We are IntechOpen,
the world's leading publisher of
Open Access books
Built by scientists, for scientists

6,900
Open access books available

185,000
International authors and editors

200M
Downloads

154
Countries delivered to

Our authors are among the

TOP 1%
most cited scientists

12.2%
Contributors from top 500 universities

BOOK CITATION INDEX
CLARIVATE ANALYTICS
INDEXED

WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com

Two-Dimensional Quaternionic Windowed Fourier Transform

Mawardi Bahri[1] and Ryuichi Ashino[2]
[1]*Department of Mathematics, Hasanuddin University*
Tamalanrea Makassar
[2]*Mathematical Sciences, Osaka Kyoiku University*
Kashiwara, Osaka, 582-8582
[1]*Indonesia*
[2]*Japan*

1. Introduction

Signal processing is a fast growing area today and the desired effectiveness in utilization of bandwidth and energy makes the progress even faster. Special signal processors have been developed to make it possible to implement the theoretical knowledge in an efficient way. Signal processors are nowadays frequently used in equipment for radio, transportation, medicine, and production, etc.

One of the basic problems encountered in signal representations using conventional Fourier transform (FT) is the ineffectiveness of the Fourier kernel to represent and compute location information. One method to overcome such a problem is the windowed Fourier transform (WFT). Recently, Gröchenig (2001); Gröchenig & Zimmermann (2001); Weisz (2008) have extensively studied the WFT and its properties from a mathematical point of view. Kemao (2007); Zhong & Zeng (2007) applied the WFT as a tool of spatial-frequency analysis, which is able to characterize the local frequency at any location in a fringe pattern.

On the other hand the quaternion Fourier transform (QFT), which is a nontrivial generalization of the real and complex Fourier transform (FT) using quaternion algebra, has been of interest to researchers, for example, Hitzer (2007); Mawardi et al. (2008); Sangwine & Ell (2007). It was found that many FT properties still hold but others have to be modified. Based on the (right-sided) QFT, one can extend the classical windowed Fourier transform (WFT) to quaternion algebra while enjoying the same properties as in the classical case.

In this paper, by using the adjoint operator of the (right-sided) QFT, we derive the Plancherel theorem for the QFT. We apply it to prove the orthogonality relation and reconstruction formula of the two-dimensional quaternionic windowed Fourier transform (QWFT). Our results can be considered as an extension and continuation of the previous work of Mawardi et al. (2008). We then present several examples to show the differences between the QWFT and the WFT. Finally, we present a generalization of the QWFT to higher dimensions.

2. Basics

For convenience of further discussions, we briefly review some basic facts on quaternions. The quaternion algebra over $\mathbb{R}$, denoted by

$$\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \tag{1}$$

is an associative non-commutative four-dimensional algebra, which obeys Hamilton's multiplication rules:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1. \tag{2}$$

The quaternion conjugate of a quaternion q is defined by

$$\bar{q} = q_0 - iq_1 - jq_2 - kq_3, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R}, \tag{3}$$

and it is an anti-involution, i.e.

$$\overline{qp} = \bar{p}\bar{q}. \tag{4}$$

From (3), we obtain the norm of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{5}$$

It is not difficult to see that

$$|qp| = |q||p|, \qquad \forall p, q \in \mathbb{H}. \tag{6}$$

Using the conjugate (3) and the modulus of q, we can define the inverse of $q \in \mathbb{H} \setminus \{0\}$ as

$$q^{-1} = \frac{\bar{q}}{|q|^2}, \tag{7}$$

which shows that $\mathbb{H}$ is a normed division algebra.

It is convenient to introduce the inner product $(f, g)_{L^2(\mathbb{R}^2;\mathbb{H})}$ valued in $\mathbb{H}$ of two quaternion functions f and g as follows:

$$(f, g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})} \, d^2\boldsymbol{x}. \tag{8}$$

The associated norm is defined by

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = (f, f)_{L^2(\mathbb{R}^2;\mathbb{H})}^{1/2} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \, d^2\boldsymbol{x} \right)^{1/2}. \tag{9}$$

As a consequence of the inner product (8), we obtain the *quaternion Cauchy-Schwarz* inequality:

$$\left| \int_{\mathbb{R}^2} f\bar{g} \, d^2\boldsymbol{x} \right| \leq \left(\int_{\mathbb{R}^2} |f|^2 d^2\boldsymbol{x} \right)^{1/2} \left(\int_{\mathbb{R}^2} |g|^2 d^2\boldsymbol{x} \right)^{1/2}, \qquad \forall f, g \in L^2(\mathbb{R}^2;\mathbb{H}). \tag{10}$$

3. Quaternionic Fourier Transform (QFT)

Let us introduce the continuous (right-sided) QFT. For more details, we refer the reader to Hitzer (2007); Mawardi et al. (2008); Sangwine & Ell (2007).

3.1 Definition of QFT

Definition 3.1 (Right-sided QFT). *The QFT of $f \in L^1(\mathbb{R}^2; \mathbb{H})$ is the function $\mathcal{F}_q\{f\} : \mathbb{R}^2 \to \mathbb{H}$ given by*

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} \, d^2\boldsymbol{x}, \tag{11}$$

where $\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2$, $\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2$, and the quaternion exponential product $e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}$ is the quaternion Fourier kernel.

Theorem 3.1 (Inverse QFT). *Suppose that $f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$. Then the QFT of f is an invertible transform and its inverse is given by*

$$\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1} \, d^2\boldsymbol{\omega}, \tag{12}$$

where the quaternion exponential product $e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1}$ is called the inverse (right-sided) quaternion Fourier kernel.

4. Linear Operators on Quaternionic Hilbert Spaces

In this section, we will briefly introduce the notation of linear operator on quaternionic Hilbert spaces. In fact, it is a natural generalization of the idea of an operator on a real and complex Hilbert space.

Definition 4.1. *Let X and Y be two $\mathbb{H}$-vector spaces. The operator $T : X \longrightarrow Y$ is called a left $\mathbb{H}$-linear space if*

$$T(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) = \alpha T(\boldsymbol{x}) + \beta T(\boldsymbol{x}), \tag{13}$$

for all quaternion constants $\alpha, \beta \in \mathbb{H}$ and for all $\boldsymbol{x}, \boldsymbol{y} \in X$.

Definition 4..2. *The adjoint of $\mathbb{H}$-linear operator $T : X \longrightarrow X$ is the unique $\mathbb{H}$-linear operator $T^* : X \longrightarrow X$ such that*

$$(T\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, T^*\boldsymbol{y}), \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X. \tag{14}$$

This gives the following result.

Theorem 4.1. *The adjoint of the QFT is inverse of the QFT multiplied by $(2\pi)^2$, i.e.*

$$(\mathcal{F}_q\{f\}, g)_{L^2(\mathbb{R}^2; \mathbb{H})} = (2\pi)^2 (f, \mathcal{F}_q^{-1}\{g\})_{L^2(\mathbb{R}^2; \mathbb{H})}. \tag{15}$$

Proof. For $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ we calculate the inner product (8) to get

$$
\begin{aligned}
(\mathcal{F}_q\{f\}, g)_{L^2(\mathbb{R}^2; \mathbb{H})} &= \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \, \overline{g(\boldsymbol{\omega})} \, d^2\boldsymbol{\omega} \\
&\overset{(11)}{=} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) \, e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} \, d^2\boldsymbol{x} \, \overline{g(\boldsymbol{\omega})} \, d^2\boldsymbol{\omega} \\
&\overset{(4)}{=} \int_{\mathbb{R}^2} f(\boldsymbol{x}) \left(\int_{\mathbb{R}^2} \overline{g(\boldsymbol{\omega}) \, e^{\boldsymbol{j} x_1 \omega_1} e^{\boldsymbol{i} x_2 \omega_2}} d^2\boldsymbol{\omega} \right) d^2\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{x}) \, (2\pi)^2 \overline{\mathcal{F}_q^{-1}\{g\}(\boldsymbol{x})} \, d^2\boldsymbol{x} \\
&= (2\pi)^2 (f, \mathcal{F}_q^{-1}\{g\})_{L^2(\mathbb{R}^2; \mathbb{H})},
\end{aligned}
\tag{16}
$$

which completes the proof. $\qquad \square$

Remark 4.1. *Note that Theorem 4.1 is not valid for the (two-sided) QFT. This fact implies that the Plancherel theorem can not be established.*

Theorem 4.2 (Plancherel formula). *Suppose that $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$. Then*

$$(\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\})_{L^2(\mathbb{R}^2;\mathbb{H})} = (2\pi)^2 (f, g)_{L^2(\mathbb{R}^2;\mathbb{H})} \tag{17}$$

and

$$(\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}], \mathcal{F}_q^{-1}[\mathcal{F}_q\{g\}])_{L^2(\mathbb{R}^2;\mathbb{H})} = (2\pi)^2 (f, g)_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{18}$$

Proof. A simple calculation gives for every $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$

$$(\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\})_{L^2(\mathbb{R}^2;\mathbb{H})} \overset{(15)}{=} (2\pi)^2 (f, \mathcal{F}_q^{-1}[\mathcal{F}_q\{g\}])_{L^2(\mathbb{R}^2;\mathbb{H})}$$

$$\overset{(12)}{=} (2\pi)^2 (f, g)_{L^2(\mathbb{R}^2;\mathbb{H})}, \tag{19}$$

as desired. Equation (18) can be established in a similar manner. $\qquad\square$

4.1 Discrete QFT

Similar to the discrete Fourier transform, the discrete quaternionic Fourier transform (DQFT) and the inverse discrete quaternionic Fourier transform (IDQFT) are defined as follows.

Definition 4.3. *Let $f(m, n)$ be a two-dimensional quaternion discrete-time sequence. The DQFT of $f(m, n)$ is defined by $F(u, v) \in \mathbb{H}^{M \times N}$, where*

$$F(u, v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) e^{-i\frac{um}{M}} e^{-j\frac{vn}{N}}. \tag{20}$$

Definition 4.4. *The IDQFT is defined by*

$$f(m, n) = \frac{1}{(2\pi)^2 MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j\frac{vn}{N}} e^{i\frac{um}{M}}. \tag{21}$$

4.2 Application of DQFT

In the following, we introduce an application of the DQFT to study two-dimensional discrete linear time-varying (TV) systems. For this purpose, let us introduce the following definition.

Definition 4.5. *Consider a two-dimensional discrete linear TV system with the quaternion impulse response of the filter denoted $h(\cdot, \cdot, \cdot, \cdot)$. The output $r(\cdot, \cdot)$ of the system to the input $f(\cdot, \cdot)$ is defined by*

$$r(m, n) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} f(u, v) h(m, n, m - u, n - v). \tag{22}$$

The transfer function of the TV filter h can be obtained by

$$R(m, n, \omega_1, \omega_2) = \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} h(m, n, m', n') e^{-im'\omega_1} e^{-jn'\omega_2}. \tag{23}$$

The following simple theorem relates the DQFT to the output of a discrete linear TV band-pass filter.

Theorem 4.3. *Consider a linear TV system with the quaternion impulse response h defined by*

$$h(m, n, m', n') = e^{-i\frac{m(m-m')}{M}} e^{-j\frac{n(n-n')}{N}}, \quad \text{for } 0 \leq m \leq M - 1, 0 \leq n \leq N - 1. \tag{24}$$

If the input to this system is the quaternion signal $f(u,v)$, then its output $r(\cdot, \cdot)$ is equal to the DQFT of $f(u,v)$.

Proof. Using Definition 4.5, we obtain

$$\begin{aligned}
r(m, n) &= \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} f(u, v) h(m, n, m - u, n - v) \\
&= \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} f(u, v) e^{-i\frac{m(m-(m-u))}{M}} e^{-j\frac{n(n-(n-v))}{N}} \\
&= \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} f(u, v) e^{-i\frac{um}{M}} e^{-j\frac{vn}{N}},
\end{aligned} \tag{25}$$

which completes the proof by Definition 4.3. □

If the quaternion impulse response h is given by

$$h(m, n, m', n') = \frac{1}{(2\pi)^2 MN} e^{j\frac{n(n-n')}{N}} e^{i\frac{m(m-m')}{M}}, \tag{26}$$

then (22) implies

$$\begin{aligned}
r_2(m, n) &= \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} f(u, v) h(m, n, m - u, n - v) \\
&= \frac{1}{(2\pi)^2 MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j\frac{n(n-(n-v))}{N}} e^{i\frac{m(m-(m-u))}{M}} \\
&= \frac{1}{(2\pi)^2 MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j\frac{nv}{N}} e^{i\frac{um}{M}},
\end{aligned} \tag{27}$$

where the input to the system is quaternion signal $F(u,v)$.

Equations (24) and (26) show that the choice of the quaternion impulse response of the filter determines output characteristics of the discrete linear TV systems.

5. Quaternionic windowed Fourier Transform

In section, we introduce the QWFT presented in Mawardi et al. (2010). As we will see, not all properties of the WFT can be established for the QWFT.

5.1 2-D WFT

Although the FT is a powerful tool for the analysis of stationary signals, the FT is not well suited for the analysis of non-stationary signals. Because the FT is a global transformation with poor spatial localization Zhong & Zeng (2007). However, in practice, most natural signals are non-stationary. In order to characterize a non-stationary signal properly, the WFT is commonly used.

Definition 5.1 (WFT). *The WFT of a two-dimensional real signal $f \in L^2(\mathbb{R}^2; \mathbb{R})$ with respect to the window function $g \in L^2(\mathbb{R}^2) \setminus \{0\}$ is given by*

$$\mathcal{G}_g f(\boldsymbol{\omega}, \boldsymbol{b}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \, \overline{g_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x})} \, d^2 \boldsymbol{x}, \tag{28}$$

where the window daughter function $g_{\boldsymbol{\omega}, \boldsymbol{b}}$ is defined by

$$g_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) = g(\boldsymbol{x} - \boldsymbol{b}) e^{\sqrt{-1} \, \boldsymbol{\omega} \cdot \boldsymbol{x}}. \tag{29}$$

The window daughter function $g_{\boldsymbol{\omega}, \boldsymbol{b}}$ is also called the windowed Fourier kernel.

Most applications make use of the Gaussian window function g, which is non-negative and well localized around the origin both in spatial and frequency domains. The Gaussian window function can be represented as

$$g(\boldsymbol{x}, \sigma_1, \sigma_2) = e^{-[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2]/2}, \tag{30}$$

where σ_1 and σ_2 are the standard deviations of the Gaussian function. For fixed $\boldsymbol{\omega}_0 = u_0 \boldsymbol{e}_1 + v_0 \boldsymbol{e}_2$,

$$g_{c, \boldsymbol{\omega}_0}(\boldsymbol{x}, \sigma_1, \sigma_2) = e^{\sqrt{-1}(u_0 x_1 + v_0 x_2)} e^{-[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2]/2} \tag{31}$$

is called a *complex Gabor filter*.

5.2 Quaternionic Gabor filters

Bülow (1999; Felsberg & Sommer) extended the complex Gabor filter $g_{c, \boldsymbol{\omega}_0}(\boldsymbol{x}, \sigma_1, \sigma_2)$ to quaternions by replacing the complex kernel $e^{\sqrt{-1}(u_0 x_1 + v_0 x_2)}$ with the inverse (two-sided) quaternion Fourier kernel $e^{\boldsymbol{i} u_0 x_1} e^{\boldsymbol{j} v_0 x_2}$. He proposed the extension form

$$g_q(\boldsymbol{x}, \sigma_1, \sigma_2) = e^{\boldsymbol{i} u_0 x_1} e^{\boldsymbol{j} v_0 x_2} e^{-[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2]/2}, \tag{32}$$

which he called *quaternionic Gabor filter*, and applied it to get the local quaternionic phase of a two-dimensional real signal. Bayro-Corrochano et al. (2007) also used quaternionic Gabor filters for the preprocessing of 2D speech representations. Based on (32), the quaternionic windowed Fourier kernel can be written in the form

$$\Phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) = e^{\boldsymbol{i} u_0 x_1} g(\boldsymbol{x} - \boldsymbol{b}) e^{\boldsymbol{j} v_0 x_2}. \tag{33}$$

The extension of the WFT to quaternions using the quaternionic windowed Fourier kernel (33) is rather complicated, due to the non-commutativity of quaternion functions. Alternatively, we use the kernel of the (right-sided) QFT to define the quaternionic windowed Fourier kernel which enables us to extend the WFT to quaternions.

Definition 5.2. *For a non-zero quaternion window function $\phi \in L^2(\mathbb{R}^2; \mathbb{H}) \setminus \{0\}$, its quaternionic window daughter function is defined by*

$$\phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) = e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1} \phi(\boldsymbol{x} - \boldsymbol{b}). \tag{34}$$

For fixed $\boldsymbol{\omega}_0 = u_0 \boldsymbol{e}_1 + v_0 \boldsymbol{e}_2$, our quaternionic Gabor filter is defined by

$$G_q(\boldsymbol{x}, \sigma_1, \sigma_2) = e^{\boldsymbol{j}v_0 x_2} e^{\boldsymbol{i}u_0 x_1} e^{-[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2]/2}. \tag{35}$$

Lemma 5.1. *For $\phi_{\boldsymbol{\omega}, \boldsymbol{b}} \in L^2(\mathbb{R}^2; \mathbb{H})$, we have*

$$\|\phi_{\boldsymbol{\omega}, \boldsymbol{b}}\|^2_{L^2(\mathbb{R}^2; \mathbb{H})} = \|\phi\|^2_{L^2(\mathbb{R}^2; \mathbb{H})}. \tag{36}$$

5.3 Definition of QWFT

Definition 5.3 (QWFT). *Let $\phi \in L^2(\mathbb{R}^2; \mathbb{H}) \setminus \{0\}$ be a non-zero quaternion window function. Denote by G_ϕ, the QWFT on $L^2(\mathbb{R}^2; \mathbb{H})$. The QWFT of $f \in L^2(\mathbb{R}^2; \mathbb{H})$ with respect to ϕ is defined by*

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \int_{\mathbb{R}^2} f(\boldsymbol{x}) \overline{\phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x})} \, d^2 \boldsymbol{x} \\ &= \int_{\mathbb{R}^2} f(\boldsymbol{x}) \overline{\phi(\boldsymbol{x} - \boldsymbol{b})} e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} d^2 \boldsymbol{x}. \end{aligned} \tag{37}$$

The quaternionic window daughter function

$$\phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) = e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1} \phi(\boldsymbol{x} - \boldsymbol{b}) \tag{38}$$

is also called the quaternionic windowed Fourier kernel.

These lead to the following observations:

- Equation (37) shows that it is generated using the inverse (right-sided) QFT kernel. Note that the definition is not valid using the kernel of the (two-sided) QFT.

- If we fix $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, and $b_1 = b_2 = 0$, and take the Gaussian function as the window function of (38), then we get the quaternionic Gabor filter

$$G_q(\boldsymbol{x}, \sigma_1, \sigma_2) = e^{\boldsymbol{j}v_0 x_2} e^{\boldsymbol{i}u_0 x_1} e^{-[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2]/2}. \tag{39}$$

- Since the modulation property does not hold for the QFT, equations (37) and (38) can not be expressed in terms of the QFT.

It is easy to see that

$$G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) = \mathcal{F}_q\{f \cdot T_{\boldsymbol{b}} \bar{\phi}\}(\boldsymbol{\omega}), \tag{40}$$

where the translation operator is defined by

$$T_{\boldsymbol{b}} f = f(\boldsymbol{x} - \boldsymbol{b}). \tag{41}$$

Equation (40) clearly shows that the QWFT can be regarded as the (right-sided) QFT of the product of a quaternion-valued signal f and a quaternion conjugated and shifted quaternion window function, or as an inner product (8) of f and the quaternionic window daughter function. In contrast to the QFT basis $e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_1 x_2}$, which has an infinite spatial extension, the QWFT basis $\phi(\boldsymbol{x} - \boldsymbol{b}) e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_1 x_2}$ has a limited spatial extension due to the locality of the quaternion window function $\phi(\boldsymbol{x} - \boldsymbol{b})$.

5.4 Properties of QWFT

The following proposition describes the elementary properties of the QWFT. Its proof is straightforward.

Proposition 5.2. *Let $\phi \in L^2(\mathbb{R}^2; \mathbb{H})$ be a quaternion window function.*

(i). (Left linearity)
$$[G_\phi(\lambda f + \mu g)](\boldsymbol{\omega}, \boldsymbol{b}) = \lambda G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) + \mu G_\phi g(\boldsymbol{\omega}, \boldsymbol{b}), \tag{42}$$

for arbitrary quaternion constants $\lambda, \mu \in \mathbb{H}$.

(ii). (Parity)
$$G_{P\phi}(Pf)(\boldsymbol{\omega}, \boldsymbol{b}) = G_\phi f(\boldsymbol{\omega}, -\boldsymbol{b}), \tag{43}$$

where P is the parity operator defined by $Pf(x) = f(-x)$.

(iii). (Specific shift) *Assume that $f = f_0 + \boldsymbol{i} f_1$ and $\phi = \phi_0 + \boldsymbol{i} \phi_1$.*

$$G_\phi(T_{\boldsymbol{x}_0} f)(\boldsymbol{\omega}, \boldsymbol{b}) = e^{-\boldsymbol{i}\omega_1 x_0} \left(G_\phi f(\boldsymbol{\omega}, \boldsymbol{b} - \boldsymbol{x}_0) \right) e^{-\boldsymbol{j}\omega_2 y_0}. \tag{44}$$

Let us give alternative proofs of the orthogonality relation and reconstruction formula. We follow the idea of Gröchenig (2001) to prove the theorems.

Theorem 5..3 (Orthogonality relation)**.** *Let ϕ, ψ be quaternion window functions and $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ arbitrary. Then we have*

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \overline{G_\psi g(\boldsymbol{\omega}, \boldsymbol{b})} \, d^2\boldsymbol{\omega} \, d^2\boldsymbol{b} = (2\pi)^2 (f(\bar{\phi}, \bar{\psi})_{L^2(\mathbb{R}^2; \mathbb{H})}, g)_{L^2(\mathbb{R}^2; \mathbb{H})}. \tag{45}$$

Proof. We notice that
$$G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) = \mathcal{F}_q\{f \cdot T_{\boldsymbol{b}} \bar{\phi}\}(\boldsymbol{\omega}), \tag{46}$$

for fixed $\boldsymbol{b}$. We have known that the Plancherel theorem is valid for the (right-sided) QFT. So, applying it into the left-hand side of (45), we get

$$\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \overline{G_\psi g(\boldsymbol{\omega}, \boldsymbol{b})} \, d^2\boldsymbol{\omega} = (\mathcal{F}_q\{f \cdot T_{\boldsymbol{b}} \bar{\phi}\}, \mathcal{F}\{f \cdot T_{\boldsymbol{b}} \bar{\psi}\})_{L^2(\mathbb{R}^2; \mathbb{H})}$$

$$= (2\pi)^2 (f \cdot T_{\boldsymbol{b}} \bar{\phi}, f \cdot T_{\boldsymbol{b}} \bar{\psi})_{L^2(\mathbb{R}^2; \mathbb{H})}$$

$$= (2\pi)^2 \int_{\mathbb{R}^2} f(x) \overline{\phi(x - \boldsymbol{b})} \psi(x - \boldsymbol{b}) \overline{g(x)} \, d^2x. \tag{47}$$

If we assume that $f\bar{\phi}$ and $\psi\bar{g}$ are in $L^2(\mathbb{R}^2; \mathbb{H})$, then integrating (47) with respect to $d^2\boldsymbol{b}$ yields

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \overline{G_\psi g(\boldsymbol{\omega}, \boldsymbol{b})} \, d^2\boldsymbol{\omega} \, d^2\boldsymbol{b} = (2\pi)^2 \int_{\mathbb{R}^2} f(x) \int_{\mathbb{R}^2} \overline{\phi(x - \boldsymbol{b})} \psi(x - \boldsymbol{b}) \overline{g(x)} \, d^2x \, d^2\boldsymbol{b}$$

$$= (2\pi)^2 \int_{\mathbb{R}^2} f(x) \int_{\mathbb{R}^2} \overline{\phi(x')} \psi(x') \, d^2x' \, \overline{g(x)} \, d^2x, \tag{48}$$

which proves the theorem. $\square$

From the above theorem, we obtain the following consequences.

(i). If $\phi = \psi$, then

$$\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \, \overline{G_\phi g(\boldsymbol{\omega}, \boldsymbol{b})} \, d^2\boldsymbol{b} \, d^2\boldsymbol{\omega} = (2\pi)^2 \|\phi\|_{L^2(\mathbb{R}^2;\mathbb{H})} (f, g)_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{49}$$

This formula is quite similar to the orthogonality relation of the classical WFT, for example, see Gröchenig (2001). However, we must remember that equation (49) is a quaternion valued function.

(ii). If $f = g$, then

$$\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \, \overline{G_\psi f(\boldsymbol{\omega}, \boldsymbol{b})} \, d^2\boldsymbol{b} \, d^2\boldsymbol{\omega} = (2\pi)^2 (f(\bar{\phi}, \bar{\psi})_{L^2(\mathbb{R}^2;\mathbb{H})}, f)_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{50}$$

(iii). If $f = g$ and $\phi = \psi$, then

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})|^2 \, d^2\boldsymbol{b} \, d^2\boldsymbol{\omega} = (2\pi)^2 \|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} \|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{51}$$

(iv). If the quaternion window function is normalized so that $\|\phi\|_{L^2(\mathbb{R}^2;\mathbb{H})} = 1$, then (51) becomes

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})|^2 \, d^2\boldsymbol{b} \, d^2\boldsymbol{\omega} = (2\pi)^2 \|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{52}$$

Equation (52) shows that the QWFT is an *isometry* from $L^2(\mathbb{R}^2; \mathbb{H})$ into $L^2(\mathbb{R}^2; \mathbb{H})$. In other words, up to the factor $(2\pi)^2$, the *total energy* of a quaternion-valued signal computed in the spatial domain is equal to the total energy computed in the quaternionic windowed Fourier domain.

Theorem 5..4 (Reconstruction formula). *Let $\phi, \psi \in L^2(\mathbb{R}^2; \mathbb{H})$ be two quaternion window functions. Assume that $(\phi, \psi)_{L^2(\mathbb{R}^2;\mathbb{H})} \neq 0$. Then, every 2-D quaternion signal $f \in L^2(\mathbb{R}^2; \mathbb{H})$ can be fully reconstructed by*

$$f(\boldsymbol{x}) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \psi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) (\bar{\phi}, \bar{\psi})^{-1}_{L^2(\mathbb{R}^2;\mathbb{H})} d^2\boldsymbol{b} \, d^2\boldsymbol{\omega}. \tag{53}$$

Under the same assumptions as in (49), we obtain

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^2 \|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) \, d^2\boldsymbol{b} \, d^2\boldsymbol{\omega}. \tag{54}$$

Proof. By direct calculation, we obtain

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \, \overline{G_\psi g(\boldsymbol{\omega}, \boldsymbol{b})} \, d^2\boldsymbol{\omega} \, d^2\boldsymbol{b} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \, \psi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) \bar{g}(\boldsymbol{x}) \, d^2\boldsymbol{\omega} \, d^2\boldsymbol{b} \, d^2\boldsymbol{x}$$

$$= (\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \psi_{\boldsymbol{\omega}, \boldsymbol{b}} \, d^2\boldsymbol{\omega} \, d^2\boldsymbol{b}, g)_{L^2(\mathbb{R}^2;\mathbb{H})}, \tag{55}$$

for every $g \in L^2(\mathbb{R}^2; \mathbb{H})$. Applying (45) of Theorem 5.3 to the left-hand side of (55), we have

$$(2\pi)^2 (f(\bar{\phi}, \bar{\psi})_{L^2(\mathbb{R}^2;\mathbb{H})}, g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \psi_{\boldsymbol{\omega}, \boldsymbol{b}} \, d^2\boldsymbol{\omega} \, d^2\boldsymbol{b}, g \right)_{L^2(\mathbb{R}^2;\mathbb{H})}, \tag{56}$$

for every $g \in L^2(\mathbb{R}^2; \mathbb{H})$. Since the inner product identity (56) holds for every $g \in L^2(\mathbb{R}^n; \mathbb{H})$, we conclude that

$$(2\pi)^2 f(\bar{\phi}, \bar{\psi})_{L^2(\mathbb{R}^2; \mathbb{H})} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \psi_{\boldsymbol{\omega}, \boldsymbol{b}} \, d^2\boldsymbol{\omega} \, d^2\boldsymbol{b}. \tag{57}$$

Multiplying both sides of (57) from the right side by $(2\pi)^{-2}(\bar{\phi}, \bar{\psi})_{L^2(\mathbb{R}^2; \mathbb{H})}^{-1}$, we immediately obtain

$$f = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \psi_{\boldsymbol{\omega}, \boldsymbol{b}} \, (\bar{\phi}, \bar{\psi})_{L^2(\mathbb{R}^2; \mathbb{H})}^{-1} \, d^2\boldsymbol{\omega} \, d^2\boldsymbol{b}. \tag{58}$$

Notice also that if $\phi = \psi$, then $(\bar{\phi}, \bar{\psi})_{L^2(\mathbb{R}^2; \mathbb{H})} = \|\bar{\phi}\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2 = \|\phi\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2$. This proves (54). $\qquad\square$

Theorem 5.5 (Reproducing kernel). *Let be $\phi \in L^2(\mathbb{R}^2; \mathbb{H})$ be a quaternion window function. If*

$$\mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}') = \frac{1}{(2\pi)^2 \|\phi\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2} (\phi_{\boldsymbol{\omega}, \boldsymbol{b}}, \phi_{\boldsymbol{\omega}', \boldsymbol{b}'})_{L^2(\mathbb{R}^2; \mathbb{H})}, \tag{59}$$

then $\mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}')$ is a reproducing kernel, i.e.

$$G_\phi f(\boldsymbol{\omega}', \boldsymbol{b}') = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}') \, d^2\boldsymbol{\omega} \, d^2\boldsymbol{b}. \tag{60}$$

Proof. By inserting (53) into the definition of the QWFT (37), we obtain

$$G_\phi f(\boldsymbol{\omega}', \boldsymbol{b}') = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \overline{\phi_{\boldsymbol{\omega}', \boldsymbol{b}'}(\boldsymbol{x})} \, d^2\boldsymbol{x}$$

$$= \int_{\mathbb{R}^2} \left(\frac{1}{(2\pi)^2 \|\phi\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \, \phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) d^2\boldsymbol{b} \, d^2\boldsymbol{\omega} \right) \overline{\phi_{\boldsymbol{\omega}', \boldsymbol{b}'}(\boldsymbol{x})} \, d^2\boldsymbol{x}$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \frac{1}{(2\pi)^2 \|\phi\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2} \left(\int_{\mathbb{R}^2} \phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) \overline{\phi_{\boldsymbol{\omega}', \boldsymbol{b}'}(\boldsymbol{x})} \, d^2\boldsymbol{x} \right) d^2\boldsymbol{b} \, d^2\boldsymbol{\omega}$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}') \, d^2\boldsymbol{b} \, d^2\boldsymbol{\omega}, \tag{61}$$

which was to be proved. $\qquad\square$

5.5 Examples of the QWFT

For illustrative purposes, we will give examples of the QWFT. Let us begin with a straightforward example given in Mawardi et al. (2010).

Example 5.1. *Consider the two-dimensional first order B-spline window function defined by*

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{if } -1 \leq x_1 \leq 1 \text{ and } -1 \leq x_2 \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{62}$$

Obtain the QWFT of the function defined as follows:

$$f(\boldsymbol{x}) = \begin{cases} e^{x_1 + x_2}, & \text{if } -\infty < x_1 < 0 \text{ and } -\infty < x_2 < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{63}$$

By applying the definition of the QWFT, we have

$$G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) = \frac{1}{(2\pi)^2} \int_{-1+b_1}^{m_1} \int_{-1+b_2}^{m_2} e^{x_1+x_2} e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} dx_1 dx_2,$$
$$m_1 = \min(0, 1+b_1), \quad m_2 = \min(0, 1+b_2). \tag{64}$$

Simplifying (64) yields

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{1}{(2\pi)^2} \int_{-1+b_1}^{m_1} \int_{-1+b_2}^{m_2} e^{x_1(1-\boldsymbol{i}\omega_1)} e^{x_2(1-\boldsymbol{j}\omega_2)} d^2\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2} \int_{-1+b_1}^{m_1} e^{x_1(1-\boldsymbol{i}\omega_1)} dx_1 \int_{-1+b_2}^{m_2} e^{x_2(1-\boldsymbol{j}\omega_2)} dx_2 \\ &= \frac{1}{(2\pi)^2} \left. e^{x_1(1-\boldsymbol{i}\omega_1)}(1-\boldsymbol{i}\omega_1)\right|_{-1+b_1}^{m_1} \left. \frac{e^{x_2(1-\boldsymbol{j}\omega_2)}}{(1-\boldsymbol{j}\omega_2)}\right|_{-1+b_2}^{m_2} \\ &= \frac{(e^{m_1(1-\boldsymbol{i}\omega_1)} - e^{(-1+b_1)(1-\boldsymbol{i}\omega_1)})(e^{m_2(1-\boldsymbol{j}\omega_2)} - e^{(-1+b_2)(1-\boldsymbol{j}\omega_2)})}{(2\pi)^2(1-\boldsymbol{i}\omega_1 - \boldsymbol{j}\omega_2 + \boldsymbol{k}\omega_1\omega_2)}. \end{aligned} \tag{65}$$

Using the properties of quaternions, we obtain

$$\begin{aligned} &G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \\ &= \frac{(e^{m_1(1-\boldsymbol{i}\omega_1)} - e^{(-1+b_1)(1-\boldsymbol{i}\omega_1)})(e^{m_2(1-\boldsymbol{j}\omega_2)} - e^{(-1+b_2)(1-\boldsymbol{j}\omega_2)})(1+\boldsymbol{i}\omega_1 + \boldsymbol{j}\omega_2 - \boldsymbol{k}\omega_1\omega_2)}{(2\pi)^2(1+\omega_1^2 + \omega_2^2 + \omega_1^2\omega_2^2)}. \end{aligned} \tag{66}$$

Example 5.2. *Let the window function be the two-dimensional Haar function defined by*

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{for } 0 \le x_1 < 1/2 \text{ and } 0 \le x_2 < 1/2, \\ -1, & \text{for } 1/2 \le x_1 < 1 \text{ and } 1/2 \le x_2 < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{67}$$

Find the QWFT of the Gaussian function $f(\boldsymbol{x}) = e^{-(x_1^2 + x_2^2)}$.

From Definition 5.3, we obtain

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) \overline{\phi(\boldsymbol{x} - \boldsymbol{b})} e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} d^2\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2} \int_{b_1}^{1/2+b_1} e^{-x_1^2} e^{-\boldsymbol{i}\omega_1 x_1} dx_1 \int_{b_2}^{1/2+b_2} e^{-x_2^2} e^{-\boldsymbol{j}\omega_2 x_2} dx_2 \\ &\quad - \frac{1}{(2\pi)^2} \int_{1/2+b_1}^{1+b_1} e^{-x_1^2} e^{-\boldsymbol{i}\omega_1 x_1} dx_1 \int_{1/2+b_2}^{1+b_2} e^{-x_2^2} e^{-\boldsymbol{j}\omega_2 x_2} dx_2. \end{aligned} \tag{68}$$

By completing squares, we have

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{1}{(2\pi)^2} \int_{b_1}^{1/2+b_1} e^{-(x_1+\boldsymbol{i}\omega_1/2)^2 - \omega_1^2/4} dx_1 \int_{b_2}^{1/2+b_2} e^{-(x_2+\boldsymbol{j}\omega_2/2)^2 - \omega_2^2/4} dx_2 \\ &\quad - \frac{1}{(2\pi)^2} \int_{1/2+b_1}^{1+b_1} e^{-(x_1+\boldsymbol{i}\omega_1/2)^2 - \omega_1^2/4} dx_1 \int_{1/2+b_2}^{1+b_2} e^{-(x_2+\boldsymbol{j}\omega_2/2)^2 - \omega_2^2/4} dx_2. \end{aligned} \tag{69}$$

Making the substitutions $y_1 = x_1 + i\frac{\omega_1}{2}$ and $y_2 = x_2 + j\frac{\omega_2}{2}$ in the above expression, we immediately obtain

$$
\begin{aligned}
G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{e^{-(\omega_1^2+\omega_2^2)/4}}{(2\pi)^2} \int_{b_1+i\omega_1/2}^{1/2+b_1+i\omega_1/2} e^{-y_1^2}\, dy_1 \int_{b_2+j\omega_2/2}^{1/2+b_2+j\omega_2/2} e^{-y_2^2}\, dy_2 \\
&\quad - \frac{e^{-(\omega_1^2+\omega_2^2)/4}}{(2\pi)^2} \int_{1/2+b_1+i\omega_1/2}^{1+b_1+i\omega_1/2} e^{-y_1^2}\, dy_1 \int_{1/2+b_2+j\omega_2/2}^{1+b_2+j\omega_2/2} e^{-y_2^2}\, dy_2 \\
&= \frac{e^{-(\omega_1^2+\omega_2^2)/4}}{(2\pi)^2} \Bigg[\left(\int_0^{b_1+i\omega_1/2} (-e^{-y_1^2})\, dy_1 + \int_0^{1/2+b_1+i\omega_1/2} e^{-y_1^2}\, dy_1 \right) \\
&\quad \times \left(\int_0^{b_2+j\omega_2/2} (-e^{-y_2^2})\, dy_2 + \int_0^{1/2+b_2+j\omega_2/2} e^{-y_2^2}\, dy_2 \right) \\
&\quad - \left(\int_0^{1/2+b_1+i\omega_1/2} (-e^{-y_1^2})\, dy_1 + \int_0^{1+b_1+i\omega_1/2} e^{-y_1^2}\, dy_1 \right) \\
&\quad \times \left(\int_0^{1/2+b_2+j\omega_2/2} (-e^{-y_2^2})\, dy_2 + \int_0^{1+b_2+j\omega_2/2} e^{-y_2^2}\, dy_2 \right) \Bigg]. \quad (70)
\end{aligned}
$$

Denote $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$. Equation (70) can be written in the form

$$
\begin{aligned}
G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{e^{-(\omega_1^2+\omega_2^2)/4}}{(2\sqrt{\pi})^3} \Bigg\{ \left[-\operatorname{erf}\left(b_1 + \frac{i}{2}\omega_1 \right) + \operatorname{erf}\left(\frac{1}{2} + b_1 + \frac{i}{2}\omega_1 \right) \right] \\
&\quad \times \left[-\operatorname{erf}\left(b_2 + \frac{j}{2}\omega_2 \right) + \operatorname{erf}\left(\frac{1}{2} + b_2 + \frac{j}{2}\omega_2 \right) \right] \\
&\quad - \left[-\operatorname{erf}\left(\frac{1}{2} + b_1 + \frac{i}{2}\omega_1 \right) + \operatorname{erf}\left(1 + b_1 + \frac{i}{2}\omega_1 \right) \right] \\
&\quad \times \left[-\operatorname{erf}\left(\frac{1}{2} + b_2 + \frac{j}{2}\omega_2 \right) + \operatorname{erf}\left(1 + b_2 + \frac{j}{2}\omega_2 \right) \right] \Bigg\}. \quad (71)
\end{aligned}
$$

6. Clifford windowed Fourier Transform

In this section, we introduce the Clifford windowed Fourier transform as a generalization of two-dimensional quaternionic Fourier transform to higher dimensions. Let us start with the following definition.

Definition 6.1. *The Clifford windowed Fourier transform of a multivector function $f \in L^2(\mathbb{R}^n; Cl_{0,n})$ with respect to the non-zero Clifford window function $\phi \in L^2(\mathbb{R}^n; Cl_{0,n})$ is given by*

$$
\begin{aligned}
G_\phi^c f(\boldsymbol{\omega}, \boldsymbol{b}) &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, \overline{\phi_{\boldsymbol{b},\boldsymbol{\omega}}(\boldsymbol{x})}\, d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, \overline{\phi(\boldsymbol{x} - \boldsymbol{b})} \prod_{k=1}^{n} e^{-\boldsymbol{e}_k \omega_k x_k}\, d^n\boldsymbol{x}, \quad (72)
\end{aligned}
$$

where $\boldsymbol{\omega}, \boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \cdots, \boldsymbol{e}_n$ are the orthonormal vector basis of Clifford algebra $Cl_{0,n}$ which satisfy the following rules:

$$
\begin{aligned}
\boldsymbol{e}_i \boldsymbol{e}_j &= -\boldsymbol{e}_j \boldsymbol{e}_i \quad && \text{for} \quad i \neq j, \quad i, j = 1, 2, 3, \cdots, n \\
\boldsymbol{e}_i^2 &= -1 \quad && \text{for} \qquad\qquad i = 1, 2, 3, \cdots, n.
\end{aligned}
$$

We call

$$
\phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x}) = \prod_{k=0}^{n-1} e^{\boldsymbol{e}_{n-k} \omega_{n-k} x_{n-k}} \phi(\boldsymbol{x} - \boldsymbol{b}), \tag{73}
$$

a *Clifford windowed Fourier kernel*. Notice that the Clifford windowed Fourier transform for $n = 2$ is identical with the QWFT and that for $n = 1$ is identical with the classical windowed Fourier transform.

7. Conclusion

Using the basic concepts of quaternion algebra and its Fourier transform, we have introduced 2-D quaternionic windowed Fourier transform. Since the multiplication in quaternions is non-commutative, some properties of the classical windowed Fourier transform, such as the shift property, orthogonality relation and reconstruction formula, needed to be modified. We have shown that the construction formula can be extended to higher dimensions using the Clifford Fourier transform. Like quaternion wavelets, which are successfully applied to optical flow, it will be possible to apply the QWFT to optical flow, image features and image fusion in the future.

8. References

Bülow, T. (1999). Hypercomplex spectral signal representations for the processing and analysis of images, Ph.D. thesis, University of Kiel, Germany.

Bülow, T., Felsberg, M. & Sommer, G. (2001). Non-commutative hypercomplex Fourier transforms of multidimensional signals, In: *Geometric Computing with Clifford Algebras: Theoretical Foundations and Applications in Computer Vision and Robotics*, Sommer, G. (Ed.), pp. 187-207, Springer.

E. Bayro-Corrochano, E.; Trujillo, N.; Naranjo, M. (2007). Quaternion Fourier descriptors for the preprocessing and recognition of spoken words using images of spatiotemporal representations, *Journal of Mathematical Imaging and Vision*, Vol. 28, No. 62, pp. 179-190.

Gröchenig, K. (2001). *Foundation of Time-Frequency Analysis*, Birkhäuser, Boston.

Gröchenig, K. & Zimmermann, G. (2001). Hardy's theorem and the short-time Fourier transform of Schwartz functions, *Journal of the London Mathematical Society*, Vol. 2, No. 62, pp. 205–214.

Ghosh, P. K. & Sreenivas, T. V. (2006). Time-varying filter interpretation of Fourier transform and its variants, *Signal Processing*, Vol. 11, No. 86, pp. 3258–3263.

Hitzer, E. (2007). Quaternion Fourier transform on quaternion fields and generalizations, *Advances in Applied Clifford Algebras*, Vol. 17, No. 3, pp. 497–517.

Kemao, Q. (2007). Two-dimensional windowed Fourier transform for fringe pattern analysis: principles, applications, and implementations, *Optics and Laser Engineering*, Vol. 45, pp. 304–317.

Kuipers, J. B. (1999). *Quaternions and Rotation Sequences*, Princeton University Press, New Jersey.

Mawardi, B.; Hitzer, E.; Hayashi, A.; Ashino, R. (2008). An uncertainty principle for quaternion Fourier transform, *Computers and Mathematics with Applications*. Vol. 56, No. 9, pp. 2411–2417.

Mawardi, B.; Hitzer, E.; Ashino, R.; Vaillancourt, R. (2010). Windowed Fourier transform of two-dimensional quaternionic signals, *Applied Mathematics and Computation*. Vol. 28, No. 6, pp. 2366–2379.

Mawardi, B. (2010). A generalized windowed Fourier transform for quaternions, *Far East Journal of Applied Mathematics*, Vol. 42, No. 1, pp. 35–47.

Mawardi, B.; Adji, S.; Zhao, J. (2011). Real Clifford windowed Fourier transform, *Acta Mathematica Sinica*, in press, 2011.

Gröchenig, K. & Zimmermann, G. (2001). Hardy's theorem and the short-time Fourier transform of Schwartz functions, *Journal of the London Mathematical Society*, Vol. 2, No. 62, pp. 205–214.

Sangwine, S. J. & Ell, T. A. (2007). Hypercomplex Fourier transforms of color images, IEEE Transactions on Image Processing, Vol. 16, No. 1, pp. 22–35.

Weisz, F. (2008). Multiplier theorems for the short-time Fourier transform, *Integral Equation and Operator Theory*, Vol. 60, No. 1, pp. 133–149.

Zhong, J. & Zeng, H. (2007). Multiscale windowed Fourier transform for phase extraction of fringe pattern, *Applied Optics*, Vol. 46, No. 14, pp. 2670–2675.
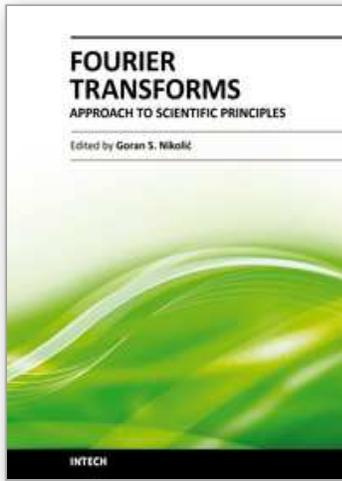
Fourier Transforms - Approach to Scientific Principles

Edited by Prof. Goran Nikolic

ISBN 978-953-307-231-9

Hard cover, 468 pages

Publisher InTech

Published online 11, April, 2011

Published in print edition April, 2011

This book aims to provide information about Fourier transform to those needing to use infrared spectroscopy, by explaining the fundamental aspects of the Fourier transform, and techniques for analyzing infrared data obtained for a wide number of materials. It summarizes the theory, instrumentation, methodology, techniques and application of FTIR spectroscopy, and improves the performance and quality of FTIR spectrophotometers.

How to reference

In order to correctly reference this scholarly work, feel free to copy and paste the following:

Mawardi Bahri and Ryuichi Ashino (2011). Two-Dimensional Quaternionic Windowed Fourier Transform, Fourier Transforms - Approach to Scientific Principles, Prof. Goran Nikolic (Ed.), ISBN: 978-953-307-231-9, InTech, Available from: http://www.intechopen.com/books/fourier-transforms-approach-to-scientific-principles/two-dimensional-quaternionic-windowed-fourier-transform

INTECH
open science | open minds