some other environmental technologies such as Wastewater treatment, Biosorption and Bio-economics. Organized around providing recent methodology, current state of modelling and techniques of parameter estimation in gasification process are presented at length. As such, this volume can be used by undergraduate and graduate students as a reference book and by the researchers and environmental engineers for reviewing the current state of knowledge on biomass and bioenergy production, biosorption and wastewater treatment.

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