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New Smoothers for Discrete-time Linear Stochastic Systems with Unknown Disturbances

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1. Introduction

We consider discrete-time linear stochastic systems with unknown inputs (or disturbances) and propose recursive algorithms for estimating states of these systems. If mathematical models derived by engineers are very accurate representations of real systems, we do not have to consider systems with unknown inputs. However, in practice, the models derived by engineers often contain modelling errors which greatly increase state estimation errors as if the models have unknown disturbances.

The most frequently discussed problem on state estimation is the optimal filtering problem which investigates the optimal estimate of state x_t at time t or x_{t+1} at time $t+1$ with minimum variance based on the observation $\mathbf{Y}_t$ of the outputs $\{y_0, y_1, \cdots, y_t\}$, i.e., $\mathbf{Y}_t = \sigma\{y_s, s = 0, 1, \cdots, t\}$ (the smallest σ-field generated by $\{y_0, y_1, \cdots, y_t\}$ (see e.g., Katayama (2000), Chapter 4)). It is well known that the standard Kalman filter is the optimal linear filter in the sense that it minimizes the mean-square error in an appropriate class of linear filters (see e.g., Kailath (1974), Kailath (1976), Kalman (1960), Kalman (1963) and Katayama (2000)). But we note that the Kalman filter can work well only if we have accurate mathematical modelling of the monitored systems.

In order to develop reliable filtering algorithms which are robust with respect to unknown disturbances and modelling errors, many research papers have been published based on the disturbance decoupling principle. Pioneering works were done by Darouach et al. (Darouach; Zasadzinski; Bassang & Nowakowski (1995) and Darouach; Zasadzinski & Keller (1992)), Chang and Hsu (Chang & Hsu (1993)) and Hou and Müller (Hou & Müller (1993)). They utilized some transformations to make the original systems with unknown inputs into some singular systems without unknown inputs. The most important preceding study related to this paper was done by Chen and Patton (Chen & Patton (1996)). They proposed the simple and useful optimal filtering algorithm, ODDO (Optimal Disturbance Decoupling Observer), and showed its excellent simulation results. See also the papers such as Caliskan; Mukai; Katz & Tanikawa (2003), Hou & Müller (1994), Hou & R. J. Patton (1998) and Sawada & Tanikawa (2002) and the book Chen & Patton (1999). Their algorithm recently has been modified by the author in Tanikawa (2006) (see Tanikawa & Sawada (2003) also).

We here consider smoothing problems which allow us time-lags for computing estimates of the states. Namely, we try to find the optimal estimate $\hat{x}_{t-L/t}$ of the state x_{t-L} based on the observation $\mathbf{Y}_t$ with $L > 0$. We often classify smoothing problems into the following three types. For the first problem, the fixed-point smoothing, we investigate the optimal estimate

$\hat{x}_{k/t}$ of the state x_k for a fixed k based on the observations $\{\mathbf{Y}_t, t = k+1, k+2, \cdots\}$. Algorithms for computing $\hat{x}_{k/t}, t = k+1, k+2, \cdots$, recursively are called fixed-point smoothers. For the second problem, the fixed-interval smoothing, we investigate the optimal estimate $\hat{x}_{t/N}$ of the state x_t at all times $t = 0, 1, \cdots, N$ based on the observation $\mathbf{Y}_N$ of all the outputs $\{y_0, y_1, \cdots, y_N\}$. Fixed-interval smoothers are algorithms for computing $\hat{x}_{t/N}, t = 0, 1, \cdots, N$ recursively. The third problem, the fixed-lag smoothing, is to investigate the optimal estimate $\hat{x}_{t-L/t}$ of the state x_{t-L} based on the observation $\mathbf{Y}_t$ for a given $L \geq 1$. Fixed-lag smoothers are algorithms for computing $\hat{x}_{t-L/t}, t = L+1, L+2, \cdots$, recursively. See the references such as Anderson & Moore (1979), Bryson & Ho (1969), Kailath (1975) and Meditch (1973) for early research works on smoothers. More recent papers have been published based on different approaches such as stochastic realization theory (e.g., Badawi; Lindquist & Pavon (1979) and Faurre; Clerget & Germain (1979)), the complementary models (e.g., Ackner & Kailath (1989a), Ackner & Kailath (1989b), Bello; Willsky & Levy (1989), Bello; Willsky; Levy & Castanon (1986) Desai; Weinert & Yasypchuk (1983) and Weinert & Desai (1981)) and others. Nice surveys can be found in Kailath; Sayed & Hassibi (2000) and Katayama (2000).

When stochastic systems contain unknown inputs explicitly, Tanikawa (Tanikawa (2006)) obtained a fixed-point smoother for the first problem. The second and the third problems were discussed in Tanikawa (2008). In this chapter, all three problems are discussed in a comrehensive and self-contained manner as much as possible. Namely, after some preliminary results in Section 2, we derive the fixed-point smoothing algorithm given in Tanikawa (2006) in Section 3 for the system with unknown inputs explicitly by applying the optimal filter with disturbance decoupling property obtained in Tanikawa & Sawada (2003). In Section 4, we construct the fixed-interval smoother given in Tanikawa (2008) from the fixed-point smoother obtained in Section 3. In Section 5, we construct the fixed-lag smoother given in Tanikawa (2008) from the optimal filter in Tanikawa & Sawada (2003).

Finally, the new feature and advantages of the obtained results are summarized here. To the best of our knowledge, no attempt has been made to investigate optimal fixed-interval and fixed-lag smoothers for systems with unknown inputs explicitly (see the stochastic system given by (1)-(2)) before Tanikawa (2006) and Tanikawa (2008). Our smoothing algorithms have similar recursive forms to the standard optimal filter (i.e., the Kalman filter) and smoothers. Moreover, our algorithms reduce to those known smoothers derived from the Kalman filter (see e.g., Katayama (2000)) when the unknown inputs disappear. Thus, our algorithms are consistent with the known smoothing algorithms for systems without unknown inputs.

2. Preliminaries

Consider the following discrete-time linear stochastic system for $t = 0, 1, 2, \cdots$:

$$x_{t+1} = A_t x_t + B_t u_t + E_t d_t + \zeta_t, \tag{1}$$

$$y_t = C_t x_t + \eta_t, \tag{2}$$

where

$x_t \in \mathbf{R}^n$ the state vector,

$y_t \in \mathbf{R}^m$ the output vector,

$u_t \in \mathbf{R}^r$ the known input vector,
$d_t \in \mathbf{R}^q$ the unknown input vector.

Suppose that ζ_t and η_t are independent zero mean white noise sequences with covariance matrices Q_t and R_t. Let A_t, B_t, C_t and E_t be known matrices with appropriate dimensions. In Tanikawa & Sawada (2003), we considered the optimal estimate $\hat{x}_{t+1/t+1}$ of the state x_{t+1} which was proposed by Chen and Patton (Chen & Patton (1996) and Chen & Patton (1999)) with the following structure:

$$z_{t+1} = F_{t+1}\, z_t + T_{t+1}\, B_t\, u_t + K_{t+1}\, y_t, \tag{3}$$
$$\hat{x}_{t+1/t+1} = z_{t+1} + H_{t+1}\, y_{t+1}, \tag{4}$$

for $t = 0, 1, 2, \cdots$. Here, $\hat{x}_{0/0}$ is chosen to be z_0 for a fixed z_0. Denote the state estimation error and its covariance matrix respectively by e_t and P_t. Namely, we use the notations $e_t = x_t - \hat{x}_{t/t}$ and $P_t = \mathbf{E}\{e_t\, e_t{}^T\}$ for $t = 0, 1, 2, \cdots$. Here, $\mathbf{E}$ denotes expectation and T denotes transposition of a matrix. We assume in this paper that random variables e_0, $\{\eta_t\}$, $\{\zeta_t\}$ are independent. As in Chen & Patton (1996), Chen & Patton (1999) and Tanikawa & Sawada (2003), we consider state estimate (3)-(4) with the matrices $F_{t+1}, T_{t+1}, H_{t+1}$ and K_{t+1} of the forms:

$$K_{t+1} = K^1_{t+1} + K^2_{t+1}, \tag{5}$$
$$E_t = H_{t+1}\, C_{t+1}\, E_t, \tag{6}$$
$$T_{t+1} = I - H_{t+1}\, C_{t+1}, \tag{7}$$
$$F_{t+1} = A_t - H_{t+1}\, C_{t+1}\, A_t - K^1_{t+1}\, C_t, \tag{8}$$
$$K^2_{t+1} = F_{t+1}\, H_t. \tag{9}$$

The next lemma on equality (6) was obtained and used by Chen and Patton (Chen & Patton (1996) and Chen & Patton (1999)). Before stating it, we assume that E_k is a full column rank matrix. Notice that this assumption is not an essential restriction.

Lemma 2.1. *Equality (6) holds if and only if*

$$rank\,(C_{t+1}\, E_t) = rank\,(E_t)\,. \tag{10}$$

When this condition holds true, matrix H_{t+1} which satisfies (6) must have the form

$$H_{t+1} = E_t \left\{ (C_{t+1}\, E_t)^T\, (C_{t+1}\, E_t) \right\}^{-1} (C_{t+1}\, E_t)^T\,. \tag{11}$$

Hence, we have

$$C_{t+1}\, H_{t+1} = C_{t+1}\, E_t \left\{ (C_{t+1}\, E_t)^T\, (C_{t+1}\, E_t) \right\}^{-1} (C_{t+1}\, E_t)^T \tag{12}$$

which is a non-negative definite symmetric matrix. ∎

When the matrix K^1_{t+1} has the form

$$K^1_{t+1} = A^1_{t+1} \left(P_t\, C_t{}^T - H_t\, R_t \right) \left(C_t\, P_t\, C_t{}^T + R_t \right)^{-1}, \tag{13}$$

$$A^1_{t+1} = A_t - H_{t+1}\, C_{t+1}\, A_t, \tag{14}$$

we obtained the following result (Theorem 2.7 in Tanikawa & Sawada (2003)) on the optimal filtering algorithm.

Proposition 2.2. *If $C_t\, H_t$ and R_t are commutative, i.e.,*

$$C_t\, H_t\, R_t = R_t\, C_t\, H_t, \tag{15}$$

then the optimal gain matrix K^1_{t+1} which makes the variance of the state estimation error e_{t+1} minimum is determined by (13). Hence, we obtain the optimal filtering algorithm:

$$\hat{x}_{t+1/t+1} = A^1_{t+1} \{\hat{x}_{t/t} + G_t\, (y_t - C_t\, \hat{x}_{t/t})\} + H_{t+1}\, y_{t+1} + T_{t+1}\, B_t\, u_t, \tag{16}$$

$$P_{t+1} = A^1_{t+1}\, M_t\, A^1_{t+1}{}^T + T_{t+1}\, Q_t\, T_{t+1}{}^T + H_{t+1}\, R_{t+1}\, H_{t+1}{}^T, \tag{17}$$

where

$$G_t = \left(P_t\, C_t{}^T - H_t\, R_t \right) \left(C_t\, P_t\, C_t{}^T + R_t \right)^{-1}, \tag{18}$$

and

$$M_t = P_t - G_t \left(C_t\, P_t - R_t\, H_t{}^T \right). \tag{19}$$

∎

Remark 2.3. If the matrix R_t has the form

$$R_t = r_t\, I$$

with some positive number r_t for each $t = 1, 2, \cdots$, then it is obvious to see that condition (15) holds. ∎

Finally, we have the following proposition which indicates that the standard Kalman filter is a special case of the optimal filter proposed in this section (see e.g., Theorem 5.2 (page 90) in Katayama (2000)).

Proposition 2.4. *Suppose that $E_t \equiv O$ holds for all t (i.e., the unknown input term is zero). Then, Lemma 2.1 cannot be applied directly. But, we can choose $H_t \equiv O$ for all t in this case, and the optimal filter given in Proposition 2.2 reduces to the standard Kalman filter.* ∎

3. The fixed-point smoothing

Let k be a fixed time. We study an iterative algorithm to compute the optimal estimate $\hat{x}_{k/t}$ of the state x_k based on the observation $\mathbf{Y}_t, t = k+1, k+2, \cdots$, with $\mathbf{Y}_t = \sigma\{y_s, s = 0, 1, \cdots, t\}$. We define state vectors $\theta_t, t = k, k+1, \cdots$, by

$$\theta_{t+1} = \theta_t, \;\; t = k, k+1, \cdots; \quad \theta_k = x_k. \tag{20}$$

It is easy to observe that the optimal estimate $\hat{\theta}_{t/t}$ of the state θ_t based on the observation $\mathbf{Y}_t$ is identical to the optimal smoother $\hat{x}_{k/t}$ in view of the equalities $\theta_t = x_k, t = k, k+1, \cdots$.
In order to derive the optimal fixed-point smoother, we consider the following augmented system for $t = k, k+1, \cdots$:

$$\begin{bmatrix} x_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} A_t & O \\ O & I \end{bmatrix} \begin{bmatrix} x_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} B_t \\ O \end{bmatrix} u_t + \begin{bmatrix} E_t \\ O \end{bmatrix} d_t + \begin{bmatrix} I \\ O \end{bmatrix} \zeta_t, \tag{21}$$

$$y_{t+1} = [C_{t+1} \;\; O] \begin{bmatrix} x_{t+1} \\ \theta_{t+1} \end{bmatrix} + \eta_{t+1}. \tag{22}$$

Denote these equations respectively by

$$\widetilde{x_{t+1}} = \widetilde{A_t}\, \widetilde{x_t} + \widetilde{B_t}\, u_t + \widetilde{E_t}\, d_t + \widetilde{J_t}\, \zeta_t, \tag{23}$$

$$y_{t+1} = \widetilde{C_{t+1}}\, \widetilde{x_{t+1}} + \eta_{t+1}, \tag{24}$$

where

$$\widetilde{x_t} = \begin{bmatrix} x_t \\ \theta_t \end{bmatrix}, \; \widetilde{A_t} = \begin{bmatrix} A_t & O \\ O & I \end{bmatrix}, \; \widetilde{B_t} = \begin{bmatrix} B_t \\ O \end{bmatrix}, \; \widetilde{E_t} = \begin{bmatrix} E_t \\ O \end{bmatrix}, \; \widetilde{J_t} = \begin{bmatrix} I \\ O \end{bmatrix}$$

and $\widetilde{C_{t+1}} = [C_{t+1} \;\; O]$.

Here, I and O are the identity matrix and the zero matrix respectively with appropriate dimensions. By making use of the notations

$$\widetilde{H_{t+1}} = \begin{bmatrix} H_{t+1} \\ O \end{bmatrix}, \; \widetilde{T_{t+1}} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} - \widetilde{H_{t+1}}\, \widetilde{C_{t+1}},$$

we have the equalities:

$$\widetilde{C_{t+1}}\, \widetilde{E_t} = C_{t+1} E_t, \; \widetilde{T_{t+1}} = \begin{bmatrix} T_{t+1} & O \\ O & I \end{bmatrix}, \; \widetilde{A^1_{t+1}} = \widetilde{T_{t+1}}\, \widetilde{A_t} = \begin{bmatrix} A^1_{t+1} & O \\ O & I \end{bmatrix}.$$

We introduce the covariance matrix $\widetilde{P_t}$ of the state estimation error of the augmented system (23)-(24):

$$\widetilde{P_t} = \begin{bmatrix} P_t^{(1,1)} & P_t^{(1,2)} \\ P_t^{(2,1)} & P_t^{(2,2)} \end{bmatrix} = \mathbf{E}\left\{ \begin{bmatrix} x_t - \hat{x}_{t/t} \\ \theta_t - \hat{\theta}_{t/t} \end{bmatrix} \begin{bmatrix} x_t - \hat{x}_{t/t} \\ \theta_t - \hat{\theta}_{t/t} \end{bmatrix}^T \right\}. \tag{25}$$

Notice that $P_t^{(1,1)}$ is equal to P_t. Applying the optimal filter given in Proposition 2.2 to the augmented system (21)-(22), we obtain the following optimal fixed-point smoother.

Theorem 3.1. *If $C_t H_t$ and R_t are commutative, i.e.,*

$$C_t H_t R_t = R_t C_t H_t, \tag{26}$$

then we have the optimal fixed-point smoother for (21)-(22) as follows:

(i) the fixed-point smoother

$$\hat{x}_{k/t+1} = \hat{x}_{k/t} + D_t(k)\left[y_t - C_t\,\hat{x}_{t/t}\right], \tag{27}$$

(ii) the gain matrix

$$D_t(k) = P_t^{(2,1)}\,C_t^T\left(C_t\,P_t\,C_t^T + R_t\right)^{-1}, \tag{28}$$

(iii) the covariance matrix of the mean-square error

$$P_{t+1}^{(2,1)} = \left\{P_t^{(2,1)} - P_t^{(2,1)}\,C_t^T\left(C_t\,P_t\,C_t^T + R_t\right)^{-1}\left(C_t\,P_t - R_t\,H_t^T\right)\right\}A_{t+1}^{1\;T}, \tag{29}$$

$$P_{t+1}^{(2,2)} = P_t^{(2,2)} - P_t^{(2,1)}\,C_t^T\left(C_t\,P_t\,C_t^T + R_t\right)^{-1}C_t\,P_t^{(2,1)\,T}. \tag{30}$$

Here, we note that $P_k^{(2,1)} = P_k^{(2,2)} = P_k$. We notice that $\hat{x}_{t/t}$ is the optimal filter of the original system (1)-(2) given in Tanikawa & Sawada (2003).

Proof Applying the optimal filter given by (16)-(17) in Proposition (2.2) to the augmented system (23)-(24), we have

$$\widehat{\widetilde{x_{t+1/t+1}}} = \widetilde{A_{t+1}}^{1}\left\{\widehat{\widetilde{x_{t/t}}} + \widetilde{G}_t\left(y_t - C_t\,\widehat{\widetilde{x_{t/t}}}\right)\right\} + \widetilde{H_{t+1}}\,y_{t+1} + \widetilde{T_{t+1}}\,\widetilde{B}_t\,u_t. \tag{31}$$

This can be rewritten as

$$\begin{bmatrix}\hat{x}_{t+1/t+1}\\ \hat{\theta}_{t+1/t+1}\end{bmatrix} = \begin{bmatrix}A_{t+1}^1 & O\\ O & I\end{bmatrix}\left\{\begin{bmatrix}\hat{x}_{t/t}\\ \hat{\theta}_{t/t}\end{bmatrix} + \begin{bmatrix}P_t^{(1,1)}\,C_t^T - H_t R_t\\ P_t^{(2,1)}\,C_t^T\end{bmatrix}\right.$$
$$\left.\times\left(C_t\,P_t\,C_t^T + R_t\right)^{-1}\left(y_t - C_t\,\hat{x}_{t/t}\right)\right\} + \begin{bmatrix}H_{t+1}\,y_{t+1}\\ O\end{bmatrix} + \begin{bmatrix}T_{t+1}\,B_t\,u_t\\ O\end{bmatrix}.$$

Thus, we have

$$\begin{aligned}\hat{x}_{t+1/t+1} = A_{t+1}^1&\left\{\hat{x}_{t/t} + \left(P_t^{(1,1)}\,C_t^T - H_t R_t\right)\left(C_t\,P_t\,C_t^T + R_t\right)^{-1}\left(y_t - C_t\,\hat{x}_{t/t}\right)\right\}\\ &+ H_{t+1}\,y_{t+1} + T_{t+1}\,B_t\,u_t\end{aligned} \tag{32}$$

and

$$\hat{\theta}_{t+1/t+1} = \hat{\theta}_{t/t} + P_t^{(2,1)}\,C_t^T\left(C_t\,P_t\,C_t^T + R_t\right)^{-1}\left(y_t - C_t\,\hat{x}_{t/t}\right). \tag{33}$$

Here, we used the equalities

$$\begin{aligned}\widetilde{C}_t\,\widetilde{P}_t\,\widetilde{C}_t^T + R_t &= \left[C_t\;\; O\right]\begin{bmatrix}P_t^{(1,1)} & P_t^{(1,2)}\\ P_t^{(2,1)} & P_t^{(2,2)}\end{bmatrix}\begin{bmatrix}C_t^T\\ O\end{bmatrix} + R_t\\ &= C_t\,P_t\,C_t^T + R_t\end{aligned} \tag{34}$$

and

$$\begin{aligned}\widetilde{G_t} &= \left(\widetilde{P_t}\begin{bmatrix} C_t{}^T \\ O \end{bmatrix} - \widetilde{H_t}\,R_t\right)\left(\widetilde{C_t}\,\widetilde{P_t}\,\widetilde{C_t}^T + R_t\right)^{-1} \\ &= \left(\begin{bmatrix} P_t^{(1,1)} & P_t^{(1,2)} \\ P_t^{(2,1)} & P_t^{(2,2)} \end{bmatrix}\begin{bmatrix} C_t{}^T \\ O \end{bmatrix} - \begin{bmatrix} H_t \\ O \end{bmatrix} R_t\right)\left(\widetilde{C_t}\,\widetilde{P_t}\,\widetilde{C_t}^T + R_t\right)^{-1} \\ &= \begin{bmatrix} P_t^{(1,1)}\,C_t{}^T - H_t\,R_t \\ P_t^{(2,1)}\,C_t{}^T \end{bmatrix}\left(C_t\,P_t\,C_t{}^T + R_t\right)^{-1}. \end{aligned} \tag{35}$$

Thus, equalities (27)-(28) can be obtained from (33) due to $\hat{\theta}_{t/t} = \hat{x}_{k/t}$.
By using the notation $\widetilde{M_t}$ for the augmented system (23)-(24)which corresponds to the matrix M_t in Proposition (2.2), we have

$$\begin{aligned}\widetilde{M_t} &= \begin{bmatrix} M_t^{(1,1)} & M_t^{(1,2)} \\ M_t^{(2,1)} & M_t^{(2,2)} \end{bmatrix} \\ &= \widetilde{P_t} - \widetilde{G_t}\left(\widetilde{C_t}\,\widetilde{P_t} - R_t\left[H_t{}^T \;\; O\right]\right) \\ &= \begin{bmatrix} P_t^{(1,1)} & P_t^{(1,2)} \\ P_t^{(2,1)} & P_t^{(2,2)} \end{bmatrix} - \begin{bmatrix} P_t^{(1,1)}\,C_t{}^T - H_t\,R_t \\ P_t^{(2,1)}\,C_t{}^T \end{bmatrix}\left(C_t\,P_t\,C_t{}^T + R_t\right)^{-1} \\ &\qquad \times\left(\left[C_t \;\; O\right]\begin{bmatrix} P_t^{(1,1)} & P_t^{(1,2)} \\ P_t^{(2,1)} & P_t^{(2,2)} \end{bmatrix} - \left[R_t\,H_t{}^T \;\; O\right]\right). \end{aligned}$$

Thus, we have

$$M_t^{(1,1)} = P_t^{(1,1)} - \left(P_t^{(1,1)}\,C_t{}^T - H_t\,R_t\right)\left(C_t\,P_t\,C_t{}^T + R_t\right)^{-1}\left(C_t\,P_t^{(1,1)} - R_t\,H_t{}^T\right), \tag{36}$$

$$M_t^{(1,2)} = P_t^{(1,2)} - \left(P_t^{(1,1)}\,C_t{}^T - H_t\,R_t\right)\left(C_t\,P_t\,C_t{}^T + R_t\right)^{-1} C_t\,P_t^{(1,2)}, \tag{37}$$

$$M_t^{(2,1)} = P_t^{(2,1)} - P_t^{(2,1)}\,C_t{}^T\left(C_t\,P_t\,C_t{}^T + R_t\right)^{-1}\left(C_t\,P_t^{(1,1)} - R_t\,H_t{}^T\right), \tag{38}$$

and

$$M_t^{(2,2)} = P_t^{(2,2)} - P_t^{(2,1)}\,C_t{}^T\left(C_t\,P_t\,C_t{}^T + R_t\right)^{-1} C_t\,P_t^{(1,2)}. \tag{39}$$

It follows from (17) in Proposition 2.2 that

$$\begin{aligned}\widetilde{P_{t+1}} &= \widetilde{A_{t+1}^1}\,\widetilde{M_t}\,\widetilde{A_{t+1}^1}^T + \widetilde{T_{t+1}}\,\widetilde{J_{t+1}}\,Q_{t+1}\,\widetilde{J_{t+1}}^T\,\widetilde{T_{t+1}} + \widetilde{H_{t+1}}\,R_{t+1}\,\widetilde{H_{t+1}}^T \\ &= \begin{bmatrix} A_{t+1}^1 & O \\ O & I \end{bmatrix}\begin{bmatrix} M_t^{(1,1)} & M_t^{(1,2)} \\ M_t^{(2,1)} & M_t^{(2,2)} \end{bmatrix}\begin{bmatrix} A_{t+1}^1{}^T & O \\ O & I \end{bmatrix} \\ &\qquad + \begin{bmatrix} T_{t+1} & O \\ O & I \end{bmatrix}\begin{bmatrix} I \\ O \end{bmatrix} Q_{t+1}\left[I \;\; O\right]\begin{bmatrix} T_{t+1}{}^T & O \\ O & I \end{bmatrix} \\ &\qquad + \begin{bmatrix} H_{t+1} \\ O \end{bmatrix} R_{t+1}\left[H_{t+1}{}^T \;\; O\right]. \end{aligned} \tag{40}$$

Equalities (29)-(30) follow from (38)-(40). Finally, we have equalities $P_k^{(2,1)} = P_k^{(2,2)} = P_k^{(1,1)} = P_k$ by the definition of $\widetilde{P}_k$. ∎

We thus have derived the fixed-point smoothing algorithm for the state-space model which explicitly contains the unknown inputs. We can indicate that the algorithm has a rather simple form and also has consistency with both the Kalman filter and the standard optimal smoother derived from the Kalman filter as shown in the following remark.

Remark 3.2. Suppose that $E_t \equiv O$ holds for all t (i.e., the unknown input term is zero) and that $H_t \equiv O$ for all t(as in Proposition 2.4). In this case, it follows from Theorem 3.1 that

$$\hat{x}_{t+1/t+1} = A_t \left\{ \hat{x}_{t/t} + P_t\, C_t{}^T \left(C_t\, P_t\, C_t{}^T + R_t \right)^{-1} (y_t - C_t\, \hat{x}_{t/t}) \right\} + B_t\, u_t, \tag{41}$$

$$\hat{\theta}_{t+1/t+1} = \hat{\theta}_{t/t} + P_t^{(2,1)}\, C_t{}^T \left(C_t\, P_t\, C_t{}^T + R_t \right)^{-1} (y_t - C_t\, \hat{x}_{t/t}), \tag{42}$$

$$P_{t+1}^{(2,1)} = \left\{ P_t^{(2,1)} - P_t^{(2,1)}\, C_t{}^T \left(C_t\, P_t\, C_t{}^T + R_t \right)^{-1} C_t\, P_t \right\} A_t{}^T, \tag{43}$$

and

$$P_{t+1}^{(2,2)} = P_t^{(2,2)} - P_t^{(2,1)}\, C_t{}^T \left(C_t\, P_t\, C_t{}^T + R_t \right)^{-1} C_t\, {P_t^{(2,1)}}^T. \tag{44}$$

Here, we note that the state estimate $\hat{x}_{t+1/t+1}$ reduces to the state estimate $\hat{x}_{t+1/t}$ in Katayama (2000) when $H_t \equiv O$ holds. Moreover, Equalities (37)-(40) with the state estimates $\hat{x}_{t+1/t+1}$ and $\hat{x}_{t/t}$ replaced respectively by $\hat{x}_{t+1/t}$ and $\hat{x}_{t/t-1}$ are identical to those for the pair of the standard Kalman filter and the optimal fixed-point smoother in Katayama (2000). Thus, it has been shown that this algorithm reduces to the well known optimal smoother derived from the Kalman filter when the unknown inputs disappear. This indicates that our smoothing algorithm is a natural extension of the standard optimal smoother to linear systems possibly with unknown inputs. ∎

Let us introduce some notations:

$$\nu_t = y_t - C_t\, \hat{x}_{t/t}, \tag{45}$$

$$L_t = A_{t+1}^1\, (I - G_t\, C_t), \tag{46}$$

$$\Psi(t,\tau) = \begin{cases} L_{t-1}\, L_{t-2} \cdots L_\tau\,, & t > \tau \\ I & , \ t = \tau\,, \end{cases} \tag{47}$$

where the matrix G_t was defined by (18), i.e.,

$$G_t = \left(P_t\, C_t{}^T - H_t\, R_t \right) \left(C_t\, P_t\, C_t{}^T + R_t \right)^{-1}. \tag{48}$$

We then have the following results due to (27).

Corollary 3.3. *We have the equalities:*

$$\hat{x}_{k/t+1} = \hat{x}_{k/k} + \sum_{i=k}^{t} D_i(k)\nu_i = \hat{x}_{k/k} + P_k \sum_{i=k}^{t} \Psi(i,k)^T C_i{}^T \left(C_i P_i C_i{}^T + R_i \right)^{-1} \nu_i\,. \tag{49}$$

∎

Proof It is straightforward to show the first equality from (27). For the second equality, it is sufficient to prove the equality

$$D_t(k) = P_k \, \Psi(t,k)^T \, C_t{}^T \left(C_t \, P_t \, C_t{}^T + R_t \right)^{-1} \tag{50}$$

for $t \geq k$. By virtue of (46), equality (29) can be rewritten as

$$P_t^{(2,1)} = P_{t-1}^{(2,1)} \left(I - C_{t-1}{}^T \, G_{t-1}{}^T \right) A_t^{1\,T} = P_{t-1}^{(2,1)} \, L_{t-1}{}^T. \tag{51}$$

By using this equality recursively, we have

$$\begin{aligned} P_t^{(2,1)} &= P_{t-2}^{(2,1)} \, L_{t-2}{}^T \, L_{t-1}{}^T = \cdots\cdots = P_k^{(2,1)} \, L_k{}^T \, L_{k+1}{}^T \cdots L_{t-1}{}^T \\ &= P_k \, \Psi(t,k)^T. \end{aligned} \tag{52}$$

Substituting this equality into (28), we obtain

$$D_t(k) = P_k \, \Psi(t,k)^T \, C_t{}^T \left(C_t \, P_t \, C_t{}^T + R_t \right)^{-1}, \tag{53}$$

i.e., (50). ∎

Finally, we study the reduction of the estimation error by the fixed-point smoothing over the optimal filtering. Due to (27), we have

$$P_t^{(2,2)} = \mathbf{E}\left[(x_k - \hat{x}_{k/t}) \, (x_k - \hat{x}_{k/t})^T \right]. \tag{54}$$

Denote this matrix simply by $P_{k/t}$. It then follows from (30) that

$$P_{k/t+1} = P_{k/t} - P_t^{(2,1)} C_t{}^T \left(C_t P_t C_t{}^T + R_t \right)^{-1} C_t P_t^{(2,1)\,T}. \tag{55}$$

Summing up these equalities for $t = k, k+1, \cdots, s$, we have

$$P_{k/k} - P_{k/s+1} = \sum_{i=k}^{s} P_i^{(2,1)} C_i{}^T \left(C_i P_i C_i{}^T + R_i \right)^{-1} C_i P_i^{(2,1)\,T}. \tag{56}$$

Thus, the right hand side indicates the amount of the reduction of the estimation error by the fixed-point smoothing over the optimal filtering.

4. The fixed-interval smoothing

We consider the fixed-interval smoothing problem in this section. Namely, we investigate the optimal estimate $\hat{x}_{t/N}$ of the state x_t at all times $t = 0, 1, \cdots, N$ based on the observation $\mathbf{Y}_N$ of all the states $\{y_0, y_1, \cdots, y_N\}$. Applying equality (49), we easily obtain the following equality.

Lemma 4.1. *The equality*

$$\hat{x}_{t/N} = \hat{x}_{t/t+1} + P_t \, L_t{}^T \, P_{t+1}{}^{-1} \left(\hat{x}_{t+1/N} - \hat{x}_{t+1/t+1} \right) \tag{57}$$

holds for $t = 0, 1, \cdots, N-1$. ∎

Proof Using the notation

$$\tilde{v}_i = C_i{}^T \left(C_i P_i C_i{}^T + R_i \right)^{-1} v_i, \tag{58}$$

we have

$$\hat{x}_{k/t+1} = \hat{x}_{k/k} + P_k \sum_{i=k}^{t} \Psi(i,k)^T \tilde{v}_i \tag{59}$$

for $k \le t$ due to (49). In view of (59) , we also have

$$\hat{x}_{k/t+1} = \hat{x}_{k/k} + P_k \tilde{v}_k + P_k \sum_{i=k+1}^{t} \Psi(i,k)^T \tilde{v}_i = \hat{x}_{k/k+1} + P_k \sum_{i=k+1}^{t} \Psi(i,k)^T \tilde{v}_i \tag{60}$$

for $k+1 \le t$. Putting $t+1 = N$ and $k = t+1$ in equality (59), we have

$$\hat{x}_{t+1/N} = \hat{x}_{t+1/t+1} + P_{t+1} \sum_{i=t+1}^{N-1} \Psi(i,t+1)^T \tilde{v}_i. \tag{61}$$

Putting $t+1 = N$ and $k = t$ in equality (60), we have

$$\hat{x}_{t/N} = \hat{x}_{t/t+1} + P_t \sum_{i=t+1}^{N-1} \Psi(i,t)^T \tilde{v}_i = \hat{x}_{t/t+1} + P_t L_t{}^T \sum_{i=t+1}^{N-1} \Psi(i,t+1)^T \tilde{v}_i. \tag{62}$$

Substituting (61) into (62), we have

$$\hat{x}_{t/N} = \hat{x}_{t/t+1} + P_t\, L_t{}^T\, P_{t+1}{}^{-1} \left(\hat{x}_{t+1/N} - \hat{x}_{t+1/t+1} \right).$$

The above derivation is valid for $t = 0, 1, \cdots, N-2$. It is easy to observe that equality (57) also holds for $t = N-1$. ∎

It is a simple task to obtain the following Fraser-type algorithm from (57).

Theorem 4.2. *We obtain the fixed-interval smoother*

$$\hat{x}_{t/N} = \hat{x}_{t/t+1} + P_t\, L_t{}^T\, \lambda_{t+1}, \tag{63}$$

$$\lambda_t = L_t{}^T \lambda_{t+1} + C_t{}^T \left(C_t P_t C_t{}^T + R_t \right)^{-1} v_t. \tag{64}$$

for $t = N-1, N-2, \cdots, 1, 0$. Here, we have $\lambda_N = 0$. ∎

Proof For $t = 0, 1, \cdots, N$, we put

$$\lambda_t = P_t{}^{-1} \left(\hat{x}_{t/N} - \hat{x}_{t/t} \right). \tag{65}$$

We then have $\lambda_N = 0$. Substituting (65) into (57), we obtain equality (63). Then, by utilizing (63) and (65), we have

$$\lambda_t = P_t{}^{-1} \left(\hat{x}_{t/t+1} + P_t\, L_t{}^T\, \lambda_{t+1} - \hat{x}_{t/t} \right). \tag{66}$$

In view of the equality

$$\hat{x}_{t/t+1} - \hat{x}_{t/t} = P_t\, \tilde{v}_t \tag{67}$$

which follows from (27) in Tanikawa & Sawada (2003), we obtain

$$\begin{aligned}\lambda_t &= {L_t}^T \lambda_{t+1} + \tilde{\nu}_t \\ &= {L_t}^T \lambda_{t+1} + {C_t}^T \left(C_t P_t {C_t}^T + R_t \right)^{-1} \nu_t. \end{aligned} \tag{68}$$

Thus, we proved (64). ∎

Remark 4.3. When $E_t \equiv O$ holds for all t (i.e., the unknown input term is zero), we shall see that fixed-interval smoother (63)-(64) is identical to the fixed-interval smoother obtained from the standard Kalman filter (see e.g., Katayama (2000)). Thus, our algorithm is consistent with the known fixed-interval smoothing algorithm for systems without unknown inputs. This can be shown as follows. Assuming that $E_t = O$, we have $H_t = O$ for $t = 0, 1, \cdots, N$ (see Propositin 2.4). Note that in (59), i.e.,

$$\hat{x}_{k/t+1} = \hat{x}_{k/k} + P_k \sum_{i=k}^{t} \Psi(i,k)^T \tilde{\nu}_i$$

$\hat{x}_{k/t+1}$ and $\hat{x}_{k/k}$ respectively reduce to $\hat{x}_{k/t}$ and $\hat{x}_{k/k-1}$ which are respectively the optimal smoother and the optimal filter obtained from the standard Kalman filter. Then, the above equality is identical to (7.18) in Katayama (2000). Since the rest of the proof can be done in the same way as in Katayama (2000), we obtain the same smoother. ∎

5. The fixed-lag smoothing

We study the fixed-lag smoothing problem in this section. For a fixed $L > 0$, we investigate an iterative algorithm to compute the optimal state estimate $\hat{x}_{t-L/t}$ of the state x_{t-L} based on the observation $\mathbf{Y}_t$.

We consider the following augmented system:

$$\begin{bmatrix} x_{t+1} \\ x_t \\ \vdots \\ x_{t-L+1} \end{bmatrix} = \begin{bmatrix} A_t & O & \dots & O \\ I & O & \dots & O \\ & \ddots & & \\ O & & I & O \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-L} \end{bmatrix} + \begin{bmatrix} B_t \\ O \\ \vdots \\ O \end{bmatrix} u_t + \begin{bmatrix} E_t \\ O \\ \vdots \\ O \end{bmatrix} d_t + \begin{bmatrix} I \\ O \\ \vdots \\ O \end{bmatrix} \zeta_t, \tag{69}$$

$$y_{t+1} = [C_{t+1} \ O \ \dots \ O] \begin{bmatrix} x_{t+1} \\ x_t \\ \vdots \\ x_{t-L+1} \end{bmatrix} + \eta_{t+1}. \tag{70}$$

Denote these equations respectively by

$$\widetilde{x_{t+1}} = \widetilde{A_t}\, \widetilde{x_t} + \widetilde{B_t}\, u_t + \widetilde{E_t}\, d_t + \widetilde{J_t}\, \zeta_t, \tag{71}$$

$$y_{t+1} = \widetilde{C_{t+1}}\, \widetilde{x_{t+1}} + \eta_{t+1}, \tag{72}$$

where

$$\widetilde{x_t} = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-L} \end{bmatrix}, \quad \widetilde{A_t} = \begin{bmatrix} A_t & O & \dots & O \\ I & O & \dots & O \\ & \ddots & & \\ O & & I & O \end{bmatrix}, \quad \widetilde{B_t} = \begin{bmatrix} B_t \\ O \\ \vdots \\ O \end{bmatrix}, \quad \widetilde{E_t} = \begin{bmatrix} E_t \\ O \\ \vdots \\ O \end{bmatrix},$$

$$\widetilde{J_t} = \begin{bmatrix} I \\ O \\ \vdots \\ O \end{bmatrix} \quad \text{and} \quad \widetilde{C_{t+1}} = [C_{t+1} \;\; O \;\; \dots \;\; O] \, .$$

Here, I and O are the identity matrix and the zero matrix respectively with appropriate dimensions. By making use of the notations

$$\widetilde{H_{t+1}} = \begin{bmatrix} H_{t+1} \\ O \\ \vdots \\ O \end{bmatrix} \quad \text{and} \quad \widetilde{T_{t+1}} = I - \widetilde{H_{t+1}} \, \widetilde{C_{t+1}},$$

we have the equalities:

$$\widetilde{C_{t+1}} \, \widetilde{E_t} = [C_{t+1} \;\; O \;\; \dots \;\; O] \begin{bmatrix} E_t \\ O \\ \vdots \\ O \end{bmatrix} = C_{t+1} \, E_t,$$

$$\widetilde{T_{t+1}} = I - \begin{bmatrix} H_{t+1} \\ O \\ \vdots \\ O \end{bmatrix} [C_{t+1} \;\; O \;\; \dots \;\; O] = \begin{bmatrix} T_{t+1} & O & \dots & O \\ O & I & \dots & O \\ & & \ddots & \\ O & O & \dots & I \end{bmatrix},$$

$$\widetilde{A^1_{t+1}} = \widetilde{T_{t+1}} \, \widetilde{A_t} = \begin{bmatrix} T_{t+1} & O & \dots & O \\ O & I & \dots & O \\ & & \ddots & \\ O & O & \dots & I \end{bmatrix} \begin{bmatrix} A_t & O & \dots & O \\ I & O & \dots & O \\ & \ddots & & \\ O & & I & O \end{bmatrix} = \begin{bmatrix} A^1_{t+1} & O & \dots & O \\ I & O & \dots & O \\ & \ddots & & \\ O & & I & O \end{bmatrix}.$$

We introduce the covariance matrix $\widetilde{P_t}$ of the state estimation error of augmented system (71)-(72):

$$\widetilde{P_t} = \mathbf{E} \left\{ \begin{bmatrix} x_t - \hat{x}_{t/t} \\ x_{t-1} - \hat{x}_{t-1/t} \\ \vdots \\ x_{t-L} - \hat{x}_{t-L/t} \end{bmatrix} \begin{bmatrix} x_t - \hat{x}_{t/t} \\ x_{t-1} - \hat{x}_{t-1/t} \\ \vdots \\ x_{t-L} - \hat{x}_{t-L/t} \end{bmatrix}^T \right\}. \tag{73}$$

By using the notations

$$P_{t-i,t-j/t} = \mathbf{E}\left\{ (x_{t-i} - \hat{x}_{t-i/t}) \left(x_{t-j} - \hat{x}_{t-j/t} \right)^T \right\},$$
$$P_{t-i/t} = P_{t-i,t-i/t},$$

we can write

$$\widetilde{P}_t = \begin{bmatrix} P_{t/t} & P_{t,t-1/t} & \cdots & P_{t,t-L/t} \\ P_{t-1,t/t} & P_{t-1/t} & \cdots & P_{t-1,t-L/t} \\ \vdots & & \ddots & \vdots \\ P_{t-L,t/t} & P_{t-L,t-1/t} & \cdots & P_{t-L/t} \end{bmatrix}. \tag{74}$$

Here, it is easy to observe that $P_{t/t} = P_t$ holds. We also note that

$$\widetilde{C}_t \widetilde{P}_t \widetilde{C}_t^{\,T} + R_t = C_t P_{t/t} C_t^{\,T} + R_t. \tag{75}$$

From now on, we use the following notation for brevity:

$$\overline{C}_t := C_t P_t C_t^{\,T} + R_t. \tag{76}$$

Applying the optimal filter given in Proposition 2.2 to augmented system (71)-(72), we have

$$\widehat{\widetilde{x_{t+1/t+1}}} = \widetilde{A^1_{t+1}} \left\{ \widehat{\widetilde{x_{t/t}}} + \widetilde{G_t} \left(y_t - \widetilde{C}_t \widehat{\widetilde{x_{t/t}}} \right) \right\} + \widetilde{H_{t+1}} y_{t+1} + \widetilde{T_{t+1}} \widetilde{B}_t u_t, \tag{77}$$

where

$$\widetilde{G_t} = \left(\widetilde{P}_t \widetilde{C}_t^{\,T} - \widetilde{H_t} R_t \right) \left(\widetilde{C}_t \widetilde{P}_t \widetilde{C}_t^{\,T} + R_t \right)^{-1} = \begin{bmatrix} P_{t/t} C_t^{\,T} - H_t R_t \\ P_{t-1,t/t} C_t^{\,T} \\ \vdots \\ P_{t-L,t/t} C_t^{\,T} \end{bmatrix} \overline{C}_t^{\,-1}. \tag{78}$$

Identifying the component matrices of (77)-(78), we have the following optimal fixed-lag smoother.

Theorem 5.1. *If $C_t H_t$ and R_t are commutative, i.e.,*

$$C_t H_t R_t = R_t C_t H_t, \tag{79}$$

then we have the optimal fixed-lag smoother for (1)-(2) as follows:
(i) the fixed-lag smoother

$$\hat{x}_{t-j/t+1} = \hat{x}_{t-j/t} + S_t(j) \left(y_t - C_t \hat{x}_{t/t} \right) \qquad (j = 0, 1, \cdots, L-1), \tag{80}$$

(ii) the optimal filter

$$\hat{x}_{t+1/t+1} = A^1_{t+1} \left\{ \hat{x}_{t/t} + G_t \left(y_t - C_t \hat{x}_{t/t} \right) \right\} + H_{t+1} y_{t+1} + T_{t+1} B_t u_t, \tag{81}$$

with G_t defined by (18) in Proposition 2.2,
(iii) the gain matrices

$$S_t(j) = \left(P_{t-j,t/t} C_t^{\,T} - \delta_{0,j} H_t R_t \right) \overline{C}_t^{\,-1} \qquad (j = 0, 1, \cdots, L-1), \tag{82}$$

where $\delta_{i,j}$ stands for the Kronecker's delta, i.e.,

$$\delta_{i,j} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}, \tag{83}$$

(iv) the covariance matrix of the mean-square error

$$P_{t+1/t+1} = A^1_{t+1}\, M^{(0,0)}_t\, {A^1_{t+1}}^T + T_{t+1}\, Q_t\, {T_{t+1}}^T + H_{t+1}\, R_{t+1}\, {H_{t+1}}^T, \tag{84}$$

$$P_{t+1,t-j/t+1} = A^1_{t+1}\, M^{(0,j)}_t \qquad (j = 0, 1, \cdots, L-1), \tag{85}$$

$$P_{t-j,t+1/t+1} = \left(P_{t+1,t-j/t+1}\right)^T \qquad (j = 0, 1, \cdots, L-1), \tag{86}$$

$$P_{t-i,t-j/t+1} = M^{(i,j)}_t \qquad (i, j = 0, 1, \cdots, L-1), \tag{87}$$

and

$$M^{(i,j)}_t = P_{t-i,t-j/t} - \left(P_{t-i,t/t}\, {C_t}^T - \delta_{0,i}\, H_t\, R_t\right) {\overline{C}_t}^{-1} \left(C_t\, P_{t,t-j/t} - \delta_{0,j}\, R_t\, {H_t}^T\right) \qquad (i, j = 0, 1, \cdots, L). \tag{88}$$

∎

Remark 5.2. Since the equalities

$$P_{t/t} = P_t \quad (\text{ in Proposition 2.2 })$$

and

$$M^{(0,0)}_t = M_t \quad (\text{ in Proposition 2.2 })$$

hold, the part of the optimal filter in Theorem 5.1 is identical to that in Proposition 2.2. When $E_t \equiv O$ holds for all t (i.e., the unknown input term is zero), we shall see that fixed-lag smoother (80)-(88) is identical to the well known fixed-lag smoother (see e.g. Katayama (2000)) obtained from the standard Kalman filter. Thus, our algorithm is consistent with the known fixed-lag smoothing algorithm for systems without unknown inputs. This can be readily shown as in Remark 4.3. ∎

Proof of Theorem 5.1 Rewriting (77)-(78) with the component matrices explicitly, we have

$$\begin{bmatrix} \hat{x}_{t+1/t+1} \\ \hat{x}_{t/t+1} \\ \hat{x}_{t-1/t+1} \\ \vdots \\ \hat{x}_{t-L+1/t+1} \end{bmatrix} = \begin{bmatrix} A^1_{t+1}\left\{\hat{x}_{t/t} + \left(P_{t/t}{C_t}^T - H_t R_t\right){\overline{C}_t}^{-1}(y_t - C_t\hat{x}_{t/t})\right\} \\ \hat{x}_{t/t} + \left(P_{t/t}\, {C_t}^T - H_t\, R_t\right){\overline{C}_t}^{-1}(y_t - C_t\, \hat{x}_{t/t}) \\ \hat{x}_{t-1/t} + P_{t-1,t/t}\, {C_t}^T {\overline{C}_t}^{-1}(y_t - C_t\, \hat{x}_{t/t}) \\ \vdots \\ \hat{x}_{t-L+1/t} + P_{t-L+1,t/t}\, {C_t}^T {\overline{C}_t}^{-1}(y_t - C_t\, \hat{x}_{t/t}) \end{bmatrix} + \begin{bmatrix} H_{t+1}\, y_{t+1} + T_{t+1}\, B_t\, u_t \\ O \\ O \\ \vdots \\ O \end{bmatrix}. \tag{89}$$

The statements in (i)-(iii) easily follow from (89).
Let $\widetilde{M_t}$ be defined by

$$\widetilde{M_t} = \widetilde{P_t} - \widetilde{G_t}\left(\widetilde{C_t}\,\widetilde{P_t} - R_t\,\widetilde{H_t}^T\right)$$

$$= \widetilde{P_t} - \begin{bmatrix} P_{t/t}C_t^T - H_t R_t \\ P_{t-1,t/t}\,C_t^T \\ P_{t-2,t/t}\,C_t^T \\ \vdots \\ P_{t-L,t/t}\,C_t^T \end{bmatrix} \overline{C}_t^{-1} \begin{bmatrix} P_{t/t}C_t^T - H_t R_t \\ P_{t-1,t/t}\,C_t^T \\ P_{t-2,t/t}\,C_t^T \\ \vdots \\ P_{t-L,t/t}\,C_t^T \end{bmatrix}^T .$$

We also introduce component matrices of $\widetilde{M_t}$ as follows:

$$\widetilde{M_t} = \begin{bmatrix} M_t^{(0,0)} & M_t^{(0,1)} & M_t^{(0,2)} & \dots & M_t^{(0,L)} \\ M_t^{(1,0)} & M_t^{(1,1)} & M_t^{(1,2)} & \dots & M_t^{(1,L)} \\ M_t^{(2,0)} & M_t^{(2,1)} & M_t^{(2,2)} & \dots & M_t^{(2,L)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_t^{(L,0)} & M_t^{(L,1)} & M_t^{(L,2)} & \dots & M_t^{(L,L)} \end{bmatrix} .$$

Concerning $\widetilde{P_{t+1}}$, we have

$$\widetilde{P_{t+1}} = \widetilde{A_{t+1}^1}\,\widetilde{M_t}\,\widetilde{A_{t+1}^1}^T + \widetilde{T_{t+1}}\,\widetilde{J_t}\,Q_t\,\widetilde{J_t}^T\,\widetilde{T_{t+1}}^T + \widetilde{H_{t+1}}\,R_{t+1}\,\widetilde{H_{t+1}}^T$$

$$= \begin{bmatrix} A_{t+1}^1 M_t^{(0,0)} A_{t+1}^{1\,T} & A_{t+1}^1 M_t^{(0,0)} & A_{t+1}^1 M_t^{(0,1)} & \dots & A_{t+1}^1 M_t^{(0,L-1)} \\ M_t^{(0,0)} A_{t+1}^{1\,T} & M_t^{(0,0)} & M_t^{(0,1)} & \dots & M_t^{(0,L-1)} \\ M_t^{(1,0)} A_{t+1}^{1\,T} & M_t^{(1,0)} & M_t^{(1,1)} & \dots & M_t^{(1,L-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_t^{(L-1,0)} A_{t+1}^{1\,T} & M_t^{(L-1,0)} & M_t^{(L-1,1)} & \dots & M_t^{(L-1,L-1)} \end{bmatrix}$$

$$+ \begin{bmatrix} T_{t+1} Q_t T_{t+1}^T + H_{t+1} R_{t+1} H_{t+1}^T & O & O & \dots & O \\ O & O & O & \dots & O \\ O & O & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & O \end{bmatrix} .$$

The final part (iv) can be obtained from the last three equalities. ∎

6. Conclusion

In this chapter, we considered discrete-time linear stochastic systems with unknown inputs (or disturbances) and studied three types of smoothing problems for these systems. We derived smoothing algorithms which are robust to unknown disturbances from the optimal filter for stochastic systems with unknown inputs obtained in our previous papers. These smoothing algorithms have similar recursive forms to the standard optimal filters and smoothers. Moreover, since our algorithms reduce to those known smoothers derived from the Kalman filter when unknown inputs disappear, these algorithms are consistent with the known smoothing algorithms for systems without unknown inputs.

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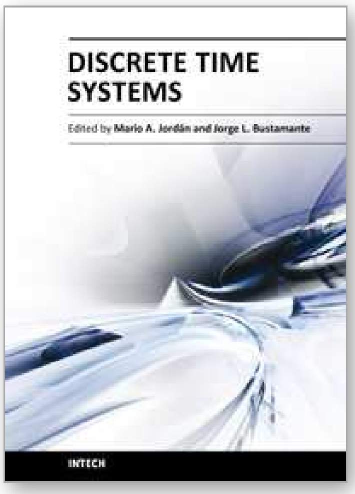

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Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.

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