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Oscillation Susceptibility of an Unmanned Aircraft whose Automatic Flight Control System Fails

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1. Introduction

Interest in oscillation susceptibility of an aircraft was generated by crashes of high performance fighter airplanes such as the YF-22A and B-2, due to oscillations that were not predicted during the aircraft development. Flying qualities and oscillation prediction, based on linear analysis, cannot predict the presence or the absence of oscillations, because of the large variety of nonlinear interactions that have been identified as factors contributing to oscillations. Pilot induced oscillations have been analyzed extensively in many papers by numerical means.
Interest in oscillation susceptibility analysis of an unmanned aircraft, whose flight control system fails, was generated by the need to elaborate an alternative automatic flight control system for the *Automatic Landing Flight Experiment* (ALFLEX) reentry vehicle for the case when the existing automatic flight control system of the vehicle fails.
The purpose of this chapter is the analysis of the oscillation susceptibility of an unmaned aircraft whose automatic flight control system fails. The analysis is focused on the research of oscillatory movement around the center of mass in a longitudinal flight with constant forward velocity (mainly in the final approach and landing phase). The analysis is made in a mathematical model defined by a system of three nonlinear ordinary differential equations, which govern the aircraft movement around its center of mass, in such a flight. This model is deduced in the second paragraph, starting with the set of 9 nonlinear ordinary differential equations governing the movement of the aircraft around its center of mass.In the third paragraph it is shown that in a longitudinal flight with constant forward velocity, if the elevator deflection outruns some limits, oscillatory movement appears. This is proved by means of coincidence degree theory and Mawhin's continuation theorem. As far as we know, this result was proved and published very recently by the authors of this chapter (research supported by CNCSIS--UEFISCSU, project number PNII – IDEI 354 No. 7/2007) and never been included in a book concerning the topic of flight control.The fourth paragraph of this chapter presents mainly numerical results. These results concern an *Aero Data Model in Research Environment* (ADMIRE) and consists in: the identification of the range of the elevator deflection for which steady state exists; the computation of the manifold of steady states; the identification of stable and unstable steady states; the simulation of successful and unsuccessful maneuvers; simulation of oscillatory movements.

2. The mathematical model

Frequently, we describe the evolution of real phenomena by systems of ordinary differential equations. These systems express physical laws, geometrical connections, and often they are obtained by neglecting some influences and quantities, which are assumed insignificant with respect to the others. If the obtained simplified system correctly describes the real phenomenon, then it has to be topologically equivalent to the system in which the small influences and quantities (which have been neglected) are also included. Furthermore, the simplified system has to be structurally stable. Therefore, when a simplified model of a real phenomenon is build up, it is desirable to verify the structural stability of the system. This task is not easy at all. What happens in general is that the results obtained in simplified model are tested against experimental results and in case of agreement the simplified model is considered to be authentic. This philosophy is also adapted in the description of the motion around the center of gravity of a rigid aircraft. According to Etkin & Reid, 1996; Cook, 1997, the system of differential equations, which describes the motion around the center of gravity of a rigid aircraft, with respect to an *xyz* body-axis system, where *xz* is the plane of symmetry, is:

$$
\begin{cases}
\dfrac{\overset{\circ}{V}}{V}\cdot\cos\alpha\cdot\cos\beta-\overset{\circ}{\beta}\cdot\cos\alpha\cdot\sin\beta-\overset{\circ}{\alpha}\cdot\sin\alpha\cdot\cos\beta=r\cdot\sin\beta-q\cdot\sin\alpha\cdot\cos\beta- \\
\qquad\qquad\qquad\dfrac{g}{V}\cdot\sin\theta+\dfrac{X}{m\cdot V} \\
\dfrac{\overset{\circ}{V}}{V}\cdot\sin\beta+\overset{\circ}{\beta}\cdot\cos\beta=p\cdot\sin\alpha\cdot\cos\beta-r\cdot\cos\alpha\cdot\cos\beta+\dfrac{g}{V}\cdot\sin\varphi\cdot\cos\theta+\dfrac{Y}{m\cdot V} \\
\dfrac{\overset{\circ}{V}}{V}\cdot\sin\alpha\cdot\cos\beta-\overset{\circ}{\beta}\cdot\sin\alpha\cdot\sin\beta+\overset{\circ}{\alpha}\cdot\cos\alpha\cdot\cos\beta=-p\cdot\sin\beta+q\cdot\cos\alpha\cdot\cos\beta+ \\
\qquad\qquad\qquad\dfrac{g}{V}\cdot\cos\varphi\cdot\cos\theta+\dfrac{Z}{m\cdot V} \\
I_x\cdot\overset{\circ}{p}-I_{xz}\cdot\overset{\circ}{r}=\left(I_y-I_z\right)\cdot q\cdot r+I_{xz}\cdot p\cdot q+L \\
I_y\cdot\overset{\circ}{q}\qquad=\left(I_z-I_x\right)\cdot p\cdot r-I_{xz}\cdot\left(p^2-r^2\right)+M \\
I_z\cdot\overset{\circ}{r}-I_{xz}\cdot\overset{\circ}{p}=\left(I_x-I_y\right)\cdot p\cdot q-I_{xz}\cdot q\cdot r+N \\
\overset{\circ}{\varphi}=p+q\cdot\sin\varphi\cdot\tan\theta+r\cdot\cos\varphi\cdot\tan\theta \\
\overset{\circ}{\theta}=q\cdot\cos\varphi-r\cdot\sin\varphi \\
\overset{\circ}{\psi}=\dfrac{q\cdot\sin\varphi+r\cdot\cos\varphi}{\cos\theta}
\end{cases}
\tag{1}
$$

The state parameters of this system are: forward velocity V, angle of attack α, sideslip angle *β,* roll rate *p*, pitch rate *q*, yaw rate *r*, Euler roll angle *φ*, Euler pitch angle *θ* and Euler yaw angle *ψ*. The constants I_x, I_y and I_z -moments of inertia about the *x*, *y* and *z*-axis, respectively;

I_{xz} - product of inertia, g -gravitational acceleration; and m - mass of the vehicle. The aero dynamical forces X, Y, Z and moments L, M, N are functions of the state parameters and the control parameters: δ_a - aileron deflection; δ_e - elevator deflection; and δ_r - rudder deflection (the body flap, speed break, δ_c, δ_{ca} are available as additional controls but, for simplicity, they are set to *0* in the analysis to follow). When the automatic flight control system is in function, then the control parameters are functions of the state parameters, describing how the flight control system works. When the automatic flight control system fails, then the control parameters are constant. This last situation will be analyzed in this chapter.

A **flight with constant forward velocity** V is defined as a flight for which V = const (i.e. $\overset{\circ}{V} = 0$).

In a flight with constant forward velocity V the following equalities hold:

$$\begin{cases} -\overset{\circ}{\beta}\cdot\cos\alpha\cdot\sin\beta-\overset{\circ}{\alpha}\cdot\sin\alpha\cdot\cos\beta=r\cdot\sin\beta-q\cdot\sin\alpha\cdot\cos\beta-\dfrac{g}{V}\cdot\sin\theta+\dfrac{X}{m\cdot V} \\ \overset{\circ}{\beta}\cdot\cos\beta=p\cdot\sin\alpha\cdot\cos\beta-r\cdot\cos\alpha\cdot\cos\beta+\dfrac{g}{V}\cdot\sin\varphi\cdot\cos\theta+\dfrac{Y}{m\cdot V} \\ -\overset{\circ}{\beta}\cdot\sin\alpha\cdot\sin\beta+\overset{\circ}{\alpha}\cdot\cos\alpha\cdot\cos\beta=-p\cdot\sin\beta+q\cdot\cos\alpha\cdot\cos\beta+\dfrac{g}{V}\cdot\cos\varphi\cdot\cos\theta+\dfrac{Z}{m\cdot V} \end{cases} \tag{2}$$

Replacing $\overset{\circ}{V}$ by 0 in the system (1), the equalities(2) are obtained.

If in a flight with constant forward velocity V one has $\beta \equiv (2n+1)\cdot \pi/2$, *then the following equalities hold:*

$$\begin{cases} (-1)^n\cdot r-\dfrac{g}{V}\cdot\sin\theta+\dfrac{X}{m\cdot V}\equiv 0 \\ \dfrac{g}{V}\cdot\sin\varphi\cdot\cos\theta+\dfrac{Y}{m\cdot V}\equiv 0 \\ (-1)^{n+1}\cdot p+\dfrac{g}{V}\cdot\cos\varphi\cdot\cos\theta+\dfrac{Z}{m\cdot V}\equiv 0 \end{cases} \tag{3}$$

Replacing $\overset{\circ}{\beta}=0$ and $\beta\equiv(2n+1)\cdot\pi/2$ in (2), the equalities (3) are obtained.

If in a flight with constant forward velocity V one has $\beta\neq(2n+1)\cdot\pi/2$, *then the following equality holds:*

$$\begin{gathered} g\left[\sin\beta\cdot\sin\varphi\cdot\cos\theta-\cos\alpha\cdot\cos\beta\cdot\sin\theta+\sin\alpha\cdot\cos\beta\cdot\cos\varphi\cdot\cos\theta\right]+\frac{Y}{m}\cdot\sin\beta+ \\ +\frac{X}{m}\cdot\cos\alpha\cdot\cos\beta+\frac{Z}{m}\cdot\sin\alpha\cdot\cos\beta\equiv 0 \end{gathered} \tag{4}$$

Equation (4) is the solvability (compatibility) condition, with respect to $\overset{\circ}{\alpha},\overset{\circ}{\beta}$, of the system (2) when $\beta\neq(2n+1)\cdot\pi/2$.

If $\beta \neq (2n+1)\cdot \pi/2$ *and equality (4) holds, then the system (2) can be solved with respect to* $\overset{\circ}{\alpha}, \overset{\circ}{\beta}$*, obtaining in this way the explicit system of differential equations, which describes the motion around the center of mass of the aircraft in a flight, with constant forward velocity* V*:*

$$\begin{cases} \overset{\circ}{\alpha} = q - p\cdot\cos\alpha\cdot\tan\beta - r\cdot\sin\alpha\cdot\tan\beta + \dfrac{g}{V\cdot\cos\beta}\cdot\left[\cos\varphi\cdot\cos\theta\cdot\cos\alpha + \right. \\ \quad \left. + \sin\theta\cdot\sin\alpha\right] + \dfrac{1}{\cos\beta}\cdot\left[\dfrac{Z}{m\cdot V}\cdot\cos\alpha - \dfrac{X}{m\cdot V}\cdot\sin\alpha\right] \\ \overset{\circ}{\beta} = p\cdot\sin\alpha - r\cdot\cos\alpha + \dfrac{1}{\cos\beta}\cdot\dfrac{g}{V}\cdot\sin\varphi\cdot\cos\theta + \dfrac{1}{\cos\beta}\cdot\dfrac{Y}{m\cdot V} \\ I_x\cdot\overset{\circ}{p} - I_{xz}\cdot\overset{\circ}{r} = \left(I_y - I_z\right)\cdot q\cdot r + I_{xz}\cdot p\cdot q + L \\ I_y\cdot\overset{\circ}{q} \qquad = \left(I_z - I_x\right)\cdot p\cdot r - I_{xz}\cdot\left(p^2 - r^2\right) + M \\ I_z\cdot\overset{\circ}{r} - I_{xz}\cdot\overset{\circ}{p} = \left(I_x - I_y\right)\cdot p\cdot q - I_{xz}\cdot q\cdot r + N \\ \overset{\circ}{\varphi} = p + q\cdot\sin\varphi\cdot\tan\theta + r\cdot\cos\varphi\cdot\tan\theta \\ \overset{\circ}{\theta} = q\cdot\cos\varphi - r\cdot\sin\varphi \\ \overset{\circ}{\psi} = \dfrac{q\cdot\sin\varphi + r\cdot\cos\varphi}{\cos\theta} \end{cases} \tag{5}$$

System (5) is obtained solving system (2) with respect to $\overset{\circ}{\alpha}$, $\overset{\circ}{\beta}$ and replacing in system (1) the equations $(1)_1$, $(1)_2$, $(1)_3$ with the obtained $\overset{\circ}{\alpha}$ and $\overset{\circ}{\beta}$.

A **longitudinal flight** is defined as a flight for which the following equalities hold:

$$\beta \equiv p \equiv r \equiv \varphi \equiv \psi \equiv 0 \text{ and } \delta_a = \delta_r = 0\,. \tag{6}$$

A longitudinal flight is possible if and only if $Y = L = N = 0$ *for* $\beta = p = r = \varphi = \psi = 0$ *and* $\delta_a = \delta_r = 0$.

This result is obtained from (1) taking into account the definition of a longitudinal flight.

The explicit system of differential equations which describes the motion of the aircraft in a longitudinal flight is:

$$\begin{cases} \overset{\circ}{V} = g\cdot\sin(\alpha - \theta) + \dfrac{X}{m}\cdot\cos\alpha + \dfrac{Z}{m}\cdot\sin\alpha \\ \overset{\circ}{\alpha} = q + \dfrac{g}{V}\cdot\cos\left(\theta - \alpha\right) - \dfrac{X}{m\cdot V}\cdot\sin\alpha + \dfrac{Z}{m\cdot V}\cdot\cos\alpha \\ \overset{\circ}{q} = \dfrac{M}{I_y} \\ \overset{\circ}{\theta} = q \end{cases} \tag{7}$$

This result is obtained from (1) taking into account the definition of a longitudinal flight.
In system (7) X, Z, M depend only on α, q, θ and δ_e. These dependences are obtained replacing in the general expression of the aerodynamic forces and moments: $\beta = p = r = \varphi = \psi = 0$ and $\delta_a = \delta_r = 0$.

The explicit system of differential equations which describes the motion around the center of gravity of the aircraft in a longitudinal flight with constant forward velocity V is:

$$\begin{cases} \overset{\circ}{\alpha} = q + \dfrac{g}{V} \cdot \cos(\theta - \alpha) - \dfrac{X}{m \cdot V} \cdot \sin\alpha + \dfrac{Z}{m \cdot V} \cdot \cos\alpha \\ \overset{\circ}{q} = \dfrac{M}{I_y} \\ \overset{\circ}{\theta} = q \end{cases} \tag{8}$$

This system is obtained from (7) taking into account $\overset{\circ}{V} = 0$.

A longitudinal flight with constant forward velocity is possible if the following equalities hold:

$$Y = L = N = 0 \textit{ for } \beta = p = r = \varphi = \psi = 0 \textit{ and } \delta_a = \delta_r = 0 \tag{9}$$

$$g \cdot \sin(\alpha - \theta) + \frac{X}{m} \cdot \cos\alpha + \frac{Z}{m} \cdot \sin\alpha = 0 \tag{10}$$

This result is obtained from system (7), taking into account the fact that $\overset{\circ}{V}$ is equal to zero.
In (8) X, Z, M depend on α, q, θ, δ_e and V. Taking into account (10), the system (8) can be written as:

$$\begin{cases} \overset{\circ}{\alpha} = q + \dfrac{g}{V} \cdot \cos(\theta - \alpha) - \dfrac{g}{V} \cdot \sin(\theta - \alpha) \cdot \tan\alpha + \dfrac{Z}{m \cdot V} \cdot \dfrac{1}{\cos\alpha} \\ \overset{\circ}{q} = \dfrac{M}{I_y} \\ \overset{\circ}{\theta} = q \end{cases} \tag{11}$$

The system (11) describes the motion around the center of gravity of an aircraft in a longitudinal flight with constant forward velocity V and defines the general nonlinear model.
In system (11) the functions $Z = Z(\alpha, q, \theta; \delta_e, V)$ and $M = M(\alpha, q, \theta; \delta_e, V)$ are considered known. When the automated flight control system fails, δ_e and V are parameters.
Frequently, in a research environment for the description of the movement around the center of the gravity of some types of aircrafts in a flight with constant forward velocity V, the explicit system of differential equations (12) is employed by Balint et al., 2009a,b,c; 2010a,b; Kaslik & Balint, 2007; Goto & Matsumoto, 2000.
The model defined by equations (12) is called *Aero Data Model In a Research Environment (ADMIRE)*.

System (12) can be obtained from (5) substituting the general aero dynamical forces and moments (see for example section 4), assuming that a and β are small and making the approximations (13).
Due to the approximations (13), the ADMIRE model defined by (12) is also called the *simplified ADMIRE model*.
The simplified system which governs the longitudinal flight with constant forward velocity V of the ADMIRE aircraft is (14).
System (14) is obtained from (12) for $\beta = p = r = \varphi = 0$, $\delta_a = \delta_r = \delta_c = \delta_{ca} = 0$ and defines the simplified nonlinear model of the motion around the center of gravity of the aircraft in a longitudinal flight with constant forward velocity V.
In system (14) $g, V, z_\alpha, z_{\delta_e}, m_\alpha, m_q, \overline{m_{\overset{\circ}{\alpha}}}, c_2, a_2, a, m_{\delta_e}$ are considered constants (see Section 4).

$$\begin{cases} \overset{\circ}{\alpha} = q - p\cdot\beta + \dfrac{g}{V}\cdot\cos\theta\cdot\cos\varphi + z_\alpha\cdot\alpha + y_\beta\cdot\beta^2 + y_p(\alpha,\beta)\cdot p\cdot\beta + y_r(\beta)\cdot r\cdot\beta + \\ \qquad + y_{\delta_a}\cdot\beta\cdot\delta_a + z_{\delta_e}\cdot\delta_e + y_{\delta_r}\cdot\beta\cdot\delta_r \\ \overset{\circ}{\beta} = p\cdot\alpha - r + \dfrac{g}{V}\cdot\sin\varphi\cdot\cos\theta - z_\alpha\cdot\alpha\cdot\beta + y_\beta\cdot\beta - y_p(\alpha,\beta)\cdot p - y_r(\beta)\cdot r + \\ \qquad + y_{\delta_a}\cdot\delta_a - z_{\delta_e}\cdot\beta\cdot\delta_e + y_{\delta_r}\cdot\delta_r \\ \overset{\circ}{p} = -i_1\cdot q\cdot r + l_\beta(\alpha)\cdot\beta + l_p\cdot p + l_r(\alpha)\cdot r + l_{\delta_a}\cdot\delta_a + l_{\delta_r}\cdot\delta_r \\ \overset{\circ}{q} = i_2\cdot p\cdot r + m_\alpha\cdot\alpha + m_q\cdot q - \overline{m_{\overset{\circ}{\alpha}}}\cdot p\cdot\beta + \overline{y_p}\cdot p\cdot\beta + \overline{y_\beta}\cdot\beta^2 + \overline{y_r}(\beta)\cdot r\cdot\beta + \\ \qquad + \dfrac{g}{V}\cdot\left(\overline{m_{\overset{\circ}{\alpha}}}\cdot\cos\theta\cdot\cos\varphi - \dfrac{c_2}{a}\cdot a_2\cdot\sin\theta\right) + \overline{y_{\delta_a}}\cdot\beta\cdot\delta_a + m_{\delta_c}\cdot\delta_c + \\ \qquad + m_{\delta_e}\cdot\delta_e + \overline{y_{\delta_r}}\cdot\beta\cdot\delta_r \\ \overset{\circ}{r} = -i_3\cdot p\cdot q + n_\beta\cdot\beta + n_p(\alpha,\beta)\cdot p + n_r(\alpha,\beta)\cdot r + n_{\delta_a}\cdot\delta_a + n_{\delta_{ca}}(\alpha)\cdot\delta_{ca} + n_{\delta_r}\cdot\delta_r \\ \overset{\circ}{\phi} = p + (q\cdot\sin\varphi + r\cdot\cos\varphi)\cdot\tan\theta \\ \overset{\circ}{\theta} = q\cdot\cos\varphi - r\cdot\sin\varphi \\ \overset{\circ}{\psi} = \dfrac{q\cdot\sin\varphi + r\cdot\cos\varphi}{\cos\theta} \end{cases} \tag{12}$$

$$\begin{cases} in\,(5)_1 \quad \cos\beta\approx 1;\ -p\cdot\cos\alpha\cdot\tan\beta\approx -p\cdot\beta;\ -r\cdot\sin\alpha\cdot\tan\beta\approx 0;\ \cos\alpha\approx 1;\sin\alpha\approx 0 \\ in\,(5)_2 \quad \cos\beta\approx 1 \\ \dfrac{I_{xz}\cdot\left(I_x + I_z - I_y\right)}{I_x\cdot I_z - I_{xz}^2}\approx 0; \qquad \dfrac{\left(I_y - I_z\right)\cdot I_z - I_{xz}^2}{I_x\cdot I_z - I_{xz}^2}\approx -i_1; \qquad \dfrac{I_{xz}}{I_y}\approx 0; \\ \dfrac{I_{xz}^2 + \left(I_x - I_y\right)\cdot I_x}{I_x\cdot I_z - I_{xz}^2}\approx -i_3; \quad \dfrac{I_{xz}\cdot\left(I_y - I_z - I_{xz}\right)}{I_x\cdot I_z - I_{xz}^2}\approx 0; \end{cases} \tag{13}$$

$$\begin{cases} \overset{\circ}{\alpha} = q + \frac{g}{V} \cdot \cos\theta + z_\alpha \cdot \alpha + z_{\delta_e} \cdot \delta_e \\ \overset{\circ}{q} = m_\alpha \cdot \alpha + m_q \cdot q + \frac{g}{V} \cdot \left(\overline{m_{\overset{\circ}{\alpha}}} \cdot \cos\theta - \frac{c_2}{a} \cdot a_2 \cdot \sin\theta \right) + m_{\delta_e} \cdot \delta_e \\ \overset{\circ}{\theta} = q \end{cases} \tag{14}$$

3. Theoretical proof of the existence of oscillatory movements

Interest in oscillation susceptibility of an aircraft is generated by crashes of high performance fighter airplanes such as the YF-22A and B-2, due to oscillations that were not predicted during the aircraft development (Mehra & Prasanth, 1998).Flying qualities and oscillation prediction are based on linear analysis and their quasi-linear extensions. These analyses cannot, in general, predict the presence or the absence of oscillations, because of the large variety of nonlinear interactions that have been identified as factors contributing to oscillations. The effects of some of these factors have been reported by Mehra et al., 1977; Shamma & Athans, 1991; Kish et al., 1997; Klyde et al., 1997. The oscillation susceptibility analysis in a nonlinear model involves the computation of nonlinear phenomena including bifurcations, which lead sometimes to large changes in the stability of the aircraft.

Interest in oscillation susceptibility analysis of an unmanned aircraft, whose flight control system fails, was generated by the elaboration of an alternative automatic flight control for the case when the existing automatic flight control system of the aircraft fails Goto & Matsumoto, 2000. Numerical results concerning this problem for the ALFLEX unmanned reentry vehicle, considered by Goto & Matsumoto, 2000, for the final approach and landing phase are reported by Kaslik et al., 2002; 2004 a; 2004 b; 2005 a; 2005 b; 2005 c; Caruntu et al., 2005.

A theoretical proof of the existence of oscillatory solutions of the ALFLEX unmanned reentry vehicle in a longitudinal flight with constant forward velocity and decoupled flight control system is reported Kaslik & Balint, 2010. Numerical results related to oscillation susceptibility analysis along the path of the longitudinal flight equilibriums of a simplified ADMIRE-model, when the automated flight control system fails are reported by Balint et al., 2009 a; 2009 b; 2009 c; 2010 a.

In this section we give a theoretical proof of the existence of oscillatory solutions in longitudinal flight in the simplified ADMIRE model of an unmanned aircraft, when the flight control system fails. This result was established by Balint et al., 2010b.

The simplified system of differential equations which governs the motion around the center of mass in a longitudinal flight with constant forward velocity of a rigid aircraft, when the automatic flight control system fails, is given by (see Balint et al., 2010b):

$$\begin{cases} \overset{\circ}{\alpha} = z_\alpha \cdot \alpha + q + \frac{g}{V} \cdot \cos\theta + z_{\delta_e} \cdot \delta_e \\ \overset{\circ}{q} = m_\alpha \cdot \alpha + m_q \cdot q + \frac{g}{V} \cdot \left(\overline{m_{\dot{\alpha}}} \cdot \cos\theta - \frac{c_2}{a} \cdot a_2 \cdot \sin\theta \right) + m_{\delta_e} \cdot \delta_e \\ \overset{\circ}{\theta} = q \end{cases} \tag{15}$$

In this system, the state parameters are: angle of attack α, pitch rate q and Euler pitch angle θ. The control parameter is the elevator angle δ_e. V is the forward velocity of the aircraft, considered constant and g is the gravitational acceleration.
The aero dynamical data appearing in (15) are given in section 4.
The following proposition (Balint et al., 2010b) addresses the existence of equilibrium states for the system (15).
Proposition 1. *$(\alpha, q\ \theta)^T$ is an equilibrium state of the system (15) corresponding to δ_e if and only if α is a solution of the equation:*

$$A \cdot \alpha^2 + B \cdot \delta_e \cdot \alpha + C \cdot \delta_e{}^2 + D = 0 \tag{16}$$

q is equal to zero and θ is a solution of the equation:

$$\cos\theta = -\frac{V}{g}\left[z_\alpha \cdot \alpha + z_{\delta_e} \cdot \delta_e\right] \tag{17}$$

where A, B, C, D are given by:

$$\begin{aligned}
A &= \left(m_\alpha - \overline{m_{\dot\alpha}} \cdot z_\alpha\right)^2 + \frac{c_2{}^2}{a^2} \cdot a_2{}^2 \cdot z_\alpha{}^2 \\
B &= 2 \cdot \left(m_\alpha - \overline{m_{\dot\alpha}} \cdot z_\alpha\right) \cdot \left(m_{\delta_e} - \overline{m_{\dot\alpha}} \cdot z_{\delta_e}\right) + 2 \cdot \frac{c_2{}^2}{a^2} \cdot a_2{}^2 \cdot z_\alpha \cdot z_{\delta_e} \\
C &= \left(m_{\delta_e} - \overline{m_{\dot\alpha}} \cdot z_{\delta_e}\right)^2 + \frac{c_2{}^2}{a^2} \cdot a_2{}^2 \cdot z_{\delta_e}{}^2 \\
D &= -\frac{g^2}{V^2} \cdot \frac{c_2{}^2}{a^2} \cdot a_2{}^2
\end{aligned} \tag{18}$$

Proof. By computation.
Proposition 2. *Equation (16) has real solutions if and only if δ_e satisfies:*

$$\left|\delta_e\right| \le \sqrt{\frac{4 \cdot A \cdot D}{B^2 - 4 \cdot A \cdot C}} \tag{19}$$

Proof. By computation.
Proposition 3. *If $z_\alpha < 0$ and α is a real solution of Eq.(16), then Eq.(17) has a solution if and only if for α the following inequality holds:*

$$\frac{1}{z_\alpha} \cdot \left[\frac{g}{V} - z_{\delta_e} \cdot \delta_e\right] \le \alpha \le -\frac{1}{z_\alpha} \cdot \left[\frac{g}{V} + z_{\delta_e} \cdot \delta_e\right] \tag{20}$$

Proof. By computation.
Remark. For $\delta_e = 0$ the solutions of Eq.(16) are:

$$\alpha = \pm\sqrt{\frac{\frac{g^2}{V^2} \cdot \frac{c_2{}^2}{a^2} \cdot a_2{}^2}{\frac{c_2{}^2}{a^2} \cdot a_2{}^2 \cdot z_\alpha{}^2 + \left(m_\alpha - \overline{m_{\dot\alpha}} \cdot z_\alpha\right)^2}} \tag{21}$$

and both verify (20) for $z_\alpha < 0$.
In the following assume that $z_\alpha < 0$ and consider $\underline{\delta_e}$, $\overline{\delta_e}$ defined as follows:

$$\underline{\delta_e} = \inf\left\{ \delta_e \middle| \delta_e < 0, \exists\ a\ real\ solution\ of\ eq.(16)\ for\ which\ (20)\ holds \right\}$$

$$\overline{\delta_e} = \sup\left\{ \delta_e \middle| \delta_e > 0, \exists\ a\ real\ solution\ of\ eq.(16)\ for\ which\ (20)\ holds \right\}$$

Let I be the closed interval $I = [\underline{\delta_e}, \overline{\delta_e}]$.

Proposition 4.

a. *If $\delta_e \in I$, then for the system (15) there exists a countable infinity of equilibriums corresponding to δ_e, namely for any $n \in Z$*

$$\left(\alpha_{1,2} = -\frac{B}{2 \cdot A} \cdot \delta_e \pm \frac{1}{2 \cdot A} \cdot \sqrt{B^2 \cdot \delta_e^{\ 2} - 4 \cdot A \cdot \left(C \cdot \delta_e^{\ 2} + D \right)};\quad q = 0; \right.$$
$$\left. \theta_{1,2} = 2 \cdot \pi \cdot n \pm \arccos\left[\frac{V}{g} \cdot \left(-z_\alpha \cdot \alpha_n^{\ \pm} - z_{\delta_e} \cdot \delta_e \right) \right] \right)$$

b. *If $\delta_e \in \partial I = \left\{ \underline{\delta_e}, \overline{\delta_e} \right\}$, then the equilibriums corresponding to δ_e are saddle-node bifurcation points.*
c. *If $\delta_e \in \overline{I}$, then for the system (15) there are no equilibriums corresponding to δ_e.*

Proof. By computation.

Proposition 4 translates into the following necessary and sufficient condition for the existence of equilibrium states for (15): $\delta_e \in I = \left[\underline{\delta_e}, \overline{\delta_e} \right]$. *At* $\delta = \underline{\delta_e}, \overline{\delta_e}$ saddle-node bifurcation occurs.

It can be easily verified that the following proposition is valid.

Proposition 5.

a. *If $\left(\alpha(t), q(t), \theta(t) \right)^T$ is a solution of the system (15), then $\theta(t)$ is a solution of the third order differential equation:*

$$\begin{aligned} &\dddot{\theta} - \left(z_\alpha + m_q \right) \cdot \ddot{\theta} + \left[z_\alpha \cdot m_q - m_\alpha + \frac{g}{V} \cdot \left(\overline{m_{\dot{\alpha}}} \cdot \sin\theta + \frac{c_2}{a} \cdot a_2 \cdot \cos\theta \right) \right] \cdot \dot{\theta} = \\ &= \frac{g}{V} \cdot \left(m_\alpha - z_\alpha \cdot \overline{m_{\dot{\alpha}}} \right) \cdot \cos\theta + \frac{g}{V} \cdot z_\alpha \cdot \frac{c_2}{a} \cdot a_2 \cdot \sin\theta + \left(m_\alpha \cdot z_{\delta_e} - z_\alpha \cdot m_{\delta_e} \right) \cdot \delta_e \end{aligned} \quad (22)$$

b. *If $\theta(t)$ is a solution of (22), then*

$$\alpha(t) = \frac{1}{m_\alpha} \cdot \left[\ddot{\theta} - m_q \cdot \dot{\theta} - \frac{g}{V} \cdot \left(\overline{m_{\dot{\alpha}}} \cdot \cos\theta - \frac{c_2}{a} \cdot a_2 \cdot \sin\theta \right) - m_{\delta_e} \cdot \delta_e \right]$$
$$q(t) = \dot{\theta}(t)$$
$$\theta(t) = \theta(t)$$

is a solution of the system (15).

Proof. By computation.

A solution $\theta(t)$ of Eq.(22) is called monotonic oscillatory solution if the derivative $\dot{\theta}(t)$ is a strictly positive or a strictly negative periodic function.

Proposition 6. a. *If there exists an increasing oscillatory solution $\theta(t)$ of Eq.(22)and $T>0$ is the period of $\dot{\theta}(t)$, then there exists $n \in N^*$ such that $\theta(t+T)=\theta(t)+2n\pi$ and there exists a $2n\pi$ - periodic solution $x(s)$ of the equation:*

$$\begin{aligned} x'' = 2(x')^2 + \left(z_\alpha + m_q\right)\cdot x' \cdot e^x + \left[z_\alpha \cdot m_q - m_\alpha + \frac{g}{V}\cdot\left(\overline{m_{\dot{\alpha}}}\cdot \sin s + \frac{c_2}{a}\cdot a_2 \cdot \cos s\right)\right]\cdot e^{2x} - \\ -\left[\frac{g}{V}\cdot\left(m_\alpha - z_\alpha \cdot \overline{m_{\dot{\alpha}}}\right)\cdot \cos s + \frac{g}{V}\cdot z_\alpha \cdot \frac{c_2}{a}\cdot a_2 \cdot \sin s + \left(m_\alpha \cdot z_{\delta_e} - z_\alpha \cdot m_{\delta_e}\right)\cdot \delta_e\right]\cdot e^{3x} \end{aligned} \tag{23}$$

satisfying

$$\int_0^{2n\pi} e^{x(s)} \cdot ds = T \tag{24}$$

b. *If for $n \in N^*$ there exists a $2n\pi$ - periodic solution $x(s)$ of Eq.(23), then there exists an increasing oscillatory solution $\theta(t)$ of Eq.(22) satisfying $\theta(t+T)=\theta(t)+2n\pi$, where T is given by (24).*

Proof. See Balint et al., 2010b.

In order to prove that Eq.(22) has an increasing oscillatory solution, it is sufficient to prove that there exists a $2n\pi$ - periodic solution of the Eq.(23).

Denoting $x_1 = x$ and $x_2 = -(z_\alpha + m_q)\cdot e^{-x} - x' \cdot e^{-2x}$, eq.(23) is replaced by the system:

$$\begin{aligned} x_1' &= -(z_\alpha + m_q)\cdot e^{x_1} - x_2 \cdot e^{2x_1} \\ x_2' &= \left[\left(-z_\alpha \cdot m_{\delta_e} + m_\alpha \cdot z_{\delta_e}\right)\cdot \delta_e + \frac{g}{V}\cdot\left(m_\alpha - z_\alpha \cdot \overline{m_{\dot{\alpha}}}\right)\cdot \cos s + \frac{g}{V}\cdot z_\alpha \cdot \frac{c_2}{a}\cdot a_2 \cdot \sin s\right]\cdot e^{x_1} - \\ &\quad -\left[z_\alpha \cdot m_q - m_\alpha + \frac{g}{V}\cdot\left(\overline{m_{\dot{\alpha}}}\cdot \sin s + \frac{c_2}{a}\cdot a_2 \cdot \cos s\right)\right] \end{aligned} \tag{25}$$

Proposition 7. *a. If there exists a decreasing oscillatory solution $\theta(t)$ of (22) and $T>0$ is the period of $\dot{\theta}(t)$, then there exists $n \in N^*$ such that $\theta(t+T)=\theta(t)-2n\pi$ and there exists a $2n\pi$ - periodic solution $x(s)$ of the equation:*

$$\begin{aligned} x'' = 2(x')^2 - \left(z_\alpha + m_q\right)\cdot x' \cdot e^x + \left[z_\alpha \cdot m_q - m_\alpha + \frac{g}{V}\cdot\left(\overline{m_{\dot{\alpha}}}\cdot \sin s + \frac{c_2}{a}\cdot a_2 \cdot \cos s\right)\right]\cdot e^{2x} + \\ +\left[\frac{g}{V}\cdot\left(m_\alpha - z_\alpha \cdot \overline{m_{\dot{\alpha}}}\right)\cdot \cos s + \frac{g}{V}\cdot z_\alpha \cdot \frac{c_2}{a}\cdot a_2 \cdot \sin s + (m_\alpha \cdot \delta_e - z_\alpha \cdot m_{\delta_e})\cdot \delta_e\right]\cdot e^{3x} \end{aligned} \tag{26}$$

satisfying

$$T = \int_0^{2n\pi} e^{x(s)} \cdot ds \tag{27}$$

b. *If for* $n \in N^*$ *, there exists a* $2n\pi$ *- periodic solution* $x(s)$ *of(26), then there exists a decreasing oscillatory solution* $\theta(t)$ *of (22), satisfying* $\theta(t+T) = \theta(t) - 2n\pi$ *with* T *given by (27).*

Proof. See Balint et al., 2010b.

It follows that in order to prove that Eq.(22) has a decreasing oscillatory solution, it is sufficient to prove that there exists a $2n\pi$ - periodic solution of the Eq.(26).

Denoting by $x_1 = x$ and $x_2 = (z_\alpha + m_q)\cdot e^{-x} + x' \cdot e^{-2x}$ Eq.(26) is replaced by the system:

$$
\begin{aligned}
x_1' &= (z_\alpha + m_q)\cdot e^{x_1} + x_2 \cdot e^{2x_1} \\
x_2' &= \left[\left(-z_\alpha \cdot m_{\delta_e} + m_\alpha \cdot z_{\delta_e}\right)\cdot \delta_e + \frac{g}{V}\cdot\left(m_\alpha - z_\alpha \cdot \overline{m_{\dot\alpha}}\right)\cdot \cos s + \frac{g}{V}\cdot z_\alpha \cdot \frac{c_2}{a}\cdot a_2 \cdot \sin s \right]\cdot e^{x_1} \\
&\quad + \left[z_\alpha \cdot m_q - m_\alpha + \frac{g}{V}\cdot\left(\overline{m_{\dot\alpha}}\cdot \sin s + \frac{c_2}{a}\cdot a_2 \cdot \cos s\right)\right]
\end{aligned} \tag{28}
$$

Hence, in order to show the existence of a decreasing oscillatory solution of Eq.(22) it is sufficient to show the existence of a $2n\pi$ - periodic solution of (28).

For a continuous $2n\pi$ *periodic function we define:*

$$
|f|_M = \max_{s\in[0,\,2n\pi]} f(s) \qquad\qquad |f|_L = \min_{s\in[0,\,2n\pi]} f(s)
$$

***Proposition** 8. If* f *is a smooth* $2n\pi$ *periodic function, then*

$$
|f|_M \le |f|_L + \frac{1}{2}\cdot \int_0^{2n\pi} \left|\dot f(s)\right|\cdot ds .
$$

Proof. See Balint et al., 2010b.

Let X, Y be two infinite dimensional Banach spaces. A linear operator $L : DomL \subset X \to Y$ is called a Fredholm operator if $Ker L$ has finite dimension and $\operatorname{Im} L$ is closed and has finite codimension. The index of a Fredholm operator L is the integer $i(L) = \dim KerL - co\dim \operatorname{Im} L$.

In the following, consider $L : DomL \subset X \to Y$ a Fredholm operator of index zero, which is not injective. Let $P : X \to X$ and $Q : Y \to Y$ be continuous projectors, such that $KerQ = \operatorname{Im} L, \operatorname{Im} P = KerL, X = KerL \oplus KerP$ and $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. The operator $L_P = L\big|_{DomL \cap KerP} : DomL \cap KerP \to \operatorname{Im} L$ is an isomorphism. Consider the operator $K_{PQ} : Y \to X$ defined by $K_{PQ} = K_P^{-1}(I - Q)$. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be a continuous nonlinear operator. We say that N is L compact if $K_{PQ}N$ is compact, QN is continuous and $QN(\overline{\Omega})$ is a bounded set in Y. Since $\operatorname{Im} Q$ is isomorphic to $KerL$, there exists an isomorphism $I : \operatorname{Im} Q \to KerL$.

Mawhin, 1972; Gaines & Mawhin, 1977 established the *Mawhin's continuation theorem : Let* $\Omega \subset X$ *be an open bounded set, let* L *be a Fredholm operator of index zero and let* N *be* $L-$ *compact on* $\overline{\Omega}$. *Assume*:

a. $Lx \ne \lambda Nx$ *for any* $\lambda \in (0, 1)$ *and* $x \in \partial\Omega \cap DomL$.

b. $QNx \ne 0$ *for any* $x \in KerL \cap \partial\Omega$.

c. *Brouwer degree* $\deg_B(IQN, \Omega \cap KerL, 0) \neq 0$.

Then $Lx = Nx$ *has at least one solution in* $DomL \cap \overline{\Omega}$.

Consider the Eq.(22) and denote by:

$$\begin{aligned}
&\tau = -(z_\alpha + m_q) \\
&\delta = z_\alpha \cdot m_q - m_\alpha + \frac{g}{V} \cdot \left(\overline{m_{\dot{\alpha}}} \cdot \sin\theta + \frac{c_2}{a} \cdot a_2 \cdot \cos\theta \right) \\
&\gamma = \left(m_\alpha \cdot z_{\delta_e} - z_\alpha \cdot m_{\delta_e} \right) \cdot \delta_e + \frac{g}{V} \cdot z_\alpha \cdot \frac{c_2}{a} \cdot a_2 \cdot \sin\theta \\
&\varepsilon = -\frac{g}{V} \cdot \left(m_\alpha - z_\alpha \cdot \overline{m_{\dot{\alpha}}} \right)
\end{aligned} \tag{29}$$

With these notations Eq.(22) can be written as:

$$\dddot{\theta} + \tau \cdot \ddot{\theta} + \delta \cdot \dot{\theta} = \gamma - \varepsilon \cdot \cos\theta \tag{30}$$

As concerns the quantities τ, δ, γ, we make the following assumptions:

$$\tau > 0, \quad \delta > 0 \text{ and } \tau^2 > 4\delta \tag{31}$$

Remark that the above inequalities can be assured as follows:

$$\begin{aligned}
&z_\alpha + m_q < 0 \quad (\Leftrightarrow \tau > 0) \\
&z_\alpha \cdot m_q - m_\alpha > \frac{g}{V} \cdot \sqrt{\overline{m_{\dot{\alpha}}}^2 + \frac{c_2^{\,2}}{a^2} \cdot a_2^{\,2}} \quad (\Rightarrow \delta > 0) \\
&(z_\alpha + m_q)^2 > 4 \cdot (z_\alpha \cdot m_q - m_\alpha) + \frac{4g}{V} \cdot \sqrt{\overline{m_{\dot{\alpha}}}^2 + \frac{c_2^{\,2}}{a^2} \cdot a_2^{\,2}} \quad (\Rightarrow \tau^2 > 4\delta)
\end{aligned} \tag{32}$$

Remark also that since $z_\alpha < 0$ for γ the following inequality holds:

$$\left(m_\alpha \cdot z_{\delta_e} - z_\alpha \cdot m_{\delta_e} \right) \cdot \delta_e - \frac{g}{V} \cdot z_\alpha \cdot \left| \frac{c_2}{a} \cdot a_2 \right| \geq \gamma \geq \left(m_\alpha \cdot z_{\delta_e} - z_\alpha \cdot m_{\delta_e} \right) \cdot \delta_e + \frac{g}{V} \cdot z_\alpha \cdot \left| \frac{c_2}{a} \cdot a_2 \right| \tag{33}$$

In terms of $\tau, \delta, \gamma, \varepsilon$ the systems (25) and (28) can be written in the forms:

$$\begin{cases} x_1' = \tau \cdot e^{x_1} - x_2 \cdot e^{2x_1} \\ x_2' = \left[\gamma(s) - \varepsilon \cdot \cos s \right] \cdot e^{x_1} - \delta(s) \end{cases} \tag{34}$$

and

$$\begin{cases} x_1' = -\tau \cdot e^{x_1} + x_2 \cdot e^{2x_1} \\ x_2' = \left[\gamma(s) - \varepsilon \cdot \cos s \right] \cdot e^{x_1} + \delta(s) \end{cases} \tag{35}$$

respectively.

Theorem 1. *If the inequalities (32) hold and*

$$\left(m_\alpha \cdot z_{\delta_e} - z_\alpha \cdot m_{\delta_e}\right)\cdot \delta_e + \frac{g}{V}\cdot z_\alpha \cdot \left|\frac{c_2}{a}\cdot a_2\right| > |\varepsilon| \tag{36}$$

then for any $n \in N^*$ *the system (2.20) has at least one* $2n\pi$ *-periodic solution.*
Proof. See Balint et al., 2010b.
Theorem 2. *If inequalities (32) hold and*

$$\left(m_\alpha \cdot z_{\delta_e} - z_\alpha \cdot m_{\delta_e}\right)\cdot \delta_e - \frac{g}{V}\cdot z_\alpha \cdot \left|\frac{c_2}{a}\cdot a_2\right| < -|\varepsilon| \tag{37}$$

then for any $n \in N^*$ *the system (2.21) has at least one* $2n\pi$ *-periodic solution.*
Proof. See Balint et al., 2010b.
The conclusion of this section can be summarized as:
Theorem 3. *If inequalities (32) and (36) hold, then for any* $n \in N^*$ *equation (22) has at least one solution* $\theta(t)$*, such that its derivative* $\dot{\theta}(t)$ *is a positive* $2n\pi$ *-periodic function (i.e.* $\theta(t)$ *is an increasing oscillatory solution).*
If inequalities (32) and (37) hold, then for any $n \in N^*$ *equation (22) has at least one solution* $\theta(t)$*, such that its derivative* $\dot{\theta}(t)$ *is a negative periodic function (i.e.* $\theta(t)$ *is a decreasing oscillatory solution).*

4. Numerical examples

To describe the flight of ADMIRE (Aero Data Model in a Research Environment) aircraft with constant forward velocity V, the system of differential equations (12) is employed:
where:

$z_\alpha = a\cdot C_N^\alpha \quad z_{\delta_e} = a\cdot C_N^{\delta_e} \quad y_\beta = a\cdot C_y^\beta \quad y_r(\beta) = a\cdot C_y^r(\beta) \quad y_{\delta_r} = a\cdot C_y^{\delta_r}$

$y_{\delta_a} = a\cdot C_y^{\delta_a} \quad y_p(\alpha,\beta) = a\cdot C_y^p(\alpha,\beta) \quad m_\alpha = a_2\cdot\left(C_m^\alpha - c_1\cdot C_N^\alpha + c_2\cdot a\cdot C_T^\alpha + C_m^{\mathring{\alpha}}\cdot a\cdot C_N^\alpha\right)$

$m_{\delta_e} = a_2\cdot\left(C_m^{\delta_e} - c_1\cdot C_N^{\delta_e} + C_m^\alpha\cdot a\cdot C_N^{\delta_e}\right) \quad m_{\delta_c} = a_2\cdot C_m^{\delta_c} \quad m_q = a_2\cdot\left(C_m^q + C_m^{\mathring{\alpha}}\right) \quad m_{\mathring{\alpha}} = a_2\cdot C_m^{\mathring{\alpha}}$

$l_\beta(\alpha) = a_1\cdot C_l^\beta(\alpha) \quad l_p = a_1\cdot C_l^p \quad l_r(\alpha) = a_1\cdot C_l^r(\alpha) \quad l_{\delta_r} = a_1\cdot C_l^{\delta_r} \quad l_{\delta_a} = a_1\cdot C_l^{\delta_a}$

$n_\beta = a_3\cdot\left(C_n^\beta + c_3\cdot C_y^\beta\right) \quad n_p(\alpha,\beta) = a_3\cdot\left(C_n^p(\alpha,\beta) + c_3\cdot C_y^p(\alpha,\beta)\right) \quad n_{\delta_{ca}}(\alpha) = a_3\cdot c_3\cdot C_n^{\delta_{ca}}(\alpha)$

$n_r(\alpha,\beta) = a_3\cdot\left(C_n^r(\alpha,\beta) + c_3\cdot C_y^r(\beta)\right) \quad n_{\delta_r} = a_3\cdot\left(C_n^{\delta_r} + c_3\cdot C_y^{\delta_r}\right) \quad n_{\delta_a} = a_3\cdot\left(C_n^{\delta_a} + c_3\cdot C_y^{\delta_a}\right)$

$\overline{y_\beta} = a_2\cdot c_2\cdot a\cdot C_y^\beta \quad \overline{y_r} = a_2\cdot c_2\cdot a\cdot C_y^r(\beta) \quad \overline{y_p} = a_2\cdot c_2\cdot a\cdot C_y^p(\alpha,\beta)$

$\overline{y_{\delta_r}} = a_2\cdot c_2\cdot a\cdot C_y^{\delta_r} \quad \overline{y_{\delta_a}} = a_2\cdot c_2\cdot a\cdot C_y^{\delta_a} \quad C_T^\alpha = -0.157[rad^{-1}] \quad C_l^p = -0.28[rad^{-1}]$

$C_N^\alpha = 3.295[rad^{-1}] \quad C_l^r(\alpha) = (0.344\cdot\alpha + 0.02)[rad^{-1}] \quad C_N^{\delta_e} = 1.074[rad^{-1}]$

$C_n^\beta = 0.0907[rad^{-1}] \quad C_m^\alpha = 0.267[rad^{-1}] \quad C_n^{\delta_r} = -0.0846[rad^{-1}] \quad C_m^{\delta_e} = -0.426[rad^{-1}]$

$C_n^{\delta_a} = 0.051[rad^{-1}] \quad C_m^{\delta_c} = 0.2[rad^{-1}] \quad C_n^{\delta_{ca}}(\alpha) = -0.49\cdot\alpha + 0.0145[rad]$

$C_m^{\overset{\circ}{\alpha}} = -0.44[rad^{-1}]$ $C_m^q = -1.45[rad^{-1}]$ $C_1^{\beta}(\alpha) = 0.896 \cdot \alpha^2 - 0.47 \cdot \alpha - 0.04\,[rad]$

$C_y^{\beta} = 0.804[rad^{-1}]$ $C_y^{\delta_r} = -0.185[rad^{-1}]$ $C_y^{\delta_a} = 0.122[rad^{-1}]$ $C_y^r(\beta) = 2.725 \cdot \beta^2\,[rad]$

$C_y^p(\alpha,\beta) = (6.796 \cdot \alpha + 0.315) \cdot \beta^2 + (0.237 \cdot \alpha - 0.498) \cdot 10^{-3}\,[rad]$

$C_n^r(\alpha,\beta) = (1.572 \cdot \alpha^2 - 0.368 \cdot \alpha - 1.07) \cdot \beta^2 - 0.005[rad]$

$C_l^{\delta_r} = 0.024[rad];$ $C_l^{\delta_a} = 0.192[rad];$ $C_n^p(\alpha,\beta) = (2.865 \cdot \alpha + 0.3) \cdot \beta^2\,[rad]$

$H = 500[m]$ $M = 0.25$ $a_s = 338[m \cdot s^{-1}]$ $\rho = 1.16[kg \cdot m^{-3}]$

$g = 9.81[m \cdot s^{-2}]$ $V = M \cdot a_s = 84.5[m \cdot s^{-1}]$ $g / V = 0.116[s^{-1}]$

$S = 45[m^2]$ $\bar{c} = 5.2[m]$ $b = 10[m]$ $\overline{m} = 9100[kg]$ $x_G = 1.3[m]$ $z_e = -0.15$

$I_x = 21000[kg \cdot m^2]$ $I_y = 81000[kg \cdot m^2]$ $I_z = 101000[kg \cdot m^2]$

$a = -0.485[s^{-1}]$ $a_1 = 88.743[s^{-1}]$ $a_2 = 11.964[s^{-1}]$ $a_3 = 18.45[s^{-1}]$

$c_1 = 0.25$ $c_2 = -0.029$ $c_3 = 0.13$ $i_1 = 0.952$ $i_2 = 0.987$ $i_3 = 0.594$

All the other derivatives are equal to zero.

The system which governs the longitudinal flight with constant forward velocity V of the ADMIRE aircraft, when the automatic flight control fails, is:

$$\begin{cases} \overset{\circ}{\alpha} = q + \dfrac{g}{V} \cdot \cos\theta + z_\alpha \cdot \alpha + z_{\delta_e} \cdot \delta_e \\ \overset{\circ}{q} = m_\alpha \cdot \alpha + m_q \cdot q + \dfrac{g}{V} \cdot \left(\overline{m}_{\overset{\circ}{\alpha}} \cdot \cos\theta - \dfrac{c_2}{a} \cdot a_2 \cdot \sin\theta \right) + m_{\delta_e} \cdot \delta_e \\ \overset{\circ}{\theta} = q \end{cases} \tag{38}$$

When the automatic flight control system is in function, then δ_e in (38) is given by:

$$\delta_e = k_\alpha \cdot \alpha + k_q \cdot q + k_p \cdot \theta \tag{39}$$

with $k_\alpha = -0.401;$ $k_q = -1.284$ and $k_p = 1 \div 8$.

System (38) is obtained from the system (12) for $\beta = p = r = \varphi = 0$ $\delta_a = \delta_r = \delta_c = \delta_{ca} = 0$.
The equilibriums of (38) are the solutions of the nonlinear system of equations:

$$\begin{cases} q + \dfrac{g}{V} \cdot \cos\theta + z_\alpha \cdot \alpha + z_{\delta_e} \cdot \delta_e = 0 \\ m_\alpha \cdot \alpha + m_q \cdot q + \dfrac{g}{V} \cdot \left(\overline{m}_{\dot{\alpha}} \cdot \cos\theta - \dfrac{c_2}{a} \cdot a_2 \cdot \sin\theta \right) + m_{\delta_e} \cdot \delta_e = 0 \\ q = 0 \end{cases} \tag{40}$$

System (40) defines the equilibriums manifold of the longitudinal flight with constant forward velocity V of the ADMIRE aircraft.

It is easy to see that (40) implies:

$$A \cdot \alpha^2 + B \cdot \delta_e \cdot \alpha + C \cdot \delta_e{}^2 + D = 0 \tag{41}$$

where A,B,C,D are given by:

$$\begin{aligned}
A &= \left(m_\alpha - \overline{m}_{\dot{\alpha}} \cdot z_\alpha\right)^2 + \frac{c_2{}^2}{a^2} \cdot a_2{}^2 \cdot z_\alpha{}^2 \\
B &= 2 \cdot \left(m_\alpha - \overline{m}_{\dot{\alpha}} \cdot z_\alpha\right) \cdot \left(m_{\delta_e} - \overline{m}_{\dot{\alpha}} \cdot z_{\delta_e}\right) + 2 \cdot \frac{c_2{}^2}{a^2} \cdot a_2{}^2 \cdot z_\alpha \cdot z_{\delta_e} \\
C &= \left(m_{\delta_e} - \overline{m}_{\dot{\alpha}} \cdot z_{\delta_e}\right)^2 + \frac{c_2{}^2}{a^2} \cdot a_2{}^2 \cdot z_{\delta_e}{}^2 \\
D &= -\frac{g^2}{V^2} \cdot \frac{c_2{}^2}{a^2} \cdot a_2{}^2
\end{aligned}$$

Solving Eq.(41) two solutions $a_1 = a_1(\delta_e)$ and $a_2 = a_2(\delta_e)$ are obtained. Replacing in (17) $a_1 = a_1(\delta_e)$ and $a_2 = a_2(\delta_e)$ the corresponding $\theta_1 = \theta_1(\delta_e) + 2k\pi$ and $\theta_2 = \theta_2(\delta_e) + 2k\pi$ are obtained $(k \in Z)$. Hence a part of the equilibrium manifold $\mathcal{M}_V$ $(k = 0)$ is the union of the following two pieces:

$$\mathcal{P}_1 = \left\{\left(\alpha_1\left(\delta_e\right), 0, \theta_1\left(\delta_e\right)\right) \vdots\ \delta_e \in I\right\}; \qquad \mathcal{P}_2 = \left\{\left(\alpha_2\left(\delta_e\right), 0, \theta_2\left(\delta_e\right)\right) \vdots\ \delta_e \in I\right\}.$$

The interval I where δ_e varies follows from the condition that the angles $\alpha_1(\delta_e)$ and $\alpha_2(\delta_e)$ have to be real.

Using the numerical values of the parameters for the ADMIRE model aircraft and the software MatCAD Professional it was found that:

$\underline{\delta_e}$ = -0.04678233231992 [rad] and $\overline{\delta_e} = 0.04678233231992$[rad].

The computed $\alpha_1(\delta_e)$, $\theta_1(\delta_e)$, $\alpha_2(\delta_e)$, $\theta_2(\delta_e)$ are represented on Fig.1, 2.

Fig.1 shows that $\alpha_1\left(\underline{\delta_e}\right) = \alpha_2\left(\underline{\delta_e}\right)$, $\alpha_1\left(\overline{\delta_e}\right) = \alpha_2\left(\overline{\delta_e}\right)$ and $\alpha_1\left(\delta_e\right) > \alpha_2\left(\delta_e\right)$ for $\delta_e \in \left(\underline{\delta_e}, \overline{\delta_e}\right)$.

Fig.2 shows that $\theta_1\left(\underline{\delta_e}\right) = \theta_2\left(\underline{\delta_e}\right)$, $\theta_1\left(\overline{\delta_e}\right) = \theta_2\left(\overline{\delta_e}\right)$ and $\theta_1\left(\delta_e\right) < \theta_2\left(\delta_e\right)$ for $\delta_e \in \left(\underline{\delta_e}, \overline{\delta_e}\right)$.

The eigenvalues of the matrix $A(\underline{\delta_e})$ are: λ_1= - 22.6334; λ_2= - 1.5765; λ_3= 1.0703 x 10^{-8} ≈ 0 .

For $\delta_e > \underline{\delta_e}$ the equilibriums of $\mathcal{P}_1$ are exponentially stable and those of $\mathcal{P}_2$ are unstable. These facts were deduced computing the eigenvalues of $A(\delta_e)$.

More precisely, it was obtained that the eigenvalues of $A(\delta_e)$ are negative at the equilibriums of $\mathcal{P}_1$ and two of the eigenvalues are negative and the third is positive at the equilibriums of $\mathcal{P}_2$. Consequently, δ_e is a turning point. Maneuvers on $\mathcal{P}_1$ are successful and on $\mathcal{P}_2$ are not successful, Fig.3, 4.

Moreover, numerical tests show that when $\delta_e{}', \delta_e{}'' \in \left(\underline{\delta_e}, \overline{\delta_e}\right)$, the maneuver $\delta_e{}' \to \delta_e{}''$ transfers the ADMIRE aircraft from the state in which it is at the moment of the maneuver in the asymptotically stable equilibrium $\left(\alpha_1\left(\delta_e{}''\right), 0, \theta_1\left(\delta_e{}''\right)\right)$.

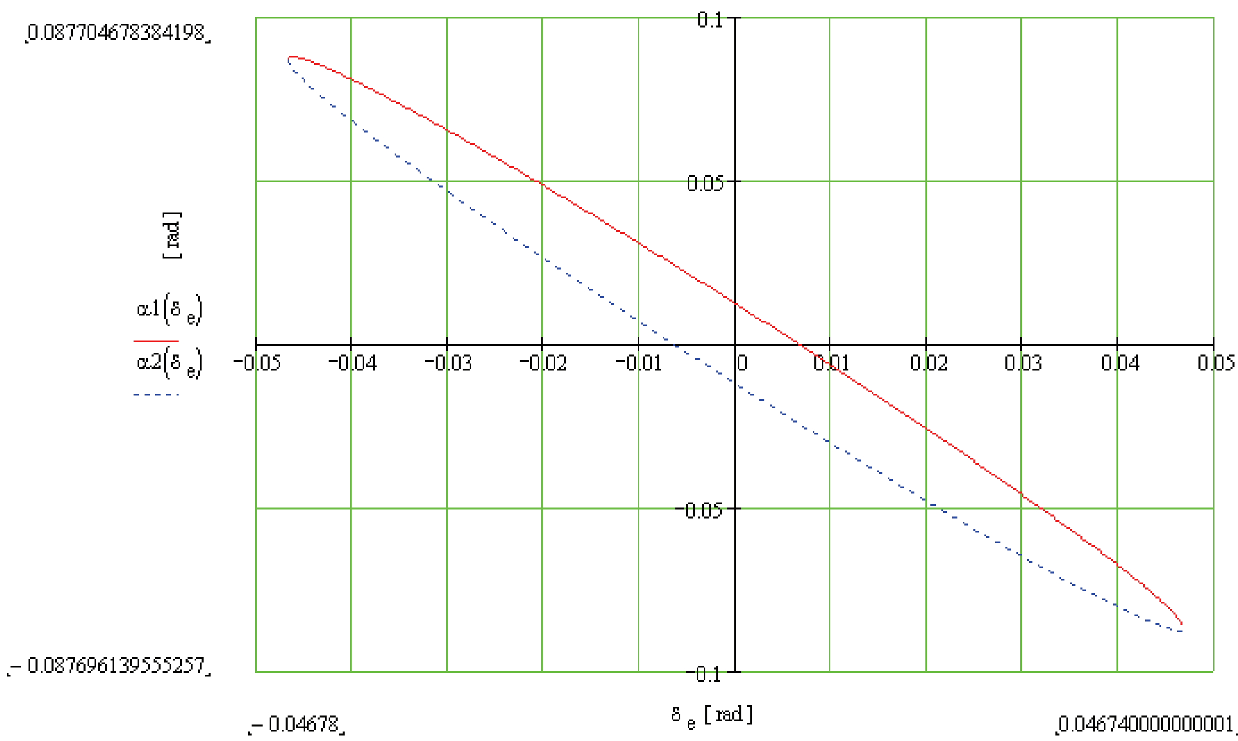

Fig. 1. The $a_1(\delta_e)$ and $a_2(\delta_e)$ coordinates of the equilibriums on the manifold M_V.

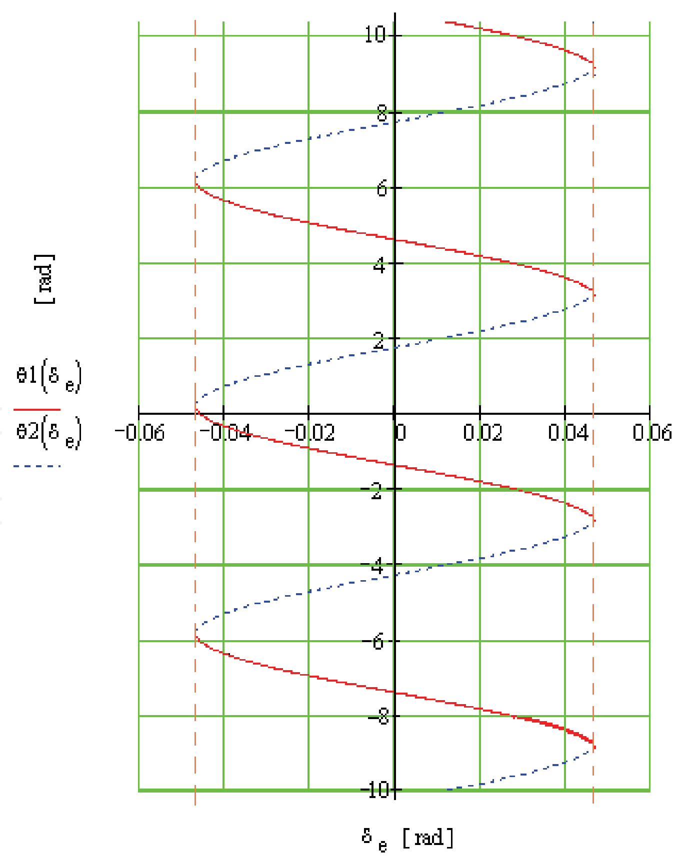

Fig. 2. The $\theta_1(\delta_e)+2k\pi$ and $\theta_2(\delta_e)+2k\pi$ coordinates of the equilibriums on the manifold $\mathcal{M}_V$.

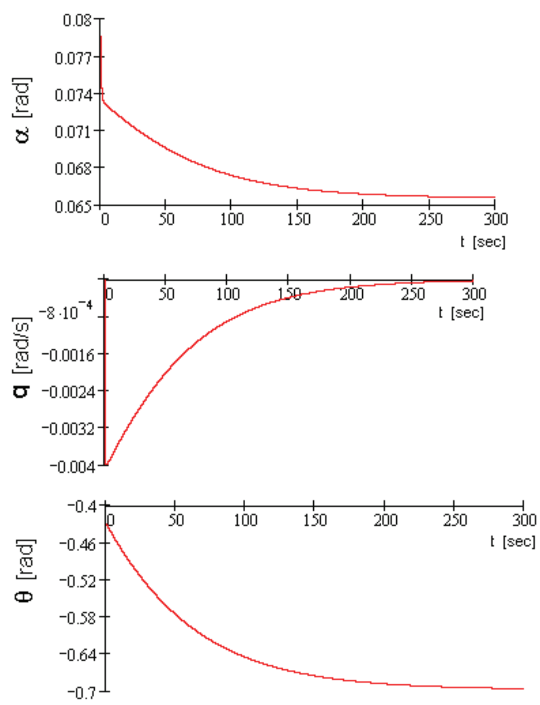

Fig. 3. A successful maneuver on $\mathcal{P}_1$:
a_1^1 = 0.078669740237840 [rad]; q_1^1= 0 [rad/s] ; θ_1^1= 0.428832005303479 [rad] →
a_1^2 = 0.065516737567037 [rad]; q_1^2= 0 [rad/s] ; θ_1^2= - 0.698066723826469 [rad]

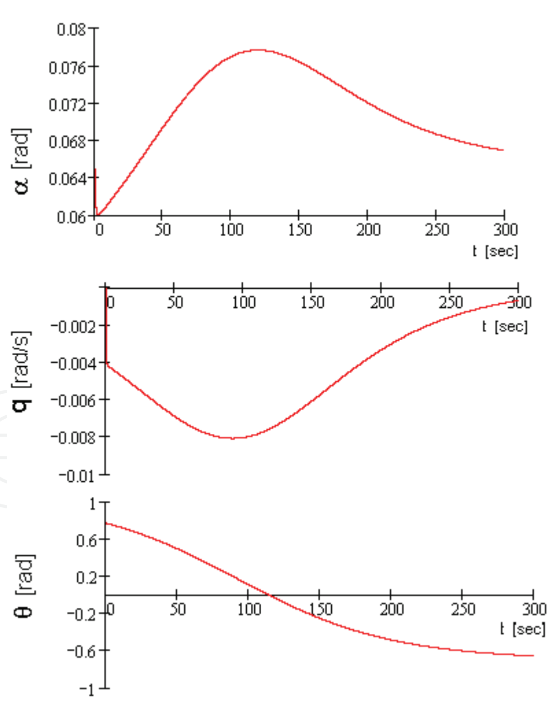

Fig. 4. An unsuccessful maneuver on $\mathcal{P}_2$:
a_2^1 = 0.064883075974905 [rad]; q_2^1= 0 [rad/s] ; θ_2^1= 0.767462467841413 [rad] →
a_1^2 = 0.065516737567037 [rad]; q_1^2= 0 [rad/s] ; θ_1^2= 0.698066723826469 [rad] instead of
a_2^1 = 0.064883075974905 [rad]; q_2^1= 0 [rad/s] ; θ_2^1= 0.767462467841413 [rad] →
a_2^2 = 0.046845089090947 [rad]; q_2^2= 0 [rad/s] ; θ_2^2= 1.036697186364400 [rad].

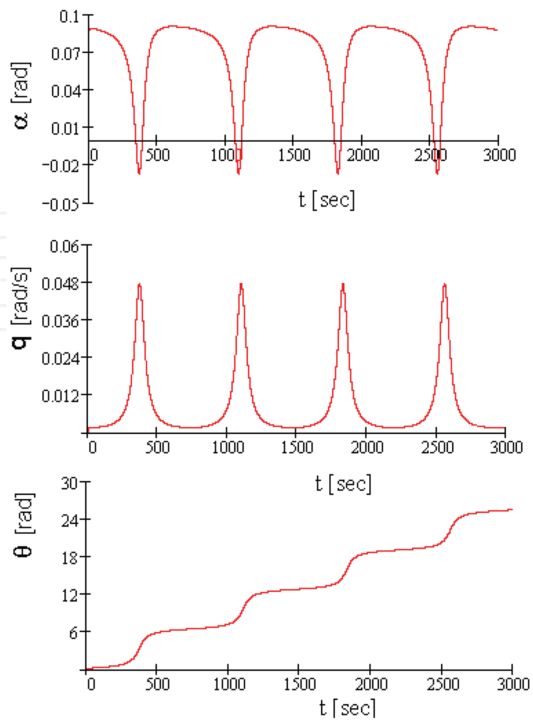

Fig. 5. Oscillation when δ_e = - 0.05 [rad] and the starting point is :
α_1 = 0.086974288419088 [rad]; q_1 = 0 [rad/sec]; θ1= 0.159329728679884[rad].

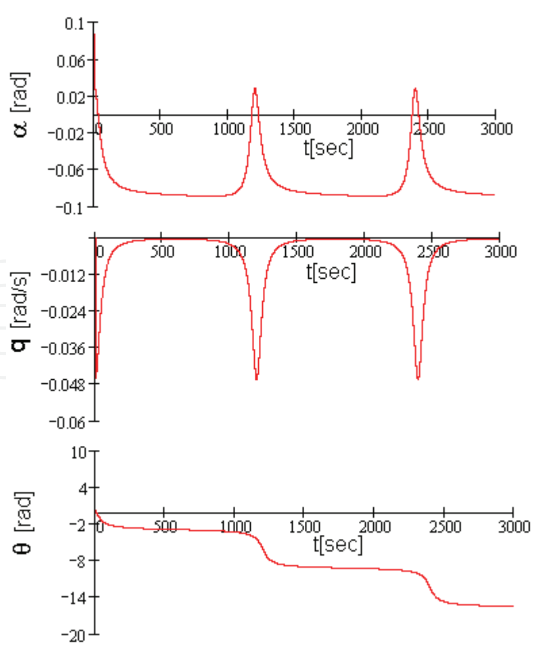

Fig. 6. Oscillation when δ_e = 0.048 [rad] and the starting point is :
α_1 = 0.086974288419088 [rad]; q_1 = 0 [rad/sec]; θ1= 0.159329728679884[rad].

The behavior of the ADMIRE aircraft changes when the maneuver $\delta_e' \to \delta_e''$ is so that $\delta_e' \in \left(\underline{\delta_e}, \overline{\delta_e}\right)$ and $\delta_e'' \notin \left(\underline{\delta_e}, \overline{\delta_e}\right)$. Computation shows that after such a maneuver α and q oscillate with the same period and θ tends to $+\infty$ or $-\infty$. (Figs.5, 6)
The oscillation presented in Figs. 5,6 is a non catastrophic bifurcation, because if δ_e is reset, then equilibrium is recovered, as it is illustrated in Fig.7.

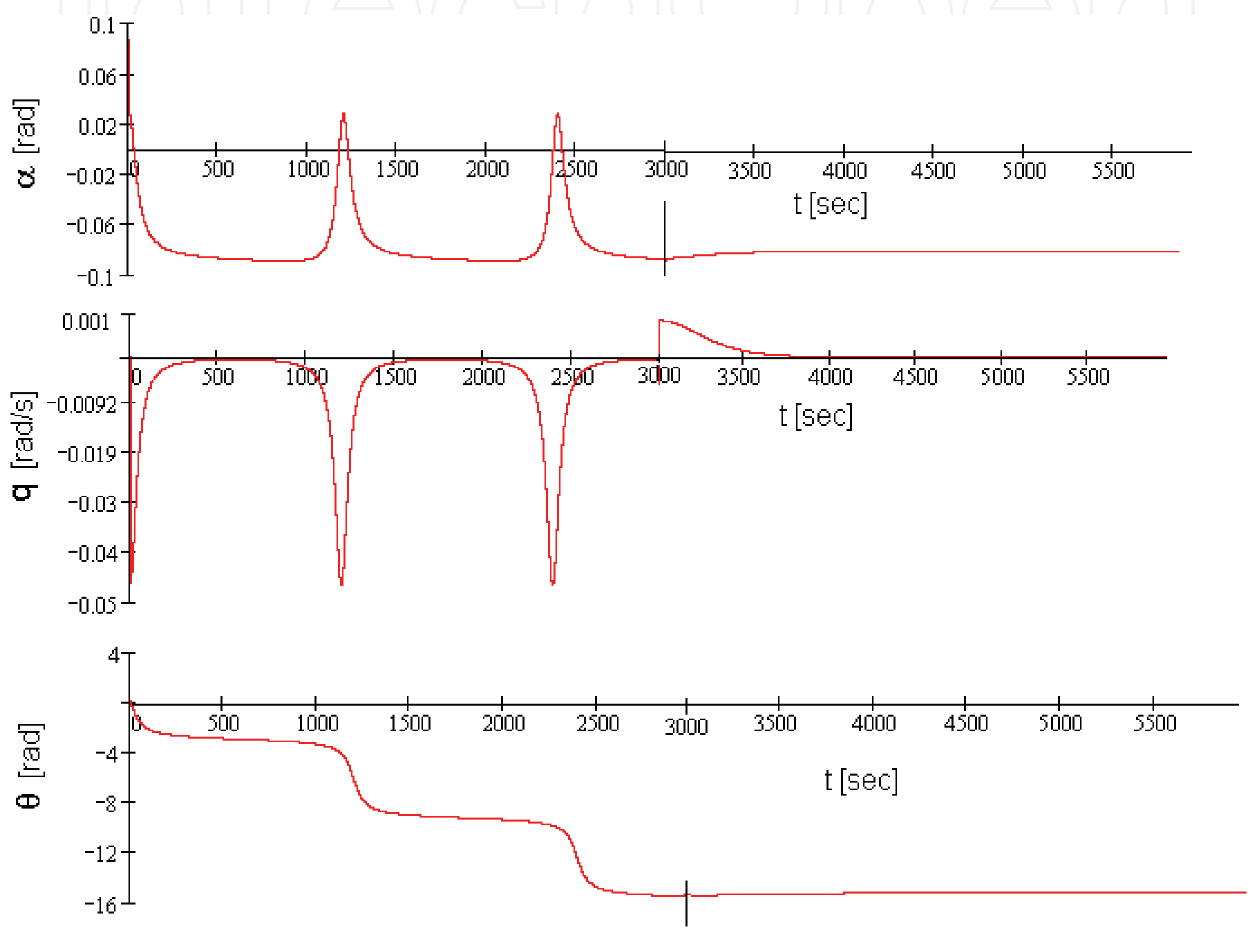

Fig. 7. Resetting $\delta_e = 0.048[rad] < \delta_{eo}$ after 3000 [s] of oscillations to $\delta_e = \delta_{eo}$, equilibrium is recovered.

7. Conclusion

For an unmanned aircraft whose automatic flight control system during a longitudinal flight with constant forward velocity fails, the following statements hold:

1. If the elevator deflection is in the range given by formula (19), then the movement around the center of mass is stationary or tends to a stationary state.
2. If the elevator deflection exceeds the value given by formula (36), then the movement around the center of mass becomes oscillatory decreasing and when the elevator

deflection is less than the value given by formula (37), then the movement around the center of mass becomes oscillatory increasing.

3. This oscillatory movement is not catastrophic, because if the elevator deflection is reset in the range given by (19), then the movement around the center of mass becomes stationary.
4. Numerical investigation of the oscillation susceptibility (when the automatic flight control system fails) in the general non linear model of the longitudinal flight with constant forward velocity reveals similar behaviour as that which has been proved theoretically and numerically in the framework of the simplified model. As far as we know, in the general non linear model of the longitudinal flight with constant forward velocity the existence of the oscillatory solution never has been proved theoretically.
5. A task for a new research could be the proof of the existence of the oscillatory solutions in the general model.

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Nonlinear problems in flight control have stimulated cooperation among engineers and scientists from a range of disciplines. Developments in computer technology allowed for numerical solutions of nonlinear control problems, while industrial recognition and applications of nonlinear mathematical models in solving technological problems is increasing. The aim of the book Advances in Flight Control Systems is to bring together reputable researchers from different countries in order to provide a comprehensive coverage of advanced and modern topics in flight control not yet reflected by other books. This product comprises 14 contributions submitted by 38 authors from 11 different countries and areas. It covers most of the currents main streams of flight control researches, ranging from adaptive flight control mechanism, fault tolerant flight control, acceleration based flight control, helicopter flight control, comparison of flight control systems and fundamentals. According to these themes the contributions are grouped in six categories, corresponding to six parts of the book.

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