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Discrete Time Mixed LQR/H∞ Control Problems

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1. Introduction

This chapter will consider two discrete time mixed LQR/ H_∞ control problems. One is the discrete time state feedback mixed LQR/ H_∞ control problem, another is the non-fragile discrete time state feedback mixed LQR/ H_∞ control problem. Motivation for mixed LQR/ H_∞ control problem is to combine the LQR and suboptimal H_∞ controller design theories, and achieve simultaneously the performance of the two problems. As is well known, the performance measure in optimal LQR control theory is the quadratic performance index, defined in the time-domain as

$$J := \sum_{k=0}^{\infty}(x^T(k)Qx(k) + u^T(k)Ru(k)) \tag{1}$$

while the performance measure in H_∞ control theory is H_∞ norm, defined in the frequency-domain for a stable transfer matrix $T_{zw}(z)$ as

$$\|T_{zw}(z)\|_\infty := \sup_{w\in[0,2\pi]} \sigma_{\max}[T_{zw}(e^{jw})]$$

where, $Q \geq 0$, $R > 0$, $\sigma_{\max}[\bullet]$ denotes the largest singular value.

The linear discrete time system corresponding to the discrete time state feedback mixed LQR/ H_∞ control problem is

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k) \tag{2.a}$$

$$z(k) = C_1 x(k) + D_{12} u(k) \tag{2.b}$$

with state feedback of the form

$$u(k) = Kx(k) \tag{3}$$

where, $x(k) \in R^n$ is the state, $u(k) \in R^m$ is the control input, $w(k) \in R^q$ is the disturbance input that belongs to $L_2[0,\infty)$, $z(k) \in R^p$ is the controlled output. A, B_1, B_2, C_1 and D_{12} are known matrices of appropriate dimensions. Let $x(0) = x_0$.

The closed loop transfer matrix from the disturbance input w to the controlled output z is

$$T_{zw}(z) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & 0 \end{array}\right] := C_K(zI - A_K)^{-1}B_K$$

where, $A_K := A + B_2K$, $B_K := B_1$, $C_K := C_1 + D_{12}K$.

Recall that the discrete time state feedback optimal LQR control problem is to find an admissible controller that minimizes the quadratic performance index (1) subject to the systems (2) (3) with $w = 0$, while the discrete time state feedback H_∞ control problem is to find an admissible controller such that $\|T_{zw}(z)\|_\infty < \gamma$ subject to the systems (2)(3) for a given number $\gamma > 0$. While we combine the two problems for the systems (2)(3) with $w \in L_2[0,\infty)$, the quadratic performance index (1) is a function of the control input $u(k)$ and disturbance input $w(k)$ in the case of $x(0)$ being given and γ being fixed. Thus, it is not possible to pose a mixed LQR/ H_∞ control problem that is to find an admissible controller that achieves the minimization of quadratic performance index (1) subject to $\|T_{zw}(z)\|_\infty < \gamma$ for the systems (2)(3) with $w \in L_2[0,\infty)$ because the quadratic performance index (1) is an uncertain function depending on the uncertain disturbance input $w(k)$. In order to eliminate this difficulty, the design criteria of state feedback mixed LQR/ H_∞ control problem should be replaced by the design criteria

$$\sup_{w \in L_{2+}} \inf_K \{J\} \text{ subject to } \|T_{zw}(z)\|_\infty < \gamma$$

because for all $w \in L_2[0,\infty)$, the following inequality always exists

$$\inf_K \{J\} \le \sup_{w \in L_{2+}} \inf_K \{J\}$$

The stochastic problem corresponding to this problem is the combined LQG/ H_∞ control problem that was first presented by Bernstein & Haddad (1989). This problem is to find an admissible fixed order dynamic compensator that minimizes the expected cost function of the form

$$J = \lim_{t \to \infty} \mathrm{E}(x^T Q x + u^T R u) \text{ subject to } \|T_{zw}\|_\infty < \gamma .$$

Here, the disturbance input w of this problem is restricted to be white noise. Since the problem of Bernstein & Haddad (1989) involves merely a special case of fixing weighting matrices Q and R, it is considered as a mixed H_2 / H_∞ problem in special case. Doyle et al. (1989b) considered a related output feedback mixed H_2 / H_∞ problem (also see Doyle et al., 1994). The two approaches have been shown in Yeh et al. (1992) to be duals of one another in some sense. Also, various approaches for solving the mixed H_2 / H_∞ problem are presented (Rotea & Khargonekar , 1991; Khargonekar & Rotea, 1991; Zhou et al., 1994; Limebeer et al. 1994; Sznaier ,1994; Rotstein & Sznaier, 1998 ; Sznaier et al. , 2000) . However, no approach has involved the mixed LQR/ H_∞ control problem until the discrete time state feedback controller for solving this problem was presented by Xu (1996). Since then, several approaches to the mixed LQR / H_∞ control problems have been presented in Xu (2007, 2008).

The first goal of this chapter is to, based on the results of Xu (1996,2007), present the simple approach to discrete time state feedback mixed LQR / H_∞ control problem by combining the Lyapunov method for proving the discrete time optimal LQR control pro-blem with an

extension of the discrete time bounded real lemma, the argument of compl-etion of squares of Furuta & Phoojaruenchanachi (1990) and standard inverse matrix man-ipulation of Souza & Xie (1992).
On the other hand, unlike the discrete time state feedback mixed LQR / H_∞ control problem, state feedback corresponding to the non-fragile discrete time state feedback mixed LQR/ H_∞ control problem is a function of controller uncertainty $\Delta F(k)$, and is given by

$$u(k) = \hat{F}_\infty x(k)\,,\ \hat{F}_\infty = F_\infty + \Delta F(k) \tag{4}$$

where, $\Delta F(k)$ is the controller uncertainty.
The closed-loop transfer matrix from disturbance input w to the controlled output z and quadratic performance index for the closed-loop system (2) (4) is respectively

$$\hat{T}_{zw}(z) = \left[\begin{array}{c|c} A_{\hat{F}_\infty} & B_{\hat{F}_\infty} \\ \hline C_{\hat{F}_\infty} & 0 \end{array}\right] := C_{\hat{F}_\infty}(zI - A_{\hat{F}_\infty})^{-1} B_{\hat{F}_\infty}$$

and

$$\hat{J} := \sum_{k=0}^{\infty} \{\left\|Q^{1/2} x(k)\right\|^2 + \left\|R^{1/2} u(k)\right\|^2 - \gamma^2 \|w\|^2\}$$

where, $A_{\hat{F}_\infty} := A + B_2 \hat{F}_\infty$, $B_{\hat{F}_\infty} := B_1$, $C_{\hat{F}_\infty} := C_1 + D_{12}\hat{F}_\infty$, $\gamma > 0$ is a given number.

Note that the feedback matrix $\hat{F}_\infty$ of the considered closed-loop system is a function of the controller uncertainty $\Delta F(k)$, this results in that the quadratic performance index (1) is not only a function of the controller F_∞ and disturbance input $w(k)$ but also a function of the controller uncertainty $\Delta F(k)$ in the case of $x(0)$ being given and γ being fixed. We can easily know that the existence of disturbance input $w(k)$ and controller uncertainty $\Delta F(k)$ makes it impossible to find $\sup_{w \in L_{2+}} \inf_K \{J\}$, while the existence of controller uncertainty $\Delta F(k)$ also makes it difficult to find $\sup_{w \in L_{2+}} \{J\}$. In order to eliminate these difficulties, the design criteria of non-fragile discrete time state feedback mixed LQR/ H_∞ control problem should be replaced by the design criteria

$$\sup_{w \in L_{2+}} \{\hat{J}\} \text{ subject to } \|T_{zw}(z)\|_\infty < \gamma\,.$$

Motivation for non-fragile problem came from Keel & Bhattacharyya (1997). Keel & Bhattacharyya (1997) showed by examples that optimum and robust controllers, designed by using the H_2 , H_∞, l^1, and μ formulations, can produce extremely fragile controllers, in the sense that vanishingly small perturbations of the coefficients of the designed controller destabilize the closed-loop system; while the controller gain variations could not be avoided in most applications.This is because many factors, such as the limitations in available computer memory and word-length capabilities of digital processor and the A/D and D/A converters,result in the variation of the controller parameters in controller implementation. Also, the controller gain variations might come about because of external effects such as temperature changes.Thus, any controller must be insensitive to the above-mentioned contoller gain variation. The question arised from this is how to design a controller that is insensitive, or non-fragile to error/uncertainty in controller parameters for a given plant. This

problem is said to be a non-fragile control problem. Recently, the non-fragile controller approach has been used to a very large class of control problems (Famularo et al. 2000, Haddad et al. 2000, Yang et al 2000, Yang et al. 2001 and Xu 2007).
The second aim of this chapter is to, based on the results of Xu (2007), present a non-fragile controller approach to the discrete-time state feedback mixed LQR/ H_∞ control problem with controller uncertainty.
This chapter is organized as follows. In Section 2, we review several preliminary results, and present two extensions of the well known discrete time bounded real lamma. In Section 3, we define the discrete time state feedback mixed LQR/ H_∞ control problem. Based on this definition, we present the both Riccati equation approach and state space approach to the discrete time state feedback mixed LQR/ H_∞ control problem. In Section 4, we intro-duce the definition of non-fragile discrete time state feedback mixed LQR/ H_∞ control problem, give the design method of a non-fragile discrete time state feedback mixed LQR / H_∞ controller, and derive the necessary and sufficient conditions for the existence of this controller. In Section 5, we give two examples to illustrate the design procedures and their effectiveness, respectively. Section 6 gives some conclusions.
Throughout this chapter, A^T denotes the transpose of A, A^{-1} denotes the inverse of A, A^{-T} is the shorthand for $(A^{-1})^T$, $G^{\sim}(z)$ denotes the conjugate system of $G(z)$ and is the shorthand for $G^T(z^{-1})$, $L_2(-\infty,+\infty)$ denotes the time domain Lebesgue space, $L_2[0,+\infty)$ denotes the subspace of $L_2(-\infty,+\infty)$, $L_2(-\infty,0]$ denotes the subspace of $L_2(-\infty,+\infty)$, L_{2+} is the shorthand for $L_2[0,+\infty)$ and L_{2-} is the shorthand for $L_2(-\infty,0]$.

2. Preliminaries

This section reviews several preliminary results. First, we consider the discerete time Riccati equation and discrete time Riccati inequality, respectively

$$X = A^T X(I + RX)^{-1} A + Q \tag{5}$$

and

$$A^T X(I + RX)^{-1} A + Q - X < 0 \tag{6}$$

with $Q = Q^T \geq 0$ and $R = R^T > 0$.
We are particularly interested in solution s X of (5) and (6) such that $(I + RX)^{-1} A$ is stable. A symmetric matrix X is said to the stabilizing solution of discrete time Riccati equation (5) if it satisfies (5) and is such that $(I + RX)^{-1} A$ is stable. Moreover, for a sufficiently small constant $\delta > 0$, the discrete time Riccati inequality (6) can be rewritten as

$$X = A^T X(I + RX)^{-1} A + Q + \delta I \tag{7}$$

Based on the above relation, we can say that if a symmetric matrix X is a stabilizing solution to the discrete time Riccati equation (7), then it also is a stabilizing solution to the discrete time Riccati inequality (6). According to the concept of stabilizing solution of discrete time Riccati equation, we can define the stabilizing solution X to the discrete time Riccati inequality (6) as follow: if there exists a symmetric solution X to the discrete time Riccati inequality (6) such that $(I + RX)^{-1} A$ is stable, then it is said to be a stabilizing solution to the discrete time Riccati inequality (6) .

If A is invertible, the stabilizing solution to the discerete time Riccati equation (5) can be obtained through the following simplectic matrix

$$S := \begin{bmatrix} A + RA^{-T}Q & -RA^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix} \tag{8}$$

Assume that S has no eigenvalues on the unit circle, then it must have n eigenvalues in $|\lambda_i| < 1$ and n in $|\lambda_i| > 1$ ($i = 1,2,\cdots,n,n+1,\cdots,2n$). If n eigenvectors corresponding to n eigenvalues in $|\lambda_i| < 1$ of the simplectic matrix (8) is computed as

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix}$$

then a stabilizing solution to the discerete time Riccati equation (5) is given by

$$X = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}^{-1}$$

Secondly, we will introduce the well known discrete time bounded real lemma (see Zhou et al. , 1996; Iglesias & Glover, 1991; Souza & Xie, 1992) .

Lemma 2.1 (Discrete Time Bounded Real Lemma)

Suppose that $\gamma > 0$, $M(z) = \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] \in RH_\infty$, then the following two statements are equivalent:

i. $\|M(z)\|_\infty < \gamma$.
ii. There exists a stabilizing solution $X \geq 0$ ($X > 0$ if (C, A) is observable) to the discrete time Riccati equation

$$A^T XA - X + \gamma^{-2}(A^T XB + C^T D)U_1^{-1}(B^T XA + D^T C) + C^T C = 0$$

such that $U_1 = I - \gamma^{-2}(D^T D + B^T XB) > 0$.

In order to solve the two discrete time state feedback mixed LQR/ H_∞ control problems considered by this chapter, we introduce the following reference system

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k) \quad \hat{z}(k) = \begin{bmatrix} C_1 \\ \Omega^{\frac{1}{2}} \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} x(k) + \begin{bmatrix} D_{12} \\ \Omega^{\frac{1}{2}} \begin{bmatrix} 0 \\ I \end{bmatrix} \end{bmatrix} u(k) \tag{9}$$

where, $\Omega = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$ and $\hat{z}(k) = \begin{bmatrix} z(k) \\ z_0(k) \end{bmatrix}$.

The following lemma is an extension of the discrete time bounded real lemma.

Lemma 2.2 Given the system (2) under the influence of the state feedback (3), and suppose that $\gamma > 0$, $T_{zw}(z) \in RH_\infty$; then there exists an admissible controller K such that $\|T_{zw}(z)\|_\infty < \gamma$ if there exists a stabilizing solution $X_\infty \geq 0$ to the discrete time Riccati equation

$$A_K^T X_\infty A_K - X_\infty + \gamma^{-2} A_K^T X_\infty B_K U_1^{-1} B_K^T X_\infty A_K + C_K^T C_K + Q + K^T RK = 0 \tag{10}$$

such that $U_1 = I - \gamma^{-2} B_K^T X_\infty B_K > 0$.

Proof: Consider the reference system (9) under the influence of the state feedback (3), and define T_0 as

$$T_0(z) := \left[\begin{array}{c:c} A_K & B_K \\ \hdashline \Omega^{1/2}\begin{bmatrix} I \\ K \end{bmatrix} & 0 \end{array}\right]$$

then the closed-loop transfer matrix from disturbance input w to the controlled output $\hat{z}$ is $T_{\hat{z}w}(z) = \begin{bmatrix} T_{zw}(z) \\ T_0(z) \end{bmatrix}$. Note that $\gamma^2 I - T_{\hat{z}w}^{\sim} T_{\hat{z}w} > 0$ is equivalent to

$$\gamma^2 I - T_{zw}^{\sim} T_{zw} > T_0^{\sim} T_0 > 0 \text{ for all } w \in L_2[0,\infty) \;,$$

and $T_{zw}(z) \in RH_\infty$ is equivalent to $T_{\hat{z}w}(z) \in RH_\infty$, so $\|T_{\hat{z}w}(z)\|_\infty < \gamma$ implies $\|T_{zw}(z)\|_\infty < \gamma$ Hence, it follows from Lemma 2.1. Q.E.D.

To prove the result of non-fragile discrete time state feedback mixed LQR/ H_∞ control problem, we define the inequality

$$A_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} - X_\infty + \gamma^{-2} A_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} + C_{\hat{F}_\infty}^T C_{\hat{F}_\infty} + Q + \hat{F}_\infty^T R \hat{F}_\infty < 0 \tag{11}$$

where, $U_1 = I - \gamma^{-2} B_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} > 0$.

In terms of the inequality (11), we have the following lemma:

Lemma 2.3 Consider the system (2) under the influence of state feedback (4) with controler uncertainty, and suppose that $\gamma > 0$ is a given number, then there exists an admissible non-fragile controller F_∞ such that $\|T_{zw}\|_\infty < \gamma$ if for any admissible uncertainty $\Delta F(k)$, there exists a stabilizing solution $X_\infty \ge 0$ to the inequality (11) such that $U_1 = I - \gamma^{-2} B_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} > 0$.

Proof: Suppose that for any admissible uncertainty $\Delta F(k)$, there exists a stabilizing solution $X_\infty \ge 0$ to the inequality (11) such that $U_1 = I - \gamma^{-2} B_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} > 0$. This implies that the solution $X_\infty \ge 0$ is such that $A_{\hat{F}_\infty} + \gamma^{-2} B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty}$ is stable.

Let $A_{F_\infty} = A + B_2 F_\infty$ and $C_{F_\infty} = C_1 + D_{12} F_\infty$; then we can rewrite (11) as

$$\begin{aligned} & A_{F_\infty}^T X_\infty A_{F_\infty} - X_\infty + \gamma^{-2} A_{F_\infty}^T X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{F_\infty} + C_{F_\infty}^T C_{F_\infty} + Q \\ & + F_\infty^T R F_\infty - (A^T U_3 B_2 + F_\infty^T U_2) U_2^{-1} (B_2^T U_3 A + U_2 F_\infty) + \Delta N_F < 0 \end{aligned}$$

where, $U_2 = B_2^T U_3 B_2 + I + R$, $U_3 = \gamma^{-2} X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty + X_\infty$,
$\Delta N_F = (A^T U_3 B_2 + F_\infty^T U_2 + \Delta F^T(k) U_2) U_2^{-1} (B_2^T U_3 A + U_2 F_\infty + U_2 \Delta F(k))$.

Since $\Delta F(k)$ is an admissible norm-bounded time- varying uncertainty, there exists a time-varying uncertain number $\delta(k) > 0$ satisfying

$$\begin{aligned} & A_{F_\infty}^T X_\infty A_{F_\infty} - X_\infty + \gamma^{-2} A_{F_\infty}^T X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{F_\infty} + C_{F_\infty}^T C_{F_\infty} + Q + F_\infty^T R F_\infty \\ & - (A^T U_3 B_2 + F_\infty^T U_2) U_2^{-1} (B_2^T U_3 A + U_2 F_\infty) + \Delta N_F + \delta(k) I = 0 \end{aligned} \tag{12}$$

Note that $A_{\hat{F}_\infty} + \gamma^{-2} B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty}$ is stable for any admissible uncertainty $\Delta F(k)$. This implies that $A_{F_\infty} + \gamma^{-2} B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{F_\infty}$ is stable.
Hence, $(U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{F_\infty}, A_{F_\infty})$ is detectable. Then it follows from standard results on Lyapunov equations (see Lemma 2.7 a), Iglesias & Glover 1991) and the equation (12) that A_{F_∞} is stable. Thus, $A_{\hat{F}_\infty} = A_{F_\infty} + B_2 \Delta F(k)$ is stable for any admissible uncertainty $\Delta F(k)$.
Define $V(x(k)) := x^T(k) X_\infty x(k)$, where, x is the solution to the plant equations for a given input w, then it can be easily established that

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} \{-\Delta V(x(k)) + x^T(k+1) X_\infty x(k+1) - x^T(k) X_\infty x(k)\} \\ &= \sum_{k=0}^{\infty} \{-\Delta V(x(k)) - \|z\|^2 + \gamma^2 \|w\|^2 - \gamma^2 \left\| U_1^{1/2} (w - \gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x) \right\|^2 \\ &\quad + x^T (A_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} - X_\infty + \gamma^{-2} A_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} + C_{\hat{F}_\infty}^T C_{\hat{F}_\infty}) x\} \end{aligned}$$

Add the above zero equality to J to get

$$\begin{aligned} J &= \sum_{k=0}^{\infty} \{-\Delta V(x(k)) - \|z\|^2 + \gamma^2 \|w\|^2 - \gamma^2 \left\| U_1^{1/2} (w - \gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x) \right\|^2 \\ &\quad + x^T (A_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} - X_\infty + \gamma^{-2} A_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} + C_{\hat{F}_\infty}^T C_{\hat{F}_\infty} + Q + \hat{F}_\infty^T R \hat{F}_\infty) x\} \end{aligned}$$

Substituting (11) for the above formula,we get that for any $u(k)$ and $w(k)$ and $x(0) = 0$,

$$J < -\|z\|_2^2 + \gamma^2 \|w\|_2^2 - \gamma^2 \left\| U_1^{1/2} (w - \gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x) \right\|_2^2$$

Note that $\|z_0\|_2^2 = \sum_{k=0}^{\infty} \hat{x}^T(k) \Omega \hat{x}(k)$, and define that $r := w - \gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x$, we get

$$\|\hat{z}\|_2^2 - \gamma^2 \|w\|_2^2 < -\gamma^2 \left\| U_1^{1/2} r \right\|_2^2$$

Suppose that Γ is the operator with realization

$$\begin{aligned} x(k+1) &= (A + B_2 \hat{F}_\infty) x(k) + B_{\hat{F}_\infty} w(k) \\ r(k) &= -\gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x(k) + w(k) \end{aligned}$$

which maps w to r.
Since Γ^{-1} exists (and is given by $x(k+1) = (A + B_2 \hat{F}_\infty + \gamma^{-2} B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty}) x(k) + B_{\hat{F}_\infty} r(k)$, $w(k) = \gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x(k) + r(k)$), we can write

$$\|\hat{z}\|_2^2 - \gamma^2 \|w\|_2^2 < -\gamma^2 \left\| U_1^{1/2} r \right\|_2^2 = -\gamma^2 \|\Gamma w\|_2^2 \le \kappa \|w\|_2^2$$

for some positive κ .This implies that there exists an admissible non-fragile controller such that $\|T_{\hat{z}w}\|_\infty < \gamma$. Note that $\gamma^2 I - T_{\hat{z}w}^{\sim} T_{\hat{z}w} > 0$ is equivalent to

$$\gamma^2 I - T_{zw}^{\sim} T_{zw} > T_0^{\sim} T_0 > 0 \text{ for all } w \in L_2[0,\infty)$$

so $\|T_{\hat{z}w}\|_\infty < \gamma$ implies $\|T_{zw}\|_\infty < \gamma$, and we conclude that there exists an admissible non-fragile controller such that $\|T_{zw}\|_\infty < \gamma$. Q. E. D.

3. State Feedback

In this section, we will consider the discrete time state feedback mixed LQR/ H_∞ control problem. This problem is defined as follows: Given the linear discrete-time systems (2)(3) with $w \in L_2[0,\infty)$ and $x(0) = x_0$ and the quadratic performance index (1), for a given number $\gamma > 0$, determine an admissible controller K that achieves

$$\sup_{w \in L_{2+}} \inf_K \{J\} \text{ subject to } \|T_{zw}(z)\|_\infty < \gamma .$$

If this controller K exists, it is said to be a discrete time state feedback mixed LQR/ H_∞ controller.
Here, we will discuss the simplified versions of the problem defined in the above. In order to do this, the following assumptions are imposed on the system
Assumption 1 (C_1, A) is detectable.
Assumption 2 (A, B_2) is stabilizable.
Assumption 3 $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.
The solution to the problem defined in the above involves the discrete time Riccati equation

$$A^T X_\infty A - X_\infty - A^T X_\infty \hat{B}(\hat{B}^T X_\infty \hat{B} + \hat{R})^{-1} \hat{B}^T X_\infty A + C_1^T C_1 + Q = 0 \tag{13}$$

where, $\hat{B} = \begin{bmatrix} \gamma^{-1} B_1 & B_2 \end{bmatrix}$, $\hat{R} = \begin{bmatrix} -I & 0 \\ 0 & R+I \end{bmatrix}$. If A is invertible, the stabilizing solution to the discrete time Riccati equation (13) can be obtained through the following simplectic matrix

$$S_\infty := \begin{bmatrix} A + \hat{B}\hat{R}^{-1}\hat{B}^T A^{-T}(C_1^T C_1 + Q) & -\hat{B}\hat{R}^{-1}\hat{B}^T A^{-T} \\ -A^{-T}(C_1^T C_1 + Q) & A^{-T} \end{bmatrix}$$

In the following theorem, we provide the solution to discrete time state feedback mixed LQR/ H_∞ control problem.
Theorem 3.1 There exists a state feedback mixed LQR/ H_∞ controller if the discrete time Riccati equation (13) has a stabilizing solution $X_\infty \geq 0$ and $U_1 = I - \gamma^{-2} B_1^T X_\infty B_1 > 0$.
Moreover, this state feedback mixed LQR/ H_∞ controller is given by

$$K = -U_2^{-1} B_2^T U_3 A$$

where, $U_2 = R + I + B_2^T U_3 B_2$, and $U_3 = X_\infty + \gamma^{-2} X_\infty B_1 U_1^{-1} B_1^T X_\infty$.

In this case, the state feedback mixed LQR/ H_∞ controller will achieve

$$\sup_{w\in L_{2+}} \inf_K\{J\} = x_0^T(X_\infty + \gamma^{-2}X_w - X_z)x_0 \text{ subject to } \|T_{zw}\|_\infty < \gamma .$$

where, $\hat{A}_K = A_K + \gamma^{-2}B_K U_1^{-1}B_K^T X_\infty A_K$, $X_w = \sum_{k=0}^{\infty}\{(\hat{A}_K^k)^T A_K^T X_\infty B_K U_1^{-2} B_K^T X_\infty A_K \hat{A}_K^k\}$, and $X_z = \sum_{k=0}^{\infty}\{(\hat{A}_K^k)^T C_K^T C_K \hat{A}_K^k\}$.

Before proving Theorem 3.1, we will give the following lemma.

Lemma 3.1 Suppose that the discrete time Riccati equation (13) has a stabilizing solution $X_\infty \ge 0$ and $U_1 = I - \gamma^{-2}B_1^T X_\infty B_1 > 0$, and let $A_K = A + B_2K$ and $K = -U_2^{-1}B_2^T U_3 A$; then A_K is stable.

Proof: Suppose that the discrete time Riccati equation (13) has a stabilizing solution $X_\infty \ge 0$ and $U_1 = I - \gamma^{-2}B_1^T X_\infty B_1 > 0$. Observe that

$$\hat{B}^T X_\infty \hat{B} + \hat{R} = \begin{bmatrix} \gamma^{-1}B_1^T \\ B_2^T \end{bmatrix} X_\infty \begin{bmatrix} \gamma^{-1}B_1 & B_2 \end{bmatrix} + \begin{bmatrix} -I & 0 \\ 0 & R+I \end{bmatrix} = \begin{bmatrix} -U_1 & \gamma^{-1}B_1^T X_\infty B_2 \\ \gamma^{-1}B_2^T X_\infty B_1 & B_2^T X_\infty B_2 + R + I \end{bmatrix}$$

Also, note that $U_1 = I - \gamma^{-2}B_1^T X_\infty B_1 > 0$, $U_3 = X_\infty + \gamma^{-2}X_\infty B_1 U_1^{-1} B_1^T X_\infty$, and $U_2 = R + I + B_2^T U_3 B_2$; then it can be easily shown by using the similar standard matrix manipulations as in the proof of Theorem 3.1 in Souza & Xie (1992) that

$$(\hat{B}^T X_\infty \hat{B} + \hat{R})^{-1} = \begin{bmatrix} -U_1^{-1} + U_1^{-1}\hat{B}_1 U_2^{-1}\hat{B}_1^T U_1^{-1} & U_1^{-1}\hat{B}_1 U_2^{-1} \\ U_2^{-1}\hat{B}_1^T U_1^{-1} & U_2^{-1} \end{bmatrix}$$

where, $\hat{B}_1 = \gamma^{-1}B_1^T X_\infty B_2$.

Thus, we have

$$A^T X_\infty \hat{B}(\hat{B}^T X_\infty \hat{B} + \hat{R})^{-1}\hat{B}^T X_\infty A = -\gamma^{-2}A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A + A^T U_3 B_2 U_2^{-1} B_2^T U_3 A$$

Rearraging the discrete time Riccati equation (13), we get

$$\begin{aligned}
X_\infty &= A^T X_\infty A + \gamma^{-2}A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A - A^T U_3 B_2 U_2^{-1} B_2^T U_3 A + C_1^T C_1 + Q \\
&= A^T X_\infty A + \gamma^{-2}A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A + C_1^T C_1 + Q - A^T U_3 B_2 U_2^{-1} B_2^T (X_\infty + \gamma^{-2}X_\infty B_1 U_1^{-1} B_1^T X_\infty)A \\
&\quad - A^T (X_\infty + \gamma^{-2}X_\infty B_1 U_1^{-1} B_1^T X_\infty) B_2 U_2^{-1} B_2^T U_3 A \\
&\quad + A^T U_3 B_2 U_2^{-1}[R + I + B_2^T (X_\infty + \gamma^{-2}X_\infty B_1 U_1^{-1} B_1^T X_\infty) B_2] U_2^{-1} B_2^T U_3 A \\
&= (A^T X_\infty A - A^T U_3 B_2 U_2^{-1} B_2^T X_\infty A - A^T X_\infty B_2 U_2^{-1} B_2^T U_3 A + A^T U_3 B_2 U_2^{-1} B_2^T X_\infty B_2 U_2^{-1} B_2^T U_3 A) \\
&\quad + (C_1^T C_1 + A^T U_3 B_2 U_2^{-1} U_2^{-1} B_2^T U_3 A) + A^T U_3 B_2 U_2^{-1} R U_2^{-1} B_2^T U_3 A + Q \\
&\quad + (\gamma^{-2}A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A - \gamma^{-2}A^T U_3 B_2 U_2^{-1} B_2^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A \\
&\qquad - \gamma^{-2}A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty B_2 U_2^{-1} B_2^T U_3 A + \gamma^{-2}A^T U_3 B_2 U_2^{-1} B_2^T X_\infty B_1 U_1^{-1} B_1^T X_\infty B_2 U_2^{-1} B_2^T U_3 A) \\
&= (A - B_2 U_2^{-1} B_2^T U_3 A)^T X_\infty (A - B_2 U_2^{-1} B_2^T U_3 A) + (C_1 - D_{12} U_2^{-1} B_2^T U_3 A)^T (C_1 - D_{12} U_2^{-1} B_2^T U_3 A) \\
&\quad + K^T R K + Q + \gamma^{-2}(A - B_2 U_2^{-1} B_2^T U_3 A)^T X_\infty B_1 U_1^{-1} B_1^T X_\infty (A - B_2 U_2^{-1} B_2^T U_3 A)
\end{aligned}$$

that is,

$$A_K^T X_\infty A_K - X_\infty + \gamma^{-2} A_K^T X_\infty B_K U_1^{-1} B_K^T X_\infty A_K + C_K^T C_K + Q + K^T RK = 0 \tag{14}$$

Since the discrete time Riccati equation (13) has a stabilizing solution $X_\infty \geq 0$, the discrete time Riccati equation (14) also has a stabilizing solution $X_\infty \geq 0$. This implies that $\hat{A}_K = A_K + \gamma^{-2} B_K U_1^{-1} B_K^T X_\infty A_K$ is stable. Hence $(U_1^{-1} B_K^T X_\infty A_K, A_K)$ is detectable. Based on this, it follows from standard results on Lyapunov equations (see Lemma 2.7 a), Iglesias & Glover 1991) that A_K is stable.Q. E. D.
Proof of Theorem 3.1: Suppose that the discrete time Riccati equation (13) has a stabilizing solution $X_\infty \geq 0$ and $U_1 = I - \gamma^{-2} B_1^T X_\infty B_1 > 0$. Then, it follows from Lemma 3.1 that A_K is stable. This implies that $T_{zw}(z) \in RH_\infty$. By using the same standard matrix manipulations as in the proof of Lemma 3.1, we can rewrite the discrete time Riccati equation (13) as follows:

$$A^T X_\infty A - X_\infty + \gamma^{-2} A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A - A^T U_3 B_2 U_2^{-1} B_2^T U_3 A + C_1^T C_1 + Q = 0$$

or equivalently,

$$A_K^T X_\infty A_K - X_\infty + \gamma^{-2} A_K^T X_\infty B_K U_1^{-1} B_K^T X_\infty A_K + C_K^T C_K + Q + K^T RK = 0$$

Thus, it follows from Lemma 2.2 that $\|T_{zw}(z)\|_\infty < \gamma$.
Define $V(x(k)) = x^T(k) X_\infty x(k)$, where X_∞ is the solution to the discrete time Riccati equation (13), then taking the difference $\Delta V(x(k))$ and completing the squares we get

$$\begin{aligned}
\Delta V(x(k)) &= x^T(k+1) X_\infty x(k+1) - x^T(k) X_\infty x(k) \\
&= x^T(k)(A_K^T X_\infty A_K - X_\infty) x(k) + x^T(k) A_K^T X_\infty B_K w(k) \\
&\quad + w^T(k) B_K^T X_\infty A_K x(k) + w^T(k) B_K^T X_\infty B_K w(k) \\
&= -\|z\|^2 + \gamma^2 \|w\|^2 - \gamma^2 \left\| U_1^{1/2}(w - \gamma^{-2} U_1^{-1} B_K^T X_\infty A_K x) \right\|^2 \\
&\quad + x^T(A_K^T X_\infty A_K - X_\infty + \gamma^{-2} A_K^T X_\infty B_K U_1^{-1} B_K^T X_\infty A_K + C_K^T C_K) x
\end{aligned}$$

Based on the above, the cost function J can be rewritten as:

$$\begin{aligned}
J = \sum_{k=0}^{\infty} \hat{x}^T(k) \Omega \hat{x}(k) = \sum_{k=0}^{\infty} \{ -\Delta V(x(k)) - \|z\|^2 + \gamma^2 \|w\|^2 - \gamma^2 \left\| U_1^{1/2}(w - \gamma^{-2} U_1^{-1} B_K^T X_\infty A_K x) \right\|^2 \\
+ x^T(A_K^T X_\infty A_K - X_\infty + \gamma^{-2} A_K^T X_\infty B_K U_1^{-1} B_K^T X_\infty A_K + C_K^T C_K + Q + K^T RK) x \}
\end{aligned} \tag{15}$$

On the other hand, it follows from the similar argumrnts as in the proof of Theorem 3.1 in Furuta & Phoojaruenchanachai (1990) that

$$\begin{aligned}
& A_K^T X_\infty A_K - X_\infty + \gamma^{-2} A_K^T X_\infty B_K U_1^{-1} B_K^T X_\infty A_K + C_K^T C_K + Q + K^T RK \\
&= A^T X_\infty A - X_\infty + \gamma^{-2} A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A - A^T U_3 B_2 U_2^{-1} B_2^T U_3 A + C_1^T C_1 + Q \\
&\qquad + (K + U_2^{-1} B_2^T U_3 A)^T U_2 (K + U_2^{-1} B_2^T U_3 A)
\end{aligned}$$

At the same time note that

$$-\gamma^{-2}A^TX_\infty B_1U_1^{-1}B_1^TX_\infty A+A^TU_3B_2U_2^{-1}B_2^TU_3A$$
$$=A^TX_\infty\hat{B}\begin{bmatrix}-U_1^{-1}+U_1^{-1}\hat{B}_1U_2^{-1}\hat{B}_1^TU_1^{-1} & U_1^{-1}\hat{B}_1U_2^{-1}\\ U_2^{-1}\hat{B}_1^TU_1^{-1} & U_2^{-1}\end{bmatrix}\hat{B}^TX_\infty A$$
$$=A^TX_\infty\hat{B}(\hat{B}^TX_\infty\hat{B}+\hat{R})^{-1}\hat{B}^TX_\infty A$$

We have

$$\begin{aligned}&A_K^TX_\infty A_K-X_\infty+\gamma^{-2}A_K^TX_\infty B_KU_1^{-1}B_K^TX_\infty A_K+C_K^TC_K+Q+K^TRK\\&=A^TX_\infty A-X_\infty-A^TX_\infty\hat{B}(\hat{B}^TX_\infty\hat{B}+\hat{R})^{-1}\hat{B}^TX_\infty A+C_1^TC_1+Q\\&\quad+(K+U_2^{-1}B_2^TU_3A)^TU_2(K+U_2^{-1}B_2^TU_3A)\end{aligned}$$

Also, noting that the discrete time Riccati equation (13) and substituting the above equality for (15), we get

$$\begin{aligned}J=\sum_{k=0}^{\infty}\hat{x}^T(k)\Omega\hat{x}(k)=\sum_{k=0}^{\infty}\{&-\Delta V(x(k))-\|z\|^2+\gamma^2\|w\|^2-\gamma^2\left\|U_1^{1/2}(w-\gamma^{-2}U_1^{-1}B_K^TX_\infty A_Kx)\right\|^2\\&+\left\|U_2^{1/2}(K+U_2^{-1}B_2^TU_3A)x\right\|^2\}\end{aligned}\tag{16}$$

Based on the above, it is clear that if $K=-U_2^{-1}B_2^TU_3A$, then we get

$$\inf_K\{J\}=x_0^TX_\infty x_0-\|z\|_2^2+\gamma^2\|w\|_2^2-\gamma^2\left\|U_1^{1/2}(w-\gamma^{-2}U_1^{-1}B_K^TX_\infty A_Kx)\right\|_2^2\tag{17}$$

By letting $w(k)=\gamma^{-2}U_1^{-1}B_K^TX_\infty A_Kx(k)$ for all $k\geq 0$, we get that $x(k)=\hat{A}_K^kx_0$ with $\hat{A}_K$ which belongs to $L_2[0,+\infty)$ since $\hat{A}_K=A-\hat{B}(\hat{B}^TX_\infty\hat{B}+\hat{R})^{-1}\hat{B}^TX_\infty A$ is stable. Also, we have

$$\|w(k)\|_2^2=\gamma^{-4}x_0^TX_wx_0,\ \|z(k)\|_2^2=x_0^TX_zx_0$$

Then it follows from (17) that

$$\sup_{w\in L_{2+}}\inf_K\{J\}=x_0^T(X_\infty+\gamma^{-2}X_w-X_z)x_0$$

Thus we conclude that there exists an admissible state feedback controller such that

$$\sup_{w\in L_{2+}}\inf_K\{J\}=x_0^T(X_\infty+\gamma^{-2}X_w-X_z)x_0\ \text{subject to}\ \|T_{zw}\|_\infty<\gamma \qquad \text{Q.E.D.}$$

4. Non-fragile controller

In this section, we will consider the non-fragile discrete-time state feedback mixed LQR/ H_∞ control problem with controller uncertainty. This problem is defined as follows: Consider the system (2) (4) satisfying Assumption 1-3 with $w\in L_2[0,\infty)$ and $x(0)=x_0$, for a given number $\gamma>0$ and any admissible controller uncertainty, determine an admissible non-fragile controller F_∞ such that

$$\sup_{w \in L_{2+}} \{\hat{J}\} \text{ subject to } \|T_{zw}(z)\|_\infty < \gamma .$$

where, the controller uncertainty $\Delta F(k)$ considered here is assumed to be of the following structure:

$$\Delta F(k) = H_K F(k) E_K$$

where, H_K and E_K are known matrices of appropriate dimensions. $F(k)$ is an uncertain matrix satisfying

$$F^T(k) F(k) \le I$$

with the elements of $F(k)$ being Lebesgue measurable.
If this controller exists, it is said to be a non-fragile discrete time state feedback mixed LQR/ H_∞ controller.
In order to solve the problem defined in the above, we first connect the its design criteria with the inequality (11).
Lemma 4.1 Suppose that $\gamma > 0$, then there exists an admissible non-fragile controller F_∞ that achieves

$$\sup_{w \in L_{2+}} \{\hat{J}\} = x_0^T X_\infty x_0 \text{ subject to } \|T_{zw}\|_\infty < \gamma$$

if for any admissible uncertainty $\Delta F(k)$, there exists a stabilizing solution $X_\infty \ge 0$ to the inequality (11) such that $U_1 = I - \gamma^{-2} B_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} > 0$.
Proof: Suppose that for any admissible uncertainty $\Delta F(k)$, there exists a stabilizing solution $X_\infty \ge 0$ to the inequality (11) such that $U_1 = I - \gamma^{-2} B_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} > 0$. This implies that the solution $X_\infty \ge 0$ is such that $A_{\hat{F}_\infty} + \gamma^{-2} B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty}$ is stable. Then it follows from Lemma 2.3 that $\|T_{zw}\|_\infty < \gamma$. Using the same argument as in the proof of Lemma 2.3, we get that $A_{\hat{F}_\infty}$ is stable and J can be rewritten as follows:

$$\begin{aligned} J = \sum_{k=0}^{\infty} \{ & -\Delta V(x(k)) - \|z\|^2 + \gamma^2 \|w\|^2 - \gamma^2 \left\| U_1^{1/2} (w - \gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x) \right\|^2 \\ & + x^T (A_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} - X_\infty + \gamma^{-2} A_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} + C_{\hat{F}_\infty}^T C_{\hat{F}_\infty} + Q + \hat{F}_\infty^T R \hat{F}_\infty) x \} \end{aligned} \tag{18}$$

Substituting (11) for (18) to get

$$J < x_0^T X_\infty x_0 - \|z\|_2^2 + \gamma^2 \|w\|_2^2 - \gamma^2 \left\| U_1^{1/2} (w - \gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x) \right\|_2^2 \tag{19a}$$

or

$$\hat{J} < x_0^T X_\infty x_0 - \|z\|_2^2 - \gamma^2 \left\| U_1^{1/2} (w - \gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x) \right\|_2^2 \tag{19b}$$

By letting $w = \gamma^{-2} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x$ for all $k \ge 0$, we get that $x(k) = \hat{A}_{\hat{F}_\infty}^k x_0$ with $\hat{A}_{\hat{F}_\infty} = A_{\hat{F}_\infty} + \gamma^{-2} B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty}$ which belongs to $L_2[0,+\infty)$ since $\hat{A}_{\hat{F}_\infty}$ is stable. It follows from (19b) that $\sup\{\hat{J}\}_{w\in L_{2+}} = x_0^T X_\infty x_0$. Thus, we conclude that there exists an admissible non-fragile controller such that $\sup\{\hat{J}\}_{w\in L_{2+}} = x_0^T X_\infty x_0$ subject to $\|T_{zw}\|_\infty < \gamma$. Q. E. D.

Remark 4.1 In the proof of Lemma 4.1, we let $w = \gamma^{-2}U_1^{-1}B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} x$ for all $k \ge 0$ to get that $x(k) = \hat{A}_{\hat{F}_\infty}^k x_0$ with $\hat{A}_{\hat{F}_\infty} = A_{\hat{F}_\infty} + \gamma^{-2}B_{\hat{F}_\infty}U_1^{-1}B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty}$ which belongs to $L_2[0,+\infty)$ since $\hat{A}_{\hat{F}_\infty}$ is stable. Also, we have

$$\|w\|_2^2 = \gamma^{-4}x_0^T X_w x_0\,,\ \|z\|_2^2 = x_0^T X_z x_0\,.$$

Then it follows from (19a) that

$$J < x_0^T(X_\infty + \gamma^{-2}X_w - X_z)x_0 \tag{20}$$

where, $X_w = \sum_{k=0}^{\infty}\{(\hat{A}_{\hat{F}_\infty}^k)^T A_{\hat{F}_\infty}^T X_\infty B_1 U_1^{-2} B_1^T X_\infty A_{\hat{F}_\infty}\hat{A}_{\hat{F}_\infty}^k\}$, and $X_z = \sum_{k=0}^{\infty}\{(\hat{A}_{\hat{F}_\infty}^k)^T C_{\hat{F}_\infty}^T C_{\hat{F}_\infty}\hat{A}_{\hat{F}_\infty}^k\}$.

Note that $\hat{A}_{\hat{F}_\infty}$ depends on the controller uncertainty $\Delta F(k)$, thus it is difficult to find an upper bound of either of X_w and X_z. This implies that the existence of controller uncertainty $\Delta F(k)$ makes it difficult to find $\sup_{w\in L_{2+}}\{J\}$ by using (20). Thus, it is clear that the existence of the controller uncertainty makes the performance of the designed system become bad.

In order to give necessary and sufficient conditions for the existence of an admissible non-fragile controller for solving the non-fragile discrete-time state feedback mixed LQR/ H_∞ control problem, we define the following parameter-dependent discrete time Riccati equation:

$$A^T X_\infty A - X_\infty - A^T X_\infty \hat{B}(\hat{B}^T X_\infty \hat{B} + \hat{R})^{-1}\hat{B}^T X_\infty A + \rho^2 E_K^T E_K + C_1^T C_1 + Q_\delta = 0 \tag{21}$$

where, $\hat{B} = \begin{bmatrix}\gamma^{-1}B_1 & B_2\end{bmatrix}$, $\hat{R} = \begin{bmatrix}-I & 0\\ 0 & I+R\end{bmatrix}$, $Q_\delta = Q + \delta I$ with $\delta > 0$ being a sufficiently small constant, ρ is a given number satisfying $\rho^2 I - H_K^T U_2 H_K > 0$, $U_1 = I - \gamma^{-2}B_1^T X_\infty B_1 > 0$, $U_2 = B_2^T U_3 B_2 + I + R$ and $U_3 = X_\infty + \gamma^{-2}X_\infty B_1 U_1^{-1} B_1^T X_\infty$. If A is invertible, the parameter-dependent discrete time Riccati equation (21) can be solved by using the following symplectic matrix

$$\hat{S}_\infty := \begin{bmatrix} A + \hat{B}\hat{R}^{-1}\hat{B}^T A^{-T}(\rho^2 E_K^T E_K + C_1^T C_1 + Q_\delta) & -\hat{B}\hat{R}^{-1}\hat{B}^T A^{-T} \\ -A^{-T}(\rho^2 E_K^T E_K + C_1^T C_1 + Q_\delta) & A^{-T}\end{bmatrix}$$

The following theorem gives the solution to non-fragile discrete time state feedback mixed LQR/ H_∞ control problem.

Theorem 4.1 There exists a non-fragile discrete time state feedback mixed LQR/ H_∞ controller iff for a given number ρ and a sufficiently small constant $\delta > 0$, there exists a stabilizing solution $X_\infty \ge 0$ to the parameter-dependent discrete time Riccati equation (21) such that $U_1 = I - \gamma^{-2}B_1^T X_\infty B_1 > 0$ and $\rho^2 I - H_K^T U_2 H_K > 0$.

Moreover, this non-fragile discrete time state feedback mixed LQR/ H_∞ controller is

$$F_\infty = -U_2^{-1} B_2^T U_3 A$$

and achieves $\sup\{\hat{J}\}_{w\in L_{2+}} = x_0^T X_\infty x_0$ subject to $\|T_{zw}\|_\infty < \gamma$.
Proof: *Sufficiency*: Suppose that for a given number ρ and a sufficiently small constant $\delta > 0$, there exists a stabilizing solution $X_\infty \geq 0$ to the parameter-dependent Riccati equation (21) such that $U_1 = I - \gamma^{-2} B_1^T X_\infty B_1 > 0$ and $\rho^2 I - H_K^T U_2 H_K > 0$. This implies that the solution $X_\infty \geq 0$ is such that $A - \hat{B}(\hat{B}^T X_\infty \hat{B} + \hat{R})^{-1} \hat{B}^T X_\infty A$ is stable. Define respectively the state matrix and controlled output matrix of closed-loop system

$$A_{\hat{F}_\infty} = A + B_2(-U_2^{-1} B_2^T U_3 A + H_K F(k) E_K)$$
$$C_{\hat{F}_\infty} = C_1 + D_{12}(-U_2^{-1} B_2^T U_3 A + H_K F(k) E_K)$$

and let $A_{F_\infty} = A - B_2 U_2^{-1} B_2^T U_3 A$ and $\overline{F}_\infty = -U_2^{-1} B_2^T U_3 A + H_K F(k) E_K$, then it follows from the square completion that

$$\begin{aligned} & A_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} - X_\infty + \gamma^{-2} A_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} + C_{\hat{F}_\infty}^T C_{\hat{F}_\infty} + Q + \overline{F}_\infty^T R \overline{F}_\infty \\ &= A^T X_\infty A - X_\infty + \gamma^{-2} A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A + C_1^T C_1 + Q + \overline{F}_\infty^T B_2^T U_3 A + A^T U_3 B_2 \overline{F}_\infty + \overline{F}_\infty^T U_2 \overline{F}_\infty \\ &= A^T X_\infty A - X_\infty + \gamma^{-2} A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A + C_1^T C_1 + Q - A^T U_3 B_2 U_2^{-1} B_2^T U_3 A + \Delta\overline{N} \end{aligned} \tag{22}$$

where, $\Delta\overline{N} = E_K^T F^T(k) H_K^T U_2 H_K F(k) E_K$.
Noting that $\rho^2 I - H_K^T U_2 H_K > 0$, we have

$$\Delta\overline{N} = -E_K^T F^T(k)(\rho^2 I - H_K^T U_2 H_K) F(k) E_K + \rho^2 E_K^T F^T(k) F(k) E_K \leq \rho^2 E_K^T E_K \tag{23}$$

Considering (22) and (23) to get

$$\begin{aligned} & A_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} - X_\infty + \gamma^{-2} A_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} + C_{\hat{F}_\infty}^T C_{\hat{F}_\infty} + Q + \overline{F}_\infty^T R \overline{F}_\infty \\ & \leq A^T X_\infty A - X_\infty + \gamma^{-2} A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A + C_1^T C_1 + Q + \rho^2 E_K^T E_K - A^T U_3 B_2 U_2^{-1} B_2^T U_3 A \end{aligned} \tag{24}$$

Also, it can be easily shown by using the similar standard matrix manipulations as in the proof of Theorem 3.1 in Souza & Xie (1992) that

$$A^T X_\infty \hat{B}(\hat{B}^T X_\infty \hat{B} + \hat{R})^{-1} \hat{B}^T X_\infty A = -\gamma^{-2} A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A + A^T U_3 B_2 U_2^{-1} B_2^T U_3 A$$

This implies that (21) can be rewritten as

$$A^T X_\infty A - X_\infty + \gamma^{-2} A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A + C_1^T C_1 + Q_\delta - A^T U_3 B_2 U_2^{-1} B_2^T U_3 A + \rho^2 E_K^T E_K = 0 \tag{25}$$

Thus, it follows from (24) and (25) that there exists a non-negative-definite solution to the inequality

$$A_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} - X_\infty + \gamma^{-2} A_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty} + C_{\hat{F}_\infty}^T C_{\hat{F}_\infty} + Q + \overline{F}_\infty^T R \overline{F}_\infty < 0$$

Note that $A-\hat{B}(\hat{B}^T X_\infty \hat{B}+\hat{R})^{-1}\hat{B}^T X_\infty A = A_{F_\infty} + \gamma^{-2} B_1 U_1^{-1} B_1^T X_\infty A_{F_\infty}$ is stable and $\Delta F(k)$ is an admissible uncertainty, we get that $A_{\hat{F}_\infty} + \gamma^{-2} B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty}$ is stable. By Lemma 4.1, there exists a non- fragile discrete time state feedback mixed LQR/ H_∞ controller.

Necessity: Suppose that there exists a non-fragile discrete time state feedback mixed LQR/ H_∞ controller. By Lemma 4.1, there exists a stabilizing solution $X_\infty \ge 0$ to the inequality (11) such that $U_1 = I - \gamma^{-2} B_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} > 0$, i.e., there exists a symmetric non-negative-definite solution X_∞ to the inequality (11) such that $A_{\hat{F}_\infty} + \gamma^{-2} B_{\hat{F}_\infty} U_1^{-1} B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty}$ is stable and $U_1 = I - \gamma^{-2} B_{\hat{F}_\infty}^T X_\infty B_{\hat{F}_\infty} > 0$ for any admissible uncertainty $\Delta F(k)$.

Rewriting (11) to get

$$\begin{aligned}&A_{F_\infty}^T X_\infty A_{F_\infty} - X_\infty + \gamma^{-2} A_{F_\infty}^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A_{F_\infty} + C_{F_\infty}^T C_{F_\infty} + Q + F_\infty^T R F_\infty + \Delta\hat{N} < 0 \\ &\Delta\hat{N} = (A^T U_3 B_2 + F_\infty^T U_2)\Delta F(k) + \Delta F^T(k)(B_2^T U_3 A + U_2 F_\infty) + \Delta F^T(k) U_2 \Delta F(k)\end{aligned} \tag{26}$$

Note that $\rho^2 I - H_K^T U_2 H_K > 0$ and

$$\begin{aligned}\Delta\hat{N} &= \rho^2 E_K^T F^T(k) F(k) E_K + (A^T U_3 B_2 + F_\infty^T U_2) H_K (\rho^2 I - H_K^T U_2 H_K)^{-1} H_K^T \\ &\quad \times (B_2^T U_3 A + U_2 F_\infty) - ((A^T U_3 B_2 + F_\infty^T U_2) H_K (\rho^2 I - H_K^T U_2 H_K)^{-1} - E_K^T F^T(k)) \\ &\quad \times (\rho^2 I - H_K^T U_2 H_K)((\rho^2 I - H_K^T U_2 H_K)^{-1} H_K^T (B_2^T U_3 A + U_2 F_\infty) - F(k) E_K) \\ &\le \rho^2 E_K^T E_K + (A^T U_3 B_2 + F_\infty^T U_2) H_K (\rho^2 I - H_K^T U_2 H_K)^{-1} H_K^T (B_2^T U_3 A + U_2 F_\infty)\end{aligned} \tag{27}$$

It follows from (26) and (27) that

$$\begin{aligned}&A_{F_\infty}^T X_\infty A_{F_\infty} - X_\infty + \gamma^{-2} A_{F_\infty}^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A_{F_\infty} + C_{F_\infty}^T C_{F_\infty} + Q + F_\infty^T R F_\infty + \rho^2 E_K^T E_K \\ &+(A^T U_3 B_2 + F_\infty^T U_2) H_K (\rho^2 I - H_K^T U_2 H_K)^{-1} H_K^T (B_2^T U_3 A + U_2 F_\infty) < 0\end{aligned} \tag{28}$$

Using the argument of completion of squares as in the proof of Theorem 3.1 in Furuta & Phoojaruenchanachai (1990), we get from (28) that $F_\infty = -U_2^{-1} B_2^T U_3 A$, where X_∞ is a symmetric non- negative-definite solution to the inequality

$$A^T X_\infty A - X_\infty + \gamma^{-2} A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A + C_1^T C_1 + Q - A^T U_3 B_2 U_2^{-1} B_2^T U_3 A + \rho^2 E_K^T E_K < 0$$

or equivalently, X_∞ is a symmetric non-negative-definite solution to the parameter-dependent discrete time Riccati equation

$$A^T X_\infty A - X_\infty + \gamma^{-2} A^T X_\infty B_1 U_1^{-1} B_1^T X_\infty A + C_1^T C_1 + Q_\delta - A^T U_3 B_2 U_2^{-1} B_2^T U_3 A + \rho^2 E_K^T E_K = 0 \tag{29}$$

Also, we can rewrite that Riccati equation (29) can be rewritten as

$$A^T X_\infty A - X_\infty - A^T X_\infty \hat{B}(\hat{B}^T X_\infty \hat{B} + \hat{R})^{-1} \hat{B}^T X_\infty A + \rho^2 E_K^T E_K + C_1^T C_1 + Q_\delta = 0 \tag{30}$$

by using the similar standard matrix manipulations as in the proof of Theorem 3.1 in Souza & Xie (1992). Note that $A-\hat{B}(\hat{B}^T X_\infty \hat{B}+\hat{R})^{-1}\hat{B}^T X_\infty A = A_{F_\infty}+\gamma^{-2}B_1U_1^{-1}B_1^T X_\infty A_{F_\infty}$ and $\Delta F(k)$ is an admissible uncertainty, the assumption that $A_{\hat{F}_\infty}+\gamma^{-2}B_{\hat{F}_\infty}U_1^{-1}B_{\hat{F}_\infty}^T X_\infty A_{\hat{F}_\infty}$ is stable implies that $A-\hat{B}(\hat{B}^T X_\infty \hat{B}+\hat{R})^{-1}\hat{B}^T X_\infty A$ is stable Thus, we conclude that for a given number ρ and a sufficiently small number $\delta > 0$, the parameter-dependent discrete time Riccati equation (30) has a stabilizing solution X_∞ and $U_1 = I-\gamma^{-2}B_1^T X_\infty B_1 > 0$ and $\rho^2 I - H_K^T U_2 H_K > 0$. Q. E. D.

5. Numerical examples

In this section, we present two examples to illustrate the design method given by Section 3 and 4, respectively.

Example 1 Consider the following discrete-time system in Peres and Geromel (1993)

$$\begin{aligned} x(k+1) &= Ax(k)+B_1w(k)+B_2u(k) \\ z(k) &= C_1x(k)+D_{12}u(k) \end{aligned}$$

where,

$$A=\begin{bmatrix}0.2113 & 0.0087 & 0.4524\\ 0.0824 & 0.8096 & 0.8075\\ 0.7599 & 0.8474 & 0.4832\end{bmatrix},\ B_2=\begin{bmatrix}0.6135 & 0.6538\\ 0.2749 & 0.4899\\ 0.8807 & 0.7741\end{bmatrix},$$

$$C_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix},\ D_{12}=\begin{bmatrix}0&0\\0&0\\0&0\\1&0\\0&1\end{bmatrix} \text{ and } B_1=I.$$

In this example, we will design the above system under the influence of state feedback of the form (3) by using the discrete-times state feedback mixed LQR/ H_∞ control method displayed in Theorem 3.1. All results will be computed by using MATLAB. The above system is stabilizable and observable, and satisfies Assumption 3, and the eigenvalues of matrix A are $p_1 = 1.6133$, $p_2 = 0.3827$, $p_3 = -0.4919$;thus it is open-loop unstable.

Let $\gamma = 2.89$, $R=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $Q=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, we solve the discrete-time Riccati equation (13) to get

$$X_\infty=\begin{bmatrix}2.9683 & 1.1296 & 0.1359\\ 1.1296 & 6.0983 & 2.4073\\ 0.1359 & 2.4073 & 4.4882\end{bmatrix}>0,$$

$$U_1 = I - \gamma^{-2} B_1^T X_\infty B_1 = \begin{bmatrix} 0.6446 & -0.1352 & -0.0163 \\ -0.1352 & 0.2698 & -0.2882 \\ -0.0163 & -0.2882 & 0.4626 \end{bmatrix} > 0 .$$

Thus the discrete-time state feedback mixed LQR/ H_∞ controller is

$$K = \begin{bmatrix} -0.3640 & -0.5138 & -0.3715 \\ -0.2363 & -0.7176 & -0.7217 \end{bmatrix}.$$

Example 2 Consider the following discrete-time system in Peres and Geromel (1993)

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k) \\ z(k) &= C_1 x(k) + D_{12} u(k) \end{aligned}$$

under the influences of state feedback with controller unceratinty of the form (4), where, A, B_1, B_2, C_1 and D_{12} are the same as ones in Example 1; the controller uncertainty $\Delta F(k)$ satisfies

$$\Delta F(k) = E_K F(k) E_K \; , \; F^T(k) F(k) \le I$$

where, $E_K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $H_K = \begin{bmatrix} 0.0100 & 0 & 0 \\ 0 & 0.0100 & 0 \end{bmatrix}$.

In this example, we illustrate the proposed method by Theorem 4.1 by using MATLAB. As stated in example 1, the system is stabilizable and observable, and satisfies Assumption 3, and is open-loop unstable.

Let $\gamma = 8.27$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\rho = 3.7800$, and $\delta = 0.0010$, then we solve the parameter-dependent discrete-time Riccati equation (21) to get

$$X_\infty = \begin{bmatrix} 18.5238 & 3.8295 & 0.1664 \\ 3.8295 & 51.3212 & 23.3226 \\ 0.1664 & 23.3226 & 22.7354 \end{bmatrix} > 0 ,$$

$$U_1 = I - \gamma^2 B_1^T X_\infty B_1 = \begin{bmatrix} 0.7292 & -0.0560 & -0.0024 \\ -0.0560 & 0.2496 & -0.3410 \\ -0.0024 & -0.3410 & 0.6676 \end{bmatrix} > 0 , \; U_2 = \begin{bmatrix} 609.6441 & 723.0571 \\ 723.0571 & 863.5683 \end{bmatrix},$$

$$\rho^2 I - H_K^T U_2 H_K = \begin{bmatrix} 14.2274 & -0.0723 & 0 \\ -0.0723 & 14.2020 & 0 \\ 0 & 0 & 14.2884 \end{bmatrix} > 0 .$$

Based on this, the non-fragile discrete-time state feedback mixed LQR/ H_∞ controller is

$$F_\infty = \begin{bmatrix} -0.4453 & -0.1789 & -0.0682 \\ -0.1613 & -1.1458 & -1.0756 \end{bmatrix}$$

6. Conclusion

In this chapter, we first study the discrete time state feedback mixed LQR/ H_∞ control problem. In order to solve this problem, we present an extension of the discrete time bounded real lemma. In terms of the stabilizing solution to a discrete time Riccati equation, we derive the simple approach to discrete time state feedback mixed LQR/ H_∞ control problem by combining the Lyapunov method for proving the discrete time optimal LQR control problem with the above extension of the discrete time bounded real lemma, the argument of completion of squares of Furuta & Phoojaruenchanachi (1990) and standard inverse matrix manipulation of Souza & Xie (1992).A related problem is the standard H_∞ control problem (Doyle et al., 1989a; Iglesias & Glover, 1991; Furuta & Phoojaruenchanachai, 1990; Souza & Xie, 1992; Zhou et al. 1996), another related problem is the H_∞ optimal control problem arisen from Basar & Bernhard (1991). The relations among the two related problem and mixed LQR/ H_∞ control problem can be clearly explained by based on the discrete time reference system (9)(3). The standard H_∞ control problem is to find an admissible controller K such that the H_∞-norm of closed-loop transfer matrix from disturbance input w to the controlled output z is less than a given number $\gamma > 0$ while the H_∞ optimal control roblem arisen from Basar & Bernhard (1991) is to find an admissible controller such that the H_∞-norm of closed-loop transfer matrix from disturbance input w to the controlled output z_0 is less than a given number $\gamma > 0$ for the discre time reference system (9)(3). Since the latter is equivalent to the problem that is to find an admissible controller K such that $\sup_{w \in L_{2+}} \inf_K \{\hat{J}\}$, we may recognize that the mixed LQR/ H_∞ control problem is a combination of the standard H_∞ control problem and H_∞ optimal control problem arisen from Basar & Bernhard (1991). The second problem considered by this chapter is the non-fragile discrete-time state feedback mixed LQR/ H_∞ control problem with controller uncertainty. This problem is to extend the results of discrete-time state feedback mixed LQR/ H_∞ control problem to the system (2)(4) with controller uncertainty. In terms of the stabilizing solution to a parameter-dependent discrete time Riccati equation, we give a design method of non-fragile discrete-time state feedback mixed LQR/ H_∞ controller, and derive necessary and sufficient conditions for the existence of this non-fragile controller.

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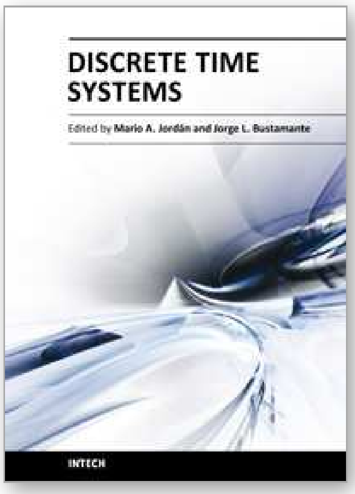

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Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.