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Approach to Fundamental Properties of the Henstock-Fourier Transform

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1. Introduction

The Riemann integral was designed to solve different problems in different areas of mathematics. Unfortunately, the Riemann integral has some shortcomings: the derivative of a function is not necessarily Riemann integrable, it lacks of "good" convergence theorems,.... To correct these defects, in the year 1902, H. Lebesgue designed an integral (Lebesgue integral) which is more general than Riemann's, it has better convergence theorems, and it allows integration over other type of sets different from intervals. However, the derivative of a function does not need to be Lebesgue-integrable. On the other hand, every function which is Improper-Riemann-Integrable is not necessarily Lebesgue-integrable.
A. Denjoy (1912) and O. Perron (1914) developed more general integrals than Lebesgue's. In both integrals, any derivative of a differentiable function is integrable. Both integrals are equivalent but they are difficult to construct. (Gordon, 1994)
Jaroslav Kurzweil (1957), a Czech mathematician, and Ralph Henstock built independently equivalent integrals (Gordon, 1994) which generalize the Lebesgue integral, and it has as "good" convergence theorems as Lebesgue and the derivative of a differentiable function is Henstock-Kurzweil integrable, including the improper Riemann integral. In addition, the construction follows the same pattern as the construction of the Riemann integral. This also facilitates its teaching.
This new integral provided new research lines:

- Construction of new types of integrals by following the Riemann approach.
- Generalization of this concept for functions of several variables, and for functions with values within a Banach space.

In addition, this integral (Henstock-Kurzweil) can be applied to the differential equations theory, integral equations theory, Fourier analysis, probability, statistics, etc.
In the Lebesgue-integrable functions space, we can define a norm with which this space becomes a Banach space with good properties.
Today, Lebesgue integral is the main integral used in various areas of mathematics, for example Fourier analysis. However, many functions (e.g. functions that have a "bad" oscillatory behavior) which are not Lebesgue-integrable are Henstock-Kurzweil-integrable. Therefore, it seems a natural way to study Fourier analysis by using this integral.

Recall that if f is integrable "in some sense", on $\mathbb{R}$, its Fourier transform in $s \in \mathbb{R}$, is defined as

$$\widehat{f}(s) = \int_{-\infty}^{\infty} e^{-ixs} f(x) dx. \tag{1}$$

In the Lebesgue space on $\mathbb{R}$, $L(\mathbb{R})$, the Fourier transform is a bounded linear transformation, whose codomain is the space of continuous functions on $\mathbb{R}$ which "vanish at infinity". It has good algebraic and analytical properties, which have wide applications in mathematics and other sciences.
Four important properties of the Fourier transform in space $L(\mathbb{R})$ are:

i For all $s \in \mathbb{R}$, the Fourier transform exists , because the function $\exp(-ixs)$ is a bounded measurable function.

ii $\widehat{f}$ is continuous on $\mathbb{R}$.

iii Riemann-Lebesgue Lemma: $\lim_{s\to\pm\infty} \widehat{f}(s) = 0$.

iv The Dirichlet-Jordan theorem is valid. This theorem provides us the pointwise inversion for functions of bounded variation on $\mathbb{R}$.

The first study of the Fourier transform using the Henstock-Kurzweil integral was made by E. Talvila in 2002, (Talvila, 2002). He shows important properties of the Fourier transform in the space of Henstock-Kurzweil integrable functions on $\mathbb{R}$, $HK(\mathbb{R})$. However, this study is incomplete, our purpose is to study other properties. We will call *Henstock-Fourier transform* to the Fourier transform definite on $HK(\mathbb{R})$.
Two important differences between the Henstock-Fourier transform and the Fourier transform are:

- This transform does not always exist. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0 \end{cases}$$

  belongs to $HK(\mathbb{R})$, but its Henstock-Fourier transform is not defined in $s = 1$.

- The Riemann-Lebesgue Lemma is not always valid. For example, the function $g(x) = \exp(ix^2)$ (Talvila, 2002) is such that $\widehat{g}(s) = \sqrt{\pi}\exp(i(\pi - s^2)/4)$, however, this later function is not tend to zero when s tend to infinity.

We begin the chapter exposing some fundamental concepts concerning the Henstock-Kurzweil integral, after we show that the intersection of $HK(\mathbb{R})$ and the space of bounded variation functions over $\mathbb{R}$, $HK(\mathbb{R}) \cap BV(\mathbb{R})$, does not have inclusion relations with $L(\mathbb{R})$, for this, we exhibit a wide family of functions in $HK(\mathbb{R}) \cap BV(\mathbb{R})$, which is not in $L(\mathbb{R})$. This makes the study of the Henstock-Fourier transform in this space interesting. Subsequently, in base of $HK(\mathbb{R}) \cap BV(\mathbb{R})$, we prove fundamental properties such as continuity, the Riemann-Lebesgue Lemma, and the Dirichlet-Jordan Theorem.

2. The Henstock-Kurzweil integral

For compact intervals in $\mathbb{R}$, the Henstock-Kurzweil integral is defined in the following way:

Definition 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a function, we will say that f is **Henstock-Kurzweil integrable** if there exists $A \in \mathbb{R}$, which satisfies the following:*
for each $\varepsilon > 0$ exists a function $\gamma_\varepsilon : [a,b] \to (0,\infty)$ such that for every partition labeled as $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$, where $t_i \in [x_{i-1}, x_i]$, if

$$[x_{i-1}, x_i] \subset [t_i - \gamma_\varepsilon(t_i), t_i + \gamma_\varepsilon(t_i)] \quad \text{for } i = 1, 2, ..., n, \tag{2}$$

then

$$|\Sigma_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A| < \varepsilon.$$

The function γ_ε is commonly called **gauge,** and if the partition P complies with the condition (2), we will say that it is γ_ε-**fine**. The number A is named as the integral of f over $[a,b]$ and it is denoted as

$$A = \int_a^b f = \int_a^b f(x)\, dx.$$

If f is defined over an interval of the way $[a,\infty]$, we condition it to $f(\infty) = 0$. In this case, given a gauge function $\gamma_\varepsilon : [a,\infty] \to (0,\infty)$, where $\gamma_\varepsilon(\infty) \in \mathbb{R}^+$, we will say that the labeled partition $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n+1}$ is γ_ε-**fine** if:

a) $x_0 = a,\ x_{n+1} = \infty$.

b) $[x_{i-1}, x_i] \subset [t_i - \gamma_\varepsilon(t_i), t_i + \gamma_\varepsilon(t_i)]$, for $i = 1, 2, ..., n$

c) $[x_n, \infty] \subset [1/\gamma_\varepsilon(\infty), \infty]$.

Thus, the function will be integrable if it satisfies Definition 2.1, and also the condition of that the partition P be γ_ε-**fine** according to the previous incises. In addition, these conditions cause that the last term of $\Sigma_{i=1}^{n+1} f(t_i)(x_i - x_{i-1})$ is zero and thus this sum is finite. For functions defined over intervals $[-\infty, a]$ and $[-\infty, +\infty]$ we do similar considerations.
Through the theory of this integral we have that $f : [-\infty, \infty] \to \mathbb{R}$ is an integrable function, if and only if, f is an integrable function over the intervals $[a,\infty]$ y $[-\infty, a]$. In this case

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^{a} f + \int_a^{\infty} f. \tag{3}$$

We denote as

$$HK(I) = \{f : I \to \mathbb{R} \mid f \text{ is Henstock-Kurzweil integrable on } I\}.$$

Some features of $HK(I)$ are the following:

1. It is a vector space, i.e.: the sum of functions and the product by scalars of Henstock-Kurzweil integrable functions are integrable. The integral is a linear functional over this space.
2. It contains the Riemann and Lebesgue integrable functions. Also, the functions whose Riemann or Lebesgue improper integrals exist, and their values coincide.
3. It generalizes the Fundamental Theorem of Calculus, in the sense that every derivative function is integrable. This does not happen with Riemann and Lebesgue integrals. In this case we have:

$$\int_a^b f' = f(b) - f(a).$$

4. Since we know, in Riemann's integral, if two functions are integrable, then their product is also integrable. In the case of the integral of HK, this is not true. Nevertheless, the product of a HK-integrable function by a bounded variation function, is in fact integrable.
5. The HK integral is not an absolute integral. This asseveration is in the sense that if f is HK-integrable, it does not imply that $|f|$ is so. When $|f|$ and f are integrable, we say that f is absolutely HK-integrable.
6. The space of the absolutely HK-integrable lebesgue is $L(I)$, the space of the functions integrable functions.

As we shall see, the properties (4) and (5) produce important differences in the behavior of the Henstock-Fourier transform with respect to the Fourier transform.

2.1 Notation and important theorems for Henstock-Kurzweil integral

Let I be a finite or infinite close interval. We work on the following subspaces:

- $HK(I) = \{f \mid f \text{ is Henstock-Kurzweil integrable on } I\}$.
- $HK_{loc}(\mathbb{R}) = \{f \mid f \in \mathcal{HK}(I), \text{ for each finite close interval } I\}$.
- $BV(I) = \{f \mid f \text{ is of bounded variation on } I\}$.
- If $f \in BV(I)$, $V_I f$ is the total variation of f on I.
- $BV([\pm\infty]) = \{f \mid f \in BV([a,\infty]) \cap BV([-\infty,b]), \text{ for some } a,b \in \mathbb{R}\}$.
- $BV_0([\pm\infty]) = \{f \in BV([\pm\infty]) \mid \lim_{|x|\to\infty} f(x) = 0\}$.
- $L(I) = \{f \mid f \text{ is Lebesgue integrable on } I\}$.

Some of the most important theorems of the Henstock-Kurzweil integral will be used in the proof of our results are as follows.

Theorem 2.1 (Fundamental Theorem I.)**.** *(Bartle, 2001) If $f : [a,b] \to \mathbb{R}$ has a primitive F on $[a,b]$, then $f \in HK([a,b])$ and*

$$\int_a^b f = F(b) - F(a).$$

This theorem guarantees that the derivative of any function on $[a,b]$ is always Henstock-Kurzweil integrable. This result is not valid for Lebesgue integral.

Theorem 2.2 (Fundamental Theorem II.)**.** *(Bartle, 2001) Let a I be a finite o infinite interval. If $f \in HK([a,b])$ then any indefinite integral F is continuous on I and exists a null $Z \subset [a,b]$ such that*

$$F'(x) = f(x) \qquad \text{for all } x \in I - Z.$$

Theorem 2.3 (Multiplier Theorem.)**.** *(Bartle, 2001) Let $[a,b]$ a finite interval, $f \in HK([a,b])$, $\varphi \in BV([a,b])$ and $F(x) = \int_a^x f(t)$, for $x \in [a,b]$, then, the product $f\varphi \in HK([a,b])$ and*

$$\int_a^b f\varphi = \int_a^b \varphi\, dF = F(b)\varphi(b) - \int_a^b F\, d\varphi, \tag{4}$$

where the second and third integrals are Riemann-Stieltjes integrals.

If $a \in \mathbb{R}$ and $b = \infty$, (4) has the following form

$$\int_a^{\infty} f\varphi = \lim_{b\to\infty}\left[F(b)\varphi(b) - \int_a^b Fd\varphi\right]. \tag{5}$$

Similary, if the integration is over the intervals $[-\infty, a]$ or $[-\infty, \infty]$, we have the respective limits in (4).

Theorem 2.4 (Dominated Convergence Theorem.)**.** *(Bartle, 2001) Let $[a, b]$ a interval (finite or infinite), let $\{f_n\}$ be sequence in $HK([a, b])$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ a.e. on $[a, b]$. Suppose that there exist functions $\alpha, \omega \in HK([a, b])$ such that*

$$\alpha(x) \leq f_n(x) \leq \omega(x) \text{ a.e. on } [a, b], \text{ and for all } n \in \mathbb{N}.$$

Then $f \in HK([a, b])$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx.$$

This theorem is an extension to the Henstock Kurzweil integral of a Dominated Convergence Theorem (DCT) for the Lebesgue integral.

Theorem 2.5 (Hake Theorem.)**.** *(Bartle, 2001) $f \in HK([a, \infty])$, if and only if, $f \in HK([a, c])$ for every compac interval $[a, c]$ with $c \in [a, \infty)$, and there exist $A \in \mathbb{R}$ such that $\lim_{c\to\infty}\int_a^c f(t)dt = A$. In this case, $\int_a^{\infty} f(t)dt = A$.*

There are versions of the Hake's Theorem for functions on $[-\infty, \infty]$ and $[-\infty, a]$.

Theorem 2.6 (Chartier-Dirichlet's Test.)**.** *(Bartle, 2001) Let $f, \varphi : [a, \infty] \to \mathbb{R}$ and suppose that:*

- *$f \in HK([a, c])$ for all $c \geq a$ and $F(x) := \int_a^x f$ is bounded on $[a, \infty)$.*
- *φ is monotone on $[a, \infty]$ and $\lim_{x\to\infty}\varphi(x) = 0$.*

Then $f\varphi \in HK([a, \infty])$.

Theorem 2.7 (Characterization of Absolute Integrability.)**.** *(Bartle, 2001) Let $f \in HK([a, b])$. Then $|f|$ is Henstock-Kurzweil integrable, if and only if, the indefinite integral $F(x) = \int_a^x f$ has bounded variation on $[a, b]$, in this case,*

$$\int_a^b |f| = V_{[a,b]}F.$$

Theorem 2.8 (Comparison Test for Absolute Integrability.)**.** *(Bartle, 2001) If $f, g \in HK([a, b])$ and $|f(x)| \leq g(x)$ for $x \in [a, b]$, then $f \in L([a, b])$. More over, we have*

$$\left|\int_a^b f\right| \leq \int_a^b |f| \leq \int_a^b g.$$

3. The $HK(I) \cap BV(I)$ subspace

If I is a finite interval, we know that:

$$BV(I) \subset L(I) \subset HK(I),$$

and consequently $HK(I) \cap BV(I) \subset L(I)$.
Now, if I is unbounded, the first two observations which we have are

$$BV(I) \nsubseteq L(I), \tag{6}$$

and

$$L(I) \nsubseteq HK(I) \cap BV(I). \tag{7}$$

Really it is easy to demonstrate that the function $f(x) = 1/x$ defined in $[1, \infty]$, is of bounded variation, with

$$V_{[1,\infty]} f = 1,$$

and

$$\int_1^\infty \frac{1}{x}\, dx = \infty.$$

This implies that (6) is true.
To verify (7), we consider the function $f : [0, \infty] \to \mathbb{R}$ defined like

$$f(x) = \begin{cases} \sqrt{x}\sin(1/x), & \text{si } x \in (0,1], \\ 0, & \text{si } x = 0,\ x \in (1, \infty] \end{cases}$$

which is in $L(\,[0,\infty]\,) \setminus BV(\,[0,\infty]\,)$.

Next, we will prove that: $HK(I) \cap BV(I) \nsubseteq L(I)$.

Proposition 3.1. *(Mendoza et al., 2008) [Theorem 2.1] Let $\varphi : [a, \infty] \to \mathbb{R}$ be a non-negative function, which is decreasing to zero when $x \to \infty$. If $\varphi \notin HK([a,\infty]\,)$, then the functions: $\varphi(t)\sin(t)$ and $\varphi(t)\cos(t)$ are in $HK(\,[a,\infty]\,) \setminus L(\,[a,\infty]\,)$.*

Proof: We will demonstrate that $\varphi(t)\sin(t) \notin L(\,[a,\infty]\,)$. The proof that $\varphi(t)\cos(t) \notin L(\,[a,\infty]\,)$ can be done in a similar way.
Suppose that n_0 is the first natural number for which $a < (1+4n_0)\pi/4$. For $t \in [a, \infty]$ we have

$$|\sin t| \geq \frac{1}{\sqrt{2}} \text{ if and only if } t \in \cup_{k=n_0}^{\infty} [\,(1+4k)\pi/4,\ (3+4k)\pi/4].$$

Let $n \in \mathbb{N}$ with $n \geq n_0$, since $(3+4n)\pi/4 < (1+n)\pi$, we have that:

$$\begin{aligned} \int_a^{(1+n)\pi} \varphi(t)|\sin t|dt &\geq \frac{1}{\sqrt{2}} \sum_{k=n_0}^{n} \int_{(1+4k)\pi/4}^{(3+4k)\pi/4} \varphi(t)\, dt \\ &\geq \frac{1}{\sqrt{2}} \sum_{k=n_0}^{n} \int_{(1+4k)\pi/4}^{(3+4k)\pi/4} \varphi((3+4k)\pi/4)\, dt \\ &= \frac{\pi}{2\sqrt{2}} \sum_{k=n_0}^{n} \varphi((3+4k)\pi/4) \\ &\geq \frac{\pi}{2\sqrt{2}} \sum_{k=n_0}^{n} \varphi((1+k)\pi). \end{aligned} \tag{8}$$

On the other hand,

$$\begin{aligned}\int_a^{(1+n)\pi}\varphi(t)\,dt &= \int_a^{n_0\pi}\varphi(t)\,dt+\int_{n_0\pi}^{(1+n)\pi}\varphi(t)\,dt\\ &= \int_a^{n_0\pi}\varphi(t)\,dt+\sum_{k=n_0}^{n}\int_{k\pi}^{(1+k)\pi}\varphi(t)\,dt\\ &\leq \int_a^{n_0\pi}\varphi(t)\,dt+\pi\sum_{k=n_0}^{n}\varphi(k\pi). \qquad (9)\end{aligned}$$

Since $\varphi \notin HK(\,[a,\infty]\,)$, then $\int_a^{\infty}\varphi(t)\,dt=\infty$ and from (9) it follows

$$\sum_{k=n_0}^{\infty}\varphi(k\pi)=\infty. \qquad (10)$$

Using (10) and approaching $n\to\infty$ in (8), we conclude that $\varphi(t)\sin(t)\notin L(\,[a,\infty]\,)$.
For any $x\in[a,\infty)$,

$$\left|\int_a^x \sin(t)\,dt\right|\leq 2 \quad\text{and}\quad \left|\int_a^x \cos(t)\,dt\right|\leq 2.$$

Then according to Chartier-Dirichlet Test (2.6), we have that: $\varphi(t)\sin(t)$ and $\varphi(t)\cos(t)$ are in $HK[a,\infty]$. ■

Example 3.1. *For any* $a>0$,

$$\frac{\sin(t)}{t}\ \in\ HK(\,[a,\infty]\,)\setminus L(\,[a,\infty]\,).$$

Proposition 3.2. *(Mendoza et al., 2008) [Corollary 2.2,Theorem 2.2] Let* $1>\alpha>0$. *The function* $f_\alpha:[\pi^{1/\alpha},\infty]\to\mathbb{R}$ *defined as*

$$f_\alpha(t)=\frac{\sin(t^\alpha)}{t}$$

satisfies:

(a) $f_\alpha\in HK[\pi^{1/\alpha},\infty]\setminus L(\,[\pi^{1/\alpha},\infty]\,)$.

(b) $f_\alpha\in BV(\,[\pi^{1/\alpha},\infty]\,)$.

Proof: **(a)** Let $c>\pi^{1/\alpha}$. Doing a change of variable $u=t^\alpha$ we have that

$$\int_{\pi^{1/\alpha}}^{c}\frac{\sin(t^\alpha)}{t}dt=\frac{1}{\alpha}\int_{\pi}^{c^\alpha}\frac{\sin(u)}{u}du.$$

Since $\sin(u)/u\in HK[\pi,\infty]$, we have that:

$$\lim_{c\to\infty}\int_{\pi^{1/\alpha}}^{c}\frac{\sin(t^\alpha)}{t}dt \ \text{ exists},$$

thus $f_\alpha\in HK[\pi^{1/\alpha},\infty]$. Moreover since

$$\int_{\pi^{1/\alpha}}^{c}\left|\frac{\sin(t^\alpha)}{t}\right|dt=\frac{1}{\alpha}\int_{\pi}^{c^\alpha}\left|\frac{\sin(u)}{u}\right|du.$$

and $\sin(u)/u \notin L(\,[\pi,\infty]\,)$, then $f_\alpha \notin L[\pi^{1/\alpha},\infty]$.
(b) Let $x \in (\pi^{1/\alpha},\infty)$. We know that $f'_\alpha \in HK(\,[\pi^{1/\alpha},x]\,)$. Now since

$$f'_\alpha(t) = \frac{\alpha\cos(t^\alpha)}{t^{2-\alpha}} - \frac{\sin(t^\alpha)}{t^2},$$

we have that

$$|f'_\alpha(t)| \leq \frac{\alpha}{t^{2-\alpha}} + \frac{1}{t^2}. \tag{11}$$

The function $g(t) = \frac{\alpha}{t^{2-\alpha}} + \frac{1}{t^2} \in HK(\,[\pi^{1/\alpha},x]\,)$, then by (11) and Theorem 2.8, we conclude that: $f'_\alpha \in L(\,[\pi^{1/\alpha},x]\,)$ and

$$\begin{aligned}\int_{\pi^{1/\alpha}}^{x} |f'_\alpha| &\leq \int_{\pi^{1/\alpha}}^{x} \left(\frac{\alpha}{t^{2-\alpha}} + \frac{1}{t^2}\right) dt \\ &= \left(\frac{1}{\alpha-1}\right)\left[x^{\alpha-1} - \pi^{\frac{\alpha-1}{\alpha}}\right] - \frac{1}{x} + \frac{1}{\pi^{1/\alpha}}.\end{aligned}$$

Consequently by Theorem 2.7,

$$V_{[\pi^{1/\alpha},x]} f_\alpha \leq \left(\frac{1}{\alpha-1}\right)\left[x^{\alpha-1} - \pi^{\frac{\alpha-1}{\alpha}}\right] - \frac{1}{x} + \frac{1}{\pi^{1/\alpha}}.$$

Therefore, as $1-\alpha > 0$ we have that

$$V_{[\pi^{1/\alpha},\infty]} f \leq \frac{1}{(1-\alpha)\pi^{(1-\alpha)/\alpha}} + \frac{1}{\pi^{1/\alpha}}.$$

Thus, $f_\alpha \in BV(\,[\pi^{1/\alpha},\infty]\,)$. ■
Analogy, we can to prove that for $1 > \alpha > 0$, the function $g_\alpha : [-\infty,-\pi^{1/\alpha}] \to \mathbb{R}$ defined as

$$g_\alpha(t) = \frac{\sin(-t)^\alpha}{-t}$$

belongs to $HK([-\infty,-\pi^{1/\alpha}]) \cap BV([-\infty,-\pi^{1/\alpha}]) \setminus L([-\infty,-\pi^{1/\alpha}])$.
Let $h \in BV([-\pi^{1/\alpha},\, \pi^{1/\alpha}])$. For $1 > \alpha > 0$, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} h(x), & \text{if } x \in (-\pi^{1/\alpha},\, \pi^{1/\alpha}), \\ \dfrac{\sin|t|^\alpha}{|t|}, & \text{if } x \in (-\infty,\, -\pi^{1/\alpha}] \cup [\pi^{1/\alpha},\, \infty) \end{cases}$$

is in $HK(\mathbb{R}) \cap BV(\mathbb{R}) \setminus L(\mathbb{R})$. With this example and Proposition 3.1, we have the following theorem.

Theorem 3.1. *(Mendoza et al., 2009) [Theorem 2.4] There exists a function f in $HK(\mathbb{R}) \cap BV(\mathbb{R}) \setminus L(\mathbb{R})$.*

Now, since $BV(\mathbb{R}) \subset BV(\,[\pm\infty]\,)$, we have immediately the next corollary.

Corollary 3.1. *(Mendoza et al., 2009) [Corollary 2.5] $HK(\mathbb{R}) \cap BV(\,[\pm\infty]\,) \not\subseteq L(\mathbb{R})$.*

We observe that $BV(\mathbb{R}) \subset BV(\,[\pm\infty]\,)$ properly, because instead of the function h in $BV([-\pi^{1/\alpha}, \pi^{1/\alpha}])$ we can to take a function in $HK([-\pi^{1/\alpha}, \pi^{1/\alpha}]) \setminus BV([-\pi^{1/\alpha}, \pi^{1/\alpha}])$.
Also we observe that if $f \in BV(\mathbb{R})$ then, by Multiplier Theorem (2.3), $f(t)\sin t/t \in HK([0,\infty])$.
To conclude this section, we know that $f \in HK(\mathbb{R})$ implies that $f(\pm\infty) = 0$, If in addition $f \in BV(\mathbb{R})$ then $\lim_{|x|\to\infty} f(x)$ exists. Therefore, we have the following lemma.

Lemma 3.1. *If $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, then $\lim_{|x|\to\infty} f(x) = 0$ and f is bounded.*

4. Existence and continuity of $\widehat{f}(s)$

4.1 Existence

A part from the Proposition 2.1 b) in (Talvila, 2002), say us that: If $f \in HK_{loc}(\mathbb{R}) \cap BV_0(\,[\pm\infty]\,)$, then $\widehat{f}(s)$ exists for all $s \in \mathbb{R}$. If $s \neq 0$, then the result is true. However with these conditions, it is not necessarily true the existence of $\widehat{f}(0)$. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in (-1, 1), \\ \frac{1}{x}, & \text{if } x \in (-\infty, -1] \cup [1, \infty) \end{cases}$$

is in $HK_{loc}(\mathbb{R}) \cap BV_0(\,[\pm\infty]\,)$ but $\widehat{f}(0)$ does not exist.
In order to have the existence of $\widehat{f}(0)$, we need that $f \in HK(\mathbb{R})$.
We will demonstrate that the Henstock-Fourier transform exist in $HK(\mathbb{R}) \cap BV(\,[\pm\infty]\,)$, for every $s \in \mathbb{R}$.

Theorem 4.1. *(Mendoza et al., 2009) [Theorem 3.1] If $f \in HK(\mathbb{R}) \cap BV(\,[\pm\infty]\,)$, then $\widehat{f}(s)$ exists for all $s \in \mathbb{R}$.*

Proof: The result is true for $s = 0$ because $f \in HK(\mathbb{R})$. Now let $s \neq 0$, since $HK(\mathbb{R}) \cap BV(\,[\pm\infty]\,) \subset HK_{loc}(\mathbb{R}) \cap BV_0(\,[\pm\infty]\,)$ then by (Talvila, 2002) [Proposition **2.1 (b)**] it follows that $\widehat{f}(s)$ exists. However for the sake of completes, here we will give proof of it:
The condition $f \in BV_0(\,[\pm\infty]\,)$ implies that $\lim_{|x|\to\infty} f(x) = 0$ and there exists $a < 0$, $b > 0$ such that f is of bounded variation on $(-\infty, a] \cup [b, \infty)$.
Let us prove that $f(x)e^{-ixs} \in HK([b, \infty))$. The functions φ_1, φ_2 defined as

$$\varphi_1(x) = V_{[b,x]}f - V_{[b,\infty)}f, \qquad \varphi_2(x) = [V_{[b,x]}f - f(x)] - V_{[b,\infty)}f$$

are increasing on $[b, \infty)$ and satisfies that $\lim_{x\to\infty} \varphi_1(x) = \lim_{x\to\infty} \varphi_2(x) = 0$ and $f = \varphi_1 - \varphi_2$. Therefore, since

$$\left| \int_b^x e^{-ius} du \right| = \left| \frac{1}{is}(e^{-ibs} - e^{-ixs}) \right| \leq \frac{2}{s} \qquad \text{for all } x \in [b, \infty),$$

we have by the Chartier-Dirichlet Test (2.6), that $\varphi_1(x)e^{-ixs}$, $\varphi_2(x)e^{-ixs} \in HK([b, \infty))$. Thus $f(x)e^{-ixs} \in HK([b, \infty))$.
In the same way we can to prove that $f(x)e^{-ixs} \in HK((-\infty, a])$. ■

4.2 Continuity

We know that the continuity of the Lebesgue-Fourier transform, on $\mathbb{R}$, is consequence of the dominated convergence Theorem and that the Lebesgue integral is absolute. Now for to prove the continuity of the Henstock-Fourier transform we don't can use the same arguments, because the Henstock - Kurzweil integral is not absolute.

Theorem 4.2. *(Mendoza et al., 2009) [Theorem 4.1] Let f be a function with support in a compact interval such that $f \in HK(\mathbb{R})$. Then $\widehat{f}$ is continuous on $\mathbb{R}$.*

Proof: We consider $[a, b] \subseteq \mathbb{R}$ such that $f(x) = 0$ for all $x \in \mathbb{R} \setminus [a, b]$. Take $t \in \mathbb{R}$ and let $\{t_n\}_{n\in\mathbb{N}} \subseteq (t-1, t+1)$ such that $t_n \to t$. For every $n \in \mathbb{N}$, define $\alpha_n(x) = e^{-ixt_n}$. Then

$$\lim_{n\to\infty} \alpha_n(x) \;=\; \lim_{n\to\infty} e^{-ixt_n} \;=\; e^{-ixt} \qquad \text{for all } x \in [a, b].$$

On the other hand, for every $n \in \mathbb{N}$, α_n is of bounded variation on $[a, b]$ and $V_{[a,b]}\alpha_n \leq 2\max\{|t-1|, |t+1|\}(b-a)$.
Thus according to (Talvila, 1999) [Corollary **3.2**],

$$\lim_{n\to\infty} \int_a^b f(x)e^{-ixt_n}dx \;=\; \lim_{n\to\infty} \int_a^b f(x)\alpha_n(x)dx \;=\; \int_a^b f(x)e^{-ixt}dx.$$

Hence $\lim_{n\to\infty} \widehat{f}(t_n) \;=\; \widehat{f}(t)$. ■

Theorem 4.3. *(Mendoza et al., 2009) [Theorem 4.2] If $f \in HK(\mathbb{R}) \cap BV(\,[\pm\infty]\,)$, then $\widehat{f}$ is continuous on $\mathbb{R} \setminus \{0\}$.*

Proof: Let $t_0 \in \mathbb{R} \setminus \{0\}$ and consider $a < 0$ and $b > 0$ such that $f \in BV(-\infty, a] \cap BV[b, \infty)$. If we show that $\widehat{f\chi_{(-\infty,a]}}$, $\widehat{f\chi_{[a,b]}}$ and $\widehat{f\chi_{[b,\infty)}}$ are continuous in t_0, then $\widehat{f}$ is continuous in t_0, because

$$\widehat{f}(t) = \widehat{f\chi_{(-\infty,a]}}(t) + \widehat{f\chi_{[a,b]}}(t) + \widehat{f\chi_{[b,\infty)}}(t) \quad \text{for all } t \in \mathbb{R}.$$

By the Theorem 4.2, $\widehat{f\chi_{[a,b]}}$ is continuous in t_0. To prove that $\widehat{f\chi_{(-\infty,a]}}$ and $\widehat{f\chi_{[b,\infty)}}$ are continuous in t_0 we will use (Talvila, 2002) [Proposition **6(a)**]. The conditions f is Henstock - Kurzweil integrable on $\mathbb{R}$ and f is of bounded variation on $(-\infty, a] \cup [b, \infty)$ implies that $\lim_{|x|\to\infty} f(x) = 0$. Now since $t_0 \neq 0$, there exists $K > 0$ and $\delta > 0$ such that if $|t - t_0| < \delta$, then $\frac{1}{|t|} < K$. Thus for all $|t - t_0| < \delta$,

$$\left|\int_u^v e^{-ixt}dx\right| \leq \frac{2}{|t|} \;<\; 2K \qquad \text{for all } [u, v] \subseteq \mathbb{R}.$$

Therefore, by (Talvila, 2002) [Proposition **6(a)**], $\widehat{f\chi_{(-\infty,a]}}$ and $\widehat{f\chi_{[b,\infty)}}$ are continuous in t_0. ■

5. The Riemann-Lebesgue lemma

A generalization of the Riemann-Lebesgue Lemma was given, still in the context of the Lebesgue integral, by Bachman (Bachman et al., 1991) [Theorem 4.4.1], assuring that for any $-\infty \leq a < b \leq \infty$,

$$\lim_{|s|\to\infty} \int_a^b h(xs)f(x)dx = 0, \qquad \text{for each } f \in L^1(\mathbb{R}), \tag{12}$$

provided $h : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function satisfying

$$\lim_{|r|\to\infty} \frac{1}{r} \int_0^r h(s)ds = 0.$$

In this section, we show a similar generalization for the Henstock-Fourier transform. In the case of a compact interval, Talvila (Talvila, 2001) showed that the Fourier transform $\hat{f}$ of a function $f \in HK(I) \setminus L^1(I)$ has the asymptotic behavior:

$$\hat{f}(s) = o(s), \text{ as } |s| \to \infty.$$

Titchmarsh (Titchmarsh, 1999) proved it is the best possible approximation for improper Riemann integrable functions. Next, we show too a generalization from this result for the Henstock-Fourier transform.

5.1 The case of a compact interval

The following theorem implies as a corollary the result in (Talvila, 2001).

Theorem 5.1. *(Mendoza et al., 2010) Let $[a,b]$ be a compact interval. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable with bounded derivative, and such that $\varphi(w) - \varphi(0) = o(w)$, as $|w| \to \infty$. Then,*

$$\int_a^b \varphi(wt)f(t)dt = o(w) \quad \text{as } |w| \to \infty,$$

for each $f \in HK([a,b])$.

Proof: For $w \in \mathbb{R}$, we define $\varphi_w : \mathbb{R} \to \mathbb{R}$ with $\varphi_w(t) = \varphi(wt)$. Moreover, $F(x) := \int_a^x f(t)dt$. Being F continuous and φ' a bounded measurable function, then $F\varphi'^1([a,b]) \subset HK([a,b])$. Also, $f \in HK([a,b])$ and $\varphi_w \in BV([a,b])$, implying $f\varphi_w \in HK([a,b])$. Furthermore, from the Multiplier Theorem (2.3),

$$\int_a^b f(t)\varphi_w(t)dt = F(b)\varphi_w(b) - \int_a^b F(t)\frac{d\varphi_w(t)}{dt}dt.$$

Therefore, for $w \neq 0$,

$$\left| \frac{1}{w}\int_a^b f(t)\varphi(wt)dt \right| \leq \left| \frac{F(b)\varphi(wb)}{w} \right| + \left| \int_a^b F(t)\varphi'(wt)dt \right|. \tag{13}$$

The Fundamental Theorem I (2.1), and the hypotheses for φ imply

$$\lim_{|w|\to\infty} \frac{1}{w}\int_0^w \varphi'(t)dt = \lim_{|w|\to\infty} \frac{\varphi(w) - \varphi(0)}{w} = 0.$$

In consequence,

$$\lim_{|w|\to\infty} \frac{F(b)\varphi(wb)}{w} = 0. \tag{14}$$

Seeing also that $F \in L^1([a,b])$, it follows that equation (12) is valid with f and h substituted for F and φ', respectively. This together with equations (13) and (14) give the result. ■

A direct consequence of the previous theorem is the result of Talvila (Talvila, 2001).

Corollary 5.1. *Let $[a,b]$ be a compact interval. For each $f \in HK([a,b]) \setminus L^1([a,b])$ the Fourier transform has the asymptotic behavior $\hat{f}(s) = o(s)$, as $|s| \to \infty$.*

5.2 The unbounded interval case

Theorem 5.2. *(Mendoza et al., 2010) Let $\varphi \in HK_{loc}(\mathbb{R})$ be fixed. Suppose in addition that $\Phi(x) = \int_0^x \varphi(t)dt$ is bounded on $\mathbb{R}$. Then, for each $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$,*

$$\lim_{|w|\to\infty} \int_{-\infty}^{\infty} f(t)\varphi(wt)dt = 0.$$

Proof: For $\omega \in \mathbb{R}$, we define $\varphi_w(t) = \varphi(wt)$. Since $\varphi \in HK_{loc}(\mathbb{R})$ then φ and φ_w are in $HK([0,b]$, for $b > 0$. Because $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, f is the sum of two monotone functions with limit 0 in infinity. As Φ is bounded in $[0,\infty)$, by the above and from the Chartier-Dirichlet Test (2.6), we have that $f\varphi_w \in HK([0,\infty])$.
For $w \neq 0$, $\Phi_w(t) = (1/w)\Phi(wt)$ is a primitive of φ_w, bounded and continuous on $[0,\infty)$. Because $f \in BV([0,b])$, for $b > 0$, it follows from the Multiplier Theorem (2.3) that

$$\int_0^b f(t)\varphi(wt)dt = \frac{f(b)}{w}\Phi(wb) - \frac{1}{w}\int_0^b \Phi(wt)df(t) \tag{15}$$

The hypotheses for φ imply that $|\Phi(x)| \leq M$, for each $x > 0$, for some constant M.
Now we use theorems (Rudin, 1987) [Theorem 3.8] and Theorem 2.7 to obtain,

$$\left|\int_0^b \Phi(wt)df(t)\right| \leq MV(f;[0,b]),$$

implying, from (15), that

$$\left|\int_0^b f(t)\,\varphi(\omega t)\,dt\right| \leq \frac{M}{|\omega|}\left(|f(b)| + V(f;[0,b])\right). \tag{16}$$

Since $f \in HK([0,\infty)) \cap BV([0,\infty))$, $\lim_{b\to\infty} V(f;[0,b])) = V(f;[0,\infty])$ and $\lim_{b\to\infty} f(b) = 0$. From (16) and Hake's Theorem (2.5) it follows that

$$\left|\int_0^\infty f(t)\varphi(wt)dt\right| \leq \frac{M}{|w|}V(f;[a,\infty].$$

Taking $|w| \to \infty$, we get

$$\lim_{|w|\to\infty} \int_0^{\infty} f(t)\varphi(wt)dt = 0.$$

A similar argument is valid for the interval $[-\infty,0]$, which yield the result. ■
The trigonometric functions $sin(t)$ and $cos(t)$ obeys the hypotheses the Theorem 5.2. Thus, the result of Mendoza-Escamilla-Sánchez (Mendoza et al., 2009) is a particular case of this theorem.

Corollary 5.2. *For each $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, $\lim_{|s|\to\infty} \hat{f}(s) = 0$.*

6. The Dirichlet-Jordan theorem

A fundamental problem for the Fourier Transform is its pointwise inversion, which means to recover the function at given points from its Fourier transform. As is known, the Dirichlet-Jordan Theorem in $L(\mathbb{R})$ solves the pointwise inversion for functions of bounded variation. This theorem tells us that if $f \in L(\mathbb{R}) \cap BV(\mathbb{R})$ then, for each $x \in \mathbb{R}$,

$$\lim_{M\to\infty}\frac{1}{2\pi}\int_{-M}^{M}e^{ixs}\widehat{f}(s)ds=\frac{1}{2}\{f(x+0)+f(x-0)\}. \tag{17}$$

We demonstrate a similar result to (17) for the Henstock-Fourier transform at $HK(\mathbb{R})\cap BV(\mathbb{R})$. We will use the Sine Integral function, wich is defined as $Si(x)=\frac{2}{\pi}\int_0^x\frac{\sin t}{t}dt$, and has the properties:

1. $Si(0)=0$, $\lim_{x\to\infty}Si(x)=1$ and
2. $Si(x)\leq Si(\pi)$ for all $x\in[0,\infty]$.
3. If $b>a>0$ and $M>0$, then $\left|\int_a^b\frac{\sin Mt}{t}dt\right|\leq\frac{2}{M}(\frac{1}{a}+\frac{1}{b})$.

Lemma 6.1. *(Mendoza, 2011) Let $\delta>0$. If $f\in HK([\delta,\infty])\cap BV([\delta,\infty])$ then*

$$\lim_{M\to\infty}\int_{\delta}^{\infty}\frac{f(t)}{t}sinMtdt=0$$

Proof: By the Multiplier Theorem (2.3) and the property 3 of the Sine Integral function, it is easy to see that

$$\left|\int_{\delta}^{\infty}\frac{\sin Mt}{t}f(t)dt\right|\leq\frac{2}{M\delta}+\frac{4}{M\delta}V_f([\delta,\infty]).$$

Therefore, making tend M to infinity, we have the result. ■

Lemma 6.2. *(Mendoza, 2011) Let $\delta>0$. If $f\in HK(\mathbb{R})\cap BV(\mathbb{R})$, then*

$$\lim_{\varepsilon\to0}\int_{\delta}^{\infty}f(t)\frac{\sin\varepsilon t}{t}dt=0.$$

Proof: By the Multiplier Theorem 2.3 and by Lemma 3.1, we have

$$\begin{aligned}\left|\int_{\delta}^{\infty}\frac{\sin\varepsilon t}{t}f\right| &\leq \lim_{b\to\infty}\left\{\left|f(b)\int_{\delta}^{b}\frac{\sin\varepsilon t}{t}dt\right|+\left|\int_{\delta}^{b}\left(\int_{\delta}^{u}\frac{\sin\varepsilon t}{t}\right)df\right|\right\}\\ &\leq \left|\int_{\delta}^{\infty}\left(\int_{\delta\varepsilon}^{u\varepsilon}\frac{\sin t}{t}\right)df\right|.\end{aligned}$$

How for each $u\in[a,\infty):\lim_{\varepsilon\to\infty}\int_{\delta\varepsilon}^{u\varepsilon}\frac{\sin t}{t}dt=0$; $\left|\int_{\delta\varepsilon}^{u\varepsilon}\frac{\sin t}{t}dt\right|\leq\pi Si(\pi)$ for all $\varepsilon>0$; and $\pi(Si)(\pi)\in L(df)$, then, by the Lebesgue Dominated Convergence Theorem 2.4, we obtain the result. ■

Lemma 6.3. *(Mendoza, 2011) Suppose that $f\in HK(\mathbb{R})\cap BV(\mathbb{R})$ and $\beta,\gamma\in\mathbb{R}$ are such that $[\beta,\gamma]\cap(\mathbb{R}\setminus\{0\}=[\beta,\gamma]$. For all $s\in[\beta,\gamma]$ we have*

$$\lim_{\substack{a\to-\infty\\ b\to\infty}}\int_{\beta}^{\gamma}e^{ixs}\int_{a}^{b}f(t)e^{-ist}dtds=\int_{\beta}^{\gamma}e^{ixs}\int_{-\infty}^{\infty}f(t)e^{-ist}dt\,ds. \tag{18}$$

Proof: For c fixed, let $\widehat{f}_{cb}(s) = \int_c^b f(t)e^{-ist}dt$, $\widehat{f}_c(s) = \int_c^\infty f(t)e^{-ist}dt$, wich are continuous at $\mathbb{R} \setminus \{0\}$. We know that there exists $S > 0$ such that $|f(t)| \leq S$ for all $t \in \mathbb{R}$ and that for any $b > c$: $(V_f([c,b]) \leq (V_f([c,\infty])$ and $f \in L([c,b])$. By the Multiplier Theorem (2.3), for each $s \in [\beta,\gamma]$, we have

$$\begin{aligned}\left|\int_c^b f(t)e^{-ist}dt\right| &\leq \left|f(b)\left\{\frac{e^{-isb}-e^{-isc}}{-is}\right\}\right| + \left|\int_c^b \left\{\frac{e^{-ist}-e^{-isc}}{-is}\right\} df(t)\right| \\ &\leq \frac{2}{|\beta|}\left\{S + \left|\int_c^b df(t)\right|\right\} \\ &\leq \frac{2}{|\beta|}\left\{S + V_f([c,\infty])\right\} = N_c.\end{aligned}$$

The previous inequality tells us that for any $b > c$ and all $s \in [\beta,\gamma]$: $\left|e^{ixs}\widehat{f}_{cb}(s)\right| \leq N_c$. Applying the Theorem of Hake (2.5): $\lim_{b\to\infty}\widehat{f}_{cb}(s) = \widehat{f}_c(s)$. Then, by the Dominated Convergence Theorem 2.4

$$\lim_{b\to\infty}\int_\beta^\gamma e^{ixs}\int_c^b f(t)e^{-ist}dt\,ds = \int_\beta^\gamma e^{ixs}\int_c^\infty f(t)e^{-ist}dt\,ds.$$

To get the result, we conducted a similar process, now taking the interval $[a,c]$ and making tend a to minus infinity. ■

Because we do not know if $e^{ixs}\widehat{f}$ is integrable around 0, our theorem is as follows:

Theorem 6.1 (Dirichlet-Jordan Theorem for $HK(\mathbb{R})$.)**.** *(Mendoza, 2011) If $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$ then, for each $x \in \mathbb{R}$*

$$\lim_{\substack{M\to\infty\\ \varepsilon\to 0}}\frac{1}{2\pi}\int_{\varepsilon<|s|<M} e^{ixs}\widehat{f}(s)ds = \frac{1}{2}\{f(x+0)+f(x-0)\}. \tag{19}$$

In terms of the Henstock-Kurzweil integral, by the Hake's Theorem (2.5), the above expression (19), shall be equal to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixs}\widehat{f}(s)ds = \frac{1}{2}\{f(x+0)+f(x-0)\}.$$

Proof: Suppose that $\delta > 0$ and let $F(x,t) = f(x-t)+f(x+t)$. By the Fubini Theorem for the Lebesgue integral (Apostol, 1974) [Theorem 15.7] at $[-M,-\varepsilon]\times[a,b]$ and $[\varepsilon,M]\times[a,b]$ and by Lemma 6.3

$$\begin{aligned}\int_{\varepsilon<|s|<M} e^{ixs}\int_{-\infty}^{\infty} f(t)e^{-ist}dt\,ds &= \int_{-\infty}^{\infty} f(t)\left(\int_{-M}^{-\varepsilon}+\int_{\varepsilon}^{M}\right)e^{is(x-t)}ds\,dt \\ &= 2\int_0^\infty \frac{F(x,t)}{t}(\sin Mt - \sin\varepsilon t)dt \\ &= 2\int_\delta^\infty \frac{F(x,t)}{t}(\sin Mt - \sin\varepsilon t)dt \\ &+ 2\int_0^\delta \frac{F(x,t)}{t}(\sin Mt - \sin\varepsilon t)dt\end{aligned}$$

In $[\delta, \infty]$ by Lemma 6.1 and Lemma 6.2, we obtain

$$\lim_{\substack{M\to\infty,\\ \varepsilon\to 0}} \int_{\delta}^{\infty} \frac{F(x,t)}{t}(\sin Mt - \sin \varepsilon t)dt = 0. \tag{20}$$

In $[0, \delta]$, the DCT (2.4) implies that

$$\lim_{\varepsilon\to 0} \int_0^{\delta} \frac{F(x,t)}{t} \sin \varepsilon t \, dt = 0. \tag{21}$$

Integrating by parts

$$\begin{aligned} \int_0^{\delta} [F(x,t)] \frac{\sin Mt}{t} dt &= [F(x,\delta)] \left(\int_0^{\delta M} \frac{\sin t}{t} dt \right) \\ &- \int_0^{\delta} \left(\int_0^{tM} \frac{\sin u}{u} du \right) d\left[F(x,t)\right]. \end{aligned}$$

Since $\lim_{M\to\infty} \left(\int_0^{Mt} \frac{\sin u}{u} du \right) = \frac{\pi}{2}$ and applying the CDT (2.4) to the last integral, we infer that

$$\begin{aligned} \lim_{M\to\infty} \int_0^{\delta} [F(x,t)] \frac{\sin Mt}{t} dt &= \frac{\pi}{2} F(x,\delta) \\ &- \frac{\pi}{2} \{(F(x,\delta)) - (F(x,0))\} \\ &= \pi \frac{[f(x-0) + f(x+0)]}{2}. \end{aligned}$$

Combining (20), (21) and the above expression, we obtain the result. ■

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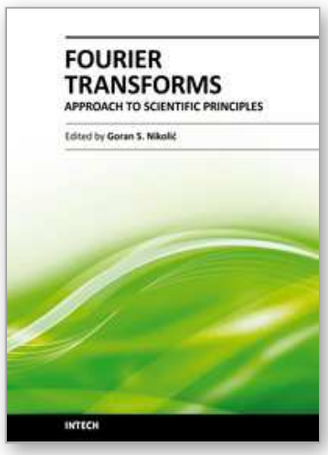

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This book aims to provide information about Fourier transform to those needing to use infrared spectroscopy, by explaining the fundamental aspects of the Fourier transform, and techniques for analyzing infrared data obtained for a wide number of materials. It summarizes the theory, instrumentation, methodology, techniques and application of FTIR spectroscopy, and improves the performance and quality of FTIR spectrophotometers.

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