18

Fourier Transform on Group-Like Structures and Applications

Massoud Amini[1], Mehrdad Kalantar[2], Hassan Myrnouri[3] and Mahmood M. Roozbahani[4]

[1,4]*Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran 14115-134,*

[2]*School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, ON K1S 5B6,*

[3]*Department of Mathematics, Islamic Azad University, Lahijan Branch, Lahijan*

[1,3,4]*Iran*

[2]*Canada*

1. Introduction

The Fourier analysis was invented by the French mathematician, Joseph Fourier, in the early nineteenth century (28). The Fourier transform (FT) is an operation that transforms a function of a real variable in a given domain into another function in another domain (there are generalizations to complex or several real variables). The domains differ from one application to another. In signal processing, typically the original domain is time and the target domain is frequency. The transform captures those frequencies present in the original function. The Fourier transform and its generalizations are part of the Fourier analysis (23). The Fourier transform on discrete structures such as finite groups (DFT), opens up the issue of the time complexity of the algorithm which computes FT. Particularly, efficient computation of a fast Fourier transform (FFT) is essential for high-speed computing (8). There is a vast literature on the theory of Fourier transform on groups, including Fourier transform on locally compact abelian groups (27), on compact groups (34), and finite groups (73). In this chapter, we review the generalizations of the Fourier transform on group-like structures, including inverse semigroups (43), hypergroups (10), and groupoids (70). This is a vast subject with an extensive literature, but here a personal view based on the authors' research is presented. References to the existing literature is given, as needed. The Fourier transform on inverse semigroups are introduced in (47). The basics of the theory of Fourier transform on commutative hypergroups are discussed in (10). The Fourier transform on arbitrary compact hypergroups is first studied in (74). The section on the Fourier transform of unbounded measures on commutative hypergroups is taken from (2). An application of the quantum Fourier transform (QFT) on finite commutative hypergroups in quantum computation is given in (4). Finally, the Fourier transform on compact groupoids is discussed in (1), (3). The case of abelian groupoids (57) is considered in (6).

2. Outline

3. Classical Fourier transform

Fourier transform has a long history. Some variants of the discrete (cosine) Fourier transform were used by Alexis Clairaut in 1754 to do some astronomical calculations, and the discrete (sine) Fourier transform in 1759 by Joseph Louis Lagrange to compute the coefficients of a trigonometric series for a vibrating string. The latter used a discrete Fourier transform of order 3 to study the solution of a cubic. Finally Carl Friedrich Gauss used a full discrete Fourier transform in 1805 to find trigonometric interpolation of asteroid orbits.
Although d'Alembert and Gauss had already used trigonometric series to study the heat equation, it was first in the 1807 seminal paper of Joseph Fourier that the idea of expanding all (continuous) functions by trigonometric series was explored.
Broadly speaking, the Fourier transform is a systematic way to decompose generic functions into a superposition of symmetric functions (72). In this broad sense, even the decomposition of a real function into its odd and even parts is an instance of a Fourier series. Similarly (72)

if complex function $w = f(z)$ is a harmonic of order j, that is $f(e^{2\pi i/n}z) = e^{2\pi ij/n}f(z)$ for all $z \in \mathbb{C}$ then

$$f(z) = \sum_{j=0}^{n-1} \frac{1}{n} \sum_{k=0}^{n-1} f(e^{2\pi ik/n}z)e^{-2\pi ijk/n}.$$

In general, the monomial $f(z) = z^n$ has rotational symmetry of order n and each continuous function $f : \mathbb{T} \to \mathbb{C}$ could be expanded (in an appropriate norm) as $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta.$$

One important feature of this expansion is that it could be considered as a generalization of the case where a complex analytic function f on the closed unit disk $\mathbb{D}$ is expanded in its Taylor series.
For a function $f : \mathbb{R}^d \to \mathbb{C}$, under some very natural conditions we have the dual formulas

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi ix\xi}d\xi, \quad \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi ix\xi}dx.$$

Consider an integrable (real or complex) function on the interval $[0, 2\pi]$ with Fourier coefficients $\hat{f}(n)$ as above. Then an important classical problem is the problem of convergence of the corresponding Fourier series. More precisely, if

$$S_N(f)(t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int}$$

then it is desirable to have (necessary and) sufficient conditions on f so that the sequence $\{S_N(f)\}$ converges to f in a given function topology.
For norm convergence, we know that if $f \in L^p[0, 2\pi]$ for $1 < p < \infty$, then $\{S_N(f)\}$ converges to f in L^p-norm (for $p = 2$, this is Riesz-Fisher theorem). The pointwise convergence is more delicate. There are many known sufficient conditions for $\{S_N(f)\}$ to converge to f at a given point x, for example if the function is differentiable at x; or even if the function has a discontinuity of the first kind at x and the left and right derivatives at x exist and are finite, then $\{S_N(f)(x)\}$ will converge to $\frac{1}{2}(f(x^+) + f(x^-))$ (but by the Gibbs phenomenon, it has large oscillations near the jump). By a more general sufficient condition (called the Dirichlet-Dini Criterion) if f is 2π-periodic and locally integrable and

$$\int_0^{\pi} \left|\frac{f(x+t) - f(x-t)}{2} - \ell\right| \frac{dt}{t} < \infty$$

then $\{S_N(f)(x)\}$ converges to ℓ. In particular, if f is of Holder class $\alpha > 0$, then $\{S_N(f)(x)\}$ converges everywhere to $f(x)$. Indeed, in the latter case, the convergence is uniform (by Dini-Lipschitz test). If f is only continuous, the sequence of n-th averages of $\{S_N(f)\}$ converge uniformly to f, that is the Fourier series is uniformly Cesaro summable (also the Gibbs phenomenon disappears in the pointwise convergence of the Cesaro sum).
In general, for a given continuous function f the Fourier series converges almost anywhere to f (this holds even for a square integrable function: Carleson theorem, or L^p function: Hunt theorem) and when f is of bounded variation, the Fourier series converges everywhere to f (by Dini test). If a function is of bounded variation and Holder of class $\alpha > 0$, then the Fourier series is absolutely convergent.

However, the family of continuous functions whose Fourier series converges at a given x is of first Baire category in the Banach space of continuous functions on the circle. This means that for most continuous functions the Fourier series does not converge at a given point. Kolmogorov constructed a concrete example of a function in L^1 whose Fourier series diverges almost everywhere (later examples showed that this may happen everywhere). More generally, for any given set E of measure zero, there exists a continuous function f whose Fourier series fails to converge at any point of E (Kahane-Katznelson theorem).

4. Fourier transform on groups

The Fourier transform could be defined in general on any locally compact Hausdorff group G. This is done using the (left) Haar measure, a (left) translation invariant positive Borel measure λ on G whose existence and uniqueness (up to positive scalars) is proved in abstract harmonic analysis (27, 2.10). If $f \in L^1(G) := L^1(G, \lambda)$ and $\pi : G \to \mathcal{B}(\mathcal{H}_\pi)$ is an irreducible (unitary) representation of G (we write $\pi \in \hat{G}$) then we may define the Fourier transform of f

$$\hat{f}(\pi) = \int_G f(x)\pi(x^{-1})d\lambda(x).$$

Similarly the Fourier-Stieljes transform of a bounded Borel measure $\mu \in M(G)$ is defined by

$$\hat{\mu}(\pi) = \int_G \pi(x^{-1})d\mu(x),$$

for $\pi \in \hat{G}$.
The Fourier transform is well studied in the two special cases where G is abelian or compact (finite). We give more details in the next two sections.

4.1 Abelian groups

When G is abelian, each irreducible representation is one dimensional (27, 3.6) and could be identified with a (continuous) character $\chi \in \hat{G}$. A character is a continuous group homomorphism $\chi : G \to \mathbb{T}$. In this case, the dual object $\hat{G}$ is itself a locally compact abelian group and one could employ the Fourier transform to prove the Pontrjagin duality, that is $(\hat{G})\hat{} \simeq G$ as topological groups.
If $f \in L^1(G)$ then the Fourier transform of f is defined on $\hat{G}$ by

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}d\lambda(x)$$

and $\hat{f} \in C_0(\hat{G})$ (Riemann-Lebesgue lemma) and $\|\hat{f}\|_\infty \leq \|f\|_1$, but the Fourier transform is not an isometry from $L^1(G)$ to $C_0(\hat{G})$ (even for $G = \mathbb{R}$). However we have the inversion formula: if $B(G)$ is the linear span of the positive definite functions on G (27, p. 84) and $f \in B(G) \cap L^1(G)$ then $\hat{f} \in L^1(\hat{G})$ and with a suitable normalization of the Haar measure ϖ of $\hat{G}$

$$f(x) = \int_{\hat{G}} \hat{f}(\chi)\chi(x)d\varpi(\chi),$$

for almost all $x \in G$. By the Pontrjagin duality, this means that $f(x) = (\hat{f})\hat{}(x^{-1})$ for λ-a.e. $x \in G$.

When G is compact and abelian and $f \in L^2(G)$ then the Fourier-Plancherel transform of f is defined on the discrete abelian group $\hat{G}$ by

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}d\lambda(x),$$

which converges by Cauchy-Schwartz inequality (similar definition works for $f \in L^p(G)$, $1 < p < \infty$, by Holder inequality). In this case $\hat{f} \in \ell^2(\hat{G})$ ($f \in \ell^q(\hat{G})$, $\frac{1}{p} + \frac{1}{q} = 1$, respectively) and $\|\hat{f}\|_2 = \|f\|_2$, hence the Fourier-Plancherel transform is an isometry from $L^2(G)$ to $\ell^2(\hat{G})$.

4.2 Compact and finite groups

If G is a compact Hausdorff group, each irreducible representation $\pi \in \hat{G}$ is finite dimensional (27, 5.2), say with dimension d_π, and the linear span of the matrix elements of π (coefficients of π) is a finite dimensional space $\mathcal{E}_\pi$ of dimension at most d_π^2. Moreover the linear span $\mathcal{E}(G)$ of the union $\cup_{\pi\in\hat{G}}\mathcal{E}_\pi$ (trigonometric functions on G) is uniformly dense in $C(G)$ and $L^2(G) = \bigoplus_{\pi\in\hat{G}} \mathcal{E}_\pi$ (Peter-Weyl theorem). If $f \in L^1(G)$, the Fourier transform of f

$$\hat{f}(\pi) = \int_G f(x)\pi(x^{-1})d\lambda(x).$$

defines an element $\hat{f} \in \bigoplus_{\pi\in\hat{G}} \mathbb{M}_{d_\pi}(\mathbb{C})$ with coefficients $\hat{f}(\pi)_{ij} = \int_G f(x)\langle\pi(x^{-1})e_i, e_j\rangle d\lambda(x)$, hence if $f \in L^1(G) \cap L^2(G)$,

$$f = \sum_{\pi\in\hat{G}} d_\pi tr(\hat{f}(\pi)\pi(.))$$

in L^2-norm with $\|f\|_2^2 = \sum_{\pi\in\hat{G}} d_\pi tr(\hat{f}(\pi)^*\hat{f}(\pi))$. If χ_π is the character associated to the irreducible representation $\pi \in \hat{G}$ by $\chi_\pi = tr(\pi(.))$, then $\{\chi_\pi : \pi \in \hat{G}\}$ is an orthonormal basis of $ZL^2(G)$ consisting of central functions in $L^2(G)$ (27, 5.23).
Given a complex-valued function f on a finite group G, we may present f as

$$f = \sum_{g\in G} f(g)g,$$

viewing f as an element of the group algebra $\mathbb{C}G$. The Fourier basis of $\mathbb{C}G$ comes from the decomposition of the semisimple algebra $\mathbb{C}G$ as the direct sum of its minimal left ideals

$$\mathbb{C}G = \oplus_{i=1}^n M_i,$$

and taking a basis for each M_i. When $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the cyclic group of order n, an element f of the group algebra $\mathbb{C}\mathbb{Z}_n$ is a signal, sampled at n evenly spaced points in time. The minimal left ideals of $\mathbb{C}\mathbb{Z}_n$ are one dimensional, and the Fourier basis is unique (up to scaling factors), and is indeed the usual basis of exponential functions given by the classical discrete Fourier transform. Hence the expansion of f in a Fourier basis corresponds to a re-expression of f in terms of the frequencies that comprise f (45).
The efficiency of computing the Fourier transform of an arbitrary function on G depends on the choice of basis in which f is expanded. A naive computation requires $|G|^2$ operations, where an operation is a complex multiplication followed by a complex addition. Much better results are obtained for different groups (21), (48), (49), (50), and (51) (see also (68) and references therein). For example, computing the Fourier transform requires no more than $O(|G|log^k|G|)$ operations on the cyclic group $G = \mathbb{Z}_n$ (13; 19), symmetric group $G = S_n$ (47), and hyper-octahedral group $G = B_n$ (67), with $k = 1, 2, 4$, respectively. On the other hand, there is no known $O(|G|log^c|G|)$ algorithms for matrix groups over a finite field (48).

4.2.1 Discrete Fourier transform

The classical discrete Fourier transform of a sequence $\{f(j)\}_{0\leq j\leq n-1}$ is defined

$$\hat{f}(k) = \sum_{j=0}^{n-1} f(j)e^{-2ikj/n},$$

for $0 \leq k \leq n-1$, and

$$f(j) = \frac{1}{n}\sum_{k=0}^{n-1} \hat{f}(k)e^{2ikj/n},$$

for $0 \leq j \leq n-1$.
For $G = \mathbb{Z}_n$, the complex irreducible representations χ_k of $\mathbb{Z}/n\mathbb{Z}$ are one dimensional,

$$\chi_k(j) = e^{-2ikj/n},$$

and the group algebra $\mathbb{C}\mathbb{Z}_n$ has only one natural basis,

$$b_k = \frac{1}{n}\sum_{j=0}^{n-1} (e^{2ikj/n})j,$$

and $\hat{f}(k) = \hat{f}(\chi_k)$.

4.2.2 Fast Fourier transform

The classical FFT has revolutionized signal processing. Applications include fast waveform smoothing, fast multiplication of large numbers, and efficient waveform compression, to name just a few (14). FFTs for more general groups have applications in statistical processing. For example, the FFT on Z_2^k (76) allows for efficient 2^k-factorial analysis. That is, it allows for the efficient statistical analysis of an experiment in which each of k variables may take on one of two states. The FFT on S_n allows for an efficient statistical analysis of votes cast in an election involving n candidates (20).
A direct calculation of the Fourier transform of an arbitrary function on $\mathbb{Z}_n$ requires $n^2 = |\mathbb{Z}_n|^2$ operations. The classical fast discrete Fourier transform (of Cooley-Tukey (19)) expressed in the group language (49) is as follows: Suppose $n = pq$ where p is a prime. Then $\mathbb{Z}_n$ has a subgroup H isomorphic to $\mathbb{Z}_q$. By the reversal decomposition $\mathbb{Z}_n = \cup_{0\leq j\leq p-1}(j+H)$. For $0 \leq k \leq n-1$,

$$\hat{f}(k) = \hat{f}(\chi_k) = \sum_{j=0}^{n-1} f(j)\chi_k(j) = \sum_{j=0}^{p-1} \chi_k(j) \sum_{h\in H} f(j+h)\chi_k(h).$$

Now the inner sums in the last term are Fourier coefficients on H. This reduces the problem of computing the Fourier transform on G to the same computation on H. When $n = 2^k$ (a k-bit number in digital terms), the Fourier transform on $\mathbb{Z}_{2^k}$ could be computed in $2^k + k2^{k+1} = O(nlogn)$ operations.
The reduction argument involved in the above FFT algorithm could basically be applied to any finite group (or semigroup). However, since the irreducible representations are not one dimensional for non abelian case, more is needed to be explored in the general case. In particular, one needs to understand the way (finite dimensional) irreducible representations behave when restricted to subgroups.

The basic idea of Cooley-Tukey is adapted by Clausen (18) to create an FFT for the symmetric group S_n. A standard reference for representation theory of the symmetric group is (38). The Clausen algorithm for FFT on the symmetric group S_n requires a set of inequivalent, irreducible, chain-adapted representations relative to the chain $S_n > S_{n-1} > \cdots > S_1 = \{e\}$, where e is the identity of S_n and S_k is the subgroup of S_n consisting of permutations which fix $k+1$ through n. Two such sets of representations are provided by the Young seminormal (62) and orthogonal forms (77). (for generalizations to seminormal representations of Iwahori-Hecke algebras see (35) and (62)).
A partition of a nonnegative integer k is a weakly decreasing sequence λ of nonnegative integers whose sum is k. Two partitions are equal if they only differ in number of zeros they contain. We write $\lambda \vdash k$. A complete set of (equivalence classes of) irreducible representations for S_n is indexed by the partitions of n (38). A partition corresponds to its Young diagram, consisting of a table whose i-th row has $\lambda(i)$ boxes. If we fill these boxes with numbers from $\{1, 2, ...n\}$ such that each number appears exactly once and column (row) entries increase from top to bottom (left to right), we get a standard tableau of shape λ. Now S_n acts on tableaux by permuting their entries. If L is a standard tableau of shape λ, and $L(i)$ is the box in L in the position (k, ℓ), we put $|L(i)| = \ell - k$ and define the action of the basic permutation $\sigma = \sigma_i = (i-1, i)$ on the vector space V^λ generated by a basis $\{v_L\}$ indexed by all standard tableaux L of shape λ by

$$\sigma v_L = (|L(i)| - |L(i-1)|)^{-1} v_L + (1 + (|L(i)| - |L(i-1)|)^{-1}) v_{\sigma L},$$

with the convention that $v_{\sigma L} = 0$, if σL is not standard. Young showed that these actions exhaust $\hat{S}_n = \{\rho^\lambda : \lambda \vdash n\}$. Moreover if λ^- is the set of all partitions of $n-1$ obtained by removing a corner (a box with no box to the right or below) of λ then $V^\lambda = \bigoplus_{\mu \in \lambda^-} V^\mu$ and if we order the basis of V^λ with the last letter ordering of standard tableaux,

$$\rho^\lambda|_{S_{n-1}} = \bigoplus_{\mu \in \lambda^-} \rho^\mu.$$

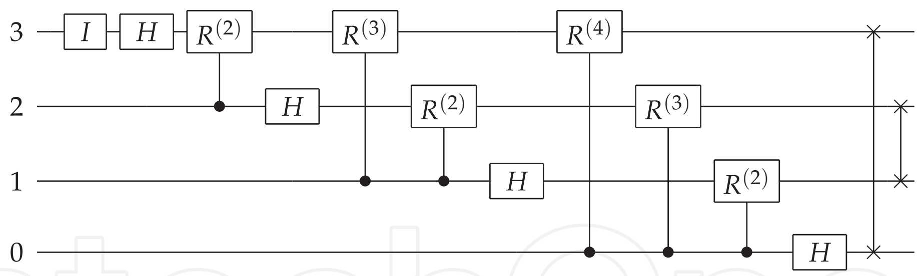
Now Clausen algorithm uses the above Young seminormal representations to find FFT for S_n (18). If $S_n = \cup_{1 \le i \le n} \tau_i S_{n-1}$ is the transversal decomposition of S_n into left cosets of the subgroup S_{n-1}, where $\tau_i = \sigma_{i+1}\sigma_{i+2} \ldots \sigma_n$, for $i < n$ and $\tau_n = e$, the identity of S_n, then for $\lambda \vdash n$,

$$\hat{f}(\rho^\lambda) = \sum_{\sigma \in S_n} f(\sigma)\rho^\lambda(\sigma) = \sum_{i=1}^{n} \rho^\lambda(\tau_i) \sum_{\sigma \in S_{n-1}} f(\tau_i \sigma)\rho^\lambda(\sigma).$$

Therefore the minimum number of needed operations is at most $\frac{2}{3}n(n+1)^2 n! = O(n! log^3 n!)$.

4.2.3 Quantum Fourier transform

In quantum computing, the quantum Fourier transform (QFT) is the quantum analogue of the discrete Fourier transform applied to quantum bits (qubits). Mathematically, QFT as a unitary operator is nothing but the Fourier-Plancherel transform on the Hilbert space $\ell^2(G)$, for some appropriate (abelian) group G. For n-qubits, this group could be considered to be $G = \mathbb{Z}_{2^n} = \mathbb{Z}/2^n\mathbb{Z}$, but in practice it could be any finite group.
QFT is an integral part of many famous quantum algorithms, including factoring and discrete logarithm, the quantum phase estimation (estimating the eigenvalues of a unitary matrix) and the hidden subgroup problem (HSP).



Scheme 1. *Quantum Circuit of 4-qubit Quantum Fourier Transform*

The quantum Fourier transform can be performed efficiently on a quantum computer, with a particular decomposition into a product of simpler unitary matrices. Using a canonical decomposition, the discrete Fourier transform on n-qubits can be implemented as a quantum circuit consisting of only $O(n^2)$ Hadamard gates and controlled phase shift gates (58). There are also QFT algorithms requiring only $O(nlogn)$ gates (32). This is exponentially faster than the classical discrete Fourier transform (DFT) on n bits, which takes $O(n2^n)$ gates. However, QFT can not give a generic exponential speedup for any task which requires DFT.
The QFT on $\mathbb{Z}_N$ maps a quantum state $\sum_{j=0}^{N-1} x_j|j\rangle$ to the quantum state $\sum_{k=0}^{N-1} y_k|k\rangle$ with $y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j\omega^{jk}$, where $\omega = e^{2\pi i/N}$ is a primitive N-th root of unity. This is a surjective isometry on the finite dimensional Hilbert space $\ell^2(\mathbb{Z}_N)$ whose corresponding unitary matrix is $\mathfrak{F}_N = \frac{1}{\sqrt{N}}[\omega^{jk}]_{0\le j,k\le N-1}$. The case $N = 2^n$ handles the transformation of n-qubits.
Using the Hadamard and phase gates

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R^{(k)} = \begin{pmatrix} 1 & 0 \\ 0 & 2\pi i/2^k \end{pmatrix}$$

one could implement the QFT over $n = 4$ qubits efficiently as in the above circuit (46) (see also (15)).

5. Fourier transform on monoids and inverse semigroups

A semigroup is a nonempty set S together with an associative binary operation. If S has an identity element, it is called a monoid. A (finite dimensional) representation of S (of dimension d) is a homomorphism from S to the semigroup $\mathbb{M}_d(\mathbb{C})$ with matrix multiplication. An inverse semigroup is a semigroup S such that for each $x \in S$ there is a unique $y = x^* \in S$ such that $xyx = x$ and $yxy = y$. The set $E = \{xx^* : x \in S\}$ is a commutative subsemigroup of S.
For a finite inverse semigroup S, the semigroup algebra $\mathbb{C}S$ with convolution

$$f * g(s) = \sum_{r,t\in S, rt=s} f(r)g(t)$$

is semisimple (53), hence representations of S are (equivalent to) a direct sum of irreducible and null representations of S. By Wedderburn theorem, the set $\hat{S}$ of (equivalence classes of) irreducible representations of S is finite and the map

$$\bigoplus_{\rho\in\hat{S}} \rho : \mathbb{C}S \to \bigoplus_{\rho\in\hat{S}} \mathbb{M}_{d_\rho}(\mathbb{C})$$

is an isomorphism of algebras sending f to $\bigoplus_{\rho\in\hat{S}}\sum_{s\in S} f(s)\rho(s)$. This isomorphism is indeed the Fourier transform on S, that is $\hat{f}(\rho) = \sum_{s\in S} f(s)\rho(s)$, for $f \in \mathbb{C}S$.
A standard example of finite inverse semigroups is the rook monoid R_n consisting of all injective partial functions on $\{1,\ldots,n\}$ under the operation of partial function composition. R_n is isomorphic to the semigroup of all $n \times n$ matrices with all 0 entries except at most one 1 in each row or column (corresponding to the set of all possible placements of non-attacking rooks on an $n \times n$ chessboard). The rook monoid plays the role of the symmetric group for finite groups in the category of finite inverse semigroups. Each finite inverse semigroup S is isomorphic to a subsemigroup of R_n, for $n = |S|$ (43). The (fast) Fourier transform on R_n is studied in details in (45), which is applied to the analysis of partially ranked data. We give a brief account of FFT on rook monoids in the next section, and refer the interested reader to (45) for details.

5.1 Fast Fourier transform on rook monoids

In this section we give a fast algorithm to compute $\hat{f}$ for $f \in \mathbb{C}R_n$ mimicing the Clausen algorithm for FFT on S_n (45). There is a much faster algorithm based on groupoid bases (45, 7.2.3), but the present approach has the advantage that explicitly employs the reversal decomposition.
As in S_n, let $\sigma_i = (i-1,i) \in R_n$ and put $\tau_i = \sigma_{i+1}\sigma_{i+2}\ldots\sigma_n$, for $i < n$ and $\tau_n = e$, the identity of R_n. Moreover, put $\tau^i = \sigma_n\sigma_{n-1}\ldots\sigma_{i+1}$, for each i, and let R_k be the subsemigroup of R_n consisting of those partial permutations σ with $\sigma(j) = j$, for $j > k$. Then Halverson has characterized $\hat{R}_n$ using the semigroup chain $R_n > R_{n-1} > \cdots > R_1$, similar to the Young seminormal representations of S_n (35). Considering the transversal decomposition of R_n into cosets of R_{n-1}, one should note that two distinct left cosets of R_{n-1} do not necessarily have the same cardinality. Indeed, we have (45, 2.4.4)

$$|R_n| = 2n|R_{n-1}| - (n-1)^2|R_{n-2}|.$$

This is because R_n consists of those rook matrices having all zeros in column and row 1, those having 1 in position $(\alpha,1)$ for $1 \le \alpha \le n$, or in position $(1,\alpha)$ for $2 \le \alpha \le n$, counting twice those with ones in positions $(\alpha,1)$ and $(1,\beta)$ for $2 \le \alpha,\beta \le n$. This suggests the following decomposition for $n \ge 3$ and $\rho \in \hat{R}_n$,

$$\begin{aligned}\hat{f}(\rho) = &\sum_{i=1}^{n} \rho(\tau_i) \sum_{\sigma\in R_{n-1}} f(\tau_i\sigma)\rho(\sigma) + \rho([n]) \sum_{\sigma\in R_{n-1}} f([n]\sigma)\rho(\sigma)\\ &+ \sum_{i=1}^{n-1} \rho(\tau^i) \sum_{\sigma\in R_{n-2}} f(\sigma\tau^i)\rho(\sigma),\end{aligned}$$

where $[n]$ is the identity of R_{n-1} considered as an element of R_n (not defined at n). This decomposition follows from dividing elements $\sigma \in R_n$ into three parts: those with $\sigma(n) = i$ for some $1 \le i \le n$, those not defined at n with $\sigma(i) = n$ for some $1 \le i \le n-1$, and those not defined at n with $\sigma(i) \ne n$ for all $1 \le i \le n$ (45, 9.1.1). This leads the upper bound

$$2\sum_{\rho\in\hat{R}_n}\sum_{i=1}^{n} 2(n-i)d_\rho^2 + \sum_{\rho\in\hat{R}_n} d_\rho^2 + \sum_{\rho\in\hat{R}_n} (2n-1)d_\rho^2,$$

where twice the first sum calculates the maximum number of operations involved in calculation of the matrix products in the first and third terms of the above decomposition, the second sum calculates the maximum number of operations involved in calculation of the matrix products in the second term, and the last sum calculates the maximum number of operations involved in calculation of all matrix sums involved in the whole decomposition. This upper bound is at most

$$2n(n-1)|R_n| + |R_n| + (2n-1)|R_n|.$$

This leads to the upper bound $n2^n|R_n| = O(|R_n|^{1+\varepsilon})$ for the minimum number of operations needed to calculate the FFT on R_n (45, 9.2.3), which could be improved as $(\frac{3}{4}n^2 + \frac{2}{3}n^3)|R_n| = O(|R_n|log^3|R_n|)$ using more sophisticated decompositions (45, 7.2.3).

6. Fourier transform on hypergroups

Fourier and Fourier-Stieltjes transforms play a central role in the theory of absolutely integrable functions and bounded Borel measures on a locally compact abelian group G (70). They are particularly important because they map the group algebra $L^1(G)$ and measure algebra $M(G)$ onto the Fourier algebra $A(G)$ and Fourier-Stieltjes algebra $B(G)$, respectively. There are important Borel measures on a locally compact, non-compact, abelian group G which are unbounded. A typical example is a left Haar measure. L. Argabright and J. de Lamadrid in (7) explored a generalized Fourier transform of unbounded measures on locally compact abelian groups. This theory has recently been successfully applied to the study of quasi-crystals (8).

A version of generalized Fourier transform is defined for a class of commutative hypergroups (10). Some of the main results (7) are stated and proved in (10) for hypergroups, but the important connection with Fourier and Fourier-Stieltjes spaces are not investigated in (10) (in contrast with the group case, these may fail to be closed under pointwise multiplication for general hypergroups.) Recently these spaces are studied in (5), and under some conditions, they are shown to be Banach algebras. Section 6.2 investigates the relation between transformablity of unbounded measures on strong commutative hypergroups and these spaces. In the next section we study unbounded measures on locally compact (not necessarily commutative) hypergroups. The main objective of this section is the study of translation bounded measures. These are studied in (10) in commutative case. The main results (Theorem 6.18 and Corollary 6.20) are stated and proved in section 6.2.1. The former states that an unbounded measure on a strong commutative hypergroup is transformable if and only if its convolution with any positive definite function of compact support is positive definite. The latter gives the condition for the transform of this measure to be a function.

A hypergroup is a triple $(K, \bar{\ }, *)$ where K is a locally compact space with an involution $\bar{\ }$ and a convolution $*$ on $M(K)$ such that $(M(K), *)$ is an algebra and for $x, y \in K$,

(i) $\delta_x * \delta_y$ is a probability measure on K with compact support,

(ii) the map $(x, y) \in K^2 \mapsto \delta_x * \delta_y \in M(K)$ is continuous,

(iii) the map $(x, y) \in K^2 \mapsto \text{supp}\,(\delta_x * \delta_y) \in \mathcal{C}(K)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(K)$ of nonvoid compact sets in K,

(iv) K admits an *identity* e satisfying $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$,

(v) $(\delta_x * \delta_y)^{-} = \delta_{\bar{y}} * \delta_{\bar{x}}$,

(vi) $e \in \text{supp}\,(\delta_x * \delta_y)$ if and only if $x = \bar{y}$.

For $\mu \in M(K)$ and Borel subset $E \subseteq K$ put $\bar{\mu}(E) = \mu(\bar{E})$, where $\bar{E} = \{\bar{x} : x \in E\}$. This is an involution on $M(K)$ making it a Banach $*$-algebra. A representation of K is a $*$-representation π of $M(K)$ such that $\pi(\delta_e) = id$ and $\pi \mapsto \pi(\mu)$ is weak-weak operator-continuous from $M_+(K)$ to $\mathcal{B}(\mathcal{H}_\pi)$. When π is irreducible, we write $\pi \in \hat{K}$. Define the conjugation operator D_π on $\mathcal{H}_\pi$ by

$$D_\pi(\sum_{i=1}^{d_\pi} \alpha_i \xi_i^\pi) = \sum_{i=1}^{d_\pi} \bar{\alpha}_i \xi_i^\pi,$$

and put $\bar{\pi} = D_\pi \pi D_\pi$. For $\mu \in M(K)$, the Fourier-Stieltjes transform of μ is defined by $\hat{\mu}(\pi) = \bar{\pi}(\mu)$. Then $\hat{\mu} \in \mathcal{E}(K) := \bigoplus_{\pi \in \hat{K}} \mathcal{B}(\mathcal{H}_\pi)$ and the Fourier-Stieltjes transform from $M(K)$ to $\mathcal{E}(K)$ is one-one.

6.1 Compact hypergroups

Let K be a compact hypergroup and $\hat{K}$ denote the set of equivalence classes of all continuous irreducible representations of K. For $\pi \in \hat{K}$, let $\{\xi_i^\pi\}_{i=1}^{d_\pi}$ be an orthonormal basis for the corresponding (finite dimensional) Hilbert space $\mathcal{H}_\pi$ and put

$$\pi_{i,j}(x) = \langle \pi(x)\xi_i^\pi, \xi_j^\pi \rangle \quad (1 \le i, j \le d_\pi).$$

Let $Trig_\pi(K) = span\{\pi_{i,j} : 1 \le i, j \le d_\pi\}$ and $Trig(K) = span\{\pi_{i,j} : \pi \in \hat{K}, 1 \le i, j \le d_\pi\}$. Then $dimTrig_\pi(K) = d_\pi^2$ and there is $k_\pi \ge d_\pi$ such that

$$\int_K \pi_{i,j}\bar{\sigma}_{r,s} dm = k_\pi^{-1} \delta_{\pi,\sigma}\delta_{i,r}\delta_{j,s} \quad (\pi, \sigma \in \hat{K}) \text{ (74, 2.6)}.$$

Also $\{k_\pi^{\frac{1}{2}} \pi_{i,j} : \pi \in \hat{K}, 1 \le i, j \le d_\pi\}$ is an orthonormal basis of $L^2(K)$ and

$$Trig(K) = \oplus_{\pi \in \hat{K}} Trig_\pi(K) \text{ (74, 2.7)}.$$

In particular, $Trig(K)$ is norm dense in both $C(K)$ and $L^2(K)$ (74, 2.13, 2.9).
For each $f \in L^2(K)$ we have the Fourier series expansion

$$f = \sum_{\pi \in \hat{K}} \sum_{i,j=1}^{d_\pi} k_\pi \langle f, \pi_{i,j} \rangle \pi_{i,j}$$

where the series converges in L^2-norm.
Consider the $*$-algebra

$$\mathcal{E}(\hat{K}) := \prod_{\pi \in \hat{K}} B(\mathcal{H}_\pi)$$

with coordinatewise operations. For $f = (f_\pi) \in \mathcal{E}(\hat{K})$ and $1 \le p < \infty$ put

$$\|f\|_p := \Big(\sum_{\pi \in \hat{K}} k_\pi \|f_\pi\|_p^p\Big)^{\frac{1}{p}}, \quad \|f\|_\infty := sup_{\pi \in \hat{K}} \|f_\pi\|_\infty,$$

where the right hand side norms are operator norms as in (34, D.37, 36(e)). Define $\mathcal{E}_p(\hat{K})$, $\mathcal{E}_\infty(\hat{K})$, and $\mathcal{E}_0(\hat{K})$ as in (34, 28.24). These are Banach spaces with isometric involution (34, 28.25), (10). Also $\mathcal{E}_0(\hat{K})$ is a C^*-algebra (34, 28.26). For each $\mu \in M(K)$, define $\hat{\mu} \in \mathcal{E}_\infty(\hat{K})$ by $\hat{\mu}(\pi) = \bar{\pi}(\mu)$, then $\mu \mapsto \hat{\mu}$ is a norm-decreasing $*$-isomorphism of $M(K)$ into $\mathcal{E}_\infty(\hat{K})$. Similarly one can define a norm-decreasing $*$-isomorphism $f \mapsto \hat{f}$ of $L^1(K)$ onto a dense subalgebra of $\mathcal{E}_0(\hat{K})$ (74, 3.2, 3.3). Also there is an isometric isomorphism $g \mapsto \hat{g}$ of $L^2(K)$ onto $\mathcal{E}_2(\hat{K})$. Each $g \in L^2(K)$ has a Fourier expansion

$$g = \sum_{\pi \in \hat{K}} \sum_{i,j=1}^{d_\pi} k_\pi \langle \hat{g}(\pi)\xi_i^\pi, \xi_j^\pi \rangle \pi_{i,j},$$

where the series converges in L^2-norm (74, 3.4).
For $\mu \in M(K)$ and $\pi \in \hat{K}$, we set $a_\pi = \bar{\pi}(\mu)^*$, and write

$$\mu \approx \sum_{\pi \in \hat{K}} k_\pi tr(a_\pi \pi).$$

If $\mu = f dm$, where $f \in L^1(K)$, then we write

$$f \approx \sum_{\pi \in \hat{K}} k_\pi tr(a_\pi \pi).$$

If moreover $\sum_{\pi \in \hat{K}} k_\pi \|a_\pi\|_1 < \infty$, we write $f \in A(K)$ and put

$$\|f\|_A = \sum_{\pi \in \hat{K}} k_\pi \|\hat{f}(\pi)\|_1.$$

$A(K)$ is a Banach space with respect to this norm, and $f \mapsto \hat{f}$ is an isometric isomorphism of $A(K)$ onto $\mathcal{E}_1(\hat{K})$. Also for each $f \in A(K)$ with $f \simeq \sum_{\pi \in \hat{K}} k_\pi tr(a_\pi \pi)$ we have

$$f(x) = \sum_{\pi \in \hat{K}} k_\pi tr(a_\pi \pi(x)),$$

m-a.e. (74, 4.2). If moreover f is positive definite, we have

$$f(e) = \|f\|_u := \sum_{\pi \in \hat{K}} k_\pi tr(\hat{f}(\pi)),$$

where the series converges absolutely (74, 4.4). If we denote the set of all continuous positive definite functions on K by $P(K)$, then $f \in P(K)$ if and only if $f \in A(K)$ and each operator $\hat{f}(\pi)$ is positive definite (74, 4.6) and $A(K) = span(P(K)) = L^2(K) * L^2(K)$ (74, 4.8, 4.9).

6.2 Commutative hypergroups

6.2.1 Fourier transform of unbounded measures

There is a fairly successful theory of Fourier transform on commutative hypergroups (10) which goes quite parallel to its group counterpart (except that there is no Pontrjagin duality for commutative hypergroups in general). The dual object $\hat{K}$ is not a hypergroup for some commutative hypergroups K, and even when $\hat{K}$ is a commutative hypergroup, its dual object is not necessarily K (10, Section 2.4). We do not review the standard results on the Fourier transform on commutative hypergroups and refer the interested reader to (10). Instead, here we develop a theory of Fourier transform for unbounded measures on commutative hypergroups. This is analogous to the Argabright-Lamadrid theory on abelian groups (7).
Let K be a locally compact hypergroup (a convo in the sense of (36)). We denote the spaces of bounded continuous functions and continuous functions of compact support on K by $C_b(K)$

and $C_c(K)$, respectively. The latter is an inductive limit of Banach spaces $C_A(K)$ consisting of functions with support in A, where A runs over all compact subsets of K. This in particular implies that if X is a Banach space, a linear transformation $T : C_c(K) \to X$ is continuous if and only if it is locally bounded, that is for each compact subset $A \subseteq K$ there is $\beta = \beta_A > 0$ such that

$$\|T(f)\| \leq \beta \|f\|_\infty \quad (f \in C_A(K)).$$

Also a version of closed graph theorem is valid for $C_c(K)$, namely T is locally bounded if and only if it has a closed graph (12). Throughout $C_c(K)$ is considered with the inductive limit topology (*ind*). Following (12) we have the following definitions.

Definition 6.1. A measure on K is an element of $C_c(K)^*$. The space of all measures on K is denoted by $M(K)$. The subspaces of bounded and compactly supported measures are denoted by $M_b(K)$ and $M_c(K)$, respectively.

Definition 6.2. For $\mu, \nu \in M(K)$, we say that μ is convolvable with ν if for each $f \in C_c(K)$, the map $(x, y) \mapsto f(x * y)$ is integrable over $K \times K$ with respect to the product measure $|\mu| \times |\nu|$. In this case $\mu * \nu$ is defined by

$$\int_K f d(\mu * \nu) = \int_K \int_K f(x * y) d\mu(x) d\nu(y) = \int_K \int_K \int_K f(t) d(\delta_x * \delta_y)(t) d\mu(x) d\nu(y).$$

Let $C(\nu)$ denote the set of all measures convolvable with ν. When K is a measured hypergroup with a left Haar measure m, a locally integrable function f is called convolvable with ν if $fm \in C(\nu)$. In this case, we put $f * \nu = fm * \nu$. Similarly if $\nu \in C(fm)$ then $\nu * f = \nu * fm$. The next two lemmas are a straightforward calculation and we omit the proof. Here Δ denotes the modular function of K. Also for a function f on K, we define $\bar{f}$ and $\tilde{f}$ by

$$\bar{f}(x) = f(\bar{x}), \ \tilde{f}(x) = \overline{f(\bar{x})} \quad (x \in K).$$

We denote the complex conjugate of f by f^-. For $\mu \in M(K)$, μ^-, $\bar{\mu}$ and $\tilde{\mu}$ are defined similarly.

Lemma 6.3. *If K is a measured hypergroup, $\mu \in M(K)$, and f is locally integrable on K, then*
(i) $(\mu * f)(x) = \int_K f(\bar{y} * x) d\mu(y) = \int_K f d(\bar{\mu} * \delta_x)$,
(ii) $(f * \mu)(x) = \int_K f(x * \bar{y}) \bar{\Delta}(y) d\mu(y) = \int_K f d(\delta_x * \Delta\bar{\mu})$,
locally almost everywhere. If $f \in C_c(K)$, the above formulas are valid everywhere and define continuous (not necessarily bounded) functions on K.

If we want to avoid the modular function in the second formula, we should define the convolution of functions and measures differently, but then this definition does not completely match with the formula for convolution of measures (see (10)).

Lemma 6.4. *If K is a measured hypergroup, $\mu, \nu \in M(K)$, $f \in C_c(K)$, and $\mu \in C(\nu)$, then $\bar{\mu} \in C(f)$, $\bar{\mu} * f \in L^1(\nu)$, and*

$$\int_K f d\mu * \nu = \int_K \bar{\mu} * f d\nu.$$

Definition 6.5. A measure $\mu \in M(K)$ is called left translation bounded if for each compact subset $A \subseteq K$

$$\ell_\mu(A) := \sup_{x \in K} |\bar{\mu} * \delta_x|(A) < \infty.$$

Similarly μ is called right translation bounded if for each compact subset $A \subseteq K$

$$r_\mu(A) := \sup_{x \in K} |\delta_x * \Delta\bar{\mu}|(A) < \infty.$$

We denote the set of left and right translation bounded measures on K by $M_{\ell b}(K)$ and $M_{rb}(K)$, respectively.

Proposition 6.6. *Let $\mu \in M(K)$, then*
(i) μ is left translation bounded if and only if $\mu * f \in C_b(K)$, for each $f \in C_c(K)$. In this case for each compact subset $A \subseteq K$,

$$\|\mu * f\|_\infty \leq \ell_\mu(A)\|f\|_\infty \quad (f \in C_A(K)).$$

(ii) μ is right translation bounded if and only if $f * \mu \in C_b(K)$, for each $f \in C_c(K)$. In this case for each compact subset $A \subseteq K$,

$$\|f * \mu\|_\infty \leq r_\mu(A)\|f\|_\infty \quad (f \in C_A(K)).$$

Proof. We prove (i), (ii) is proved similarly. If $f \in C_A(K)$, then by Lemma 6.3,

$$|(\mu * f)(x)| \leq \|f\|_\infty |\bar{\mu} * \delta_x|(A) \leq \|f\|_\infty \ell_\mu(A),$$

for each $x \in K$. Conversely, if $\mu * f$ is bounded for each $f \in C_c(K)$, then $f \mapsto \mu * f$ defines a linear map from $C_c(K)$ into $C_b(K)$. We claim that it has a closed graph. If $f_\alpha \to 0$ in (ilt), then there is a compact subset $A \subseteq K$ such that eventually $supp(f_\alpha) \subseteq A$ and $f_\alpha \to 0$, uniformly on A. If $\mu * f_\alpha \to g$, uniformly on K, then $|(\mu * f_\alpha)(x)| \leq \|f\|_\infty |\bar{\mu} * \delta_x|(A) \to 0$, for each $x \in K$. Hence $g = 0$. By closed graph theorem, for each compact subset $B \subseteq K$, there is $k_B > 0$ such that

$$\|\mu * f\|_\infty \leq k_B \|f\|_\infty \quad (f \in C_B(K)).$$

Now let $A \subseteq K$ be compact and choose $B \subseteq K$ compact with $B^\circ \supseteq A$. By Lemmas 6.3 and 6.4, for each $x \in K$,

$$\begin{aligned}
|\bar{\mu} * \delta_x|(A) &= \sup\{|\int_K g d(\bar{\mu} * \delta_x)| : \|g\|_\infty \leq 1, supp(g) \subseteq A\} \\
&\leq \sup\{|\int_K g d(\bar{\mu} * \delta_x)| : \|g\|_\infty \leq 1, supp(g) \subseteq B\} \\
&= \sup\{|\mu * g(x)| : \|g\|_\infty \leq 1, supp(g) \subseteq B\} \leq k_B. \quad \square
\end{aligned}$$

Example 6.7. All elements of $M_b(K)$ and $L^p(K, m)$ are translation bounded. Also each hypergroup has a left-translation invariant measure n (36, 4.3C). We have

$$(n * f)(x) = \int_K (\delta_x * \bar{f}) dn \leq \int_K \bar{f} dn,$$

for each $x \in K$ and $f \in C_c^+(K)$. Hence $n \in M_{\ell b}(K)$. Similarly $\Delta n \in M_{rb}(K)$.

Lemma 6.8. *(i) If $\theta \in M_c(K)$ then $\theta \in C(\mu)$ and $\mu \in C(\theta)$, for each $\mu \in M(K)$.*
(ii) If $\nu \in M(K)$, then
$a)$ $\nu \in M_{\ell b}(K)$ if and only if $\mu \in C(\nu)$, for each $\mu \in M_b(K)$.
$b)$ $\nu \in M_{rb}(K)$ if and only if $\nu \in C(\mu)$, for each $\mu \in M_b(K)$.

Proof. (i) follows from the fact that $|\bar{\theta}| * |f| \in C_c(K)$ for each $f \in C_c(K)$ (36, 4.2F). If $f \in C_c(K)$, $\nu \in M_{\ell b}(K)$ and $\mu \in M_b(K)$, then $|\nu| * |\bar{f}| \in C_b(K)$, hence

$$\int_K \int_K |f(x * y)| d|\mu|(x) d|\nu|(y) = \int_K (|\nu| * |\bar{f}|) d|\mu| < \infty,$$

so $\mu \in C(\nu)$. Conversely, if $\nu \notin M_{\ell b}(K)$ then there is $f \in C_c^+(K)$ such that $\nu * f$ is unbounded. Hence there is $\mu \in M_b^+(K)$ with $\nu * f \notin L^1(K)$, i.e.

$$\int_K \bar{f}(x * y) d\bar{m}(x) d\nu(y) = \int_K (\nu * f) d\mu = \infty,$$

that is $\bar{\mu} \notin C(\nu)$. This proves (iia). (iib) is similar. □

Corollary 6.9. *If $\nu \in M_{\ell b}(K)$ then for each $\mu \in M_b(K)$ and $f \in C_c(K)$ we have $\mu * f \in L^1(K, \nu)$ and*

$$\int_K (\mu * f) d\nu = \int_K (\nu * \bar{f}) d\mu.$$

Following (12, chap. 8), we have the following associativity result which follows from the above lemma and a straightforward application of Fubini's Theorem.

Theorem 6.10. *(i) If $\mu, \nu \in M(K)$, $\mu \in C(\nu)$, and $\theta \in M_c(K)$, then $\theta \in C(\mu) \cap C(\mu * \nu)$ and $\theta * \mu \in C(\nu)$, and*

$$(\theta * \mu) * \nu = \theta * (\mu * \nu).$$

(ii) If $\nu \in M_{\ell b}(K)$, $\mu_1, \mu_2 \in M_b(K)$ then $\mu_1 * (\mu_2 * \nu) = (\mu_1 * \mu_2) * \nu$.

6.2.2 Transformable measures

In this section we assume that K is a commutative hypergroup such that $\hat{K}$ is a hypergroup, namely K is strong (10, 2.4.1). We don't assume that $(\hat{K})\hat{} = K$, unless otherwise specified. We denote the Haar measure on K and $\hat{K}$ by m_K and $m_{\hat{K}}$, respectively. For $\mu \in M_b(K)$, $\hat{\mu} \in C_b(\hat{K})$ is defined by

$$\hat{\mu}(\gamma) = \int_K \overline{\gamma(x)} d\mu(x) \quad (\gamma \in \hat{K}).$$

We usually identify $\hat{\mu}$ with $\hat{\mu} m_{\hat{K}}$. Put $\check{\mu} = (\hat{\mu})\bar{}$.

Definition 6.11. (10, 2.3.10) A measure $\mu \in M(K)$ is called transformable if there is $\hat{\mu} \in M^+(\hat{K})$ such that

$$\int_K (f * \tilde{f}) d\mu = \int_{\hat{K}} |\check{f}| d\hat{\mu} \quad (f \in C_c(K)).$$

We denote the space of transformable measures on K by $M_t(K)$ and put $\hat{M}_t(\hat{K}) = \{\hat{\mu} : \mu \in M_t(K)\}$. Also we put $C_2(K) = span\{f * \tilde{f} : f \in C_c(K)\}$. Then above relation could be rewritten as

$$\int_K g d\mu = \int_{\hat{K}} \check{g} d\hat{\mu} \quad (g \in C_2(K)),$$

or

$$\int_K f * g d\mu = \int_{\hat{K}} \check{f} \check{g} d\hat{\mu} \quad (f, g \in C_c(K)).$$

Example 6.12. The Haar measure m_K is transformable and $\hat{m}_K = \pi_K$ is the Levitan-Plancherel measure on $\hat{K}$ (10, 2.2.13). Also all bounded measures are transformable. If $\mu \in M_b(K)$ and $\hat{\mu} \geq 0$ on $supp(\pi_K)$, then the transform of μ in the above sense is $\hat{\mu}\pi_K$, where $\hat{\mu}$ is the Fourier-Stieltjes transform of μ. In particular, $\hat{\delta}_e = \pi_K$ (10). Finally, if f is a bounded positive definite function on K, then by Bochner's Theorem (10, 4.1.16), there is a unique $\sigma \in M_b^+(K)$ such that $f = \check{\sigma}$. Then it is easy to see that $f m_K \in M_t(K)$ and $(f m_K)\hat{} = \tilde{\sigma}$.

Lemma 6.13. *$C_2(K)$ is dense in $C_c(K)$.*

Proof. Let $\{u_\alpha\}$ be a bounded approximate identity of $L^1(K, m_K)$ consisting of elements of $C_c(K)$ with $u_\alpha \geq 0$, $u_\alpha = \bar{u}_\alpha$, and $\int_K u_\alpha dm_K = 1$ such that $supp(u_\alpha)$ is contained in a compact neighborhood V of e for each α (29). For $f \in C_c(K)$ with $supp(f) = W$, we have $supp(f), supp(f * u_\alpha) \subseteq W * V =: U$. By uniform continuity of f, given $\varepsilon > 0$, there is α_0 such that $|f(x * y) - f(x)| < \varepsilon$, for each $x \in U$ and $y \in supp(u_\alpha)$ with $\alpha \geq \alpha_0$.

$$|(f * u_\alpha - f)(x)| \leq \int_K |f(x * y) - f(x)||u_\alpha(y)| dm_K(y) = \int_V \varepsilon u_\alpha dm_K = \varepsilon,$$

for $\alpha \geq \alpha_0$. □

By the above lemma and an argument similar to (7, Thm. 2.1) we have:

Theorem 6.14. *(Uniqueness Theorem) If $\mu \in M_t(K)$ then μ and $\hat{\mu}$ determine each other uniquely and the map $\mu \mapsto \hat{\mu}$ is an isomorphism of $M_t(K)$ onto $\hat{M}_t(\hat{K})$.*

The proof of the next result is straightforward.

Lemma 6.15. *For each $\mu \in M_t(K)$, the following measure are transformable with the given transform.*
(i) $\hat{\bar{\mu}} = \bar{\hat{\mu}}$,
(ii) $(\tilde{\mu})\hat{} = (\hat{\mu})^-$,
(iii) $(\mu^-)\hat{} = (\hat{\mu})\tilde{}$,
(iv) $(\delta_x * \mu)\hat{} = \bar{x}\hat{\mu} \quad (x \in K)$,
(v) $(\gamma\mu)\hat{} = \delta_{\tilde{\gamma}} * \hat{\mu} \quad (\gamma \in \hat{K})$.

If $\mu \in M_t(K)$, then by definition we have

$$\hat{C}_2(\hat{K}) := \{\hat{g} : g \in C_2(K)\} \subseteq L^2(\hat{K}, \hat{\mu}).$$

Next lemma is proved with the same argument as in (7, Prop. 2.2). We bring the proof for the sake of completeness.

Lemma 6.16. *$\hat{C}_2(\hat{K}) \subseteq L^2(\hat{K}, \hat{\mu})$ is dense.*

Proof. Since $C_c(\hat{K}) \subseteq L^2(\hat{K}, \hat{\mu})$ is dense, we need to show that given $\varphi \in C_c(\hat{K})$ and $\varepsilon > 0$, there is $g \in C_2(K)$ such that $\int_{\hat{K}} |\varphi - \hat{g}|^2 d\hat{\mu} < \varepsilon^2$.
Put $A = supp(\varphi)$. If $|\hat{\mu}|(A) = 0$, we take $g = 0$. Assume that $|\hat{\mu}|(A) > 0$. By (10, 2.2.4(iv)), $\hat{C}_c(\hat{K})$ is dense in $C_0(\hat{K})$, so there is $f \in C_c(K)$ such that

$$|\hat{f}(\gamma) - 1| < \varepsilon \|\varphi\|_\infty^{-1} (|\hat{\mu}|(A))^{\frac{-1}{2}} =: \delta \quad (\gamma \in A).$$

We may assume that $\delta < \frac{1}{2}$, so $|\hat{f}| > \frac{1}{2}$, and so $\int_{\hat{K}} |\hat{f}|^2 d\hat{\mu} \neq 0$. Choose $h \in C_c(K)$ such that $\|\hat{h} - \varphi\|_\infty < \varepsilon (\int_{\hat{K}} |\hat{f}|^2 d\hat{\mu})^{\frac{-1}{2}}$. Put $g = h * f \in C_2(K)$, then

$$\begin{aligned}\Big(\int_{\hat{K}} |\hat{g} - \varphi|^2 d\hat{\mu}\Big)^{\frac{1}{2}} &= \Big(\int_{\hat{K}} |\hat{h}\hat{f} - \varphi|^2 d\hat{\mu}\Big)^{\frac{1}{2}} \\ &\leq \Big(\int_{\hat{K}} |\hat{h}\hat{f} - \varphi\hat{f}|^2 d\hat{\mu}\Big)^{\frac{1}{2}} + \Big(\int_{\hat{K}} |\varphi\hat{f} - \varphi|^2 d\hat{\mu}\Big)^{\frac{1}{2}} < 2\varepsilon. \quad \square\end{aligned}$$

Lemma 6.17. *If $\mu \in M_t(K)$ and $g \in C_2(K)$, then $\hat{g} \in L^1(\hat{K}, \hat{\mu})$ and $g * \mu = (\hat{g}\hat{\mu})\check{}$.*

Proof. The first assertion follows from definition of $\hat{\mu}$. For the second, since K is unimodular, given $x \in K$,

$$(g * \mu)(x) = \int_K g d(\delta_x * \bar{\mu}) = \int_{\hat{K}} \check{g}(\gamma)\gamma(\bar{x}) d\hat{\mu}$$
$$= \int_{\hat{K}} \hat{g}(\gamma)\gamma(x) d\hat{\mu} = (\hat{g}\hat{\mu})\check{}. \quad \square$$

Now we are ready to prove the main result of this section.

Theorem 6.18. *For $\mu \in M(K)$, the following are equivalent:*
(i) $\mu \in M_t(K)$,
(ii) $g * \mu \in B(K)$, for each $g \in C_2(K)$.

Proof. If $g \in C_2(K)$, then $\hat{g} \in L^1(\hat{K}, \hat{\mu})$, hence $\hat{g}\hat{\mu} \in M_b(\hat{K})$. Therefore, by the above lemma and (10, 4.1.15), $g * \mu = (\hat{g}\hat{\mu})\check{} \in B(K)$. Conversely if $g * \mu \in B(K)$, for each $g \in C_2(K)$, then by Bochner's Theorem, there is a unique $\nu_g \in M_b(\hat{K})$ such that

$$g * \mu(x) = \int_{\hat{K}} \gamma(x) d\nu_g(\gamma) \quad (x \in K).$$

For each $f \in L^1(K, m)$,

$$\int_K f(g * \mu) dm = \int_K \int_{\hat{K}} f(x)\gamma(x) d\nu_g(\gamma) dm(x) = \int_{\hat{K}} \check{f} d\nu_g.$$

Fix $g, h \in C_2(K)$, then for each $f \in C_c(K)$,

$$\int_K (f * \bar{g})(h * \mu) dm = \int_K (f * \bar{g} * \bar{h}) d\mu = \int_K (f * \bar{h} * \bar{g}) d\mu = \int_K (f * \bar{h})(g * \mu) dm.$$

Hence

$$\int_{\hat{K}} \check{f}\hat{g} d\nu_h = \int_{\hat{K}} \check{f}\hat{h} d\nu_g.$$

By density of $\hat{C}_c(\hat{K})$ in $C_0(\hat{K})$, we get $\hat{g} d\nu_h = \hat{h} d\nu_g$. Define $\hat{\mu}$ on $\hat{K}$ by

$$\int_{\hat{K}} \psi d\hat{\mu} = \int_{\hat{K}} \frac{\psi}{\hat{h}} d\hat{\nu}_h,$$

where $h \in C_2(K)$ is such that $\hat{h} > 0$ on $supp(\psi)$. This is a well defined linear functional by what we just observed. It is easy to see that this is locally bounded on $C_c(\hat{K})$. Also

$$\int_{\hat{K}} \psi \hat{g} d\hat{\mu} = \int_{\hat{K}} \frac{\psi \hat{g}}{\hat{h}} d\nu_h = \int_{\hat{K}} \psi d\nu_g,$$

that is $\hat{g} d\hat{\mu} = d\nu_g$, and so $\hat{g} \in L^1(\hat{K}, \hat{\mu})$ and

$$\int_K g d\mu = (\bar{g} * \mu)(e) = \int_{\hat{K}} d\nu_{\hat{g}} = \int_{\hat{K}} \check{g} d\hat{\mu}.$$

Finally, since $\hat{h} > 0$ on $supp(\psi)$, we have $\int_{\hat{K}} \psi d\hat{\mu} \geq 0$, for $\psi \geq 0$. These all together show that $\mu \in M^+(\hat{K})$ is the transform of μ and we are done. $\square$

In (5) the authors introduced the concept of tensor hypergroups and showed that for a tensor hypergroup K, the Fourier space $A(K)$ is a Banach algebra. The following lemma follows from Plancherel Theorem exactly as in the group case (26, 3.6.2°).

Lemma 6.19. *If K is a commutative, strong, tensor hypergroup, then $A(K)$ is isometrically isomorphic (through Fourier transform) to $L^1(\hat{K})$.*

The last result of this paper is a direct consequence of the above lemma and Theorem 6.18 (see the proof of (7, Theorem 2.4)).

Corollary 6.20. *If K is a commutative, strong, tensor hypergroup, then for each $\mu \in M(K)$, the following are equivalent:*
(i) $\mu \in M_t(K)$ and $\hat{\mu}$ is a function,
(ii) $g * \mu \in A(K)$, for each $g \in C_2(K)$.

6.3 Finite hypergroups

Peter Shor in his seminal paper presented efficient quantum algorithms for computing integer factorizations and discrete logarithms. These algorithms are based on an efficient solution to the hidden subgroup problem (HSP) for certain abelian groups. HSP was already appeared in Simon's algorithm implicitly in form of distinguishing the trivial subgroup from a subgroup of order 2 of $\mathbb{Z}_{2^n}$.
The efficient algorithm for the abelian HSP uses the Fourier transform. Other methods have been applied by Mosca and Ekert (52). The fastest currently known (quantum) algorithm for computing the Fourier transform over abelian groups was given by Hales and Hallgren (32). Kitaev (39) has shown us how to efficiently compute the Fourier transform over any abelian group (see also (37)).
For general groups, Ettinger, Hoyer and Knill (25) have shown that the HSP has polynomial query complexity, giving an algorithm that makes an exponential number of measurements. Several specific non-abelian HSP have been studied by Ettinger and Hoyer (24), Rotteler and Beth (69), and Puschel, Rotteler, and Beth (61). Ivanyos, Mangniez, and Santha (37) have shown how to reduce certain non-abelian HSP's to an abelian HSP. The non-abelian HSP for normal subgroups is solved by Hallgren, Russell, and Ta-Shma (33).
As for the Graph Isomorphism Problem (GIP), which is a special case of HSP for the symmetric group S_n, Grigni, Schulman, Vazirani and Vazirani (31) have shown that measuring representations is not enough for solving GIP. However, they show that the problem can be solved when the intersection of the normalizers of all subgroups of G is large. Similar negative results are obtained by Ettinger and Hoyer (24). At the positive side, Beals (9) showed how to efficiently compute the Fourier transform over the symmetric group S_n (see also (40)).

6.3.1 Hidden subgroup problem

Definition 6.21. (Hidden Subgroup Problem (HSP)). Given an efficiently computable function $f : G \to S$, from a finite group G to a finite set S, that is constant on (left) cosets of some subgroup H and takes distinct values on distinct cosets, determine the subgroup H.

An efficient quantum algorithms for abelian groups is given in the next page. Note that the resulting distribution over ρ is independent of the coset cH arising after the first stage. Thus, repetitions of this experiment result in the same distribution over $\hat{G}$. Also by the principle of delayed measurement, measuring the second register in the first step can in fact be delayed until the end of the experiment.
In the next algorithm, one wishes the resulting distribution to be independent of the actual coset cH and depend only on the subgroup H. This is guaranteed by measuring only the name of the representation ρ and leaving the matrix indices unobserved. The fact that $O(log(|G|))$ samples of this distribution are enough to determine H with high probability is proved in (33).

1. Prepare the state

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |f(g)\rangle$$

and measure the second register, the resulting state is

$$\frac{1}{\sqrt{|H|}} \sum_{h \in H} |ch\rangle |f(ch)\rangle$$

where c is an element of G selected uniformly at random.
2. Compute the Fourier transform of the "coset" state above, resulting in

$$\frac{1}{\sqrt{|H|.|G|}} \sum_{\rho \in \hat{G}} \sum_{h \in H} \rho(ch) |\rho\rangle |f(ch)\rangle$$

where $\hat{G}$ denotes the Pontryagin dual of G, namely the set of homomorphisms $\rho : G \to \mathbb{T}$.
3. Measure the first register and observe a homomorphism ρ.

Algorithm 1. *Algorithm for Abelian Hidden Subgroup Problem*

1. Prepare the state $\sum_{g \in G} |g\rangle |f(g)\rangle$ and measure the second register $|f(g)\rangle$. The resulting state is $\sum_{h \in H} |ch\rangle |f(ch)\rangle$ where c is an element of G selected uniformly at random. As above, this state is supported on a left coset cH of H.
2. Let $\hat{G}$ denote the set of irreducible representations of G and, for each $\rho \in \hat{G}$, fix a basis for the space on which ρ acts. Let d_ρ denote the dimension of ρ. Compute the Fourier transform of the coset state, resulting in

$$\sum_{\rho \in \hat{G}} \sum_{1 \le i,j \le d_\rho} \frac{\sqrt{d_\rho}}{\sqrt{|H|.|G|}} \sum_{h \in H} \rho(ch) |\rho, i, j\rangle |f(ch)\rangle$$

3. Measure the first register and observe a representation ρ.

Algorithm 2. *Algorithm for Non-abelian Hidden Normal Subgroup Problem*

A finite hypergroup is a set $K = \{c_0, c_1, \ldots, c_n\}$ together with an $*$-algebra structure on the complex vector space $\mathbb{C}K$ spanned by K which satisfies the following axioms. The product of elements is given by the structure equations

$$c_i * c_j = \sum_k n_{i,j}^k c_k,$$

with the convention that summations always range over $\{0, 1, \ldots, n\}$. The axioms are

1. $n_{i,j}^k \in \mathbb{R}$ and $n_{i,j}^k \ge 0$,
2. $\sum_k n_{i,j}^k = 1$,
3. $c_0 * c_i = c_i * c_0 = c_i$,
4. $K^* = K$, $n_{i,j}^0 \ne 0$ if and only if $c_i^* = c_j$,

for each $0 \le i, j, k \le n$.
If $c_i^* = c_i$, for each i, then the hypergroup is called hermitian. If $c_i * c_j = c_j * c_i$, for each i, j, then the hypergroup is called commutative. Hermitian hypergroups are automatically commutative.
In harmonic analysis terminology, we have a convolution structure on the measure algebra $M(K)$. This means that we can convolve finitely additive measures on K and, for $x, y \in K$, the convolution $\delta_x * \delta_y$ is a probability measure. Indeed $\delta_{c_i} * \delta_{c_j}\{c_k\} = n_{i,j}^k$. We follow the convention of harmonic analysis texts and denote the involution by $x \mapsto \bar{x}$ (instead of x^*), and the identity element by e (instead of c_0). For a function $f : K \to \mathbb{C}$, and sets $A, B \subseteq K$ we put

$$f(x * y) = \sum_{z \in K} f(z)(\delta_x * \delta_y)\{z\}, \quad (x, y \in K),$$

and

$$A * B = \cup\{supp(\delta_x * \delta_y) : x \in A, y \in B\}.$$

A finite hypergroup K always has a left Haar measure (positive, left translation invariant, finitely additive measure) $\omega = \omega_K$ given by

$$\omega\{x\} = \big((\delta_{\bar{x}} * \delta_x)\{e\}\big)^{-1} \quad (x \in K).$$

A function $\rho : K \to \mathbb{C}$ is called a character if $\rho(e) = 1, \rho(x * y) = \rho(x)\rho(y)$, and $\rho(\bar{x}) = \overline{\rho(x)}$. In contrast with the group case, characters are not necessarily constant on conjugacy classes. Let K be a finite commutative hypergroup, then $\hat{K}$ denotes the set of characters on K. In this case, for $\mu \in M(K)$ and $f \in \ell^2(K)$, we put

$$\hat{\mu}(\rho) = \sum_{x \in K} \rho(x)\mu\{x\}, \quad \hat{f}(\rho) = \sum_{x \in K} f(x)\rho(x)\omega\{x\} \quad (\rho \in \hat{K}).$$

Hence $\hat{f} = (f\omega)\hat{}$.

6.3.2 Subhypergroups

If $H \subseteq K$ is a subhypergroup (i.e. $\bar{H} = H$ and $H * H \subseteq H$), then $\hat{\omega}_H = \chi_{H^\perp}$ (10, 2.1.8), where the right hand side is the indicator (characteristic) function of

$$H^\perp = \{\rho \in \hat{K} : \rho(x) = 1 \ (x \in H)\}.$$

If K/H is the coset hypergroup (which is the same as the double coset hypergroup $K//H$ in finite case (10, 1.5.7)) with hypergroup epimorphism (quotient map) $q : K \to H/K$ (10, 1.5.22), then $(K/H)\hat{} \simeq H^\perp$ (with isomorphism map $\chi \mapsto \chi \circ q$) (10, 2.2.26, 2.4.8). Moreover, for each $\mu \in M(K)$, $q(\mu * \omega_H) = q(\mu)$ (10, 1.5.12). We say that K is strong if $\hat{K}$ is a hypergroup with respect to some convolution satisfying

$$(\rho * \sigma)\check{} = \check{\rho}\check{\sigma} \quad (\rho, \sigma \in \hat{K}),$$

where

$$\check{k}(x) = \sum_{\rho \in \hat{K}} k(\rho)\rho(x)\pi\{\rho\} \quad (x \in K, k \in \ell^2(\hat{K}, \pi))$$

is the inverse Fourier transform. In this case, for $\rho, \sigma \in \hat{K}$, we have $\rho \in \sigma * H^\perp$ if and only if $Res_H\rho = Res_H\sigma$, where $Res_H : \hat{K} \to \hat{H}$ is the restriction map (10, 2.4.15). Also H is strong and $\hat{K}/H^\perp \simeq \hat{H}$ (10, 2.4.16). Moreover $(\hat{K})\hat{} \simeq K$ (10, 2.4.18).

6.3.3 Fourier transform

Let us quote the following theorem from (10, 2.2.13) which is the cornerstone of the Fourier analysis on commutative hypergroups.

Theorem 6.22. *(Levitan) If K is a finite commutative hypergroup with Haar measure ω, there is a positive measure π on $\hat{K}$ (called the Plancherel measure) such that*

$$\sum_{x\in K} |f(x)|^2\omega\{x\} = \sum_{\rho\in\hat{K}} |\hat{f}(\rho)|^2\pi\{\rho\} \quad (f \in \ell^2(K,\omega)).$$

Moreover $supp(\pi) = \hat{K}$ and $\pi\{\rho\} = \pi\{\bar{\rho}\}$. In particular the Fourier transform $\mathfrak{F}$ is a unitary map from $\ell^2(K,\omega)$ onto $\ell^2(\hat{K},\pi)$.

In quantum computation notation,

$$\mathfrak{F} : |x\rangle \mapsto \frac{1}{\tau(x)} \sum_{\rho\in\hat{K}} \rho(x)\pi\{\rho\}|\rho\rangle,$$

where

$$\tau(x) = \big(\sum_{\rho\in\hat{K}} |\rho(x)|^2\pi^2\{\rho\}\big)^{\frac{1}{2}} \quad (x \in K).$$

When K is a group, $\tau(x) = |\hat{K}|^{\frac{1}{2}}$, for each $x \in K$. It is essential for quantum computation purposes to associate a unitary matrix to each quantum gate. however, if we write the matrix of $\mathfrak{F}$ naively using the above formula we don't get a unitary matrix. The reason is that, in contrast with the group case, the discrete measures on ℓ^2 spaces are not counting measure. More specifically, when K is a group, $\ell^2(K) = \bigoplus_{x\in K}\mathbb{C}$, where as here $\ell^2(K,\omega) = \bigoplus_{x\in K}\omega\{x\}^{\frac{1}{2}}\mathbb{C}$ and $\ell^2(K) = \bigoplus_{\rho\in\hat{K}}\pi\{\rho\}^{\frac{1}{2}}\mathbb{C}$. The exponent $\frac{1}{2}$ is needed to get the same inner product on both sides. If we use change of bases $|x\rangle' = \omega\{x\}^{\frac{1}{2}}|x\rangle$ and $|\rho\rangle' = \pi\{\rho\}^{\frac{1}{2}}|\rho\rangle$, the Fourier transform can be written as

$$\mathfrak{F} : |x\rangle' \mapsto \omega\{x\}^{\frac{1}{2}} \sum_{\rho\in\hat{K}} \rho(\bar{x})\pi\{\rho\}^{\frac{1}{2}}|\rho\rangle',$$

and the corresponding matrix turns out to be unitary.

6.3.4 Examples

There ar a variety of examples of (commutative hypergroups) whose dual object is known. One might hope to relate the HSP on a (non-abelian) group G to the HSHP on a corresponding commutative hypergroup like $\hat{G}$ (see next example). The main difficulty is to go from a function f which is constant on cosets of some subgroup $H \leq G$ to a function which is constant on cosets of a subhypergroup of $\hat{G}$. The canonical candidate $\hat{f}$ fails to be constant on costs of $H^{\perp} \leq \hat{G}$.
We list some of the examples of commutative hypergroups and their duals, hoping that one can get such a relation in future.

Example 6.23. If G is a finite group, then $\hat{G} := (G^G)\hat{}$ is a commutative strong (and so Pontryagin (10, 2.4.18)) hypergroup (10, 8.1.43). The dual hypergroups of the Dihedral group D_n and the (generalized) Quaternion group Q_n are calculated in (10, 8.1.46,47).

Example 6.24. If G is a finite group and H is a (not necessarily normal) subgroup of G then the double coset space $G//H$ (which is basically the same as the homogeneous space G/H in the finite case) is a hypergroup whose dual object is $A(\hat{G}, H)$ (10, 2.2.46). It is easy to put conditions on H so that $G//H$ is commutative.

There are also a vast class of special hypergroups (see chapter 3 of (10) for details) which are mainly infinite hypergroups, but one might mimic the same constructions to get similar finite hypergroups in some cases.
There are not many finite hypergroups whose character table is known (75). Here we give two classical examples (of order two and three) and compute the corresponding Fourier matrix.

Example 6.25 (Ross)**.** The general form of an hypergroup of order 2 is known. It is denoted by $K = \mathbb{Z}_2(\theta)$ and consists of two elements 0 and 1 with multiplication table

$*$	δ_0	δ_1
δ_0	δ_0	δ_1
δ_1	δ_1	$\theta\delta_0 + (1-\theta)\delta_1$

and Haar measure and character table

	0	1
ω	1	$\frac{1}{\theta}$
χ_0	1	1
χ_1	1	$-\theta$

When $\theta = 1$ we get $K = \mathbb{Z}_2$. The dual hypergroup is again $\mathbb{Z}_2(\theta)$ with the Plancherel measure

	χ_0	χ_1
π	$\frac{\theta}{1+\theta}$	$\frac{1}{1+\theta}$

The unitary matrix of the corresponding Fourier transform is given by

$$\mathfrak{F}_2 = \frac{1}{\sqrt{1+\theta^2}} \begin{pmatrix} \theta & 1 \\ 1 & -\theta \end{pmatrix}$$

Example 6.26 (Wildberger)**.** The general form of hypergroups of order 3 is also known. We know that it is always commutative, but in this case, the Hermitian and non Hermitian case should be treated separately. Let $K = \{0, 1, 2\}$ be a Hermitian hypergroup of order three and put $\omega_i = \omega\{i\}$, for $i = 0, 1, 2$. Then the multiplication table of K is

$*$	δ_0	δ_1	δ_2
δ_0	δ_0	δ_1	δ_2
δ_1	δ_1	$\frac{1}{\omega_1}\delta_0 + \alpha_1\delta_1 + \beta_1\delta_2$	$\gamma_1\delta_1 + \gamma_2\delta_2$
δ_2	δ_2	$\gamma_1\delta_1 + \gamma_2\delta_2$	$\frac{1}{\omega_2}\delta_0 + \beta_2\delta_1 + \alpha_2\delta_2$

where

$$\beta_1 = \frac{\gamma_1\omega_2}{\omega_1},\ \beta_2 = \frac{\gamma_2\omega_1}{\omega_2},\ \alpha_1 = 1 - \frac{1+\gamma_1\omega_2}{\omega_1},\ \alpha_2 = 1 - \frac{1+\gamma_2\omega_1}{\omega_2}\ \gamma_2 = 1-\gamma_1,$$

and γ_1, ω_1 and ω_2 are arbitrary parameters subject to conditions $0 \le \gamma_1 \le 1, \omega_1 \ge 1, \omega_2 \ge 1$, and

$$1 + \gamma_1\omega_2 \le \omega_1$$
$$1 + (1-\gamma_1)\omega_1 \le \omega_2.$$

The Plancherel measure and character table are given by

	π	0	1	2
χ_0	$\frac{s_1}{t}$	1	1	1
χ_1	$\frac{s_2}{t}$	1	x	z
χ_2	$\frac{s_3}{t}$	1	y	v

where

$$x = \frac{\alpha_1-\gamma_1}{2} + \frac{D}{2\omega_2}, \quad y = \frac{\alpha_1-\gamma_1}{2} - \frac{D}{2\omega_2}$$
$$z = \frac{\alpha_2-\gamma_2}{2} - \frac{D}{2\omega_2}, \quad v = \frac{\alpha_2-\gamma_2}{2} + \frac{D}{2\omega_2}$$
$$D = \sqrt{(1+\gamma_1\omega_2-\gamma_2\omega_1)^2 + 4\gamma_2\omega_1}$$

and

$$s_1 = x^2v^2 + \frac{y^2}{\omega_2} + \frac{z^2}{\omega_1} - (y^2z^2 + \frac{x^2}{\omega_2} + \frac{v^2}{\omega_1})$$
$$s_2 = y^2 + \frac{v^2}{\omega_1} + \frac{1}{\omega_2} - (v^2 + \frac{y^2}{\omega_2} + \frac{1}{\omega_1})$$
$$s_3 = z^2 + \frac{x^2}{\omega_2} + \frac{1}{\omega_1} - (x^2 + \frac{z^2}{\omega_1} + \frac{1}{\omega_1})$$
$$t = x^2v^2 + y^2 + z^2 - (x^2 + y^2z^2 + v^2).$$

Let $\pi_i = \pi\{\chi_i\} = \frac{s_i}{t}$ and $w_{ij} = \sqrt{\omega_i\pi_j}$, for $i, j = 0, 1, 2$, then the Fourier transform is given by the unitary matrix

$$\mathfrak{F}_3 = \begin{pmatrix} w_{00} & w_{10} & w_{20} \\ w_{01} & xw_{11} & zw_{21} \\ w_{02} & yw_{12} & vw_{22} \end{pmatrix}$$

One concrete example is the normalized Bose Mesner algebra of the square. In this case, $\omega_1 = 1, \omega_2 = 2, \gamma_1 = \beta_1 = \alpha_1 = \alpha_2 = 0, \gamma_2 = 1$, and $\beta_2 = \frac{1}{2}$. A simple calculation gives $D = 2, x = 1, y = z = -1, v = 0$, and if we put $\pi_1 = \frac{1}{4}$, we get $\pi_2 = \frac{1}{4}$ and $\pi_3 = \frac{1}{2}$. In this case, the Fourier transform matrix is

$$\mathfrak{F}_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

In the non-Hermitian case, the multiplication table of K is

$*$	δ_0	δ_1	δ_2
δ_0	δ_0	δ_1	δ_2
δ_1	δ_1	$\gamma\delta_1 + (1-\gamma)\delta_2$	$\alpha\delta_0 + \gamma\delta_1 + \gamma\delta_2$
δ_2	δ_2	$\alpha\delta_0 + \gamma\delta_1 + \gamma\delta_2$	$(1-\gamma)\delta_1 + \gamma\delta_2$

where $\gamma = \frac{1-\alpha}{2}$, and α is an arbitrary parameter with $0 < \alpha \leq 1$. When $\alpha = 1$, we get $K = \mathbb{Z}_3$. The dual hypergroup is again K and the Plancherel measure and character table are given by

	π	0	1	2
χ_0	$\frac{s_1}{t}$	1	1	1
χ_1	$\frac{s_2}{t}$	1	z	$\bar{z}$
χ_2	$\frac{s_2}{t}$	1	$\bar{z}$	z

where

$$z = \frac{-\alpha \pm i\sqrt{\alpha^2 + 2\alpha}}{2}.$$

$$s_1 = 2 - \omega_1(\alpha^2 + \alpha), \quad s_2 = \omega_1 - 1, \quad t = \omega_1(2 - \alpha^2 - \alpha).$$

Put $\pi_i = \pi\{\chi_i\}$ and $w_{ij} = \sqrt{\omega_i \pi_j}$, for $i, j = 0, 1, 2$, then the Fourier transform is given by the unitary matrix

$$\mathfrak{F}_3 = \begin{pmatrix} w_{00} & w_{10} & w_{20} \\ w_{01} & zw_{11} & \bar{z}w_{21} \\ w_{02} & \bar{z}w_{12} & zw_{22} \end{pmatrix}$$

As a concrete example, let us put $\omega_1 = \omega_2 = 2, \gamma = \frac{1}{4}$ and $\alpha = \frac{1}{2}$ to get $z = \frac{-1+i\sqrt{5}}{4}$ and $\pi_1 = \frac{1}{5}$, $\pi_2 = \pi_3 = \frac{2}{5}$. In this case, the Fourier transform matrix is

$$\mathfrak{F}_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \frac{-1+i\sqrt{5}}{4} & \frac{-1-i\sqrt{5}}{4} \\ \sqrt{2} & \frac{-1-i\sqrt{5}}{4} & \frac{-1+i\sqrt{5}}{4} \end{pmatrix}$$

6.3.5 Hidden subhypergroup problem

In this section we give an algorithm for solving hidden sub-hypergroup problem (HSHP) for abelian (strong) hypergroups. This algorithm is efficient for those finite commutative hypergroups whose Fourier transform is efficiently calculated. It is desirable that, following Kitaev (39), one shows that the Fourier transform could be efficiently calculated on each finite commutative hypergroup. This could be difficult, as there is yet no complete structure theory for finite commutative hypergroups (see chapter 8 of (10)).

Definition 6.27. (Hidden Sub-hypergroup Problem (HSHP)). Given an efficiently computable function $f : K \to S$, from a finite hypergroup K to a finite set S, that is constant on (left) cosets of some subhypergroup H and takes distinct values λ_c on distinct cosets $c * H$, for $c \in K$. Determine the subhypergroup H.

Lemma 6.28. *Let K be commutative and H be a sub-hypergroup of K and $\rho \in \hat{K}$, then the following are equivalent.*
(i) $\rho \in H^{\perp}$,
(ii) $\sum_{m \in c*H} \omega\{m\}\rho(\bar{m}) \neq 0$, *for each* $c \in K$,
(iii) $\sum_{m \in c*H} \omega\{m\}\rho(m) \neq 0$, *for some* $c \in K$.

Proof. $(i) \Rightarrow (ii)$ If $\rho \in H^{\perp}$ and $q : K \to K/H$ is the quotient map, then given $c \in K$, $q(\mu * \omega_H) = q(\mu)$ for $\mu = \delta_c\omega \in M(K)$. But clearly

$$q(\delta_c\omega) = \delta_{c*H}\omega = \sum_{m \in c*H} \delta_m\omega.$$

Hence $\rho(\delta_{c*H})\omega = \rho \circ q(\delta_c\omega) \neq 0$, where the last equality is because $\rho \circ q \in (K/H)\hat{}$ and a character is never zero.
$(iii) \Rightarrow (i)$ If $\rho \notin H^{\perp}$ then the multiplicative map $\rho \circ q$ should be identically zero on K/H (otherwise it is a character and $\rho \in H^{\perp}$). Hence $\sum_{m \in c*H} \rho(m)\omega = \rho(\delta_{c*H})\omega = 0$, for each $c \in K$. □

---

1. Prepare the state $|\chi_0\rangle'|0\rangle$.
2. Apply $\mathfrak{F}^{-1}$ to the first register to get

$$\sum_{x \in K} \omega\{x\}^{\frac{1}{2}}|x\rangle'|0\rangle.$$

3. Apply the black box to get

$$\sum_{x \in K} \omega\{x\}^{\frac{1}{2}}|x\rangle'|f(x)\rangle,$$

and measure the second register, to get

$$\frac{\sqrt{|K|}}{\sqrt{|c * H|}} \sum_{m \in c*H} \omega\{m\}^{\frac{1}{2}}|m\rangle'|\lambda_c\rangle,$$

where c is an element of K selected uniformly at random, and λ_c is the value of f on the coset $c * H$.
4. Apply $\mathfrak{F}$ to the first register to get

$$\frac{\sqrt{|K|}}{\sqrt{|c * H|}} \sum_{m \in c*H} \sum_{\rho \in \hat{K}} \omega\{m\}\pi\{\rho\}^{\frac{1}{2}}\rho(m)|\rho\rangle'|\lambda_c\rangle = \frac{\sqrt{|K|}}{\sqrt{|c * H|}} \sum_{\rho \in \hat{K}} \pi\{\rho\}^{\frac{1}{2}} \sum_{m \in c*H} \omega\{m\}\rho(m)|\rho\rangle'|\lambda_c\rangle$$

5. Measure the first register and observe a character ρ.

Algorithm 3. *Algorithm for Abelian Hidden Subhypergroup Problem*

Theorem 6.29. *If the Fourier transform could be efficiently calculated on a finite commutative hypergroup K, then the above algorithm solves HSHP for K in polynomial time.*

Note that in the above algorithm, the resulting distribution over ρ is independent of the coset $c * H$ arising after the first step. Also note that by Lemma 6.28, the character observed in step 3 is in $H^{\perp}$.

7. Fourier transform on groupoids

7.1 Abelian groupoids

The structure of abelian groupoids is recently studied by the third author in (57). Our basic reference for groupoids is (70).

A groupoid G is a small category in which each morphism is invertible. The unit space $X = G^{(0)}$ of G is is the subset of elements $\gamma\gamma^{-1}$ where γ ranges over G. The range map $r : G \to G^{(0)}$ and source map $s : G \to G^{(0)}$ are defined by $r(\gamma) = \gamma\gamma^{-1}$, and $s(\gamma) = \gamma^{-1}\gamma$, for $\gamma \in G$. We set $G^u = r^{-1}(u)$ and $G_u = s^{-1}(u)$. The loop space $G_u^u = \{\gamma \in G | r(\gamma) = s(\gamma)\}$ is a called the isotropy group of G at u.

Definition 7.1. An abelian groupoid is a groupoid whose isotropy groups are abelian.

Definition 7.2. (57) An equivalence relation R on X is r-discrete proper if its graph R is closed in $X \times X$ and the quotient map $q : X \to X/R$ is a local homeomorphism.

If the equivalence relation R on X is r-discrete proper, then the groupoid R is proper in the sense of (30, page 14), that is, the inverse image of every compact subset of $X \times X$ is compact in R. For a groupoid Γ with unit space $X = \Gamma^{(0)}$, the equivalence relation $R(\Gamma)$ and isotropy groupoid $\Gamma(X)$ are defined by $R(\Gamma) = \{(s(\gamma), r(\gamma)) : \gamma \in \Gamma\}$ and $\Gamma(X) = \{(\gamma, \sigma) : \gamma, \sigma \in \Gamma, r(\gamma) = s(\sigma)\}$.

Definition 7.3. An abelian groupoid Γ is called decomposable if $R = R(\Gamma)$ is r-discrete proper and covered with compact open R-sets.

Lemma 7.4. *The isotropy subgroupoid* $\Gamma(X)$ *is open in an decomposable abelian groupoid* Γ.

Proof. Since X is open in Γ and $r \times s$ is a continuous map when R has quotient topology, $\Gamma(X) = (r \times s)^{-1}(X)$ is open in Γ, because X is open in $R(\Gamma)$. □

Example 7.5. If Γ is an r-discrete abelian groupoid with finite unit space, then it is decomposable abelian groupoid.

7.1.1 Eigenfunctionals

A pair $(\mathcal{C}, \mathcal{D})$ is called a *regular* C^**-inclusion* if $\mathcal{D}$ is a maximal abelian C*-subalgebra of the unital C^*-algebra $\mathcal{C}$ such that $1 \in \mathcal{D}$ and (*i*) $\mathcal{D}$ has the extension property in $\mathcal{C}$; (*ii*) $\mathcal{C}$ is regular (as a $\mathcal{D}$-bimodule). Let P denote the (unique) conditional expectation of $\mathcal{C}$ onto $\mathcal{D}$. We call $(\mathcal{C}, \mathcal{D})$ a C*-diagonal if in addition, (*iii*) P is faithful.

If $\mathcal{C}$ is non-unital, $\mathcal{D}$ is called a diagonal in $\mathcal{C}$ if its unitization $\widetilde{\mathcal{D}}$ is a diagonal in $\widetilde{\mathcal{C}}$. A twist $\mathcal{G}$ is a proper $\mathbb{T}$-groupoid such that $\mathcal{G}/\mathbb{T}$ is an r-discrete equivalence relation $\mathcal{R}$. There is a one-to-one correspondence between twists and diagonal pairs of C*-algebras: Define

$$C_c(\mathcal{R}, \mathcal{G}) = \{f \in C_c(\mathcal{G}) : f(t\gamma) = tf(\gamma) \, (t \in \mathbb{T}, \gamma \in \mathcal{G})\}.$$

Then $\mathcal{G}$ is a $\mathbb{T}$-groupoid over $\mathcal{R}$ and

$$D(\mathcal{G}) = \{f \in C_c(\mathcal{R}, \mathcal{G}) : suppf \subset \mathbb{T}\mathcal{G}^0\}$$

where $\mathbb{T}\mathcal{G}^0$ denotes the isotropy group bundle of $\mathcal{G}$.

$C_c(\mathcal{R}, \mathcal{G})$ is a C^*-algebra with a distinguished abelian subalgebra $D(\mathcal{G})$. Furthermore, $D(\mathcal{G}) \cong C_0(\mathcal{G}^0)$. Now $C_c(\mathcal{R}, \mathcal{G})$ becomes a pre-Hilbert $D(\mathcal{G})$-module, whose completion $\mathbf{H}(E)$ is a

Hilbert $C_0(E^0)$-module. One may construct a $*$-homomorphism $\pi : C_c(\mathcal{R},\mathcal{G}) \to \mathcal{B}(\mathbf{H}(\mathcal{G}))$ such that $\pi(f)g = fg$ (the convolution product) for all $f, g \in C_c(\mathcal{R},\mathcal{G})$ and define $\mathcal{C}(\mathcal{G})$ to be the closure of $\pi(C_c(\mathcal{R},\mathcal{G}))$ and $\mathcal{D}(\mathcal{G})$ to be the closure of $\pi(D(\mathcal{G}))$. It turns out that $\mathcal{D}(\mathcal{G})$ is a diagonal in $\mathcal{C}(\mathcal{G})$, hence every twist gives rise to a diagonal pair of C^*-algebras. Conversely, given C^*-algebras $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{D}$ is a diagonal in $\mathcal{C}$, one may construct a twist $\mathcal{G}$ such that $\mathcal{C} = \mathcal{C}(\mathcal{G})$ and $\mathcal{D} = \mathcal{D}(\mathcal{G})$, giving a bijective correspondence between twists and diagonal pairs (see (41) for more details).

Theorem 7.6. *If Γ is an decomposable abelian groupoid, then $(C^*(\Gamma), C^*(\Gamma(X)))$ is a C^*-diagonal pair.*

Proof. Since $\mathcal{R}^{(0)} = \widehat{\Gamma(X)}$, where $\mathcal{R} := \widehat{\Gamma(X)} \rtimes_c R$, and $\widehat{\Gamma(X)} \rtimes_c R$ is a principal r-discrete groupoid, $(C^*(\widehat{\Gamma(X)} \rtimes_c R, \mathcal{G}), C_0(\widehat{\Gamma(X)}))$ is a diagonal pair (55, Theorem VIII.6), hence $(C^*(\Gamma), C^*(\Gamma(X)))$ is a C^*-diagonal, because the isomorphism between groupoid C^*-algebras preserves their diagonals. □

Another piece of structure of the pair $(C^*(\mathcal{R},\mathcal{G}) = C^*(\Gamma), C_0(\mathcal{R}^{(0)}) = C^*(\Gamma(X)))$ is the restriction map $P : f \mapsto f|_{C^*(\Gamma(X))}$ from $C^*(\Gamma)$ to $C^*(\Gamma(X))$ (65). In (54), the authors show that there is generalized conditional expectation $C^*(\Gamma) \to C^*(\Gamma(u))$.

Corollary 7.7. *If Γ is an decomposable abelian groupoid, then the restriction map $P : C^*(\Gamma) \to C^*(\Gamma(X))$ is the unique faithful conditional expectation onto $C^*(\Gamma(X))$.*

Corollary 7.8. *If Γ is an decomposable abelian groupoid, then $C^*(\Gamma(X))$ (resp. $C^*_\mu(\Gamma(X))$) is a maximal abelian sub-algebra (masa) in $C^*(\Gamma)$ (resp. $C^*_\mu(\Gamma)$).*

If Γ is a nontrivial decomposable abelian groupoid, X can not be the interior of $\Gamma(X)$, therefore $C^*(X)$ is not a maximal subalgebra of $C^*(\Gamma)$ (70, prop. II.4.7(ii)). By theorem 7.7, $C^*(\Gamma(X))$ is commutative and we can conclude that there is no $f \in C_c(\Gamma)$ with support outside $\Gamma(X)$ which is in the commutative C^*-algebra $C^*(\Gamma(X))$.

Corollary 7.9. *If Γ is an decomposable abelian groupoid, $C^*(\Gamma)$ is regular as a $C^*(\Gamma(X))$-bimodule.*

For a Banach $C^*(\Gamma(X))$-bimodule $\mathcal{M}$, An element $m \in \mathcal{M}$ is called an *intertwiner* if

$$m.C^*(\Gamma(X)) = C^*(\Gamma(X)).m.$$

If $m \in \mathcal{M}$ is an intertwiner such that for every $f \in C^*(\Gamma(X))$, $f.m \in \mathbb{C}m$, we call m a minimal intertwiner (compare intertwiners to normalizers (22, prop 3.3)). Regularity of a bimodule $\mathcal{M}$ is equivalent to norm-density of the $C^*(\Gamma(X))$-intertwiners (22, remarks 4.2).
A $C^*(\Gamma(X))$-eigenfunctional is a nonzero linear functional $\phi : \mathcal{M} \to \mathbb{C}$ such that, for all $f \in C^*(\Gamma(X))$, $g \to \phi(f * g)$, $g \to \phi(g * f)$ are multiples of ϕ. A minimal intertwiner of $\mathcal{M}^*$ is an eigenfunctional. We equip the set $\mathcal{E}_{C^*(\Gamma(X))}(\mathcal{M})$ of all $C^*(\Gamma(X))$- eigenfunctionals with the relative weak* topology $\sigma(\mathcal{M}^*, \mathcal{M})$. Its normalizer is the set $N(C^*(\Gamma(X))) = \{v \in C^*(\Gamma) : vC^*(\Gamma(X))v^* \subset C^*(\Gamma(X))$ and $v^*C^*(\Gamma(X))v \subset C^*(\Gamma(X))\}$. For $v \in N(C^*(\Gamma(X)))$, let $dom(v) := \{\phi \in \mathcal{Z} : \phi(v^*v) > 0\}$, note this is an open set in $\mathcal{Z} = \widehat{C^*(\Gamma(X))}$. There is a homeomorphism $\beta_v : dom(v) \to dom(v^*) := ran(v)$ given by

$$\beta_v(\phi)(f) = \frac{\phi(v^* * f * v)}{\phi(v^* * v)}.$$

It is easy to show that $\beta_v^{-1} = \beta_{v^*}$ (41).

Remark 7.10. Intertwiners and normalizers are closely related, at least when $C^*(\Gamma(X))$ is a masa in the unital C^*-algebra $C^*(\Gamma)$ containing $\mathcal{M}$ (22). Indeed, if $v \in C^*(\Gamma)$ is an intertwiner for $C^*(\Gamma(X))$, then $v^*v, vv^* \in C^*(\Gamma(X))' \bigcap C^*(\Gamma)$. If $C^*(\Gamma(X))$ is maximal abelian in $C^*(\Gamma)$, then v is a normalizer of $C^*(\Gamma(X))$ (22, Proposition 3.3). Conversely, for $v \in N(C^*(\Gamma(X)))$, if β_{v^*} extends to a homeomorphism of $S(v^*)$ onto $S(v)$, then v is an intertwiner. Moreover, if I is the set of intertwiners, then $N(C^*(\Gamma(X)))$ is contained in the norm-closure of I, and when $C^*(\Gamma(X))$ is a masa in $C^*(\Gamma)$, $N(C^*(\Gamma(X))) = I$ (22, proposition 3.4).

Since $C^*(\Gamma(X))$ is a C^*-subalgebra of the C^*-algebra $C^*(\Gamma)$, and it contains an approximate unit of $C^*(\Gamma)$, for each $v \in N(C^*(\Gamma(X)))$, we have $vv^*, v^*v \in C^*(\Gamma(X))$ (65, lemma 4.6).
Given an eigenfunctional $\phi \in \mathcal{E}_{C^*(\Gamma(X))}(\mathcal{M})$, the associativity of the maps $f \in C^*(\Gamma(X)) \mapsto f.\phi$ and $f \in C^*(\Gamma(X)) \mapsto \phi.f$ yields the existence of unique multiplicative linear functionals $s(\phi)$ and $r(\phi)$ on $C^*(\Gamma(X))$ satisfying $s(\phi)(f)\phi = f.\phi$ and $r(\phi)(f)\phi = \phi.f$, that is

$$\phi(g * f) = \phi(g)[s(\phi)(f)],\ \phi(f * g) = [r(\phi)(f)]\phi(g).$$

We call $s(\phi)$ and $r(\phi)$ the source and range of ϕ respectively (22, page 6). There is a natural action of the nonzero complex numbers z on $\mathcal{E}(\mathcal{M})$, sending (z, ϕ) to the functional $m \to z\phi(m)$; clearly $s(z\phi) = s(\phi)$ and $r(z\phi) = r(\phi)$. Also $\mathcal{E}(\mathcal{M}) \cup \{0\}$ is closed in the weak*-topology. Furthermore, $r : \mathcal{E}(\mathcal{M}) \to \mathcal{Z}$ and $s : \mathcal{E}(\mathcal{M}) \to \mathcal{Z}$ are continuous.

Notation 7.11. We define $\mathcal{G} = \mathcal{E}^1(C^*(\Gamma))$, where $\mathcal{E}^1(C^*(\Gamma))$ is the collection of norm-one eigenvectors for the dual action of $C^*(\Gamma(X))$ on the Banach space dual $C^*(\Gamma)^*$. For a bimodule $\mathcal{M} \subset C^*(\Gamma)$, $\mathcal{G}|\mathcal{M}$ can be defined directly in terms of the bimodule structure of $\mathcal{M}$, without explicit reference to $C^*(\Gamma)$ as in (22, remark 4.16).

The groupoid $\mathcal{G}$, with a suitable operation and the relative weak*- topology admits a natural $\mathbb{T}$-action. If $\phi, \psi \in \mathcal{E}(\mathcal{M})$ satisfy $r(\phi) = r(\psi)$ and $s(\phi) = s(\psi)$, then there exists $z \in \mathbb{C}$ such that $z \neq 0$ and $\phi = z\psi$ (22, corollary 4.10). Also $\mathcal{E}^1(\mathcal{M}) \cup \{0\}$ is weak*-compact (22, prop. 4.17). Thus, $\mathcal{E}^1(\mathcal{M})$ is a locally compact Hausdorff space. We may regard an element $m \in \mathcal{M}$ as a function on $\mathcal{E}^1(\mathcal{M})$ via $\hat{m}(\phi) = \phi(m)$. When $\mathcal{A}$ is both a norm-closed algebra and a $C^*(\Gamma(X))$-bimodule, the coordinate system $\mathcal{E}^1(\mathcal{A})$ has the additional structure of a continuous partially defined product as described in (22, remark4.14).

Notation 7.12. Let $\mathcal{R}(\mathcal{M}) := \{|\phi| : \phi \in \mathcal{E}^1(\mathcal{M})\}$. Then $\mathcal{R}(\mathcal{M})$ may be identified with the quotient $\mathcal{E}^1(\mathcal{M})\backslash\mathbb{T}$ of $\mathcal{E}^1(\mathcal{M})$ by the natural action of $\mathbb{T}$. A twist is a proper $\mathbb{T}$-groupoid $\mathcal{G}$ so that $\mathcal{G}\backslash\mathbb{T}$ is an principal r-discrete groupoid. The topology on $\mathcal{R}(C^*(\Gamma))$ is compatible with the groupoid operations, so $\mathcal{R}(C^*(\Gamma))$ is a topological equivalence relation (22).

We know that $(C^*(\Gamma), C^*(\Gamma(X)))$ is a C^*-diagonal, let $\mathcal{M} \subset C^*(\Gamma)$ be a norm-closed $C^*(\Gamma(X))$-bimodule. Then the span of $\mathcal{E}^1(\mathcal{M})$ is $\sigma(\mathcal{M}^*, \mathcal{M})$-dense in $\mathcal{M}^*$. Suppose $\mathcal{A}$ is a norm closed algebra satisfying $C^*(\Gamma(X)) \subset \mathcal{A} \subset C^*(\Gamma)$. If $\mathcal{B}$ is the C^*-subalgebra of $C^*(\Gamma)$ generated by $\mathcal{A}$, then $\mathcal{B}$ is the C^*-envelope of $\mathcal{A}$. If in addition, $\mathcal{B} = C^*(\Gamma)$, then $\mathcal{R}(C^*(\Gamma))$ is the topological equivalence relation generated by $\mathcal{R}(\mathcal{A})$ (22).
Eigenfunctionals can be viewed as normal linear functional on $\mathcal{B}^{**}$ and one may use the polar decomposition to obtain a minimal partial isometry for each eigenfunctional (22). Indeed, by the polar decomposition for linear functionals, there is a partial isometry $u^* \in C^*(\Gamma)^{**}$ and positive linear functionals $|\phi|, |\phi^*| \in C^*(\Gamma)^*$ so that $\phi = u^*.|\phi| = |\phi^*|.u^*$. Then $r(\phi) = |\phi|$ and $s(\phi) = |\phi^*|$. Moreover, uu^* and u^*u are the smallest projections in $C^*(\Gamma)^{**}$ which satisfy $u^*u.s(\phi) = s(\phi).u^*u = s(\phi)$ and $uu^*.r(\phi) = r(\phi).uu^* = r(\phi)$. For $\phi \in \mathcal{E}^1(C^*(\Gamma))$, we call the

above partial isometry u the partial isometry associated to ϕ, and denote it by v_ϕ. If $\phi \in \mathcal{Z}$, then u is a projection and we denote it by p_ϕ. The above equations show that $v_\phi^* v_\phi = p_{s(\phi)}$ and $v_\phi v_\phi^* = p_{r(\phi)}$. Moreover, given $\phi \in \mathcal{E}^1(C^*(\Gamma))$, v_ϕ may be characterized as the unique minimal partial isometry $w \in C^*(\Gamma)^{**}$ such that $\phi(w) > 0$. Recall that $\chi, \xi \in C^*(\hat{\Gamma}(X))$ satisfy $(\chi, \xi) \in \mathcal{R}(C^*(\Gamma))$ if and only if there is $\phi \in \mathcal{E}^1(C^*(\Gamma))$ with $r(\phi) = \chi$ and $s(\phi) = \xi$. For brevity, we write $\chi \sim \xi$ in this case (22).

Remark 7.13. For $\chi \in \mathcal{Z}$, we denote the GNS-representation of $C^*(\Gamma)$ associated to the unique extension of χ by $(\mathbf{H}_\chi, \pi_\chi)$. Let $\chi, \xi \in \mathcal{Z}$, then $\xi \sim \chi$ if and only if the GNS-representations π_χ and π_ξ are unitarily equivalent (22, lemma 5.8). Therefore if $\chi, \xi \in \mathcal{Z}(x)$, then $\xi \sim \chi$ if and only if $\xi = \chi$ (54, lemma 2.11).

If we set $\mathcal{M} = \{f \in C^*(\Gamma) : \chi(f^* f) = 0\}$, then $C^*(\Gamma)/\mathcal{M}$ is complete relative to the norm induced by the inner product $\langle f + \mathcal{M}, g + \mathcal{M}\rangle = \chi(g^* f)$, and thus $\mathbf{H}_\chi = C^*(\Gamma)/\mathcal{M}$.

Proposition 7.14. *Suppose $\chi \in \mathcal{Z}$ and $\phi \in \mathcal{E}^1(C^*(\Gamma))$ satisfy $\chi \sim s(\phi)$. Then there exist unit orthogonal unit vectors $\omega_1, \omega_2 \in \mathbf{H}_\chi$ such that for every $f \in C^*(\Gamma)$, $\phi(f) = \langle \pi_\chi(f)\omega_1, \omega_2\rangle$ (22).*

Theorem 7.15. *We have $\mathcal{R}(C^*(\Gamma)) \cong \mathcal{Z} \rtimes_c R(\Gamma)$, algebraically and topologically.*

Proof. Ionescu and Williams showed that every representation of Γ induced from an irreducible representation of a stability group is irreducible (30, page 296). We can extend a character $\chi \in \mathcal{Z}$ to $\chi \in \widehat{C^*(\Gamma)}$ such that $\chi|C^*(\Gamma|_{[y]\neq[x]}) = 1$, but since the extension is unique in C^*-diagonals, it will be equal to $Ind(x, \Gamma(X)_x\ , \chi)$. Let $\chi, \xi \in \widehat{\Gamma(X)}$, then $\chi \sim \xi$ if and only if the GNS-representations π_χ and π_ξ are unitarily equivalent. By (54, lemma 2.5), $Ind(x, \Gamma(X)_x, \chi)$ is unitarily equivalent to $Ind(x.k,\ \Gamma(X)_{s(k)}, \chi.k)$ in $\widehat{C^*(\Gamma)}$, hence they are in the same class in $\widehat{C^*(\Gamma)}$, that is $\chi \sim \chi.k$, where $k \in R$. But if two stability groups of Γ are not in the same orbit, none of their irreducible representation can be equivalent. This shows that the two sets are the same algebraically. Since the topology is r-discrete, they are also topologically isomorphic. □

Remark 7.16. We know that $C^*(\Gamma|_{[x]}) \cong K(L^2(X|_{[x]}, \mu)) \otimes C^*(\Gamma(x))$ (16, page 107). Also we have $L^2(X|_{[x]}, \mu) \cong L^2(R_x, \alpha_x)$ (56, page 135), therefore $\mathbf{H}_\chi = C_c(\Gamma_x) \otimes \mathbf{H}_{\pi_\chi}$ and

$$Ind(x, \Gamma(X)_x, \chi) = \pi_\chi,$$

where $\mathbf{H}_{\pi_\chi}$ is the Hilbert space constructed in $C^*(\Gamma(X)_x)$ by χ (54), but $\mathbf{H}_\chi$ is the Hilbert space constructed in $C^*(\Gamma)$ by the unique extension of χ (22). Hence

$$\frac{C^*(\Gamma|_{[x]})}{\{f \in C^*(\Gamma) : \chi(f^* * f) = 0\}} \cong C_c(\Gamma_x) \otimes \mathbf{H}_{\pi_\chi}$$

and $\pi_\chi(f)(g \otimes \omega) = Ind(x, \Gamma(X)_x, \chi)(f)(g \otimes \omega) = (f * g \otimes \omega)$.

Corollary 7.17. *The groupoid $\mathcal{G}$ above is the $\mathbb{T}$-groupoid of $\mathcal{Z} \rtimes_c R$. In other words, we have the exact sequence*

$$\mathcal{Z} \to \mathcal{Z} \times \mathbb{T} \to \mathcal{G} \to \mathcal{G}\backslash\mathbb{T} \cong \mathcal{Z} \rtimes_c R.$$

7.1.2 Fourier transform

We know that $\mathcal{Z} \subset \mathcal{E}^1(\mathcal{M})$. If Γ is an abelian group, we have $C^*(\Gamma(X)) = \mathcal{M} = C^*(\Gamma)$, hence $\mathcal{E}_{C^*(\Gamma(X))}(\mathcal{M}) \supset \widehat{C^*(\Gamma)}$, hence eigenfunctionals are generalizations of multiplicative linear functionals. However, in general, $\mathcal{E}^1(\mathcal{M})$ need not separate points of $\mathcal{M}$.
We define the open support of $f \in C^*(\mathcal{R})$ as

$$supp'(f) = \{\gamma \in \mathcal{R} : f(\gamma) \neq 0\}.$$

Let

$$C_0(\mathcal{Z}) = \{f \in C^*(\mathcal{R}, \mathcal{G}) : supp' f \subset \mathcal{Z}\}$$

(65). Since $\mathcal{R}$ is (topologically) principal, and the normalizer $N(C^*(\Gamma(X)))$ consists exactly of the elements of $C^*(\Gamma)$ whose open support is a bisection (65, Proposition 4.8), we may define $\alpha_v = \alpha_{supp'v}$, where α_S for a bisection S of an r-discrete groupoid $\mathcal{R}$ is defined in (65). We define the Weyl groupoid $\mathcal{K}$ of $(C^*(\Gamma), C^*(\Gamma(X)))$ as the groupoid of germs of

$$K(C^*(\Gamma(X))) = \{\alpha_v, v \in N(C^*(\Gamma(X)))\}.$$

By proposition 4.13 in (65), the Weyl groupoid $\mathcal{K}$ of $(C^*(\Gamma), C^*(\Gamma(X)))$ is canonically isomorphic to $\mathcal{R}$.
Following Renault, let us define $D = \{(z_1, v, z_2) \in \mathcal{Z} \times N(C^*(\Gamma(X))) \times \mathcal{Z} : v^*v(z_2) > 0$ and $z_1 = \alpha_v(z_2)\}$. Consider the quotient $\mathcal{G}(C^*(\Gamma(X))) = D/\sim$ by the equivalence relation: $(z_1, v, z_2) \sim (z_1', v', z_2')$ if and only if $z_2 = z_2'$ and there exist $f, f' \in C^*(\Gamma(X))$ with $f(z_2), f'(z_2) > 0$ such that $v * f = v' * f'$. The class of (z_1, v, z_2) is denoted by $[z_1, v, z_2]$. Then $\mathcal{G}(C^*(\Gamma(X)))$ has a natural groupoid structure over $\mathcal{Z}$, defined exactly in the same fashion as the groupoid of germs: the range and source maps are defined by $r[z_1, v, z_2] = z_1$, $s[z_1, v, z_2] = z_2$, the product by $[z_1, v, z_2][z_2, v', z] = [z_1, v * v', z]$ and the inverse by $[z_1, v, y]^{-1} = [y, v^*, z_1]$.
The map $(z_1, v, z_2) \to [z_1, \alpha_v, z_2]$ from D to $\mathcal{K}$ factors through the quotient and defines a groupoid homomorphism from $\mathcal{G}(C^*(\Gamma(X)))$ onto $W(C^*(\Gamma(X)))$. Moreover the subset $\mathcal{H} = \{[z, f, z] : f \in C^*(\Gamma(X)), f(z) \neq 0\} \subset \mathcal{G}(C^*(\Gamma(X)))\}$ can be identified with the trivial group bundle $\mathbb{T} \times \mathcal{Z}$. Since $C^*(\Gamma(X))$ is maximal abelian and contains an approximate unit of $C^*(\Gamma)$, the sequence

$$\mathcal{H} \to \mathcal{G}(C^*(\Gamma(X))) \to \mathcal{K}$$

is (algebraically) an extension (65, Proposition 4.14). Also we have a canonical isomorphism of extensions:

$$\begin{array}{ccccc}
\mathcal{H} & \longrightarrow & \mathcal{G}(C^*(\Gamma(X))) & \longrightarrow & \mathcal{K} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{T} \times \mathcal{Z} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{R}
\end{array}$$

It is easy to recover the topology of $\mathcal{G}(C^*(\Gamma(X)))$. Indeed, every $v \in N(C^*(\Gamma(X)))$ defines a trivialization of the restriction of $\mathcal{G}(C^*(\Gamma(X)))$ to the open bisection $S = supp'(v)$. Therefore $\mathcal{G}(C^*(\Gamma(X)))$ is a *locally trivial* topological twist over $\mathcal{K}$ (65, lemma 4.16).

Definition 7.18 (Fourier transform)**.** The Fourier transform is defined in (54, equaion (4.5)) as

$$\hat{f}(\chi, \gamma) = \frac{\int_{\Gamma(r(\gamma))} \overline{\chi(g)} f(g\gamma) d\beta^{r(\gamma)}(g)}{\sqrt{\omega(\gamma)}},$$

where ω is a continuous $\Gamma(X)$-invariant homomorphism from Γ to $\mathbb{R}^+$ such that for all $f \in C_c(\Gamma(X))$,

$$\int_{\Gamma(X)} f(g)d\beta^{r(\gamma)}(g) = \omega(\gamma)\int_{\Gamma(X)} f(\gamma g\gamma^{-1})d\beta^{s(\gamma)}(g).$$

Definition 7.19 (Gelfand transform)**.** Given $f \in C^*(\Gamma)$ and $(z_1, v, z_2) \in D$, define the Gelfand transform of f by

$$\hat{f}(z_1, v, z_2) = \frac{P(v^* * f)(z_2)}{\sqrt{v^* * v(z_2)}}.$$

Then $\hat{f}(z_1, v, z_2)$ depends only on its class in $\mathcal{G}(C^*(\Gamma(X)))$, $\hat{f}$ defines a continuous section of the *line bundle* $L(C^*(\Gamma(X))) := (\mathbb{C} \times \mathcal{G}(C^*(\Gamma(X))))/\mathbb{T}$, and the map $f \mapsto \hat{f}$ is linear and injective (65, Lemma 5.3). When f belongs to $C^*(\Gamma(X))$, then $\hat{f}$ vanishes off $\mathcal{Z}$ and its restriction to $\mathcal{Z}$ is the classical Gelfand transform. If v belongs to $N(C^*(\Gamma(X)))$, then the open support of $\hat{v}$ is the open bisection of $\mathcal{K}$ defined by the partial homeomorphism α_v (65, corollary 5.6). The Weyl groupoid $\mathcal{K}$ is a Hausdorff r-discrete groupoid (65, proposition 5.7). Let $N_c(C^*(\Gamma(X)))$ be the set of elements v in $N(C^*(\Gamma(X)))$ such $\hat{v}$ has compact support and let $C^*(\Gamma)_c$ be its linear span. Then $N_c(C^*(\Gamma(X)))$ is dense in $N(C^*(\Gamma(X)))$ and $C^*(\Gamma)_c$ is dense in $C^*(\Gamma)$, and the Gelfand map $\Psi : f \to \hat{f}$ defined above sends $C^*(\Gamma)_c$ bijectively onto $C_c(\mathcal{K}, \mathcal{G})$ and $C^*(\Gamma(X))_c = C^*(\Gamma(X)) \bigcap C^*(\Gamma)_c$ onto $C_c(\mathcal{Z})$. Hence the Gelfand map $\Psi : C^*(\Gamma)_c \to C_c(\mathcal{K}, \mathcal{G})$ is an $*$-algebra isomorphism (65, lemma 5.8).

7.2 Compact groupoids

All over this section, we assume that $\mathcal{G}$ is compact and the Haar system on $\mathcal{G}$ is normalized so that $\lambda_u(\mathcal{G}_u) = 1$, for each $u \in X$, where $X = \mathcal{G}^{(0)}$ is the unit space of $\mathcal{G}$. In general $\mathcal{G}$ may be non-Hausdorff, but as usual we assume that $\mathcal{G}^{(0)}$ and $\mathcal{G}^u$, $\mathcal{G}_v$, for each $u, v \in \mathcal{G}^{(0)}$ are Hausdorff.

7.2.1 Fourier transform

Let $\mathcal{G}$ be a groupoid (70, 1.1). The unit space of $\mathcal{G}$ and the range and source maps are denoted by $X = \mathcal{G}^{(0)}$, r and s, respectively. For $u, v \in \mathcal{G}^{(0)}$, we put $\mathcal{G}^u = r^{-1}\{u\}$, $\mathcal{G}_v = s^{-1}\{v\}$, and $\mathcal{G}^u_v = \mathcal{G}^u \cap \mathcal{G}_v$. Also we put $\mathcal{G}^{(2)} = \{(x, y) \in \mathcal{G} \times \mathcal{G} : r(y) = s(x)\}$.
We say that $\mathcal{G}$ is a topological groupoid (70, 2.1) if the inverse map $x \mapsto x^{-1}$ on $\mathcal{G}$ and the multiplication map $(x, y) \mapsto xy$ from $\mathcal{G}^{(2)}$ to $\mathcal{G}$ are continuous. This implies that the range and source maps r and s are continuous and the subsets $\mathcal{G}^u$, $\mathcal{G}_v$, and $\mathcal{G}^u_v$ are closed, and so compact, for each $u, v \in X$. We fix a left Haar system $\lambda = \{\lambda^u\}_{u \in X}$ and put $\lambda_u(E) = \lambda^u(E^{-1})$, for Borel sets $E \subseteq \mathcal{G}_u$, and let λ^v_u be the restriction of λ_u to the Borel σ-algebra of $\mathcal{G}^v_u$. The integrals against λ^u and λ^v_u are understood to be on $\mathcal{G}^u$ and $\mathcal{G}^v_u$, respectively. The functions on $\mathcal{G}^v_u$ are extended by zero, if considered as functions on $\mathcal{G}$. All over this section, we assume that $\lambda_u(\mathcal{G}^v_u) \neq 0$, for each $u, v \in X$. This holds in transitive groupoids. In this case, we say that $\mathcal{G}$ is *locally non-trivial* and we denote the restriction of λ_u to the σ-algebra of Borel subsets of $\mathcal{G}^v_u$ by λ^v_u.
The convolution product of two measurable functions f and g on $\mathcal{G}$ is defined by

$$f * g(x) = \int f(y)g(y^{-1}x)d\lambda^{r(x)}(y) = \int f(xy^{-1})g(y)d\lambda_{s(x)}(y).$$

A (continuous) *representation* of G is a double $(\pi, \mathcal{H}_\pi)$, where $\mathcal{H}_\pi = \{\mathcal{H}^\pi_u\}_{u \in X}$ is a continuous bundle of Hilbert spaces over X such that:

(i) $\pi(x) \in \mathcal{B}(\mathcal{H}^{\pi}_{s(x)}, \mathcal{H}^{\pi}_{r(x)})$ is a unitary operator, for each $x \in \mathcal{G}$,
(ii) $\pi(u) = id_u : \mathcal{H}^{\pi}_u \to \mathcal{H}^{\pi}_u$, for each $u \in X$,
(iii) $\pi(xy) = \pi(x)\pi(y)$, for each $(x, y) \in \mathcal{G}^{(2)}$,
(iv) $\pi(x^{-1}) = \pi(x)^{-1}$, for each $x \in \mathcal{G}$,
(v) $x \mapsto \langle \pi(x)\xi(s(x)), \eta(r(x))\rangle$ is continuous on $\mathcal{G}$, for each $\xi, \eta \in C_0(G^{(0)}, \mathcal{H}_\pi)$.

Two representations π_1, π_2 of $\mathcal{G}$ are called (unitarily) *equivalent* if there is a (continuous) bundle $U = \{U_u\}_{u\in X}$ of unitary operators $U_u \in \mathcal{B}(\mathcal{H}^{\pi}_u, \mathcal{H}^{\pi}_u)$ such that

$$U_{r(x)}\pi_1(x) = \pi_2(x)U_{s(x)} \quad (x \in \mathcal{G}).$$

We use $\mathcal{R}ep(\mathcal{G})$ to denote the category consisting of (equivalence classes of continuous) representations of $\mathcal{G}$ as objects and intertwining operators as morphisms (3, Notation 2.5). Let $\pi \in \mathcal{R}ep(\mathcal{G})$, the mappings

$$x \mapsto \langle \pi(x)\xi_{s(x)}, \eta_{r(x)}\rangle,$$

where ξ, η are continuous sections of $\mathcal{H}_\pi$ are called *matrix elements* of π. This terminology is based on the fact that if $\{e^i_u\}$ is a basis for $\mathcal{H}^{\pi}_u$, then $\pi_{ij}(x) = \langle \pi(x)e^j_{s(x)}, e^i_{r(x)}\rangle$ is the (i, j)-th entry of the (possibly infinite) matrix of $\pi(x)$. We denote the linear span of matrix elements of π by $\mathcal{E}_\pi$. By continuity of representations, $\mathcal{E}_\pi$ is a subspace of $C(\mathcal{G})$. It is clear that $\mathcal{E}_\pi$ depends only on the unitary equivalence class of π. For $u, v \in X$, $\mathcal{E}^{\pi}_{u,v}$ consists of restrictions of elements of $\mathcal{E}_\pi$ to $\mathcal{G}^v_u$. Also we put $\mathcal{E}_{u,v} = span\big(\cup_{\pi\in\hat{\mathcal{G}}} \mathcal{E}^{\pi}_{u,v}\big)$ and $\mathcal{E} = span\big(\cup_{\pi\in\hat{\mathcal{G}}} \mathcal{E}_\pi\big)$.
It follows from the Peter-Weyl theorem (1, Theorem 3.10) (note that there is a typo in (3, Theorem 3.10), and the orthonormal basis elements $\sqrt{d^{\pi}_u/\lambda_u(\mathcal{G}^v_u)}\pi^{ij}_{u,v}$ is wrongly inscribed as $\sqrt{d^{\pi}_u\lambda_u(\mathcal{G}^v_u)}\pi^{ij}_{u,v}$) that, for $u, v \in X$, if $\lambda_u(\mathcal{G}^v_u) \neq 0$, then for each $f \in L^2(\mathcal{G}^v_u, \lambda^v_u)$,

$$f = \sum_{\pi\in\hat{\mathcal{G}}} \sum_{i=1}^{d^{\pi}_v} \sum_{j=1}^{d^{\pi}_u} c^{ij}_{u,v,\pi}\pi^{ij}_{u,v} \ ,$$

where

$$c^{ij}_{u,v,\pi} = \frac{d^{\pi}_u}{\lambda_u(\mathcal{G}^v_u)} \int_{\mathcal{G}^v_u} f(x)\overline{\pi^{ij}_{u,v}(x)}d\lambda^v_u(x) \quad (1 \le i \le d^{\pi}_v, 1 \le j \le d^{\pi}_u).$$

This is a local version of the classical non commutative Fourier transform. As in the classical case, the main drawback is that it depends on the choice of the basis (which in turn gives the choice of the coefficient functions). The trick is similar to the classical case, that's to use the continuous decomposition using integrals. This is the content of the next definition. As usual, all the integrals are supposed to be on the support of the measure against which they are taken.

Definition 7.20. Let $u, v \in X$ and $f \in L^1(\mathcal{G}^v_u, \lambda^v_u)$, then the Fourier transform of f is $\mathfrak{F}_{u,v}(f) : \mathcal{R}ep(\mathcal{G}) \to \mathcal{B}(\mathcal{H}^{\pi}_v, \mathcal{H}^{\pi}_u)$ defined by

$$\mathfrak{F}_{u,v}(f)(\pi) = \int f(x)\pi(x^{-1})d\lambda^v_u(x).$$

To better understand this definition, let us go back to the group case for a moment. Let's start with a locally compact abelian group G. Then the Pontryagin dual $\hat{G}$ of G is a locally compact abelian group and for each $f \in L^1(G)$, its Fourier transform $\hat{f} \in C_0(\hat{G})$ is defined by

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx \quad (\chi \in \hat{G}).$$

The continuity of $\hat{f}$ is immediate and the fact that it vanishes at infinity is the so called *Riemann-Lebesgue lemma*. For non abelian compact groups, a similar construction exists, namely, with an slight abuse of notation, for each $f \in L^1(G)$ one has $\hat{f} \in C_0(\hat{G}, \mathcal{B}(\mathcal{H}))$, where $\hat{G}$ is the set of (unitary equivalence classes of) irreducible representations of G endowed with the Fell topology. In the groupoid case, one has a similar local interpretation. Each $f \in L^1(\mathcal{G}_u^v, \lambda_u^v)$ has its Fourier transform $\mathfrak{F}_{u,v}(f)$ in $C_0(\hat{\mathcal{G}}, \mathcal{B}_{u,v}(\mathcal{H}))$, where $\hat{\mathcal{G}}$ is the set of (unitary equivalence classes of) irreducible representations of $\mathcal{G}$ endowed again with the Fell topology, and $\mathcal{B}_{u,v}(\mathcal{H})$ is a bundle of operator spaces over $\hat{\mathcal{G}}$ whose fiber at π is $\mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$, and $C_0(\hat{\mathcal{G}}, \mathcal{B}_{u,v}(\mathcal{H}))$ is the set of all continuous sections vanishing at infinity.
Now let us discuss the properties of the Fourier transform. If we choose (possibly infinite) orthonormal bases for $\mathcal{H}_u^\pi$ and $\mathcal{H}_v^\pi$ and let each $\pi(x)$ be represented by the (possibly infinite) matrix with components $\pi_{u,v}^{ij}(x)$, then $\mathfrak{F}_{u,v}(f)$ is represented by the matrix with components $\mathfrak{F}_{u,v}(f)(\pi)^{ij} = \frac{\lambda_u(\mathcal{G}_u^v)}{d_u^\pi} c_{u,v,\pi}^{ji}$. When $f \in L^2(\mathcal{G}_u^v, \lambda_u^v)$, summing up over all indices i, j, we get the following.

Proposition 7.21. (Fourier inversion formula) *For each $u, v \in X$ and $f \in L^2(\mathcal{G}_u^v, \lambda_u^v)$,*

$$f = \sum_{\pi \in \hat{\mathcal{G}}} \frac{d_u^\pi}{\lambda_u(\mathcal{G}_u^v)} Tr\big(\mathfrak{F}_{u,v}(f)(\pi)\pi(\cdot)\big),$$

where the sum converges in the L^2 norm and

$$\|f\|_2^2 = \sum_{\pi \in \hat{\mathcal{G}}} \frac{d_u^\pi}{\lambda_u(\mathcal{G}_u^v)} Tr\big(\mathfrak{F}_{u,v}(f)(\pi)\mathfrak{F}_{u,v}(f)(\pi)^*\big).$$

We collect the properties of the Fourier transform in the following lemma. Note that in part (iii), $f^*(x) = \overline{f(x^{-1})}$, for $x \in \mathcal{G}_u^v$ and $f \in L^1(\mathcal{G}_u^v, \lambda_u^v)$.

Lemma 7.22. *Let $u, v, w \in X$, $a, b \in \mathbb{C}$, and $f, f_1, f_2 \in L^1(\mathcal{G}_u^v, \lambda_u^v)$, $g \in L^1(\mathcal{G}_v^w, \lambda_v^w)$, then for each $\pi \in \mathcal{R}ep(\mathcal{G})$,*
$(i)\mathfrak{F}_{u,v}(af_1 + bf_2) = a\mathfrak{F}_{u,v}(f_1) + b\mathfrak{F}_{u,v}(f_2)$,
$(ii)\mathfrak{F}_{u,w}(f * g)(\pi) = \mathfrak{F}_{u,v}(f)(\pi)\mathfrak{F}_{v,w}(g)(\pi)$,
$(iii)\mathfrak{F}_{v,u}(f^*)(\pi) = \mathfrak{F}_{u,v}(f)(\pi)^*$,
(iv) $\mathfrak{F}_{u,w}(\ell_x(f))(\pi) = \mathfrak{F}_{u,v}(f)(\pi)\pi(x^{-1})$ and $\mathfrak{F}_{w,v}(r_y(f))(\pi) = \pi(y)\ \mathfrak{F}_{u,v}(f)(\pi)$, whenever $x \in \mathcal{G}_v^w, y \in \mathcal{G}_u^w$.

As in the group case, there is yet another way of introducing the Fourier transform. For each finite dimensional continuous representation π of $\mathcal{G}$, let the *character* χ_π of π be the bundle of functions χ_π whose fiber χ_u^π at $u \in X$ is defined by $\chi_u^\pi(x) = \mathrm{Tr}(\pi(x))$, for $x \in \mathcal{G}_u^u$, where Tr is the trace of matrices. Note that one can not have these as functions defined on $\mathcal{G}_u^v$, since when $x \in \mathcal{G}_u^v$, $\pi(x)$ is not a square matrix in general. Also note that the values of the above character functions depend only on the unitary equivalence class of π, as similar matrices have the same trace. Now if $\pi \in \hat{\mathcal{G}}$, $x \in \mathcal{G}_u^v$, and $f \in L^1(\mathcal{G}_u^v, \lambda_u^v)$, then

$$\mathrm{Tr}\big(\mathfrak{F}_{u,v}(f)(\pi)\pi(x)\big) = \int f(y)\mathrm{Tr}(\pi(y^{-1}x))d\lambda_u^v(y) = f * \chi_u^\pi(x),$$

where in the last equality f is understood to be extended by zero to $\mathcal{G}_u$.

Corollary 7.23. *The map $P_{u,v}^\pi : L^2(\mathcal{G}_u^v, \lambda_u^v) \to \mathcal{E}_{u,v}^\pi$, $f \mapsto d_u^\pi f * \chi_u^\pi$ is a surjective orthogonal projection and for each $f \in L^2(\mathcal{G}_u^v, \lambda_u^v)$, we have the decomposition*

$$f = \sum_{\pi \in \hat{\mathcal{G}}} d_u^\pi f * \chi_u^\pi,$$

which converges in the L^2 norm.

Applying the above decomposition to the case where $u = v$ and $f = \chi_u^\pi$, we get

Corollary 7.24. *For each $u \in X$ and $\pi, \pi' \in \hat{\mathcal{G}}$,*

$$\chi_u^\pi * \chi_u^{\pi'} = \begin{cases} d_u^{\pi -1} & \text{if } \pi \sim \pi', \\ 0 & \text{otherwise.} \end{cases}$$

7.2.2 Inverse Fourier and Fourier-Plancherel transforms

Next we are aiming at the construction of the inverse Fourier transform. This is best understood if we start with yet another interpretation of the local Fourier transform. It is clear from the definition of $\mathfrak{F}_{u,v}$ that if $u, v \in X$, $\pi_1, \pi_2 \in \mathcal{R}ep(\mathcal{G})$, and $f \in L^1(\mathcal{G}_u^v, \lambda_u^v)$, then

$$\mathfrak{F}_{u,v}(f)(\pi_1 \oplus \pi_2) = \mathfrak{F}_{u,v}(f)(\pi_1) \oplus \mathfrak{F}_{u,v}(f)(\pi_2),$$

and the same is true for any number (even infinite) of continuous representations, so it follows from (1, Theorem 2.16) that $\mathfrak{F}_{u,v}(f)$ is uniquely characterized by its values on $\hat{\mathcal{G}}$, namely we can regard

$$\mathfrak{F}_{u,v} : L^1(\mathcal{G}_u^v, \lambda_u^v) \to \prod_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi),$$

where the Cartesian product is the set of all choice functions $g : \mathcal{G} \to \bigcup_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ with $g(\pi) \in \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$, for each $\pi \in \hat{\mathcal{G}}$. Consider the ℓ^∞-direct sum $\sum_{\pi \in \hat{\mathcal{G}}} \bigoplus \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$. The domain of our inverse Fourier transform then would be the algebraic sum $\sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$, consisting of those elements of the direct sum with only finitely many nonzero components. An element $g \in \mathcal{D}(\mathfrak{F}_{u,v}^{-1})$ is a choice function such that $g(\pi) \in \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$, for each $\pi \in \hat{\mathcal{G}}$ is zero, except for finitely many π's.

Definition 7.25. Let $u, v \in X$. The inverse Fourier transform

$$\mathfrak{F}_{u,v}^{-1} : \sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi) \to C(\mathcal{G}_u^v)$$

is defined by

$$\mathfrak{F}_{u,v}^{-1}(g)(x) = \sum_{\pi \in \hat{\mathcal{G}}} \frac{d_u^\pi}{\lambda_u(\mathcal{G}_u^v)} \mathrm{Tr}\big(g(\pi)\pi(x)\big) \quad (x \in \mathcal{G}_u^v).$$

To show that this is indeed the inverse map of the (local) Fourier transform we need a version of the Schur's orthogonality relations (1, Theorem 3.6).

Proposition 7.26. (Orthogonality relations) *Let $\tau, \rho \in \hat{\mathcal{G}}$, $u, v \in X$, $T \in \mathcal{B}(\mathcal{H}_\tau)$, $S \in \mathcal{B}(\mathcal{H}_\rho)$, $A \in B(\mathcal{H}_\rho, \mathcal{H}_\tau)$, and $\xi \in \mathcal{H}_\tau$, $\eta \in \mathcal{H}_\rho$, then*

$$(i) \quad \int \tau(x^{-1}) A_{r(x)} \rho(x) d\lambda_u^v(x) = \begin{cases} \frac{\lambda_u(\mathcal{G}_u^v)}{d_u^\tau} Tr(A_u) id_{\mathcal{H}_u^\tau} & \text{if } \tau = \rho, \\ 0 & \text{otherwise,} \end{cases}$$

$$(ii) \quad \int \tau(x^{-1}) \xi_{r(x)} \otimes \rho(x) \eta_{s(x)} d\lambda_u^v(x) = \begin{cases} \frac{\lambda_u(\mathcal{G}_u^v)}{d_u^\tau} \eta_u \otimes \xi_u & \text{if } \tau = \rho, \\ 0 & \text{otherwise,} \end{cases}$$

$$(iii) \quad \int Tr(T_{s(x)} \tau(x^{-1})) Tr(S_{r(x)} \rho(x)) d\lambda_u^v(x) = \begin{cases} \frac{\lambda_u(\mathcal{G}_u^v)}{d_u^\tau} Tr(T_u S_u) & \text{if } \tau = \rho, \\ 0 & \text{otherwise,} \end{cases}$$

$$(iv) \quad \int Tr(T_{s(x)} \tau(x^{-1})) \rho(x) d\lambda_u^v(x) = \begin{cases} \frac{\lambda_u(\mathcal{G}_u^v)}{d_u^\tau} T_u & \text{if } \tau = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

In some applications we need to use the orthogonality relations over $\mathcal{G}_u$ (not $\mathcal{G}_u^v$). In this case, using the normalization $\lambda_u(\mathcal{G}_u) = 1$, and essentially by the same argument we get the following result (3, Proposition 3.3).

Proposition 7.27. (Orthogonality relations) *Let $\tau, \rho \in \hat{\mathcal{G}}$, $u \in X$, $T \in \mathcal{B}(\mathcal{H}_\tau)$, $S \in \mathcal{B}(\mathcal{H}_\rho)$, $A \in B(\mathcal{H}_\rho, \mathcal{H}_\tau)$, and $\xi \in \mathcal{H}_\tau$, $\eta \in \mathcal{H}_\rho$, then*

$$(i) \quad \int \tau(x^{-1}) A_{r(x)} \rho(x) d\lambda_u(x) = \begin{cases} \frac{Tr(A_u)}{d_u^\tau} id_{\mathcal{H}_u^\tau} & \text{if } \tau = \rho, \\ 0 & \text{otherwise,} \end{cases}$$

$$(ii) \quad \int \tau(x^{-1}) \xi_{r(x)} \otimes \rho(x) \eta_{s(x)} d\lambda_u(x) = \begin{cases} \frac{1}{d_u^\tau} \eta_u \otimes \xi_u & \text{if } \tau = \rho, \\ 0 & \text{otherwise,} \end{cases}$$

$$(iii) \quad \int Tr(T_{s(x)} \tau(x^{-1})) Tr(S_{r(x)} \rho(x)) d\lambda_u(x) = \begin{cases} \frac{1}{d_u^\tau} Tr(T_u S_u) & \text{if } \tau = \rho, \\ 0 & \text{otherwise,} \end{cases}$$

$$(iv) \quad \int Tr(T_{s(x)} \tau(x^{-1})) \rho(x) d\lambda_u(x) = \begin{cases} \frac{1}{d_u^\tau} T_u & \text{if } \tau = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Now we are ready to state the properties of the local inverse Fourier transform (3, Proposition 3.4). But let us first introduce the natural inner products on its domain and range. For $f, g \in C(\mathcal{G}_u^v)$ and $h, k \in \sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ put $\langle f, g \rangle = \int \bar{f}.g d\lambda_u^v$, and

$$\langle h, k \rangle = \sum_{\pi \in \hat{\mathcal{G}}} \frac{d_u^\pi}{\lambda_u(\mathcal{G}_u^v)} \mathrm{Tr}(k(\pi) h^*(\pi)),$$

where the right hand side is a finite sum as h and k are of finite support. Also note that if $\epsilon_u : C(\mathcal{G}_u^u) \to \mathbb{C}$ is defined by $\epsilon_u(f) = f(u)$, then for each $f, g \in C(\mathcal{G}_u^v)$, we have $g * f^* \in C(\mathcal{G}_u^u)$ and $\langle f, g \rangle = \epsilon_u(g * f^*)$, where $f^* \in C(\mathcal{G}_v^u)$ is defined by $f^*(x) = \overline{f(x^{-1})}$, for $x \in \mathcal{G}_v^u$. Similarly, $h^* \in \sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ is defined by $h^*(\pi) = \bar{h}(\check{\pi})$, where $\bar{h}(\pi) = h(\pi)^*$, $\bar{\pi}(x) = \pi(x)^*$, and $\check{\pi}(x) = \pi(x^{-1})^*$, for each $\pi \in \hat{\mathcal{G}}$ and $x \in \mathcal{G}$. The star superscript denotes the conjugation of Hilbert space operators.

Proposition 7.28. *For each $u, v \in X$ and $h, k \in \sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$, $\ell \in \sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_w^\pi, \mathcal{H}_v^\pi)$ we have*
$(i)\mathfrak{F}_{u,v}\mathfrak{F}_{u,v}^{-1}(h) = h,$
$(ii)\lambda_u(\mathcal{G}_u^v)\mathfrak{F}_{u,w}^{-1}(hk) = \mathfrak{F}_{u,v}^{-1}(h) * \mathfrak{F}_{v,w}^{-1}(k),$
$(iii)\mathfrak{F}_{v,u}^{-1}(h^*) = (\mathfrak{F}_{u,v}^{-1}(h))^*,$
$(iv)\langle \mathfrak{F}_{u,v}^{-1}(h), \mathfrak{F}_{u,v}^{-1}(k) \rangle = \langle h, k \rangle.$

Next we define a norm on the domain of the inverse Fourier transform in order to get a Plancherel type theorem. Let $u, v \in X$, for $h \in \sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ we put $\|h\|_2 = \langle h, h \rangle^{\frac{1}{2}}$. This is the natural norm on the algebraic direct sum, when one endows each component $\mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ with the Hilbert space structure given by $\langle T, S \rangle = \frac{d_u^\pi}{\lambda_u(\mathcal{G}_u^v)} \mathrm{Tr}(ST^*)$. We denote the completion of $\sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ with respect to this norm by $\mathcal{L}_{u,v}^2(\mathcal{G})$. The above map is called the (local) *Fourier-Plancherel transform*. The next result is a direct consequence of (3, Proposition 3.2).

Theorem 7.29. (Plancherel Theorem) *For each $u, v \in X$ such that $\lambda_u(\mathcal{G}_u^v) \neq 0$, $\mathfrak{F}_{u,v}$ extends to a unitary $\mathfrak{F}_{u,v} : L^2(\mathcal{G}_u^v, \lambda_u^v) \to \mathcal{L}_{u,v}^2(\mathcal{G})$.*

8. Conclusion

The classical Fourier transform extends to most of the group-like structures. The fast Fourier transform could be computed on finite abelian groups and monoids. There are more sophisticated versions of the Fourier transform on compact groups and hypergroups. The theory of abelian groupoids should be developed with care, but one could safely define the Fourier transform in this case. The theory of Fourier transform on compact groups extends to compact groupoids and a local (bundle-wise) as well as a global Fourier transform is defined in this case.

9. References

[1] Amini, M. (2007). Tannaka-Krein duality for compact groupoids I, Representation theory. *Advances in Mathematics*, 214, 1, 78-91.
[2] Amini, M. (2007). Fourier transform of unbounded measures on hypergroups. *Bull. Ital. Union Math.*, 8, 819-828.
[3] Amini, M. (2010). Tannaka-Krein duality for compact groupoids II, duality. *Operators and Matrices*, 4, 573-592.
[4] Amini, M.; Kalantar, M.; Roozbehani, M.M. (2005). Hidden Sub-hypergroup Problem, Technical Report, Institute for Researches in Theoretical Physics and Mathematics, Tehran.
[5] Amini, M.; Medghalchi, A. (2004). Fourier algebras on tensor hypergrups. *Contemporary Math.*, 363, 1-14.

[6] Amini, M.; Myrnouri, H. (2010). Gelfand transform in abelian groupoid C^*-algebras. Preprint.
[7] Argabright, L.; de Lamadrid, J. (1974). *Fourier Analysis of Unbounded Measures on Locally Compact Abelian Groups*. Memoirs of the American Mathematical Society, No. 145, American Mathematical Society, Providence, R.I.
[8] Baake, M. (2002). Diffraction of weighted lattice subsets. *Canad. Math. Bull.*, 45, 4, 483-498.
[9] Beals, R. (1997). Quantum computation of Fourier transforms over symmetric groups, In: *Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing*, 48-53, El Paso, Texas.
[10] Bloom, R.; Heyer, H. (1995). *Harmonic Analysis of Probability Measures on Hypergroups*. de Gruyter Stud. Math., vol. 20, Walter de Gruyter, Berlin and Hawthorne.
[11] Bluestein, L. I. (1970). A linear filtering approach to the computation of the discrete Fourier transform.*IEEE Trans. Electroacoustics*, 18, 451-455.
[12] Bourbaki, N. (1965). *Elements de Mathematiques, Integration*. Second edition, Herman, Paris.
[13] Bracewell, R. N. (2000). *The Fourier Transform and Its Applications*. 3rd ed., McGraw-Hill, Boston.
[14] Brigham, E. O. (1988). *The Fast Fourier Transform and Applications*. Prentice Hall, Englewood Cliffs.
[15] Brown, D.E. (2007). Efficient classical simulation of the quantum Fourier transform. *New J. Phys.,* 9, 146, 1-7.
[16] Clark, L. O. (2007). Classifying the types of groupoid C^*-algebras, *J. Operator Theory*, 57, 2, 101-116.
[17] Cleve, R. (2004). A note on computing Fourier transforms by quantum programs, http://pages.cpsc.ucalgary.ca/ cleve.
[18] Clausen, M.; Baum, U. (1993). Fast Fourier transforms for symmetric groups: Theory and implementation. *Math. Comput.,* 61, 204, 833-847.
[19] Cooley, J. W.; Tukey, J. W. (1965). An algorithm for machine calculation of complex Fourier series. *Math. Comput.,* 19, 297-301.
[20] Diaconis, P. (1989). A generalization of spectral analysis with application to ranked data. *Ann. Statist.,* 17, 3, 949-979.
[21] Diaconis P.; Rockmore, D. (1990). Efficient computation of the Fourier transform on finite groups. *J. Amer. Math. Soc.,* 3, 2, 297-332.
[22] Donsig, A. P.; Pitts, D.R. (2008). Coordinate systems and bounded isomorphisms, *J. Operator Theory*, 59, 2, 359-416.
[23] Duoandikoetxea, J. (2001). *Fourier Analysis*, American Mathematical Society, Providence.
[24] Ettinger, M.; Hoyer, P. (2000). On quantum algorithms for noncommutative hidden subgroups. *Advances in Applied Mathematics,* 25, 239-251.
[25] Ettinger, M.; Hoyer, P.; Knill, R. (1999). Hidden subgroup states are almost orthogonal, Technical report, quant-ph/9901034.
[26] [E]] Eymard, P. (1964). L'algèbre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France*, 92, 181-236.
[27] Folland, G. (1995). *A Course in Abstract Harmonic Analysis*. Studies in Advanced Mathematics, CRC Press, Boca Raton.
[28] Fourier, J. B. (1822). *Theorie Analytique de la Chaleur*, Paris.

[29] Ghahramani, F.; Medghalchi, A.R. (1985-86). Compact multipliers on weighted hypergroup algebras I,II. *Math. Proc. Camb. Phil. Soc.*, 98, 493-500; 100, 145-149.
[30] Goehle, G. (2009). *Groupoid Crossed Products*. Ph.D dissertation, Dartmouth College.
[31] Grigni, M.; Schulman, L.; Vazirani, M.; Vazirani, U. (2001). Quantum mechanical algorithms for the nonAbelian hidden subgroup problem, In: *Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing*, Crete, Greece, 6-8 July.
[32] Hales, L.; Hallgren, S. (2000). An Improved Quantum Fourier Transform Algorithm and Applications, In: *Proceedings of the 41st Annual Symposium on Foundations of Computer Science*, 515-525, Redondo Beach, California, 12-14 November.
[33] Hallgren, S.; Russell, A.; Ta-Shma, A. (2000). Normal subgroup reconstruction and quantum computation using group representations, In: *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing*, 627-635, Portland, Oregon, 21-23 May.
[34] Hewitt, E.; Ross, K. A. (1970). *Abstract Harmonic Analysis*, Vol. II, Die Grundlehren der mathematischen Wissenschaften, Band 152, Springer-Verlag, Berlin.
[35] Halverson, T. (2004). Representations of the q-rook monoid. *J. Algebra*, 273, 227-251.
[36] Jewett, R.I. (1975). Spaces with an abstract convolution of measures. *Advances in Math.*, 18, 1-110.
[37] Ivanyos, G.; Magniez, F.; Santha, M. (2001). Efficient quantum algorithms for some instances of the non-abelian hidden subgroup problem, In: *Proceedings of the Thirteenth Annual ACM Symposium on Parallel Algorithms and Architectures*, 263-270, Heraklion, Crete Island, Greece, 4-6 July.
[38] James, G.; Kerber, A. (1984). *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, vol. 16, Cambridge University Press, Cambridge.
[39] Kitaev, A. Y. (1997). Quantum computations: algorithms and error correction. *Russian Mathematical Surveys*, 52, 6, 1191-1249.
[40] Kobler, J.; Schoning, U.; Toran, J. (1993). *The Graph Isomorphism Problem: Its Structural Complexity*. Birkhauser, Boston, MA.
[41] Kumjian, A. (1986). On C^*-diagonals. *Can. J. Math.*, 38, 4, 969-1008.
[42] Kumjian, A.; Muhly, P.S.; Renault J.; Williams, D.P. (1998). The Brauer group of a locally compact groupoid. *J. Amer. Math.*, 120, 901-954.
[43] Lawson, M. V. (1998). *Inverse Semigroups: The Theory of Partial Symmetries*. World Scientific, Singapore.
[44] Lomont, C. (2004). The hidden subgroup problem: review and open problems, quant-ph/0411037.
[45] Malandro, M. (2008). *Fast Fourier Transforms for Inverse Semigroups*, PhD. Thesis, Dartmouth College.
[46] Marquezino, F.L.; Portugal, R.; Sasse, F.D. (2010). Obtaining the Quantum Fourier Transform from the Classical FFT with QR Decomposition, quant-ph/1005.3730v1.
[47] Maslen, D.K. (1998). The efficient computation of Fourier transforms on the symmetric group. *Math. Comput.*, 67, 223, 1121-1147.
[48] Maslen, D.K.; Rockmore, D. N. (1995). Adapted Diameters and FFTs on Groups, Proc. 6th ACM-SIAM SODA, 1995, 253-62.
[49] Maslen, D.K.; Rockmore, D. N. (1997). Generalized FFTs-A survey of some recent results. *Proc. 1995 DIMACS Workshop on Groups and Computation*, 28, 183-238.
[50] Maslen, D.K.; Rockmore, D. N. (1997). Separation of variables and the computation of Fourier transforms on finite groups I. *J. Amer. Math. Soc.*, 10, 1, 169-214.

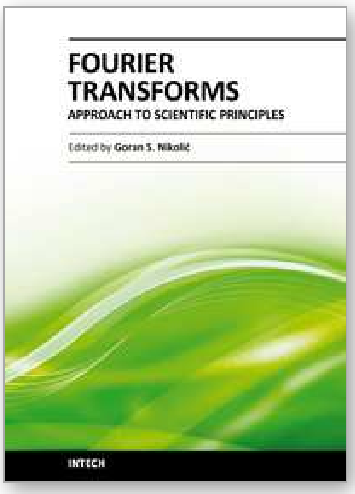
[51] Maslen, D.K.; Rockmore, D. N. (2001). The Cooley-Tukey FFT and group theory. *Notices of the AMS*, 48, 10, 1151-1161.
[52] Mosca, M.; Ekert, A. (1999). The hidden subgroup problem and eigenvalue estimation on a quantum computer, In: Proceedings if the 1st NASA International Conference on Quantum Computing and Quantum Communications, C.P. Williams (Ed.), Lecture Notes in Computer Science, 1509, 174-188, Springer-Verlag, Berlin.
[53] Munn, W. D. (1957). Matrix representations of semigroups. *Proc. Cambridge Philos. Soc.*, 53, 5-12.
[54] Muhly, P. S.; Renault, J. N.; Williams, D. P. (1996). Continuous-trace groupoid C^*-algebras III. *Trans. Amer. Math. Soc.*, 348, 9, 3621-3641.
[55] Muhly, P. S. (2003). Private communication with M. Amini.
[56] Muhly, P. S.; Williams, D. P. (1992). Continuous-trace groupoid C^*-algebras.II, *Math.Scand.*, 70, 127-145.
[57] Myrnouri, H. (2010). *Abelian Groupoids*. Ph.D. Thesis, Islamic Azad University.
[58] Nielsen, M. A.; Chuang, I. L. (2000). *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge.
[59] Parthasarathy, K. R. (2001). *Lectures on Quantum Computation and Quantum Error Correcting Codes*. Indian Statistical Institute, Delhi Center, Delhi.
[60] Preskill, J. (1998). *Lecture Notes for Physics 229: Quantum Information and Computation*, CIT.
[61] Puschel, M.; Rotteler, M. Beth, T. (1999). Fast quantum Fourier transforms for a class of non-Abelian groups, In: *Proc. 13th AAECC*, LNCS, 1719, 148-159.
[62] Ram, A. (1997). Seminormal representations of Weyl groups and Iwahori-Hecke algebras. *Proc. London Math. Soc.*, 75, 1, 99-133.
[63] Renault, J. (1980). *A Groupoid Approach to C^*-Algebras*. Lecture Notes in Mathematics 793, Springer Verlag, Berlin.
[64] Renault, J. (1983). Two applications of the dual groupoid of a C^*-algebra, In: *Lecture Notes in Mathematics*, 434-445, Springer-Verlag, New York.
[65] Renault, J. (2008). Cartan subalgebras in C^*-algebras. *Irish Math. Soc. Bulletin*. 61, 29-63.
[66] Rhodes, J.; Zalcstein, Y. (1991). Elementary representation and character theory of finite semigroups and its application, In: *Monoids and Semigroups with Applications*, 334-367, World Scitific, River Edge.
[67] Rockmore, D. N. (1995). Fast Fourier transforms for wreath products, *Appl. Comput. Harmon. Anal.*, 2, 279-292.
[68] Rockmore, D. N. (2005). Recent progress and applications in group FFTs, In: *Computational Noncommutative Algebra and Applications*, NATO Science Series, 136, 227-254, Springer, Netherlands.
[69] Rotteler, M.; Beth, T. (1998). Polynomial-time solution to the hidden subgroup problem for a class of non-abelian groups, quant-ph/9812070.
[70] Rudin, W. (1990). *Fourier Analysis on Groups*. Wiley Classics Library, John Wiley & Sons, New York.
[71] Shor, P. W. (1994). Algorithms for quantum computation: Discrete logarithms and factoring, In: *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science*, S. Goldwasser (Ed.), 124-134, IEEE Computer Society, Los Alamitos, CA.
[72] Terence Tao, Fourier transform, UCLA preprints.
[73] Terras, A. (1999). *Fourier Analysis on Finite Groups and Applications.* Cambridge University Press, Cambridge.

[74] Vrem, R.C. (1979). Harmonic analysis on compact hypergroups. *Pacific J. Math.*, 85, 239-251.

[75] Wildberger, N. J. (1993). Finite commutative hypergroups and applications from group theory to conformal field, In: *Applications of Hypergroups and Related Measure Algebras*, Contemp. Math. 183, 413-434, Proceedings Seattle Conference.

[76] Yates, F. (1937). The design and analysis of factorial experiments. *Imp. Bur. Soil Sci. Tech. Comm.*, 35.

[77] Young, A. (1977). The Collected Papers of Alfred Young, 1873-1940, University of Toronto Press, Toronto.



Fourier Transforms - Approach to Scientific Principles
Edited by Prof. Goran Nikolic

ISBN 978-953-307-231-9
Hard cover, 468 pages
Publisher InTech
Published online 11, April, 2011
Published in print edition April, 2011

InTech Europe
University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
Phone: +385 (51) 770 447
Fax: +385 (51) 686 166
www.intechopen.com

InTech China
Unit 405, Office Block, Hotel Equatorial Shanghai
No.65, Yan An Road (West), Shanghai, 200040, China
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元
Phone: +86-21-62489820
Fax: +86-21-62489821